NEW METHODS FOR COMPUTING A CLOSEST SADDLE NODE BIFURCATION AND WORST CASE LOAD POWER MARGIN FOR VOLTAGE COLLAPSE

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Abstract: Voltage collapse and blackout can occur in an electric power system when load powers vary so that the system loses stability in a saddle node bifurcation. We propose new iterative and direct methods to compute load powers at which bifurcation occurs and which are locally closest to the current operating load powers. The distance in load power parameter space to this locally closest bifurcation is an index of voltage collapse. The pattern of load power increase need not be predicted; instead the index is a worst case load power margin. The computations are illustrated in the 6 dimensional load power parameter space of a 5 bus power system. The normal vector and curvature of a hypersurface of critical load powers at which bifurcation occurs are also computed. The sensitivity of the index to parameters and controls is easily obtained from the normal vector.
Keywords: voltage collapse, saddle node bifurcation, index, sensitivity, load power margin, numerical methods

1. Introduction

We model an electric power system by differential equations of the form

$$\dot{x} = f(x,\lambda), \qquad x \in \mathbf{R}^p, \;\; \lambda \in \mathbf{R}^m \tag{1.1}$$

where λ is a parameter vector of real and reactive load powers. It is convenient to write x for the system state vector at a stable equilibrium (operating point). We assume current operating load powers λ_0 at which the corresponding equilibrium x_0 is stable. As the load powers λ vary from the current load powers λ_0, the equilibrium x will vary in the state space. At critical load powers λ_1 the system can lose stability by x disappearing in a saddle node bifurcation and we denote the set of such λ_1 in the load power parameter space $\mathbf{R}^m$ by Σ. Σ is a boundary of the feasible region for operation at the stable equilibrium x. The saddle node bifurcation instability when the load powers encounter Σ can cause catastrophic collapse of the system voltages and blackout [7,15,16]. Σ typically consists of hypersurfaces in $\mathbf{R}^m$ and their intersections. The instability can be avoided by monitoring the position of the current load powers λ_0 relative to Σ and taking corrective action if λ_0 moves too close to Σ. In particular it is useful to calculate a critical load power λ_* in Σ for which $|\lambda_* - \lambda_0|$ is a local minimum of the distance from λ_0 to Σ [17,21,26,23,8,9,10,12]. Then the line segment $\lambda_0\lambda_*$ represents a worst case load power parameter variation and $|\lambda_* - \lambda_0|$ measures the proximity to saddle node bifurcation. That is, the worst case load power margin $|\lambda_* - \lambda_0|$ is a voltage collapse index. We call the bifurcation at λ_* 'a closest saddle node bifurcation' with the understanding that the distance to bifurcation is measured in parameter space relative to the fixed value λ_0. Note that we do not require the direction of load increase to be predicted.

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We introduce a two bus power system model to illustrate these ideas in a simple context: Consider a generator slack bus with voltage 1.0, lossless line with admittance 4.0 and a PQ load.. The system state is 2 dimensional and the system operating equilibrium is $x = (\alpha, V)$ where $V\angle\alpha$ is the load voltage phasor. The parameter space is 2 dimensional and the parameter vector $\lambda = (P, Q)$ contains the real and reactive powers consumed by the load. The load flow equations are

$$\begin{aligned} 0 &= -4V\sin\alpha - P \\ 0 &= -4V^2 + 4V\cos\alpha - Q \end{aligned} \tag{1.2}$$

The operating equilibrium is given by $x_0 = (\alpha_0, V_0) = (-0.138, 0.908)$ and the current load powers are $\lambda_0 = (P_0, Q_0) = (0.5, 0.3)$ (see figure 1). The set of critical loadings giving bifurcation and voltage collapse is the hypersurface Σ which in this case is the one dimensional curve shown in figure 1. Figure 1 also shows the closest λ_* to λ_0; note that $\lambda_0\lambda_*$ is normal to Σ. Then the line segment $\lambda_0\lambda_*$ represents the worst case load power variation and the worst case load power margin $|\lambda_* - \lambda_0|$ measures the proximity to saddle node bifurcation and voltage collapse.

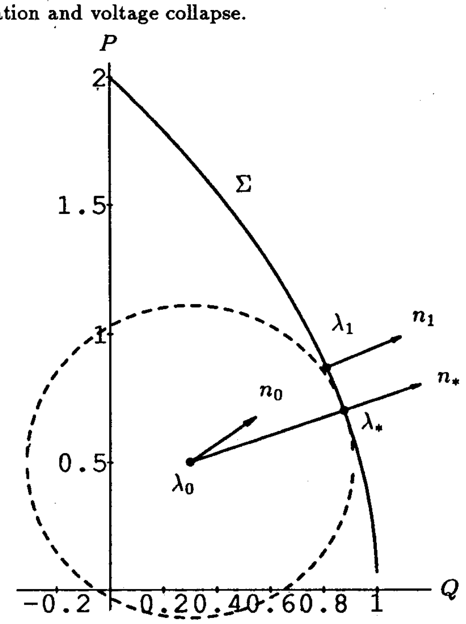

Figure 1. Load power parameter space

If a reliable load forecast specifying the direction of load increase in $\mathbf{R}^m$ is available, then a load power margin index ℓ assuming the direction of load increase can also be computed [2,5,30,23,20,1] and the combination of the load power margin index ℓ with the worst case load power margin index $|\lambda_* - \lambda_0|$ gives a useful description of the relation in parameter space of the current load powers λ_0 to the critical load powers Σ. For example, in the situation of figure 2, although the load power margin index $\ell = |\lambda_1 - \lambda_0|$ assuming the direction of load increase given by n_0 may be acceptably large, $|\lambda_* - \lambda_0|$ is dangerously small so that even a minor contingency could precipitate a voltage collapse. Of course, if the direction of load increase is not available, then only $|\lambda_* - \lambda_0|$ can be computed.

