New Lower Bounds for Orthogonal Graph Drawings *

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Abstract. An orthogonal drawing is an embedding of a graph such that edges are drawn as sequences of horizontal and vertical segments. In this paper we explore lower bounds. We find lower bounds on the number of bends when crossings are allowed, and lower bounds on both the grid-size and the number of bends for planar and plane drawings.

1 Introduction

Orthogonal graph drawings are an important tool for graph layout, e.g. for Data Flow Diagrams or Entity Relationships Diagrams. Two important measurements of the quality of a drawing are the grid-size and the number of bends. Every 4-graph has an orthogonal drawing of grid-size $\mathcal{O}(n) \times \mathcal{O}(n)$ with $\mathcal{O}(n)$ bends. Minimizing the number of bends is $\mathcal{NP}$-complete [6], and so is the question whether a graph can be embedded in a grid of prescribed size [8, 5]. Therefore, one tries to find heuristics where the obtained worst-case sizes are a priori known and small. Different algorithms have been developed, depending on the connectivity and whether the graph is planar or not. See Table 1 for an overview.

	Triconnected		Biconnected		Connected	
	Grid-size	Bends	Grid-size	Bends	Grid-size	Bends
	Nonplanar					
Simple	hp $\frac{7}{4}n-2$ *	$2n+2$ [1]	hp $\frac{7}{4}n-2$ *	$2n+2$ [1]	n [1]	$2n+2$ [1]
Multigraph	$n+1$ [1]	$2n+4$ [1]	$n+1$ [1]	$2n+4$ [1]	$\frac{4}{3}n-1$ [4]	$\frac{8}{3}n+2$ [4]
With Loops	–		$2n-1$ [4]	$2n-1$ [4]	$4n$ [4]	$4n$ [4]
	Plane					
Simple	hp $\frac{4}{3}n+2$ [4]	$\frac{4}{3}n+4$ [4]	n [11]	$2n+2$ [2]	$\frac{6}{5}n+1$ [4]	$\frac{12}{5}n+2$ [4]
Multigraph	$n+1$ [13]	$2n+4$ [14]	$n+1$ [13]	$2n+4$ [14]	$2n-1$ [4]	$4n-2$ [4]
With Loops	–		$2n+1$ [4]	$4n+4$ [4]	$2n+1$ [4]	$4n+4$ [4]

Table 1. Known algorithms. "hp" means that this is a bound on the half-perimeter. We give the (to our knowledge) first citation of each result. The results marked $*$ are by Papakostas and Tollis (private communication).

* A full version of this paper can be found in [3]. This paper was written while the author was visiting TU Berlin and working at Tom Sawyer Software.

To measure the goodness of these algorithms, we want to find graphs which need at least a certain grid-size or at least a certain amount of bends. In this paper we deal with these lower bounds. We summarize our results in Table 2.

Nonplanar Drawings		Triconnected	Biconnected	Connected	Non-Connected
Non-planar	Simple	$\frac{10}{7}n$	$\frac{10}{6}n$	$\frac{11}{6}n$	$\frac{12}{5}n$
Planar	Simple	$\frac{6}{5}n$	$\frac{10}{7}n$	$\frac{11}{7}n$	$2n$
	Multigraph	$\frac{10}{7}n$	$2n$	$\frac{7}{3}n$	$4n$
	With Loops	–	$3n$	$3n$	$6n$

Plane Drawings		Triconnected	Biconnected	Connected	Non-Connected
Simple	Grid-size	$\frac{2}{3}n+1$	$n-1$	$\frac{6}{5}(n-1)-1$	?
	Bends	$\frac{4}{3}n+4$	$2n-2$ *	$\frac{12}{5}(n-1)-2$	?
Multigraph	Grid-size	$\frac{2}{3}(n-2)+3$	$n+1$	$2n-3$	$2n-1$
	Bends	$\frac{4}{3}(n-2)+8$	$2n+4$ *	$4n-6$	$4n$
With Loops	Grid-size	–	$n+2$	$2n+1$	$4n-1$
	Bends	–	$3n$	$4n+4$	$6n$

Table 2. Lower bounds for orthogonal drawings. "–" means that this case is impossible. "?" means that we didn't find lower bounds better than for the connected case. The results marked * were already discovered by [15].

