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New Masjed Jamei–Type Inequalities for Inverse Trigonometric and Inverse Hyperbolic Functions

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Abstract: In this paper, we establish two new inequalities of the Masjed Jamei type for inverse trigonometric and inverse hyperbolic functions and apply them to obtain some refinement and extension of Mitrinović–Adamović and Lazarević inequalities. The inequalities obtained in this paper go beyond the conclusions and conjectures in the previous literature. Finally, we apply the main results of this paper to the field of mean value inequality and obtain two new inequalities on Seiffert-like means and classical means.

Keywords: conjectures; Bernoulli number and Euler number; inverse trigonometric functions; inverse hyperbolic functions; Seiffert-like means; classical means

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1. Introduction

Masjed Jamei [1] obtained the following inequality

$$(\arctan x)^2 \leq \frac{x \ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \tag{1}$$

which holds for all $x \in (-1,1)$, where $\ln(x+\sqrt{1+x^2}) = \sinh^{-1} x$. In [2], Zhu and Malešević were the first who affirmed Masjed Jamei's conjecture that the inequality (1) holds on the real axis $(-\infty,\infty)$, obtained some natural generalizations of this inequality, and represented a conjecture on a natural approach of Masjed Jamei's inequality, having been inspired by [3–9]. Using flexible analysis tools, Zhu and Malešević [10] obtained a more general concept on the natural approximation of the function $(\arctan x)^2 - \left(x \sinh^{-1} x\right)/\sqrt{1+x^2}$, and proved the above conjecture. In [10], Zhu and Malešević showed the analogue result for $\tanh^{-1} x = (1/2)\ln([(1+x)/(1-x)]$ and $\arcsin x$, which can be formulated as follows: the inequality

$$\left(\tanh^{-1} x\right)^2 \leq \frac{x \arcsin x}{\sqrt{1-x^2}} \tag{2}$$

holds for all $x \in (-1,1)$ with the best power number 2.

Masjed Jamei's inequality topics have attracted the attention of several scholars. For interested readers, please refer to the literature [11–13]. Among them, Chesneau and Bagul [11] and Chen and Malesevic [12] further explored inequalities (1) and (2), respectively. In particular, Chesneau and Bagul [11] recently drew the following conclusion that the inequality

$$\frac{(\sin x)\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \leq (\arctan x)^2 \tag{3}$$

holds for all $x \in (-\pi,\pi)$. In [11], Chesneau and Bagul raised an open question that the above inequality (3) holds for all $x \in (-\infty,\infty)$.

From the above description, we find that the results of the paper [2] give us the upper bound of the function $(\arctan x)^2$ while in [11] the authors were trying to find a lower bound for $(\arctan x)^2$ but the scope of discussion was limited to a small interval,

$(-\pi, \pi)$. In order to compensate the limitation, a conjecture was proposed in [11] that the inequality (3) may be established on $(-\infty, \infty)$.

Generally, the analogue result of the conclusion (3) can be established, that is, the following inequality

$$\frac{(\sinh x)\arcsin x}{\sqrt{1-x^2}} \le \left(\tanh^{-1} x\right)^2 \tag{4}$$

holds for all $x \in (-1, 1)$. In [12], Chen and Malesevic attempted in a new way to find the lower bounds for the function $\left(\tanh^{-1} x\right)^2$, reached the conclusion that the inequality

$$\frac{x \arcsin x}{1-\frac{1}{2}x^2} \le \left(\tanh^{-1} x\right)^2 \tag{5}$$

holds for $x \in (-1, 1)$, and raised an open question that the following inequality

$$\frac{x \arcsin x}{\left(1-\frac{41}{45}x^2\right)^{45/82}} \le \left(\tanh^{-1} x\right)^2 \tag{6}$$

holds for $x \in (-1, 1)$.

We are interested in finding better lower bounds for the functions $(\arctan x)^2$ and $\left(\tanh^{-1} x\right)^2$. A natural idea is about these three functions $\arctan x$, $\ln(x + \sqrt{1+x^2})$ and $\sqrt{1+x^2}$: what kind of structure will make them connected closely? A similar problem also relates to those three functions $\tanh^{-1} x$, $\arcsin x$ and $\sqrt{1-x^2}$. This paper will first discuss the lower bounds of the two functions $(\arctan x)^2$ and $\left(\tanh^{-1} x\right)^2$ from the development logic of mathematics itself, and then naturally apply the corresponding conclusions to the mean value theory.

The rest of the paper is arranged as follows: In Section 2, we propose two lemmas and give some concise proofs of the lemmas, named Lemma 2 and Lemma 3. Then using Lemma 2 and Lemma 3, we prove the main results of the paper, namely Theorem 1 and Theorem 2. In Section 3, applying Lemma 2 and Lemma 3, we strengthen the Mitrinovic–Adamović inequality (see [14,15]) and Lazarević inequality (see [16]). In Section 4, we apply the results of Theorems 1 and 2 to obtain new inequalities related to Seiffert-like means and classical means.

2. Main Results

2.1. Lemms

In order to prove the main results of this paper, we need to introduce the following lemmas.

Lemma 1 ([17])**.** *Let $\{a_k\}_{k=0}^{\infty}$ be a non-negative real sequence with $a_m > 0$ and $\sum_{k=m+1}^{\infty} a_k > 0$, and let*

$$S(t) = -\sum_{k=0}^{m} a_k t^k + \sum_{k=m+1}^{\infty} a_k t^k$$

be a convergent power series on the interval $(0, r)$ $(r > 0)$. (i) If $S(r^-) \le 0$ then $S(t) < 0$ for all $t \in (0, r)$. (ii) If $S(r^-) > 0$ then there is the unique $t_0 \in (0, r)$ such that $S(t) < 0$ for $t \in (0, t_0)$ and $S(t) > 0$ for $t \in (t_0, r)$.