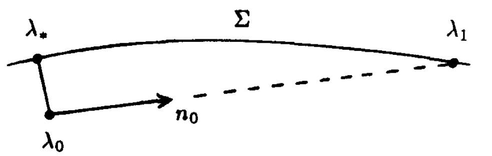

Figure 2. Insecure point λ_0 in load power parameter space

Galiana and Jarjis [17] consider the real power load flow equations with constant voltage magnitudes, define a feasibility region in a real power injection parameter space and present the idea of computing a closest instability in this parameter space. In our notation, the boundary of the feasible region would be labelled Σ. Using a conjecture that Σ is convex, Galiana and Jarjis parameterize Σ with the normal vector N to Σ and define a real power margin D which is the perpendicular distance from the operating real power injections λ_0 to the tangent hyperplane of Σ with normal N. Minimizing D with conjugate gradient methods yields a worst case real power margin and this computation is illustrated in a 6 bus system. Jarjis and Galiana [21] consider the load flow equations and define a feasibility region in a real and reactive power injection parameter space augmented with the voltage magnitudes of PV buses. A non-Euclidean worst case parameter space margin is defined and computed using Fletcher-Powell minimization. Constrained minimization in the load power parameter space is also considered and the computations are illustrated in 5 bus systems. Jung et al. [23] suggest a gradient projection optimization method to compute a worst case load power margin and Sekine et al. [26] attempt to compute a worst case load power margin by gradient descent on the determinant of the Jacobian. The use of $|\lambda_* - \lambda_0|$ and other indices in determining the costs of secure power system operation is explained in Alvarado et al. [3]. This paper explains and applies new iterative and direct methods to compute a closest saddle node bifurcation and the index $|\lambda_* - \lambda_0|$ and is based on work in [8,9,10,12].

2. Static and Dynamic Power System Models

Although we regard the power system as being modelled by differential equations of the form (1.1), Dobson [11] shows that useful computations can be done with static equations such as the load flow equations whose solutions are equilibria of the differential equations. This follows since saddle node bifurcations of the static equation solutions coincide with saddle node bifurcations of the differential equation equilibria.

For example [11], let y be a vector of load bus voltage angles and magnitudes and let δ_G be a vector of generator voltage angles. Then load flow equations may be written as

$$\begin{aligned} 0 &= f_1(\delta_G, y) \\ 0 &= f_2(\delta_G, y, \lambda) \end{aligned} \tag{2.1}$$

where f_1 describes real power balance at the generators and f_2 describes real and reactive power balance at the loads.

A differential equation model which extends (2.1) by including generator swing dynamics and load dynamics is

$$\begin{aligned} \dot{\delta}_G &= \omega \\ \dot{\omega} &= f_1(\delta_G, y) - \Delta\, \omega \\ \dot{y} &= h(f_2(\delta_G, y, \lambda), \omega) \end{aligned} \tag{2.2}$$

Here h defines any dynamic load model which depends on the real and reactive power balance at each load and frequency ω and which satisfies $h(0) = 0$ (see [7,22] for examples). Then solutions (δ_G, y) of (2.1) yield equilibria $(\delta_G, 0, y)$ of (2.2) and bifurcation of solutions of (2.1) at (δ_{G*}, y_*) implies bifurcation of equilibria of (2.2) at $(\delta_{G*}, 0, y_*)$ [11]. The point is that we can assume a differential equation model of the form (2.2) even if the precise form of h is unknown and perform computations of saddle bifurcations of (2.2) using the simpler static equations (2.1). This is useful since convincing models for load dynamics have not yet been obtained. We remark that a similar reduction has not been obtained for the Hopf bifurcation; the Hopf bifurcation seems to depend more strongly on details of underlying dynamic equations.

Much useful engineering information is contained in right and left eigenvectors of the Jacobian of equation (2.2) at the bifurcation. The right eigenvector corresponding to the zero eigenvalue specifies the pattern of voltage decline in the initial dynamic collapse [7,11] and also the asymptotic direction in which the stable operating point x approaches the closest unstable equilibrium point as the saddle node bifurcation occurs [7,11,18]. The left eigenvector corresponding to the zero eigenvalue can be used to compute the normal vector to Σ [11]. This is used below to compute the index $|\lambda_* - \lambda_0|$ and index sensitivities. This eigenvector information is not lost when we compute with the static model (2.1) instead of the differential equations (2.2) since [11] and [13] show that this information may be easily obtained from eigenvectors of the static model.

The computations below apply to any power system model of the form (1.1) or to any static power system model which is equivalent to some underlying differential equation model of the form (1.1) in the sense explained above. For this paper we choose the load flow equations parameterized by real and reactive load powers to demonstrate the computations. Note that the parameters λ are only restricted to load powers for ease of exposition; any power system controls or parameters may be included in the parameter vector λ as required and the results of the paper can easily be generalized to this case.

3. Preliminaries

The iterative method to compute a closest saddle node bifurcation has two main ingredients: the formula for the normal vector to Σ and any of the standard methods for finding the load power margin ℓ assuming a direction of load increase. We consider these in turn before describing the iterative method in section 4.