2 Definitions

Let G be a graph with n vertices. We always assume that G is a *4-graph*, i.e. it has a maximum degree of 4. G is called *4-regular* if every vertex has degree 4. By *subdivision* of an edge e we understand that we delete e, add a new vertex, and connect it with the two endpoints of e. Edges of the form (v, v) (*loops*) are not necessarily forbidden. Also, two vertices may be connected by more than one edge (*multiple edge*). Graphs without loops and multiple edges are called *simple*, graphs without loops, but possibly with multiple edges, are called *multigraphs*.

G is called *connected* if for any two vertices there is a path between them. It is called *biconnected* if for any vertex v the graph $G - \{v\}$ is connected. It is called *triconnected* if for any two vertices v, w the graph $G - \{v, w\}$ is connected. A triconnected 4-graph with more than three vertices can never have a loop.

A graph is called *planar* if it has a drawing without crossing (*planar drawing*). This defines a circular ordering of the edges incident to a vertex v (*combinatorial embedding*). A planar drawing splits the plane into different components, called *faces*. The unbounded component is called the *outerface*. The combinatorial embedding defines a planar drawing which is unique except for the choice of the outerface. A planar graph is called *plane* if both a combinatorial embedding and the outerface are specified.

An *(orthogonal) drawing* of G is an embedding of G in the plane such that all edges are drawn as sequences of horizontal and vertical line segments. It is called *planar* if no drawings of edges intersect. It is called *plane* if it is planar, if G was a plane graph, and the drawing exactly reflects the given embedding and the outerface. A point where the drawing of an edge changes its direction is called a *bend* of this edge.

A column (row) of the drawing is called *vertex-used* if it contains a vertex, *line-used* if it contains a vertical (horizontal) part of an edge, and *used* if it is vertex-used or line-used. The *width* of the drawing is the number of used columns minus 1, the height is the number of used rows minus 1. A drawing with width n_1 and height n_2 has *grid-size* $n_1 \times n_2$, *half-perimeter* $n_1 + n_2$, and *area* $n_1 \cdot n_2$.

3 Lower Bounds for Non-Planar Drawings

Very little is known about lower bounds for the grid-size. There exist 4-graphs with crossing number $\Omega(n^2)$ which therefore need a grid of the same area [16]. Leighton [9] proved that the planar tree of meshes needs $\Omega(n \log n)$ area in any orthogonal drawing. In both cases the constants are very small.

We deal here with the number of bends, and develop various lower bounds, depending on the connectivity of the graph. The only known results are for simple biconnected graphs: Storer [11] showed a lower bound of $\frac{8}{7}n$ bends, and Papakostas and Tollis improved it to $\frac{8}{5}n$ (private communication).

3.1 Lower Bounds for Small Graphs

We use the following special graphs: The *complete graph* K_5, the *octahedron* O, the *quadruple edge graph* Q, and the *double loop* L, which are shown below. We develop a few easy lemmas to get lower bounds for these graphs.

Fig. 1. K_5, the octahedron, the quadruple edge, and the double-loop.

Lemma 1. *In a drawing of a 4-regular graph, every used column has two bends. Formulated differently: if a drawing of a 4-regular graph has width w, then there are at least $2w + 2$ bends.*

Proof. If a column is vertex-used, let u be the top vertex and w be the bottom vertex in the column. By 4-regularity all connections at u and w are used. So the top connection from u must have a bend, and so must the bottom connection of w. If a column is line-used only, every begin- and endpoint of a line is a bend. Hence we have at least two bends.

In a drawing of width w there are $w+1$ used columns (by definition of width), and therefore at least $2w + 2$ bends by the above. □

Lemma 2. *A 4-regular graph has $2\lceil\sqrt{n}\rceil + 4$ bends in any drawing.*

Proof. We only sketch this proof. It is clear that we need at least either $\lceil\sqrt{n}\rceil$ columns or $\lceil\sqrt{n}\rceil$ rows to accomodate the vertices. Since we have a 4-regular graph, we need two more rows and two more columns at the extreme ends. This proves the claim together with Lemma 1. □

Lemma 3. *A simple 4-regular graph has at least 12 bends in any drawing.*

Proof. The proof is an easy (but lengthy) case analysis to show that in any embedding we must have at least 6 rows or 6 columns. We then have at least 12 bends by Lemma 1 (or analogously for rows). We skip this for brevity. □

These two lemmas imply that L needs 6 bends, Q needs 8 bends, and K_5 and O need 12 bends in any drawing.