Lemma 2. *The inequality*

$$\arctan(\sinh t) > \frac{t}{(\cosh t)^{1/3}} \tag{7}$$

holds for all $t > 0$ with the best constant $1/3$.

Proof. Let us prove Lemma 2 in two steps.

(1) Let

$$F(t)=\arctan(\sinh t)-\frac{t}{(\cosh t)^{1/3}},\ t>0.$$

Then

$$F'(t)=\frac{1}{3}\times\frac{3(\cosh t)^{1/3}-(3\cosh t-t\sinh t)}{(\cosh t)^{4/3}}.$$

We shall prove that $F'(t)>0$ for all $t>0$, which is equivalent to the following inequality

$$(\cosh t)^{1/3}>\cosh t-\frac{1}{3}t\sinh t=\frac{1}{3}(3\cosh t-t\sinh t):=\frac{1}{3}r(t) \tag{8}$$

which holds for $t>0$. Since

$$\sinh t=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}t^{2n+1},\ \cosh t=\sum_{n=0}^{\infty}\frac{1}{(2n)!}t^{2n},$$

we have

$$\begin{aligned}
r(t) &= 3\cosh t-t\sinh t=3\sum_{n=0}^{\infty}\frac{1}{(2n)!}t^{2n}-t\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}t^{2n+1}\\
&= 3\sum_{n=0}^{\infty}\frac{1}{(2n)!}t^{2n}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}t^{2n+2}\\
&= 3+3\times\frac{1}{2!}t^2+3\sum_{n=2}^{\infty}\frac{1}{(2n)!}t^{2n}-t^2-\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}t^{2n+2}\\
&= 3+\frac{1}{2}t^2-\sum_{n=2}^{\infty}\frac{1}{(2n-1)!}t^{2n}+\sum_{n=2}^{\infty}\frac{3}{(2n)!}t^{2n}\\
&= 3+\frac{1}{2}t^2-\sum_{n=2}^{\infty}\frac{2n-3}{(2n)!}t^{2n}.
\end{aligned}$$

The coefficients of the power series expansion of the function $r(t)$ change from positive to negative, and $r(\infty)=-\infty$, as known from Lemma 1, there is a unique point ξ on interval $(0,\infty)$ such that $r(\xi)=0$, $r(t)>0$ for $t\in(0,\xi)$ and $r(t)<0$ for $t\in(\xi,\infty)$. So we should only prove that the inequality (8) is true on $(0,\xi)$. Let

$$f(t)=\ln\cosh t-3\ln\left(\cosh t-\frac{1}{3}t\sinh t\right),\ 0<t<\xi.$$

Then $f(0)=0$ and

$$f'(t)=\frac{3t\cosh^2 t-3\cosh t\sinh t-t\sinh^2 t}{(\cosh t)(3\cosh t-t\sinh t)}=\frac{2t+t\cosh 2t-\frac{3}{2}\sinh 2t}{(\cosh t)(3\cosh t-t\sinh t)}.$$

By substituting the power series expansions of the functions $\sinh 2t$ and $\cosh 2t$

$$\sinh 2t=\sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}t^{2n+1},\ \cosh 2t=\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n}$$

into the function $2t + t\cosh 2t - 3(\sinh 2t)/2$, we have

$$\begin{aligned}
2t + t\cosh 2t - \frac{3}{2}\sinh 2t &= 2t + t\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n} - \frac{3}{2}\sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}t^{2n+1} \\
&= 2t + \sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n+1} - \frac{3}{2}\sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}t^{2n+1} \\
&= \sum_{n=2}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n+1} - \frac{3}{2}\sum_{n=2}^{\infty}\frac{2^{2n+1}}{(2n+1)!}t^{2n+1} \\
&= \sum_{n=2}^{\infty}\left[\frac{2^{2n}}{(2n)!} - \frac{3}{2}\frac{2^{2n+1}}{(2n+1)!}\right]t^{2n+1} \\
&= \sum_{n=2}^{\infty}\frac{(2n+1)-3}{(2n+1)!}2^{2n}t^{2n+1} \\
&= \sum_{n=2}^{\infty}\frac{2n-2}{(2n+1)!}2^{2n}t^{2n+1} > 0.
\end{aligned}$$

So $f'(t) > 0$ for $t \in (0, \xi)$. This leads to that $f(t)$ is increasing on $(0, \xi)$. Therefore $f(t) > f(0) = 0$ for $t \in (0, \xi)$. Then the inequality (8) is true, and $F'(t) > 0$ for all $t > 0$. In view of the relation $F(0) = 0$, we have that the fact $F(t) > 0$ holds for all $t > 0$.

(2) The result just proved is equivalent to the fact that the inequality

$$\frac{1}{3} > \frac{\ln\frac{t}{\arctan(\sinh t)}}{\ln(\cosh t)} := A(t)$$

holds for all $t > 0$. In this way, when obtaining $\lim_{t\to 0} A(t) = 1/3$ or $\lim_{t\to\infty} A(t) = 1/3$, we will prove that $1/3$ is the best constant in (8). Considering the fact

$$\begin{aligned}
\lim_{t\to 0} A(t) &= \lim_{t\to 0}\frac{\ln t - \ln\arctan(\sinh t)}{\ln(\cosh t)} = \lim_{t\to 0}\frac{[\ln t - \ln\arctan(\sinh t)]'}{[\ln(\cosh t)]'} \\
&= \lim_{t\to 0}\frac{\frac{1}{t} - \frac{1}{\arctan(\sinh t)}\frac{1}{1+\sinh^2 t}\cosh t}{\frac{\sinh t}{\cosh t}} \\
&= \lim_{t\to 0}\frac{\frac{1}{t} - \frac{1}{\arctan(\sinh t)}\frac{1}{\cosh^2 t}\cosh t}{\frac{\sinh t}{\cosh t}} = \lim_{t\to 0}\frac{\frac{1}{t} - \frac{1}{\arctan(\sinh t)}\frac{1}{\cosh t}}{\frac{\sinh t}{\cosh t}} \\
&= \lim_{t\to 0}\frac{\frac{1}{3}t + o(t)}{t + o(t)} = \frac{1}{3},
\end{aligned}$$

we have completed the proof of the Lemma 2. □

Lemma 3. *The inequality*

$$\frac{t}{(\cos t)^{1/3}} > \tanh^{-1}(\sin t) \tag{9}$$

holds for all $t \in (0, \pi/2)$ with the best constant $1/3$.