At a saddle node bifurcation specified by load powers $\lambda_1 \in \Sigma$ the corresponding equilibrium x_1 is degenerate and the Jacobian $f_x|_{(x_1,\lambda_1)}$ is singular. Throughout this paper, we make the generic assumptions that $f_x|_{(x_1,\lambda_1)}$ has a unique simple zero eigenvalue with a corresponding left eigenvector w_1 and that certain transversality conditions (see [11,8,12,6]) are satisfied (we write w_1 as a row vector). It follows that Σ is a smooth hypersurface near λ_1 and that a normal vector to the hypersurface at λ_1 is

$$N(\lambda_1) = \alpha w_1 f_\lambda \tag{3.1}$$

where f_λ is the Jacobian of f with respect to λ [11,6]. α is a scaling factor whose magnitude is chosen so that $|N(\lambda_1)| = 1$ and whose sign is chosen so that an increase of load in the direction $N(\lambda_1)$ leads to disappearance of the operating equilibrium. Note that f_λ is a constant $p \times m$ matrix since the load powers λ appear linearly in the load flow equations. (3.1) also follows from [17, eqn. (13)] in the case when f_λ is the identity matrix.

Suppose we specify a particular direction of future load increase from the operating load powers λ_0. That is, we specify a ray in parameter space based at λ_0 with a unit vector n_0 so that the load powers λ along the ray are given by

$$\lambda = \lambda_0 + \ell' n_0 \tag{3.2}$$

as the loading factor ℓ' assumes positive real values. There are several methods to compute the closest saddle node bifurcation *assuming* this ray of load increase. That is, we can compute a critical loading factor ℓ so that $\lambda_1 = \lambda_0 + \ell n_0 \in \Sigma$. Since n_0 is a unit vector, $\ell = |\lambda_1 - \lambda_0|$; see figure 1. ℓ is a voltage collapse index measuring the load power margin assuming the direction of load increase. For example, this computation is done by [20,5,1,33] using continuation methods and [2,23,5,30] using direct or optimization methods. The text by Seydel [27] gives an entry to the extensive numerical analysis literature on continuation and direct methods.

For this paper we use a variant [13] of the direct method in which solution of the following equations by Newton's method yields the bifurcating equilibrium x_1, load power margin ℓ and the left eigenvector w_1:

$$f(x_1, \lambda_0 + \ell n_0) = 0 \tag{3.3}$$

$$w_1 f_x|_{(x_1,\lambda_0+\ell n_0)} = 0 \tag{3.4}$$

$$w_1 c - 1 = 0 \tag{3.5}$$

Equation (3.3) states that x_1 is an equilibrium at parameter $\lambda_1 = \lambda_0 + \ell n_0$ and equation (3.4) states that the Jacobian f_x evaluated at (x_1, λ_1) is singular with left eigenvector w_1. $c \in \mathbb{R}^p$ is a fixed vector and (3.5) ensures that the left eigenvector w_1 is nonzero. The only difference between equations (3.3–3.5) and the standard methods [2,23,5,27] is that we use a left eigenvector w_1 in place of the corresponding right eigenvector. This is convenient since computing $\lambda_1 = \lambda_0 + \ell n_0$ and x_1 by solving (3.3–3.5) yields as a byproduct the left eigenvector w_1 needed to compute the normal vector to Σ with formula (3.1). The left eigenvector form of the equations also arises naturally if an optimization approach is taken as in [30]. Continuation methods are also a good choice for computing ℓ.

4. Iterative Method

Now we describe how computation of the load power margin ℓ and the computation of the normal vector of Σ may be iterated to compute the direction n_* and parameter value λ_* of a locally closest saddle node bifurcation and hence the worst case load power margin $|\lambda_* - \lambda_0|$. The procedure is as follows:

(0) Let n_0 be an initial guess for the direction n_*.
(1) Given n_{i-1}, compute the saddle node bifurcation along the ray given by n_{i-1}; that is compute ℓ_i, λ_i, x_i so that $\lambda_i = \lambda_0 + n_{i-1}\ell_i \in \Sigma$.
(2) Compute the left eigenvector w_i of $f_x|_{(x_i,\lambda_i)}$ corresponding to the zero eigenvalue.
(3) Set $n_i = N(\lambda_i) = w_i f_\lambda$.
(4) Iterate steps 1,2,3 until convergence of n_i to a value n_*. Then $\lambda_* = \lambda_0 + \ell_* n_*$.

The direction n_* of a locally closest bifurcation is parallel to the normal vector $N(\lambda_*)$ of Σ at λ_* and it follows that n_* is a fixed point of the iteration. The quickest way to grasp how the iteration works is to try it with pencil and paper in the case of Σ an ellipse and λ_0 an interior point of the ellipse. Note that the iteration converges in one step if Σ is a hyperplane.

The iteration can be understood as minimizing $|\lambda_* - \lambda_0|$ on a series of tangent hyperplane approximations to Σ. At each iteration, $n_i = N(\lambda_i)$ indicates the direction of the point closest to λ_0 on the tangent hyperplane $\mathrm{T}\Sigma_{\lambda_i}$ to Σ at λ_i [12]. The following claims are proved in [8,12]:

- If the iteration converges exponentially to a fixed point n_* then the parameter $\lambda_* = \lambda_0 + n_*\ell_*$ specifies a locally closest bifurcation.
- If λ_* is the parameter of a locally closest bifurcation and Σ is 'not too concave' at λ_* then the direction $n_* = N(\lambda_*)$ is an exponentially stable fixed point of the iteration. (The precise meaning of 'not too concave' is that the minimum principal curvature of Σ at λ_* must exceed $-|\lambda_* - \lambda_0|^{-1}$.)