3.2 Constructing bigger graphs

The lower bounds for non-connected graphs are easy to get by taking many copies of O, K_5, Q, and L, respectively. To obtain bigger connected graphs, we need to study how subdividing an edge changes the lower bound.

Lemma 4. *Subdividing an edge lowers the lower bound on the bends by at most 1.*

Proof. Assume G needs b bends, and after subdividing one edge, we get G_s. Let G_s be drawn with c bends. If we remove the vertex that came from subdividing, this adds at most one bend, so we get a drawing of G with $c + 1$ bends. Consequently $c \geq b - 1$. □

Theorem 5. *There are the following lower bounds for connected graphs:*
1. *planar simple graphs: $\frac{11}{7}n$ bends*
2. *simple graphs: $\frac{11}{6}n$ bends*
3. *planar multigraphs: $\frac{7}{3}n$ bends*
4. *planar graphs with loops: $3n$ bends*

Proof. We demonstrate only case (1) in detail, the other cases are analogous. Take an octahedron and subdivide one edge. We get a graph with 7 vertices that by Lemma 4 needs 11 bends. Take k copies and connect the vertices of degree 2. This graph is connected, has $7k$ vertices and needs $11k = \frac{11}{7}n$ bends. □

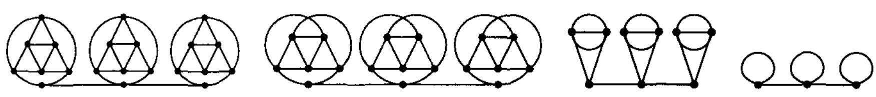

Fig. 2. Connected graphs that need many bends in any drawing.

Theorem 6. *There are the following lower bounds for biconnected graphs:*

1. *planar simple graphs:* $\frac{10}{7}n$ *bends*
2. *simple graphs:* $\frac{10}{6}n$ *bends*
3. *planar multigraphs:* $\frac{6}{3}n = 2n$ *bends*
4. *planar graphs with loops:* $3n$ *bends*

Proof. Again we demonstrate only case (1). Take an octahedron and subdivide two edges on the outerface. We get a graph with 8 vertices that by Lemma 4 needs 10 bends. Take k copies and identify the vertices of degree 2. This graph is biconnected, has $7k$ vertices and needs $10k = \frac{10}{7}n$ bends. □

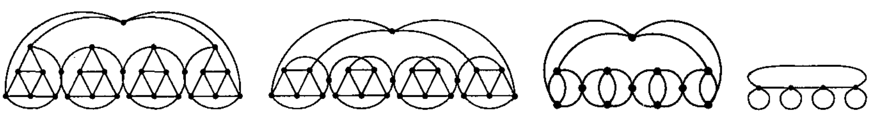

Fig. 3. Biconnected graphs that need many bends in any drawing.

Theorem 7. *There are the following lower bounds triconnected graphs:*

1. *planar simple graphs:* $\frac{6}{5}n$ *bends.*
2. *simple graphs:* $\frac{18}{13}n$ *bends.*
3. *planar multigraphs:* $\frac{18}{13}$ *bends.*

Proof. Again we demonstrate only case (1). Take an octahedron and subdivide the three edges on the outerface. We get a graph with 9 vertices that by Lemma 4 needs 9 bends. Take $2k$ copies and identify the vertices of degree 2 in such a way that the resulting graph is planar and triconnected (see also Fig. 4). The graph then has $(6 + \frac{3}{2})2k = 15k$ vertices and needs $9 \cdot 2k = \frac{18}{15}n = \frac{6}{5}n$ bends. □

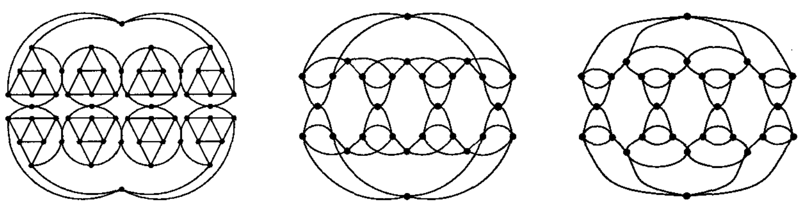

Fig. 4. Triconnected graphs that need many bends in any drawing.