Proof. Let us prove Lemma 3 in two steps.

(i) Let

$$G(t) = \frac{t}{(\cos t)^{1/3}} - \tanh^{-1}(\sin t),\ 0 < t < \frac{\pi}{2}.$$

Then

$$G'(t) = \frac{1}{3} \times \frac{3\cos t + t\sin t - 3(\cos t)^{1/3}}{(\cos t)^{4/3}}.$$

We will prove that

$$3\cos t + t\sin t > 3(\cos t)^{\frac{1}{3}}$$

or

$$(3\cos t + t\sin t)^3 > 27\cos t$$

for $t \in (0, \pi/2)$. We calculated that

$$\begin{aligned} k(t) &=: \frac{(3\cos t + t\sin t)^3 - 27\cos t}{\sin^3 t} = 27\frac{\cos^3 t}{\sin^3 t} - 27\frac{\cos t}{\sin^3 t} + t^3 + 9t^2\frac{\cos t}{\sin t} + 27t\frac{\cos^2 t}{\sin^2 t} \\ &= 27\frac{(1-\sin^2 t)\cos t}{\sin^3 t} - 27\frac{\cos t}{\sin^3 t} + t^3 + 9t^2\frac{\cos t}{\sin t} + 27t\frac{(1-\sin^2 t)}{\sin^2 t} \\ &= 27t\frac{1}{\sin^2 t} - 27t - 27\frac{\cos t}{\sin t} + t^3 + 9t^2\frac{\cos t}{\sin t}. \end{aligned}$$

The power series expansion of the function $\cot t$ can be found in ([18], § 4.3.70) and ([19], § 1.3.1.4) as follows:

$$\cot t = \frac{1}{t} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1},\ 0 < |t| < \pi. \tag{10}$$

By (10), we have

$$\frac{1}{\sin^2 t} = \csc^2 t = -(\cot t)' = \frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|t^{2n-2},\ 0 < |t| < \pi. \tag{11}$$

Substituting (10) and (11) into the above expression of $k(t)$, we have

$$\begin{aligned} & k(t) \\ = & \ 27t\left[\frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|t^{2n-2}\right] - 27t - 27\left[\frac{1}{t} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1}\right] \\ & + t^3 + 9t^2\left[\frac{1}{t} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1}\right] \\ = & \ 27\sum_{n=2}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|t^{2n-1} + 27\sum_{n=2}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1} - 9\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n+1} \\ = & \ \sum_{n=2}^{\infty}\frac{54n \times 2^{2n}}{(2n)!}|B_{2n}|t^{2n-1} - 9\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n+1} \\ = & \ \sum_{n=1}^{\infty}\frac{54(n+1) \times 2^{2n+2}}{(2n+2)!}|B_{2n+2}|t^{2n+1} - 9\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n+1} \\ = & \ \sum_{n=1}^{\infty}\left[\frac{54(n+1) \times 4}{(2n+2)!}|B_{2n+2}| - \frac{9}{(2n)!}|B_{2n}|\right]2^{2n}t^{2n+1} \\ = & \ \sum_{n=1}^{\infty}a_n 2^{2n}t^{2n+1} = \frac{2}{5}t^3 + \sum_{n=2}^{\infty}a_n 2^{2n}t^{2n+1}, \end{aligned}$$

where

$$a_n = \frac{216(n+1)}{(2n+2)!}|B_{2n+2}| - \frac{9}{(2n)!}|B_{2n}|,\ n \geq 2.$$

The following estimates of the even-indexed Bernoulli numbers B_{2n} can be found in [18,20,21]:

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n}}{2^{2n}-1} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n-1}}{2^{2n-1}-1}. \tag{12}$$

By (12) we have that for $n \geq 2$,

$$\begin{aligned} a_n &> \frac{216(n+1)}{(2n+2)!}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\frac{2^{2n+2}}{2^{2n+2}-1} - \frac{9}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n-1}}{2^{2n-1}-1} \\ &= \frac{2}{\pi^{2n+2}}\frac{216(n+1)}{2^{2n+2}-1} - \frac{9\pi^2}{\pi^{2n+2}}\frac{1}{2^{2n-1}-1} = 18 \times \frac{\left(6n-\pi^2+6\right) \times 2^{2n+2} - \left(48n-\pi^2+48\right)}{\pi^{2n+2}(4\times 2^{2n}-1)(2^{2n}-2)} \\ &= \frac{18 \times \left(6n-\pi^2+6\right)}{\pi^{2n+2}(4\times 2^{2n}-1)(2^{2n}-2)}\left[2^{2n+2} - \frac{48n-\pi^2+48}{6n-\pi^2+6}\right]. \end{aligned}$$

Now, by mathematical induction, we can prove that

$$2^{2n+2} > \frac{48n-\pi^2+48}{6n-\pi^2+6},\ n \geq 2. \tag{13}$$

It is not difficult to verify that the above formula (13) is true for $n = 2$. Suppose (13) holds for m, that is

$$2^{2m+2} > \frac{48m-\pi^2+48}{6m-\pi^2+6},\ m \geq 2. \tag{14}$$

By (14), we have

$$2^{2m+4} = 4 \cdot 2^{2m+2} > 4 \cdot \frac{48m-\pi^2+48}{6m-\pi^2+6}.$$

The inequality (13) is proved when proving

$$\frac{4\left(48m-\pi^2+48\right)}{6m-\pi^2+6} > \frac{48(m+1)-\pi^2+48}{6(m+1)-\pi^2+6}$$

or

$$\frac{A}{B} =: \frac{4\left(48m-\pi^2+48\right)}{6m-\pi^2+6} > \frac{48m-\pi^2+96}{6m-\pi^2+12} := \frac{C}{D}.$$