Note that when the iteration converges, it converges to a locally closest bifurcation which is not necessarily a globally closest bifurcation. This is a potential problem in practice, particularly if the hypersurfaces of Σ are corrugated or if λ_0 is close to several portions of Σ. One problem in analyzing the performance of the iteration is the lack of knowledge of the geometry of Σ; all that seems to be known are formulas for the normal vector [11] and curvature of Σ [8,9,12]. A conjecture of Jarjis and Galiana [21] implies that the interior of Σ is convex. Venkatasubramanian et al. [31] assume a differential-algebraic power system model and compute a 2 dimensional bifurcation diagram of a simple power system.

An initial ray direction n_0 for the iterative method may often be calculated as follows from information available at the current operating point (x_0, λ_0) [9]: If λ_0 is close enough to one of the hypersurfaces of Σ, then the algebraically largest eigenvalue of the Jacobian f_x at (x_0, λ_0) is the eigenvalue which will increase to zero as λ_0 moves towards $\lambda_* \in \Sigma$. The corresponding left eigenvector w_0 of $f_x|_{(x_0,\lambda_0)}$ approximates the left eigenvector w_* at (x_*, λ_*) so that $n_0 = w_0 f_\lambda$ approximates $n_* = w_* f_\lambda$ [11]. Note that this argument only works when λ_* is close enough to exactly one of the hypersurfaces of Σ; obtaining justifiably good estimates for n_0 in general is an open problem. However the iterative method seems robust to the choice of n_0. In the case of the two bus power system (1.2), the operating point is given by $x_0 = (\alpha_0, V_0) = (-0.138, 0.91)$ and $\lambda_0 = (P_0, Q_0) = (0.5, 0.3)$. $f_x|_{(x_0,\lambda_0)}$ has eigenvalues $(-3.994, -2.904)$ and the left eigenvector corresponding to -2.904 is $w_0 = (0.585, 0.811)$ so that $n_0 = w_0 f_\lambda = (0.585, 0.811)$.

5. Direct Method

We generalize equations (3.3–3.5), which assume a one dimensional ray of load increase, to equations for a closest saddle node bifurcation at (x_*, λ_*) [9]:

$$f(x_*, \lambda_*) = 0 \tag{5.1}$$

$$w_* f_x|_{(x_*,\lambda_*)} = 0 \tag{5.2}$$

$$k_*(\lambda_* - \lambda_0)^T - w_* f_\lambda = 0 \tag{5.3}$$

$$w_* f_\lambda (w_* f_\lambda)^T - 1 = 0 \tag{5.4}$$

Since the distance $|\lambda_* - \lambda_0|$ is a local minimum, $(\lambda_* - \lambda_0)^T$ is parallel to the normal vector $n_* = w_* f_\lambda$ to Σ at λ_* and equation (5.3) holds for some $k_* \in \mathbf{R}$. Equation (5.4) scales the normal vector n_* so that it has unit length. In particular (5.4) implies that w_* is nonzero so that (5.2) does indeed guarantee the singularity of $f_x|_{(x_*,\lambda_*)}$. For practical calculation it is convenient to replace (5.4) by

$$w_* f_\lambda \, c = 1 \tag{5.5}$$

where c is a fixed vector (c.f. (3.5)).

Equations (5.1–5.4) can be solved by Newton type methods and a solution $(x_*, w_*, \lambda_*, k_*)$ satisfies necessary conditions for a closest saddle-node bifurcation. Moreover, a solution $(x_*, w_*, \lambda_*, k_*)$ is a closest saddle bifurcation if the additional condition (6.1) explained in section 6 on the curvature of Σ at λ_* is satisfied. The Jacobian of equations (5.1–5.4) is *invertible near the solution if condition* (6.1) and transversality conditions are satisfied [8,12] so that Newton type methods are locally well defined and have second order convergence to a solution.

Equations (5.1–5.4) are (3.3–3.5) generalized to $\lambda \in \mathbf{R}^m$ by the addition of the m linear equations (5.3). In the case of a single parameter (m=1), (5.3) is a trivial equation for k which may be omitted and the equations reduce to (3.3–3.5).

Equations (5.1–5.4) may also be related to an optimization similar to that proposed by Jung [23]: Minimize $\frac{1}{2}|\lambda_* - \lambda_0|^2$ subject to the constraints (5.1), (5.2), (5.4). (Jung uses a right eigenvector instead of the left eigenvector w_*.) The Lagrangian is $L =$

$$\frac{1}{2}|\lambda_* - \lambda_0|^2 + m_1 f + m_2 (w_* f_x|_{(x_*,\lambda_*)})^T + m_3 (w_* f_\lambda (w_* f_\lambda)^T - 1)$$

$$\text{and} \qquad L_{\lambda_*} = (\lambda_* - \lambda_0)^T + m_1 f_\lambda$$

since $f_x|_{(x_*,\lambda_*)}$ and w_* are not explicit functions of λ_* in equations (1.1). Now $L_\lambda = 0$ reduces to (5.3) with Lagrange multiplier $m_1 = -w_*/k_*$.