4 Lower Bounds for Planar Drawings

4.1 Plane graphs

Assume after taking away all edges on the outerface of a plane graph H we get the graph G. Then we say that H *contains* G *on the inside.*

Lemma 8. *Let H contain G on the inside, where G has minimum degree 2. If H can be drawn in a $w \times h$-grid then G can be drawn in a $(w-2) \times (h-2)$-grid.*

Proof. Consider a drawing Γ_H of H in a $w \times h$-grid. Deleting the edges on the outerface of H we get a drawing Γ_G of G. Since G has minimum degree 2, the highest row of Γ_G must be used by a horizontal line. This line belongs to an edge e on the outerface of G. Since G is on the inside of H, e is not on the outerface of H. Γ_H reflects the embedding of H, so we must have a line drawn above e in Γ_H. Therefore Γ_H has at least one more unit in top-direction. The same holds for the other three directions. □

Triconnected graphs Kant showed lower bounds of $\frac{2}{3}(n-2)+2$ on the grid-size and $\frac{4}{3}(n-2)+2$ bends for triconnected plane graphs. We improve this slightly with the following graph class:

Definition 9. Define the graph classes $\{T'_i\}$ and $\{T_i\}$ as follows:

- T'_1 is a 3-cycle.
- T'_i is obtained by taking a copy of T'_{i-1}, and adding three vertices in the outerface. Then we add a 6-cycle between the three new vertices and the three vertices of degree 2 of T'_{i-1}, such that T'_i contains T'_{i-1} on the inside.
- T_i is obtained by taking a copy of T'_i and adding a 3-cycle between the three vertices of degree 2 of T'_i such that T_i contains T'_i on the inside.

See Fig. 5 for an illustration of this graph.

Lemma 10. *T_i needs a width and height of $2i+1$ in any plane drawing.*

Proof. We first show a lower bound for T'_i, namely, it needs a width of $2i-1$. This is shown by induction on i. Since T'_1 is a triangle, it needs a 1×1-grid. Now consider T'_i, $i \geq 2$. By construction it contains T'_{i-1} on the inside, so if we could embed T'_i with width less than $2i+1$, then by Lemma 8 we could embed T'_{i-1} in a grid of width less than $2i-1 = 2(i-1)+1$, a contradiction.

Now finally consider T_i. By construction it contains T'_i on the inside, so it needs two more units in width than T'_i, which gives a lower bound of $2i+1$ on the width. The proof for the height is similar. □

Lemma 11. *T_i needs $4i+4$ bends in any plane drawing.*

Proof. This is trivial: T_i is 4-regular and has a width of $2i+1$ in any drawing. By Lemma 1 it therefore must have $2(2i+1)+2 = 4i+4$ bends in any drawing. □

Theorem 12. *There are the following lower bounds for plane triconnected graphs:*

- *simple graphs:* $(\frac{2}{3}n+1) \times (\frac{2}{3}n+1)$*-grid and* $\frac{4}{3}n+4$ *bends.*
- *multigraphs:* $(\frac{2}{3}(n-2)+3) \times (\frac{2}{3}(n-2)+3)$*-grid and* $\frac{4}{3}(n-2)+8$ *bends.*

Proof. We are done in the simple case, since T_i has $n = 3i$ vertices. Obtain graph $\hat{T}_i$ as follows: subdivide two different edges on the outerface of T_i, and add a double edge between the two new vertices, such that the double edge encloses T_i. Some calculation shows that $\hat{T}_i$ has $3i+2$ nodes, needs a width and height of $2i+3$, and $4i+8$ bends. This proves the claim for multigraphs. □

Biconnected graphs Storer showed a lower bound of $n-2$ on the grid-size of biconnected simple graphs [11]. Tamassia, Tollis, and Vitter showed a lower bound of $2n+4$ bends for multigraphs and $2n-2$ bends for simple graphs [15]. We use their graph class to show a slightly better bound on the grid-size.

Definition 13. Define the graph classes $\{B_i'\}$ and $\{B_i\}$ as follows:

- B_1' is a double edge.
- B_i' is obtained by taking a copy of B_{i-1}', adding two vertices in the outerface, and adding a 4-cycle alternating between the two new vertices and the two vertices of degree 2 of B_{i-1}', such that B_i' contains B_{i-1}' on the inside.
- B_i is obtained by taking a copy of B_i' and adding a double edge between the two vertices of degree 2 of B_i' such that B_i contains B_i' on the inside.