In fact,

$$AD - BC = 864m^2 + m\left(2592 - 162\pi^2\right) + \left(3\pi^4 - 138\pi^2 + 1728\right) > 0$$

holds for all $m \geq 2$ due to

$$\Delta = \left(2592 - 162\pi^2\right)^2 - 4 \times 864 \times \left(3\pi^4 - 138\pi^2 + 1728\right) = 324\left(7\pi^2 - 144\right)\left(7\pi^2 - 16\right) < 0.$$

So we have obtained that $a_n > 0$ for $n \geq 2$. This leads to that $G'(t) > 0$ for all $t > 0$. In view of the relation $G(0) = 0$, we have that the fact $G(t) > 0$ holds for all $t > 0$.

(ii) The result just proved is equivalent to the fact that the inequality

$$\frac{1}{3} > \frac{\ln \frac{t}{\tanh^{-1}(\sin t)}}{\ln(\cos t)} := B(t)$$

holds for all $t > 0$. In this way, when obtaining $\lim_{t\to 0} B(t) = 1/3$ or $\lim_{t\to\infty} B(t) = 1/3$, we will prove that $1/3$ is the best constant in (9). Considering the fact

$$\begin{aligned} \lim_{t\to 0} B(t) &= \lim_{t\to 0} \frac{\ln t - \ln \tanh^{-1}(\sin t)}{\ln(\cos t)} = \lim_{t\to 0} \frac{\left[\ln t - \ln \tanh^{-1}(\sin t)\right]'}{[\ln(\cos t)]'} \\ &= \lim_{t\to 0} \frac{\frac{1}{t} - \frac{1}{\tanh^{-1}(\sin t)}\frac{1}{\cos t}}{-\frac{\sin t}{\cos t}} = \lim_{t\to 0} \frac{-\frac{1}{3}t + o(t)}{-t + o(t)} = \frac{1}{3}, \end{aligned}$$

we can complete the proof of Lemma 3. □

2.2. Main Results and Their Proofs

In order to explore better and more accurate inequalities on the relationship between the inverse hyperbolic sine function and the inverse tangent function, we consider the Taylor's formula of the following function:

$$(\arctan x)^2 - \frac{\ln^2(x+\sqrt{1+x^2})}{(1+x^2)^{\alpha}} = x^4\left(\alpha - \frac{1}{3}\right) - x^6\left(\frac{1}{2}\alpha^2 + \frac{5}{6}\alpha - \frac{1}{3}\right) + o\left(x^7\right),\ x \to 0.$$

Letting $\alpha = 1/3$, we can obtain the following expression for $x \to 0$,

$$(\arctan x)^2 - \frac{\ln^2(x+\sqrt{1+x^2})}{(1+x^2)^{1/3}} = \frac{4}{2835}x^8 - \frac{26}{8505}x^{10} + \frac{6436}{1403\,325}x^{12} - \frac{10\,678}{1804\,275}x^{14} + o\left(x^{15}\right),$$

and draw the following conclusion.

Theorem 1. *Let $x \in (-\infty, \infty)$. Then the inequality*

$$\frac{\ln^2(x+\sqrt{1+x^2})}{(1+x^2)^{1/3}} \le (\arctan x)^2 \tag{15}$$

holds with the best exponent $1/3$.

Proof. Since the functions on both sides of (15) are even functions, we can limit our discussion to the interval $(0, \infty)$ due to

$$\frac{\ln^2(x+\sqrt{1+x^2})}{(1+x^2)^{1/3}} < (\arctan x)^2,\ 0 < |x| < \infty$$

is equivalent to

$$\frac{\ln^2(x+\sqrt{1+x^2})}{(1+x^2)^{1/3}} < (\arctan x)^2,\ x > 0,$$

that is

$$\frac{\ln(x+\sqrt{1+x^2})}{(1+x^2)^{1/6}} < \arctan x,\ x > 0.$$

Let $\sinh^{-1} x = t$, $x \in (0, \infty)$. Then $x = \sinh t$, $t \in (0, \infty)$ and Theorem 1 is equivalent to Lemma 2. This completes the proof of Theorem 1. □

The second objective of this paper is to obtain the analogue result of the inequality (15).

Theorem 2. *Let $x \in (-1, 1)$. Then the inequality*

$$\frac{(\arcsin x)^2}{(1-x^2)^{1/3}} \le \left(\tanh^{-1} x\right)^2 \tag{16}$$

holds with the best exponent $1/3$.

Proof. Since the functions on both sides of (16) are even functions, we can limit our discussion to the interval $(0, 1)$. Let $\arcsin x = t$, $x \in (0, 1)$. Then $x = \sin t$, $t \in (0, \pi/2)$ and Theorem 2 is equivalent to Lemma 3. This completes the proof of Theorem 2. □

3. Some Improvements of Mitrinovic–Adamović and Lazarević Inequalities and Remarks

First, we know that the Mitrinović–Adamović inequality (see [14,15]) can be described as follows:

$$\left(\frac{\sin t}{t}\right)^3 > \cos t,\ 0 < t < \frac{\pi}{2} \tag{17}$$

or

$$\frac{\sin t}{t} > (\cos t)^{1/3},\ 0 < t < \frac{\pi}{2}. \tag{18}$$

The hyperbolic version of the above result is Lazarević's inequality (see [16,22]):

$$\left(\frac{\sinh t}{t}\right)^3 > \cosh t,\ t > 0 \tag{19}$$

or

$$\frac{\sinh t}{t} > (\cosh t)^{1/3},\ t > 0. \tag{20}$$

Now, connecting two key inequalities (9) and (7) with the above two inequalities, we can obtain the following interesting results.