To reliably solve (5.1–5.4) by Newton type methods it is essential to have good initial estimates $(x_1, w_1, \lambda_1, k_1)$ of the solution. Initial estimates x_1, w_1, λ_1 may be computed in the same way as for the iterative method [9] (and improved if required by a few iterations of the iterative method). Then k_1 may be estimated by $|\lambda_1 - \lambda_0|^{-1}$.

6. Curvature of Σ

We discuss a condition on the curvature of Σ which guarantees that solutions of the direct method equations (5.1-5.4) are locally closest bifurcations. First consider the 2 dimensional load power parameter space of figure 1 which shows a critical loading $\lambda_* \in \Sigma$ which satisfies equations (5.1–5.4) since $\lambda_* - \lambda_0$ is parallel to the normal vector of Σ at λ_*. Solutions of (5.1–5.4) are stationary in the distance from λ_0 to Σ and not necessarily local minima. The condition guaranteeing that λ_* is a local minimum of the distance from λ_0 to Σ is that the curvature of the dashed circle of radius $|\lambda_* - \lambda_0|$ exceeds the curvature of Σ at λ_*. This condition is satisfied in figure 1. For a general parameter space $\mathbf{R}^m$, the condition guaranteeing that solutions of the equations (5.1–5.4) are locally closest bifurcations is that the curvature k_* of the sphere of radius $|\lambda_* - \lambda_0|$ strictly exceeds the maximum principal curvature k_{max} of Σ at λ_* [28]:

$$k_* = |\lambda_* - \lambda_0|^{-1} > k_{max} \tag{6.1}$$

Note that condition (6.1) will hold for λ_0 sufficiently close to Σ or k_{max} negative (Σ concave at λ_*).

Thus after computing a solution of (5.1–5.4), it is necessary to compute the curvature of Σ at λ_* and check condition (6.1) to ensure that the solution corresponds to a locally closest bifurcation. We quote formulas from [8,9] for the curvature of Σ. The formulas are simplified since f_λ is a constant matrix for equations (1.1). The curvature of Σ at λ_* is given by the $(m-1) \times (m-1)$ matrix

$$\mathrm{II} = y_{\lambda_*}^T (w_* f_{xx}|_{(x_*,\lambda_*)}) \, y_{\lambda_*} \tag{6.2}$$

and k_{max} is the largest eigenvalue of II. Note that the Hessian f_{xx} is a $p \times p \times p$ tensor and that $w_* f_{xx}$ is a $p \times p$ matrix. It remains to calculate the $p \times (m-1)$ matrix y_{λ_*} appearing in (6.2). y_{λ_*} satisfies $f_x|_{(x_*,\lambda_*)} y_{\lambda_*} = f_\lambda \pi^T$ but this is not sufficient to obtain y_{λ_*} since $f_x|_{(x_*,\lambda_*)}$ has rank $n-1$. (π is the projection of $\mathbf{R}^m$ onto $T\Sigma_{\lambda_*}$ along $n_* = w_* f_\lambda$.) The additional equation required to calculate y_{λ_*} is $v_*^T (w_* f_{xx}|_{(x_*,\lambda_*)}) \, y_{\lambda_*} = 0$ where v_* is the right eigenvector of $f_x|_{(x_*,\lambda_*)}$ corresponding to the zero eigenvalue. II is also of independent interest as II and the normal vector $w_* f_\lambda$ describe the geometry of Σ in $\mathbf{R}^m$ near λ_* to second order. The formulas for II are somewhat complicated and checking the convergence of the iterative method at a solution of (5.1-5.4) might well be more practical for large systems than computing II, k_{max} and directly checking (6.1).

7. Examples with 2 and 5 Buses

Consider the two bus power system model (1.2) with operating point given by $x_0 = (\alpha_0, V_0) = (-0.138, 0.908)$ and $\lambda_0 = (P_0, Q_0) = (0.5, 0.3)$ (see figure 1). Note that P and Q enter linearly in equations (1.2) and that f_λ is the negative of the 2×2 identity matrix.

The closest bifurcation at $x_* = (\alpha_*, V_*) = (-.338, 0.530)$ and $\lambda_* = (P_*, Q_*) = (0.703, 0.877)$ was computed using the iterative method (see table 1). At each step of the iteration, the direct method equations (3.3-3.5) were used to compute the closest bifurcation along a ray based at λ_0. As claimed in section 4, the convergence of the iterative method to the solution demonstrates that λ_* is a locally closest bifurcation.

i	n_i	λ_i
0	(0.585, 0.811)	(0.869, 0.811)
1	(0.398, 0.917)	(0.744, 0.862)
2	(0.349, 0.937)	(0.713, 0.873)
3	(0.336, 0.942)	(0.705, 0.876)
4	(0.333, 0.943)	(0.703, 0.876)
5	(0.332, 0.943)	(0.703, 0.877)
6	(0.331, 0.943)	(0.703, 0.877)

Table 1. 2 bus iteration

For the direct method, initial estimates are $x_1 = (\alpha_1, V_1) = (-0.410, 0.545)$, $\lambda_1 = (P_1, Q_1) = (0.869, 0.811)$, $w_1 = (0.398, 0.917)$, $k_1 = 1.586$. Solving (5.1),(5.2),(5.3),(5.5) by Newton's method [32] gives the same solution as the iterative method. The normal vector to Σ at λ_* is $n_* = (0.331, 0.943)$. The worst case load power margin is $|\lambda_* - \lambda_0| = 0.611$ and the curvature of the dashed circle in figure 1 is k_*=1.636. Equation (6.2) gives the curvature of Σ at λ_* as 0.420. Since k_* exceeds 0.420, condition (6.1) is satisfied, confirming that the solution is a locally closest bifurcation.