The following lemma is proved exactly as in Lemma 10 and Lemma 11. The second claim was known before [15], though proved by different means.

Lemma 14. B_i *needs a width and height of* $2i+1$ *and* $4i+4$ *bends.*

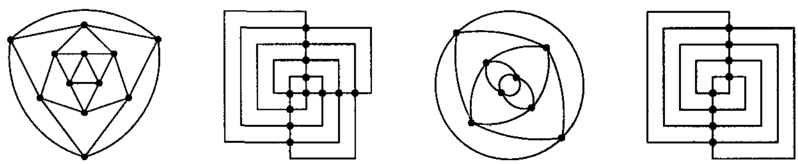

Fig. 5. T_4 and B_4, both drawn optimally.

Theorem 15. *There are the following lower bounds for plane biconnected graphs:*

- *simple graphs:* $(n-1) \times (n-1)$*-grid and* $2n-2$ *bends*
- *multigraphs:* $(n+1) \times (n+1)$*-grid and* $2n+4$ *bends*
- *graphs with loops:* $(n+2) \times (n+2)$*-grid and* $3n$ *bends*

Proof. For multigraphs we are done by Lemma 14 since B_i has $2i$ vertices.

For simple graphs, let $\bar{B}_i$ be the graph obtained from B_i by subdividing one of each of the double edges. Some calculation shows that $\bar{B}_i$ has $2i + 2$ vertices, needs a height and width of $2i + 1$ and $4i + 2$ bends in any drawing.

For graphs with loops, let $\hat{B}_i$ be the graph obtained from B_i by subdividing one edge on the outerface, and adding a loop incident to the new vertex, such that the loop encloses B_i. Some calculation shows that $\hat{B}_i$ has $2i+1$ vertices, and needs a width and height of $2i + 3$ in any drawing. This proves the lower bound on the grid-size. For the number of bends, the claim was shown in Theorem 6. □

Connected graphs The known lower bounds for connected plane graphs are $\frac{8}{3}(n-2)$ bends for multigraphs, and $4(n-2)$ bends for graphs with loops [15]. We improve these bounds and develop new ones for simple graphs.

Definition 16. Define the graph classes $\{C'_i\}$ and $\{C_i\}$ as follows:
- C'_1 is a loop.
- C'_i is obtained by taking a copy of C'_{i-1}, adding one vertex in the outerface, and adding a double edge between the new vertex and the vertex of degree 2 of C'_{i-1}, such that C'_i contains C'_{i-1} on the inside.
- C_i is obtained by taking a copy of C'_i and adding a loop at the vertex of degree 2 of C'_i such that C_i contains C'_i on the inside.

See Fig. 6 for an illustration of this graph class. The following lemma is proved exactly as in Lemma 10 and Lemma 11.

Lemma 17. *C_i needs a width and height of $2i + 1$ and $4i + 4$ bends.*

Definition 18. The graph class CM_i is essentially defined as the graph class C_i, with the exception that both loops are replaced by a quadruple edge with one edge subdivided. See also Fig. 6.

The following lemma is proved similar as in Lemma 10 and Lemma 11.

Lemma 19. *CM_i needs a width and height of $2i + 5$ and $4i + 12$ bends.*

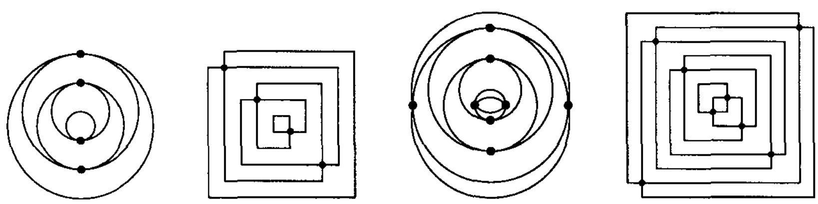

Fig. 6. C_4 and CM_4, both drawn optimally.