Theorem 3. *Let* $0 < t < \pi/2$. *Then*

$$\frac{\sin t}{t} > \frac{t}{\tanh^{-1}(\sin t)} > (\cos t)^{1/3}. \tag{21}$$

Proof. We only prove the first inequality of (21), which is

$$\tanh^{-1}(\sin t) - \frac{t^2}{\sin t} > 0,\ 0 < t < \frac{\pi}{2}.$$

In ([18], § 4.3.69, § 4.3.68), we can find the following power series expansion of the functions $1/\cos t$ and $1/\sin t$ as follows:

$$\frac{1}{\cos t} = \sum_{n=0}^{\infty} \frac{|E_{2n}|}{(2n)!} t^{2n},\ |t| < \pi, \tag{22}$$

$$\frac{1}{\sin t} = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| t^{2n-1},\ 0 < |t| < \pi. \tag{23}$$

By (22) and (23) we have

$$\begin{aligned}
&\tanh^{-1}(\sin t) - \frac{t^2}{\sin t} \\
= {}& \int_0^t \frac{1}{\cos t} dt - \frac{t^2}{\sin t} = \sum_{n=0}^{\infty} \frac{|E_{2n}|}{(2n+1)!} t^{2n+1} - t^2 \left[\frac{1}{t} + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| t^{2n-1}\right] \\
= {}& \sum_{n=2}^{\infty} \frac{|E_{2n}|}{(2n+1)!} t^{2n+1} - \sum_{n=2}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| t^{2n+1} = \sum_{n=2}^{\infty} \left[\frac{|E_{2n}|}{(2n+1)!} - \sum_{n=2}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}|\right] t^{2n+1} \\
= {}& : \sum_{n=2}^{\infty} b_n t^{2n+1},
\end{aligned}$$

where

$$b_n = \frac{|E_{2n}|}{(2n+1)!} - \frac{2^{2n}-2}{(2n)!} |B_{2n}|,\ n \geq 2.$$

The following estimates of the even-indexed Euler numbers E_{2n} can be found in [18]:

$$\frac{2^{2(n+1)}}{\pi^{2n+1}}\frac{3^{2n+1}}{3^{2n+1}+1} < \frac{|E_{2n}|}{(2n)!} < \frac{2^{2(n+1)}}{\pi^{2n+1}}. \tag{24}$$

From (12) and (24), we have

$$\begin{aligned} b_n &> \frac{1}{(2n+1)}\frac{2^{2(n+1)}}{\pi^{2n+1}}\frac{3^{2n+1}}{3^{2n+1}+1} - \frac{2^{2n}-2}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}\frac{2^{2n-1}}{2^{2n-1}-1} \\ &= 2\times\frac{6\times 6^{2n}-3\times 3^{2n}\pi(2n+1)-\pi(2n+1)}{\pi^{2n+1}(2n+1)(3\times 3^{2n}+1)} \\ &> 0. \end{aligned}$$

This completes the proof of Theorem 3. □

Theorem 4. *Let $t > 0$. Then*

$$\frac{t}{\arctan(\sinh t)} < (\cosh t)^{1/3} < \frac{\sinh t}{t}. \tag{25}$$

Theorem 5. *The contribution of the two conclusions (7) and (9) in this paper is that (9) subdivides the Mitrinović–Adamović inequality and (7) gives a lower bound for the function $(\cosh t)^{1/3}$.*

Second, by comparing the inequality (15) with the one (3), we conclude that

Theorem 6. *Let $x \in (-\infty, \infty)$. Then*

$$\frac{(\sin x)\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \le \frac{\ln^2(x+\sqrt{1+x^2})}{(1+x^2)^{1/3}} \le (\arctan x)^2. \tag{26}$$

Proof. We only prove the first inequality of (26) holds for $x > 0$. Obviously, we have

$$\left(1+x^2\right)^{1/3} \le \sqrt{1+x^2}. \tag{27}$$

At the same time, we can prove that

$$\ln(x+\sqrt{1+x^2}) > \sin x \tag{28}$$

holds for all $x \in (0, \infty)$.

First, we prove that the inequality (28) holds for $x \in (0, \pi)$. Let

$$g(x) = \ln(x+\sqrt{1+x^2}) - \sin x,\ 0 < x < \pi.$$

Then

$$g'(x) = \frac{1}{\sqrt{x^2+1}} - \cos x = \frac{1-\sqrt{x^2+1}\cos x}{\sqrt{x^2+1}} := \frac{j(x)}{\sqrt{x^2+1}},$$

where $j(x) = 1-\sqrt{x^2+1}\cos x$. Since

$$\begin{aligned} j'(x) &= \frac{\sin x + x^2\sin x - x\cos x}{\sqrt{x^2+1}} = \frac{\sin x}{\sqrt{x^2+1}}\frac{\sin x + x^2\sin x - x\cos x}{\sin x} \\ &= \frac{\sin x}{\sqrt{x^2+1}}\left(x^2+1-x\frac{\cos x}{\sin x}\right) := \frac{\sin x}{\sqrt{x^2+1}}p(x), \end{aligned}$$

where

$$p(x) = x^2+1-x\frac{\cos x}{\sin x},\ 0 < x < \pi.$$

Using (10), we have

$$\begin{aligned} p(x) &= x^2+1-x\left[\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}\right] \\ &= x^2+1-1+\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n} = x^2+\frac{2^2}{2!}|B_2|x^2+\sum_{n=2}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n} \\ &= \frac{4}{3}x^2+\sum_{n=2}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n} > 0 \end{aligned}$$

for all $x \in (0,\pi)$. This leads to that $j'(x) > 0$ on $(0,\pi)$ and $j(x)$ is increasing on $(0,\pi)$. Therefore $j(x) > j(0) = 0$, which means $g'(x) > 0$ on $(0,\pi)$ and $g(x)$ is increasing on $(0,\pi)$. So $g(x) > g(0) = 0$ holds for $x \in (0,\pi)$.