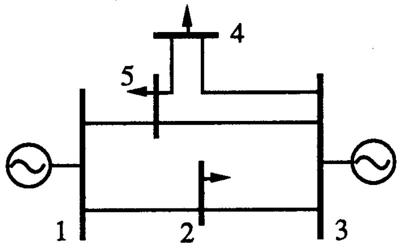

Figure 3. 5 bus system

Consider the five bus system from [23,2] shown in figure 3. This system has generators at buses 1 and 3 (bus 1 is slack), and loads at buses 2, 4 and 5. The line parameters are given in table 4 and the generator parameters are $V_1 = 1.04$, $\alpha_1 = 0$, $V_3 = 1.02$ and $P_3 = -1.1$. We choose the real and reactive powers consumed by the loads as the parameter vector $\lambda = (P_2, Q_2, P_4, Q_4, P_5, Q_5)$. The state vector $x = (\alpha_2, \alpha_3, \alpha_4, \alpha_5, V_2, V_4, V_5)$. The operating point x_0, bifurcating equilibrium x_1 along the ray of direction n_0, a closest bifurcating equilibrium x_* and the corresponding left eigenvectors w_0, w_1 and w_* are shown in table 2. The corresponding load powers λ_0, λ_1 and λ_* and the progress of the iteration of n_i are shown in table 3. These iterative method results check with the direct method results in [9].

	α_2	α_3	α_4	α_5	V_2	V_4	V_5
x_0	-.1043	-.0545	-.1759	-.0916	.9603	.9151	.9681
w_0	.3513	.4851	.7160	.3570	.0055	-.0326	.0074
x_1	-.3667	-.4189	-.9350	-.3743	.9161	.6038	.8214
w_1	.0379	.0581	.7595	.0983	.0083	.6143	.1770
x_*	-.1665	-.1622	-.4328	-.1593	.9567	.5253	.8524
w_*	.0094	.0157	.5213	.0550	.0009	.8385	.1477

Table 2. State space vectors for 5 bus iteration

	P_2	Q_2	P_4	Q_4	P_5	Q_5
λ_0	1.1500	0.6000	0.7000	0.3000	0.7000	0.4000
λ_1	1.7919	0.6101	2.0080	0.2404	1.3522	0.4136
n_0	0.3513	0.0055	0.7160	-0.0326	0.3570	0.0074
n_1	0.0379	0.0083	0.7595	0.6143	0.0983	0.1770
n_2	0.0130	0.0015	0.5957	0.7857	0.0607	0.1535
n_3	0.0103	0.0011	0.5425	0.8246	0.0563	0.1490
n_4	0.0097	0.0010	0.5272	0.8347	0.0553	0.1480
n_5	0.0095	0.0009	0.5229	0.8375	0.0551	0.1478
n_6	0.0095	0.0009	0.5217	0.8382	0.0550	0.1477
n_*	0.0094	0.0009	0.5213	0.8385	0.0550	0.1477
λ_*	1.1584	0.6008	1.1659	1.0495	0.7491	0.5320
$\lambda_* - \lambda_0$	0.0084	0.0008	0.4659	0.7495	0.0491	0.1320

Table 3. Parameter space vectors for 5 bus iteration

The curvature of Σ can be computed using the formulas of Section 6. The principal curvatures of Σ at λ_* are 0.313,0.026,0.008,0.017,0.000, $|\lambda_* - \lambda_0| = 0.8938$, and the curvature k_* of the sphere of radius $|\lambda_* - \lambda_0|$ is 1.1189. Since k_* exceeds the maximum principal curvature $k_{max} = 0.313$, condition (6.2) is satisfied. All calculations were done using [32].

Our preliminary experience suggests that the iterative method of computing $|\lambda_* - \lambda_0|$ is more robust to choice of initial conditions than the direct method of [9] which suffers from the sensitivity to initial conditions typical of Newton type methods. Moreover, the iterative method only converges to locally closest instabilities and does not require the curvature of Σ at the solution to be checked as in the direct method. However, the asymptotic convergence of the iterative method is only exponential and the direct method is faster to compute for the examples above.

8. Computational requirements

We briefly discuss the potential computational requirements for the iterative and direct methods.

Each iteration of the iterative method requires the load power margin ℓ_i and the normal vector n_i to Σ to be computed. See [4,20,5,30] for the computational requirements for ℓ_i. The main computational burden in computing n_i with equation (3.1) lies in computing w_i, the left eigenvector of the Jacobian $f_x|_{(x_i,\lambda_i)}$ corresponding to the zero eigenvalue. Direct and optimization methods of computing ℓ_i can yield w_i as a by-product (see section 3 and [13]). If continuation methods are used to compute ℓ_i, then w_i can be calculated using the inverse power method [19]. Another possibility is observing [14] that w_i is the left singular vector of $f_x|_{(x_i,\lambda_i)}$ corresponding to the zero singular value and using the inverse power method of Lof et al. [24].

The number of iterations of the iterative method depends on $|\lambda_* - \lambda_0|$ and the curvature of Σ. More precisely, the asymptotic convergence of the iterative method to λ_* is geometric with factor $\kappa|\lambda_* - \lambda_0|$ where κ is the maximum absolute principal curvature of Σ at λ_* [8,12]. Thus the asymptotic convergence to λ_* is quick if $|\lambda_* - \lambda_0|$ is small or if Σ is approximately flat near λ_*.