Definition 20. Define the graph classes $\{CS_i\}, \{CS'_i\}$ and $\{CS''_i\}$ as follows:

- CS''_1 is a 3-cycle.
- CS'_i $(i \geq 1)$ is obtained by taking a copy of CS''_i and adding two vertices in the outerface. Then we add a 4-cycle alternating between the two new vertices and two vertices of degree 2 of CS''_i, such that CS'_i contains CS''_i on the inside.
- CS_i $(i \geq 1)$ is obtained by taking a copy of CS'_i and adding one vertex in the outerface. Then we add a 3-cycle between this new vertex and the two vertices of degree 2 of CS'_i, such that CS_i contains CS'_i on the inside.
- CS''_{i+1} $(i \geq 1)$ is obtained by taking a copy of CS_i and adding two vertices in the outerface. Then we add a 3-cycle between the two new vertices and the vertex of degree 2 of CS_i, such that CS''_{i+1} contains CS_i on the inside.

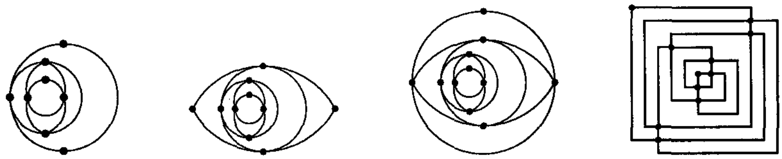

Fig. 7. CS''_2, CS'_2, and CS_2; and an optimal drawing.

Lemma 21. *CS_i needs a width and height of $6i - 1$ in any plane drawing.*

Proof. We show only the width by induction on i; showing also a lower bound for CS''_i of $6i - 5$ and a lower bound for CS'_i of $6i - 3$. CS''_1 is a triangle that needs a width of $1 = 6 \cdot 1 - 5$. Assume the claim was shown for CS''_i.

CS'_i contains CS''_i on the inside and needs two more units in width than CS''_i, which gives a lower bound of $6i - 3$. CS_i contains CS'_i on the inside and needs two more units in width than CS'_i, which gives a lower bound of $6i - 1$. Finally, CS''_{i+1} contains CS_i on the inside and needs two more units in width than CS_i, which gives a lower bound of $6i + 1 = 6(i + 1) - 5$. □

Lemma 22. *CS_i needs $12i - 2$ bends in any plane drawing.*

Proof. CS_i has two vertices of degree 2. If we remove those and connect their neighbors by an edge, we get a 4-regular graph CS^*_i, which needs the same width as CS_i, $6i - 1$. By Lemma 1 CS^*_i therefore needs $12i$ bends. CS_i results from CS^*_i by subdividing two edges, so by Lemma 4 CS_i needs $12i - 2$ bends. □

Theorem 23. *There are the following lower bounds for plane connected graphs:*

- *graphs with loops: $(2n + 1) \times (2n + 1)$-grid and $4n + 4$ bends.*
- *multigraphs: $(2n - 3) \times (2n - 3)$-grid and $4n - 4$ bends.*
- *simple graphs: $(\frac{6}{5}(n - 1) - 1) \times (\frac{6}{5}(n - 1) - 1)$-grid and $\frac{12}{5}(n - 1) - 2$ bends.*

Proof. One shows easily that C_i has i vertices, CM_i has $i + 4$ and CS_i has $5i + 1$ vertices. The results follow with Lemma 17, 19, 21 and 22. □

Non-connected graphs For brevity we skip the definition and proofs for non-connected graphs. For simple graphs we did not find graphs with better lower bounds than those for connected graphs.

Fig. 8. Non-connected 4-graphs which need a big grid and many bends.

Theorem 24. *There are the following lower bounds for plane graphs:*
- *graphs with loops:* $(4n-1) \times (4n-1)$*-grid and* $8n$ *bends.*
- *multigraphs:* $(2n-1) \times (2n-1)$*-grid and* $4n$ *bends.*

4.2 Combinatorial embedding can be chosen

To the author's knowledge no research has been done into lower bounds for planar drawings. We provide some results here which are close to optimality in the number of bends.

Triconnected Graphs For triconnected planar graphs there exists only one combinatorial embedding. Therefore, if we consider all possible choices of the outerface of T_i, then we get a lower bound for any planar drawing of T_i.

Lemma 25. *T_i needs a width and height of i and $4i-2$ bends in any planar drawing.*

Proof. Assume that T_1' had the vertices $\{v_1, v_2, v_3\}$ and that we obtained T_i' by adding the vertices $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$. Let Γ be the embedding of T_i defined in Definition 9, this induces an embedding of T_i'. We know that T_i' in this embedding needs a width of $2i-1$, and can also show that it needs $4i-3$ bends. Assume we are given some planar orthogonal drawing of T_i, this induces a planar embedding Γ'. The outerface of Γ' can have degree 3 or 4.