Then we prove that the inequality (28) holds for $x \in (0,\infty)$. When $x \in (2k\pi+\pi, 2k\pi+2\pi)$ and $k \geq 0$, we have $\sin x < 0$ and the inequality (28) is true. So we can assume $x \in (2k\pi, 2k\pi+\pi), k \geq 0$.Let $u = x - 2k\pi$. Then we obtain that the variable u changes on $(0,\pi)$, which is exactly the interval discussed in above. So we can obtain the following conclusions:

$$\ln(x+\sqrt{1+x^2}) > \ln(u+\sqrt{1+u^2}) > \sin u = \sin x,\ x > 0$$

and

$$\ln^2(x+\sqrt{1+x^2}) > (\sin x)\ln(x+\sqrt{1+x^2}),\ x > 0. \tag{29}$$

Therefore, the desired result follows easily from (27) and (29). □

Remark 1. *The inequality* (15) *is better than* (3) *on the real axis* $(-\infty,\infty)$.

Third, by comparing the inequality (16) with the one (6), we conclude that

Theorem 7. *Let* $x \in (-1,1)$. *Then*

$$\frac{x\arcsin x}{\left(1-\frac{41}{45}x^2\right)^{45/82}} \leq \frac{(\arcsin x)^2}{(1-x^2)^{1/3}} \leq (\arctan x)^2. \tag{30}$$

Proof. We only prove the first inequality of (30). Let

$$H(x) = \arcsin x - \frac{x\left(1-x^2\right)^{1/3}}{\left(1-\frac{41}{45}x^2\right)^{45/82}},\ 0 < x < 1.$$

Then

$$H'(x) = \frac{(1-x^2)^{1/6}\left(1-\frac{41}{45}x^2\right)^{127/82} - \left(1-\frac{10}{9}x^2\right)\left(1-\frac{7}{15}x^2\right)}{(1-x^2)^{2/3}\left(1-\frac{41}{45}x^2\right)^{127/82}} := \frac{h(x)}{(1-x^2)^{2/3}\left(1-\frac{41}{45}x^2\right)^{127/82}},$$

where

$$h(x) = \left(1-x^2\right)^{1/6}\left(1-\frac{41}{45}x^2\right)^{127/82} - \left(1-\frac{10}{9}x^2\right)\left(1-\frac{7}{15}x^2\right), 0 < x < 1.$$

For $x \in \left(3/\sqrt{10}, 1\right)$,

$$\left(1-\frac{10}{9}x^2\right)\left(1-\frac{7}{15}x^2\right) < 0,$$

we have $h(x) > 0$. In the following, we will prove that

$$\left(1-x^2\right)^{1/6}\left(1-\frac{41}{45}x^2\right)^{127/82} > \left(1-\frac{10}{9}x^2\right)\left(1-\frac{7}{15}x^2\right)$$

holds for $x \in \left(0, 3/\sqrt{10}\right)$.

Let

$$q(x) = \frac{1}{6}\ln\left(1-x^2\right) + \frac{127}{82}\ln\left(1-\frac{41}{45}x^2\right) - \ln\left(1-\frac{10}{9}x^2\right) - \ln\left(1-\frac{7}{15}x^2\right),\ 0 < x < \frac{3}{\sqrt{10}}.$$

Then

$$\begin{aligned} q'(x) &= \frac{548}{2025}x^5\frac{1-\frac{1225}{1233}x^2}{(1-x^2)\left(1-\frac{10}{9}x^2\right)\left(1-\frac{41}{45}x^2\right)\left(1-\frac{7}{15}x^2\right)} \\ &> \frac{548}{2025}x^5\frac{1-\frac{1225}{1233}\left(\frac{3}{\sqrt{10}}\right)^2}{(1-x^2)\left(1-\frac{10}{9}x^2\right)\left(1-\frac{41}{45}x^2\right)\left(1-\frac{7}{15}x^2\right)} \\ &= \frac{58}{2025}x^5\frac{1}{(1-x^2)\left(1-\frac{10}{9}x^2\right)\left(1-\frac{41}{45}x^2\right)\left(1-\frac{7}{15}x^2\right)} \\ &> 0. \end{aligned}$$

So the function $q(x)$ is increasing on $\left(0, 3/\sqrt{10}\right)$. Due to $q(0) = 0$ we have $q(x) > 0$ for $x \in \left(0, 3/\sqrt{10}\right)$. This leads to $h(x) > 0$ for $\left(0, 3/\sqrt{10}\right)$. So $h(x) > 0$ holds for all $x \in (0,1)$, which means $H'(x) > 0$ for all $x \in (0,1)$. In view of $H(0) = 0$, we have that $H(x) > 0$ holds for all $x \in (0,1)$. This complete the proof of the first inequality of (30). □

Remark 2. *The inequality* (16) *is better than* (5) *and* (6)*.*

4. Some Applications of Theorem 1 and Theorem 2 in the Mean Value Theory

Mean value inequality is an eternal topic in the field of mathematical inequalities. Averaging is the most common way to combine the inputs and is commonly used in voting, multicriteria and group decision making, constructing various performance scores, statistical analysis, etc. In this section, assume that x and y are two different positive numbers, $\mathcal{G}(x,y), \mathcal{L}(x,y), \mathcal{A}(x,y)$ and $\mathcal{M}_2(x,y)$ are the geometric, logarithmic, arithmetic and quadratic means, respectively, where

$$\mathcal{G}(x,y) = \sqrt{xy},\ \mathcal{L}(x,y) = \frac{y-x}{\log y - \log x},\ \mathcal{A}(x,y) = \frac{x+y}{2},\ \mathcal{M}_2(x,y) = \left(\frac{x^2+y^2}{2}\right)^{1/2}.$$