The computational requirements of the direct method are those of the direct method for the computation of ℓ_i [4] with an additional m linear equations (recall that m is the number of parameters).

9. Example of Σ with multiple hypersurfaces

We give an example of Σ being composed of two hypersurfaces Σ^1 and Σ^2 in a simple power system model. Consider a 3 bus power system consisting of an infinite bus connected by two identical, lossless transmission lines to two PQ load buses. The load buses are not directly connected to each other so that the system is essentially two decoupled infinite bus and PQ load systems. Then the load power parameter vector $\lambda = (P_1, Q_1, P_2, Q_2)$ and the parameter space $\mathbf{R}^4$ is the Cartesian product of Figure 1 with itself. There are two 3 dimensional bifurcation hypersurfaces $\Sigma^1 = \Sigma^0 \times A$ and $\Sigma^2 = A \times \Sigma^0$ where Σ^0 is the curve in Figure 1 and A is the area enclosed by the curve. Σ^1 and Σ^2 intersect in the 2 dimensional set $\Sigma^0 \times \Sigma^0$. It is unclear whether this example of multiple hypersurfaces is typical and determining the typical structure of Σ for power system models remains a challenging problem.

10. Index Sensitivities

If a voltage collapse index shows an unacceptable closeness to voltage collapse, it is useful to compute the sensitivity of the index to power system controls so that control action may be taken to optimally improve the index and the voltage stability of the system. For example, Tiranuchit and Thomas [29] compute the sensitivity of a minimum singular value index, Overbye and DeMarco [25] compute the sensitivity of an energy function index and Galiana and Jarjis [17] compute the sensitivity of the real power margin D.

The sensitivity to the operating parameters λ_0 of the load power margin index ℓ and the worst case load power margin index $|\lambda_* - \lambda_0|$ are rigorously derived in [13] and [12] respectively. Here we give an informal derivation of these index sensitivities.

First we assume a ray of load increase based at λ_0 in the direction n_0 as in (3.2) and derive the sensitivity to λ_0 of the index ℓ. (Recall that ℓ is the critical loading factor so that $\lambda_1 = \lambda_0 + \ell n_0 \in \Sigma$.) The key assertion in informally deriving the first order sensitivity of ℓ is that it is valid to approximate Σ to first order near λ_1; that is, Σ may be approximated near λ_1 by its tangent hyperplane $T\Sigma|_{\lambda_1}$ at λ_1. Now the problem reduces to finding the sensitivity to λ_0 of the distance $\ell = |\lambda_1 - \lambda_0|$ of the point λ_0 to the hyperplane $T\Sigma|_{\lambda_1}$ in the direction n as shown in figure 4. But $|\lambda_1 - \lambda_0|$ is proportional to the perpendicular distance D from λ_0 to $T\Sigma|_{\lambda_1}$:

$$\ell = |\lambda_1 - \lambda_0| = D(\cos\theta)^{-1} = D\,(N(\lambda_1).n_0)^{-1}$$

and the sensitivity $D_{\lambda_0} = -N(\lambda_1)$ since the optimum direction to move away from a hyperplane is normal to the hyperplane [17]. Hence the required sensitivity is

$$\ell_{\lambda_0} = -N(\lambda_1)\,(N(\lambda_1).n_0)^{-1} = -w_1 f_\lambda\,(w_1 f_\lambda.n_0)^{-1} \quad (10.1)$$

Informally deriving the sensitivity of the index $|\lambda_* - \lambda_0|$ is similar but easier. Let λ_* be the parameter of the closest bifurcation to λ_0 and approximate Σ near λ_* by $T\Sigma|_{\lambda_*}$. Then $|\lambda_* - \lambda_0|$ is the perpendicular distance D from λ_0 to $T\Sigma|_{\lambda_*}$ and the required sensitivity (also see [17]) is

$$|\lambda_* - \lambda_0|_{\lambda_0} = D_{\lambda_0} = -N(\lambda_*) = -\alpha w_* f_\lambda \qquad (10.2)$$

For both indices, the geometric content is clear: the optimum direction to increase the distance of a point λ_0 to a hypersurface Σ is antiparallel to the normal vector to Σ.

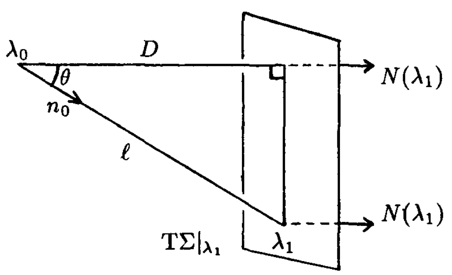

Figure 4. Geometry of sensitivity of ℓ

If the parameters λ are load powers, then the sensitivity formulas (10.1) and (10.2) are useful in determining the optimum combination of loads to shed in order to avoid voltage collapse. That is, the loads which it is most effective to shed correspond to the larger entries in the vector $N(\lambda_*)$. Shedding loads corresponding to the smaller entries tends to move λ in a direction more parallel to Σ rather than away from Σ. It is straightforward to obtain the sensitivities to any power system parameter or control by augmenting the parameter space with these parameters or controls and hence compute the optimum combination of controls to avoid voltage collapse [13].

In the 5 bus example of section 7, inspection of the components of n_* in table 3 shows that Q_4 is the load bus power most influential on the load power margin so that shedding load at bus 4 is most effective in increasing the index $|\lambda_* - \lambda_0|$.