If the degree is 3, then we may assume that the outerface of Γ' is either the outerface of Γ, or one of the three faces adjacent to it. After deletion of the edges (v_{3i}, v_{3i-1}), (v_{3i-1}, v_{3i-2}), (v_{3i-2}, v_{3i}) we have a copy of T_i' embedded as in Γ. So we need a width and height of $2i-1 \geq i$ and $4i-3$ bends of T_i'. The three deleted edges form a triangle and need a bend, so we get a lower bound of $4i-2$ bends.

If the degree is 4, we assume after possible renumbering that the outerface is $\{v_{3j-3}, v_{3j-1}, v_{3j}, v_{3j+1}\}$ for some $1 < j < i$. Splitting the graph at v_{3j-2}, v_{3j-1} and v_{3j} we get a copy of T_j' and a copy of T_{i-j+1}', embedded as in Γ. So the number of bends in this embedding is at least $4j-3+4(i-j+1)-3 = 4i-2$. At the very least the width and height of the drawing must be $\min\{2j-1, 2(i-j+1)-1\}$ which is smallest for $j = \frac{i+1}{2}$ and then equals i. □

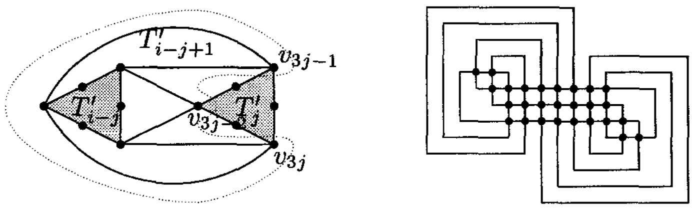

Fig. 9. T_i embedded with outerface-degree 4 leads to a bend-optimal drawing.

Theorem 26. *We have a lower bound of an $\frac{n}{3} \times \frac{n}{3}$-grid and $\frac{4}{3}n - 2$ bends for planar drawings of triconnected graphs.*

Biconnected graphs For biconnected graphs the combinatorial embedding is not unique. However, even though B_i has many different combinatorial embeddings, all give the same planar drawing, except for possible renaming of the vertices and the choice of the outerface. Therefore, by considering different choices of the outerface, we get a lower bound for planar drawings of B_i.

We do not explain the details here, and leave it to the reader to show that B_i needs an $i \times i$-grid and $4i$ bends in any planar drawing. We can then again go over to $\bar{B}_i$ to get the lower bounds for simple graphs.

Theorem 27. *There are the following lower bounds for planar biconnected graphs:*
- *multigraphs: $\frac{n}{2} \times \frac{n}{2}$ and $2n$ bends.*
- *simple graphs: $(\frac{n}{2} - 1) \times (\frac{n}{2} - 1)$ and $2n - 6$ bends.*

5 Remarks and Open Problems

In this paper we have considered lower bounds: for the number of bends in the non-planar case and on both the number of bends and the grid-size in the planar and plane case. Various results have been proved, which either give completely new lower bounds or considerably improved the old ones.

For plane graphs, the results are almost optimal, and the difference is only a small constant, if at all. For planar graphs, the results are fairly good in terms of the number of bends, but improvement should be possible for the grid-size.

Much work remains to be done for non-planar drawings. For the number of bends, there is a small gap in the factor between the lower and the upper bound. No algorithm is known that draws a planar graph with fewer bends if we allow for crossings. We suspect that such an algorithm should be possible.

An even bigger problem are lower bounds on the grid-size for non-planar drawings. The current proofs give only a fairly small constant. It would be also interesting to see more techniques for proving lower bounds on the grid-size of non-planar drawings.

Finally, we would like to pose the open problem of lower bounds for graphs of higher maximum degree. Usually, graphs with higher degree are drawn orthogonally by assigning boxes instead of points to vertices. With such a representation every planar graph can be drawn without bends (see 1D visibility representations [10, 12]). But not all graphs can be drawn without bends: such a drawing is a 2D visibility representation, and can exist only for graphs which are the union of two planar graphs. It would be interesting to see which graphs can be drawn without bends at all, and what are lower bounds for those that can't.

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