For convenience, we note that $\mathcal{M} < \mathcal{N}$ implies $\mathcal{M}(x,y) < \mathcal{N}(x,y)$, where $\mathcal{M}$ and $\mathcal{N}$ are means with two different positive numbers x and y. In order to explore the various relationships between the above means, we can take the geometric mean $\mathcal{G}(x,y)$ as a reference and make transformation $\sqrt{y/x} = e^w$ to obtain the following results:

$$\frac{\mathcal{L}(x,y)}{\mathcal{G}(x,y)} = \frac{\sinh w}{w},\ \frac{\mathcal{A}(x,y)}{\mathcal{G}(x,y)} = \cosh w,\ \frac{\mathcal{M}_2(x,y)}{\mathcal{G}(x,y)} = \sqrt{\cosh 2w},\ w > 0.$$

Then via the relationships connected with hyperbolic sine function and hyperbolic cosine function,

$$1 < \frac{\sinh w}{w} < \cosh w < \sqrt{\cosh 2w}, \tag{31}$$

we can obtain a chain of inequalities

$$\mathcal{G} < \mathcal{L} < \mathcal{A} < \mathcal{M}_2. \tag{32}$$

For the study of the classical mean inequalities, see [22–29].

The traditional method of dealing with mean inequality is to replace the relationship between two means with the relationship between two corresponding hyperbolic functions through a change of variables. In recent years, scholars have been exploring another novel methods to study the relationships between classical means and newly introduced means. Among them, the so-called Seiffert function method is worth introducing.

In 1998, Kahlig and Matkowski [30] introduced a new concept in the field of means, the ratio of a homogeneous bivariable mean $\mathcal{M}$ in $(0, \infty)$ to a classical mean $\mathcal{N}$ can be expressed as a function of $(x-y)/(x+y)$, which is called index function of $\mathcal{M}$ with respect to $\mathcal{N}$ or an $\mathcal{N}$-index of $\mathcal{M}$:

$$\frac{\mathcal{M}(x,y)}{\mathcal{N}(x,y)} = f_{\mathcal{M},\mathcal{N}}\left(\frac{x-y}{x+y}\right),$$

where $f_{\mathcal{M},\mathcal{N}}$: $(-1,1) \longrightarrow (0,2)$ is a unique single variable function (with the graph laying in a set of a butterfly shape).

Assuming $f : (0,1) \to \mathbb{R}^+$, Witkowski [31] constructed a new binary function:

$$\mathcal{M}_f(x,y) = \begin{cases} \frac{|x-y|}{2f(t)} & x \neq y \\ x & x = y \end{cases}, \tag{33}$$

where

$$t = \frac{|x-y|}{x+y}.$$

When the function $f(t)$ satisfies

$$\frac{t}{1+t} \le f(t) \le \frac{t}{1-t},$$

Witkowski [31] proved that $\mathcal{M}_f(x,y)$ is a symmetric and homogeneous mean. In this case, the function $f(t)$ with the property above produces a corresponding mean, and there is the following relationship between the two functions $f(t)$ and $\mathcal{M}_f(x,y)$:

$$f(t) = \frac{t}{\mathcal{M}_f(1-t, 1+t)}. \tag{34}$$

In this way, the two functions $f(t)$ and $\mathcal{M}_f(x,y)$ determine the one-to-one correspondence through these two relations (33) and (34). For this reason, we can rewrite $f(t) =: f_{\mathcal{M}}(t)$. In general, $f(t)$ is called the Seiffert function of the mean $\mathcal{M}$ (see [32,33]).

As a Chinese idiom says, how can you catch tiger cubs without entering the lair of the tiger? In order to make the above new method of dealing with mean value inequalities more useful, we have made the following appropriate modifications. We change the parameters sign of a mean $\mathcal{M}_f$ about two parameters with u and v, and assume that $0 < u < v$. So there must be three positive numbers x, y, and λ, so that

$$\begin{cases} u = \lambda \frac{2x}{x+y} \\ v = \lambda \frac{2y}{x+y} \end{cases}, \; 0 < x < y.$$

Then we have that $0 < t < 1$,

$$t = \frac{y-x}{x+y},$$

and

$$\begin{aligned} \mathcal{M}_f(u,v) &= \mathcal{M}_f\left(\lambda\frac{2x}{x+y},\lambda\frac{2y}{x+y}\right) \\ &= \lambda\mathcal{M}_f\left(\frac{2x}{x+y},\frac{2y}{x+y}\right) = \lambda\mathcal{M}_f(1-t,1+t) \\ &= \lambda\frac{t}{f_{\mathcal{M}}(t)}. \end{aligned} \tag{35}$$

Via (34), we can obtain that

$$f_{\mathcal{G}}(t) = \frac{t}{\sqrt{(1-t)(1+t)}} = \frac{t}{\sqrt{1-t^2}}, \tag{36}$$

$$f_{\mathcal{L}}(t) = \frac{t}{\frac{(1+t)-(1-t)}{\ln(1+t)-\ln(1-t)}} = \frac{1}{2}\ln\frac{1+t}{1-t} = \tanh^{-1} t, \tag{37}$$

$$f_{\mathcal{A}}(t) = \frac{t}{\frac{(1-t)+(1+t)}{2}} = t, \tag{38}$$

$$f_{\mathcal{M}_2}(t) = \frac{t}{\sqrt{\frac{(1-t)^2+(1+t)^2}{2}}} = \frac{t}{\sqrt{1+t^2}}. \tag{39}$$

If the following relation holds

$$\frac{t}{\sqrt{1+t^2}} < t < \tanh^{-1} t < \frac{t}{\sqrt{1-t^2}}, \tag{40}$$

then using (35), we obtain (32). The chain of inequalities (40) is not difficult to prove. That is to say, we can also prove (32) by the Seiffert function.

The first Seiffert mean $\mathcal{P}(x,y)$, the second Seiffert mean $\mathcal{T}(x,y)$, and Neuman–Sandor mean $\mathcal{R}(x,y)$ are, respectively, defined [34–38] by

$$\mathcal{P}(x,y) = \frac{|x-y|}{2\arcsin t},\ \mathcal{T}(x,y) = \frac{|x-y|}{2\arctan t},\ \mathcal{R}(x,y) = \frac{|x-y|}{2\sinh^{-1} t}.$$

Incidentally, we can find the logarithmic mean is a Seiffert-type mean:

$$\mathcal{L}(x,y) = \frac{|x-y|}{2\tanh^{-1} t}.$$

Obviously, we have

$$f_{\mathcal{P}}(t) = \arcsin t, \tag{41}$$

$$f_{\mathcal{T}}(t) = \arctan t, \tag{42}$$

$$f_{\mathcal{R}}(t) = \sinh^{-1} t. \tag{43}$$

Next, we apply the conclusions of Theorem 1 and Theorem 2 to obtain the relationships between the Seiffert-like means and the classical means mentioned above. According to the above definitions (38), (39), (42), (43) and the relationship (35), we have

$$\sinh^{-1} t = \frac{\lambda t}{\mathcal{R}},\ \arctan t = \frac{\lambda t}{\mathcal{T}}$$

and

$$\begin{aligned} f_{\mathcal{A}}(t) &= t = \frac{\lambda t}{\mathcal{A}}, \\ f_{\mathcal{M}_2}(t) &= \frac{t}{\sqrt{1+t^2}} = \frac{\lambda t}{\mathcal{M}_2}. \end{aligned}$$

So

$$\begin{aligned} \frac{\sinh^{-1} t}{\arctan t} &= \frac{\mathcal{T}}{\mathcal{R}}, \\ \left(1+t^2\right)^{1/6} &= \left(\frac{\mathcal{M}_2}{\mathcal{A}}\right)^{1/3}. \end{aligned} \tag{44}$$

The result of Theorem 1 can be subsequently converted into the following inequality:

$$\left(1+t^2\right)^{1/6} > \frac{\sinh^{-1} t}{\arctan t}, \ 0 < t < 1.$$

Note that using (44), we can obtain the fact

$$\left(\frac{\mathcal{M}_2}{\mathcal{A}}\right)^{1/3} > \frac{\mathcal{T}}{\mathcal{R}},$$

which is

$$\mathcal{M}_2\mathcal{R}^3 > \mathcal{A}\mathcal{T}^3. \tag{45}$$

Similarly, according to the above related concepts (36), (37), (41) and Formula (35), we have

$$\arcsin t = \frac{\lambda t}{\mathcal{P}}, \ \tanh^{-1} t = \frac{\lambda t}{\mathcal{L}}$$

and

$$f_{\mathcal{G}}(t) = \frac{t}{\sqrt{1-t^2}} = \frac{\lambda t}{\mathcal{G}}.$$

Then we obtain that

$$\begin{aligned} \frac{\arcsin t}{\tanh^{-1} t} &= \frac{\mathcal{L}}{\mathcal{P}}, \\ \left(1-t^2\right)^{1/6} &= \left(\frac{\mathcal{G}}{\mathcal{A}}\right)^{1/3}. \end{aligned} \tag{46}$$

The result of Theorem 2 is equivalent to the following inequality:

$$\left(1-t^2\right)^{1/6} > \frac{\arcsin t}{\tanh^{-1} t}, \ 0 < t < 1.$$

Note that using (46), we can obtain the fact

$$\left(\frac{\mathcal{G}}{\mathcal{A}}\right)^{1/3} > \frac{\mathcal{L}}{\mathcal{P}},$$

that is

$$\mathcal{G}\mathcal{L}^3 > \mathcal{A}\mathcal{P}^3. \tag{47}$$

Therefore, applying Theorems 1, 2, we obtain two novel inequalities (45) and (47) connected with Seiffert-like means and classical means.

Remark 3. *The application of the main conclusions of this paper is not limited to this section. In fact, applying fractional integral operator to the both sides of the inequalities* (15) *and* (16) *respectively, we can obtain inequalities similar to (3.1) in [39] and (2.3) in [40]. Similar exploration can become a research topic and direction in the future.*

5. Conclusions

In this paper, we obtain two new inequalities:

$$\frac{\ln^2(x+\sqrt{1+x^2})}{(1+x^2)^{1/3}} \le (\arctan x)^2, \ -\infty < x < \infty, \tag{48}$$

$$\frac{(\arcsin x)^2}{(1-x^2)^{1/3}} \le \left(\tanh^{-1} x\right)^2, \ -1 < x < 1. \tag{49}$$

On the one hand, because of

$$\frac{(\sin x)\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \le \frac{\ln^2(x+\sqrt{1+x^2})}{(1+x^2)^{1/3}} \le (\arctan x)^2, \ -\infty < x < \infty,$$

$$\frac{x\arcsin x}{1-\frac{1}{2}x^2} \le \frac{x\arcsin x}{\left(1-\frac{41}{45}x^2\right)^{45/82}} \le \frac{(\arcsin x)^2}{(1-x^2)^{1/3}} \le \left(\tanh^{-1} x\right)^2, \ -1 < x < 1,$$

the above two inequalities (48) and (49) are better than the conclusions and open problems in the previous literature. On the other hand, because of

$$(\cos t)^{1/3} < \frac{t}{\tanh^{-1}(\sin t)} < \frac{\sin t}{t}, \ 0 < t < \frac{\pi}{2},$$

$$\frac{t}{\arctan(\sinh t)} < (\cosh t)^{1/3} < \frac{\sinh t}{t}, \ t > 0,$$

the equivalent forms of the above two inequalities (48) and (49) refine and strengthen the Mitrinović–Adamović and Lazarević inequalities.

Finally, we apply the results of Theorems 1 and 2 to the theory of mean value inequalities and obtain new inequalities related to Seiffert-like means and classical means.

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