11. Conclusions

We propose new iterative and direct methods for computing a voltage collapse index which is a worst case load power margin and the distance to a locally closest saddle node bifurcation in load power parameter space. Both methods exploit a recent observation [11] that the normal vector to hypersurfaces of the critical load powers Σ is easy to calculate. The worst case load power margin is useful when the direction of load increase in load power parameter space is unknown or uncertain and, when the direction of load increase is known, it supplements load power margins which assume the direction of load increase. The computations can be done on static power system models such as the load flow equations and the results apply to a class of underlying dynamic power system models whose dynamics need not be completely specified [11].

The iterative method is easy to implement if any of the standard methods for finding load power margins assuming a direction of load increase is given. The iterative method seems simpler and more robust than the direct method but may be slower to compute. The direct method is a generalization of the extended system equations [27,2,5,23] and can be related to an optimization similar to that of [23]. The direct method has the disadvantages of requiring good initial solution estimates and a check on a condition involving the curvature of Σ.

If the operating load powers λ_0 are close to several portions of Σ it would be desirable to compute a locally closest saddle node bifurcation on each of the portions of Σ but it is not known whether this situation will arise often in practice. More needs to be known about the general structure of Σ in order that these methods can be shown effective or improved to take advantage of this structure. The formulas for the normal vector [11] and curvature of Σ [8,9] are basic steps towards describing Σ. We neglect in this paper the significant possibility of oscillatory instability via a Hopf bifurcation, but see [8,12] for an iterative method for finding a locally closest Hopf bifurcation. The methods could apply generally to finding closest saddle node instabilities of a general dynamical system with parameters.

We give examples of computing a worst case load power margin and the normal vector and curvature of Σ. The larger example has 5 buses and a 6 dimensional load power parameter space. This example is too small to illustrate the practicality of the methods on large power system models and further work is needed to turn the calculation for this example into a robust and efficient algorithm for large power system models.

The sensitivity of the worst case load power margin to parameters is easy to obtain from the normal vector to Σ and we give an informal derivation of this useful sensitivity. We also informally derive the sensitivity of the load power margin index which assumes a direction of load increase.

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Line(bus to	*bus)*	*G(p.u.)*	*B(p.u.)*
1	2	1.40056	-5.60224
1	5	1.84118	-7.48352
2	3	1.84118	-7.48352
3	4	0.70028	-2.80112
3	5	1.12985	-4.47675
4	5	0.93372	-3.43483

Table 4. Line data for 5 bus system

Ian Dobson (M'89) received the BA in Mathematics from Cambridge, England in 1978 and joined the University of Wisconsin-Madison faculty after receiving the PhD in Electrical Engineering from Cornell in 1989. His industrial experience working for the British firm EASAMS from 1978 to 1983 included writing a general simulation of switching power supplies. His current interests include voltage collapse and other power system instabilities and applications of bifurcation theory, nonlinear dynamics and chaos.

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Discussion

M. A. Pai (University of Illinois, Urbana, IL): This is a very interesting paper which yields the closest saddle node bifurcation point (CSNB) for a given stable operating point. This in some sense is analogous to the closest u.e.p. computation in angle stability studies. As one would expect the CSNB point will give conservative results regarding voltage security. This is illustrated clearly in Fig. 1. However, it may give some clue to the operator as to how to steer the same system away from λ_* by using the sensitivities. Most of the literature to date in voltage collapse has been in terms of increasing P or Q only at a given bus. This paper gives a global view of P or Q increases at all the buses. The load margin seems to be related in some sense to maximum loadability of the system. Can the authors comment on this? Also can these computations be done for an area only instead of the system as a whole. In the 6 bus example, λ_* identifies the weakest bus which is #4 in this case. It is possible that the method identifies only weakest buses in the system and it could be used to alert the operator. Application to a large scale system using this method would be helpful.

As for Hopf bifurcation, it is the opinion of the discusser that the simple swing model may mask the Hopf bifurcation due to the exciter model. The model of the machine and exciter dictate which complex pair of mode goes unstable first depending on the controllable parameters such as exciter gain, etc. This represents a fruitful area of research in parametric stability.

Adam Semlyen (University of Toronto): I would like to congratulate the authors for their interesting and instructive paper. It provides useful insight into the problem of static voltage stability analysis by giving a geometrical description and illustration by hand of a small system. It also presents a practical method for computing the margin to voltage collapse.

It is interesting to note that the initial problem formulation in this paper consists of differential equations only, rather than a system of differential and algebraic equations, as suggested in several references. I also preferred, until recently, to view some of the equations, notably those expressing the reactive power balance constraint, as being strictly algebraic. This is not because the related dynamics may not be known, but because I felt there should be no dynamics at all, since the reactive power Q (in contrast to the real power P) is an artificial concept, difficult to be viewed as representing a driving force for any dynamic phenomenon. Still, even this difficulty can be overcome if one includes network dynamics in the study, at least in principle, so that all equations can be considered to be ODEs. It is, however, normally closer to reality to attribute the existence of an approximately constant P and/or Q load to the intermittent tap changing dynamics of the transformer.

The nice thing about the uniformity in the approach (of using only ODEs for describing the system, as in equations (2.2)) is that *a static voltage stability analysis becomes meaningful in all practical cases.* The resulting bifurcation can be described as saddle-node or Hopf, which are concepts related to dynamic phenomena, even though the practical analysis is performed on algebraic equations (as (2.1)). The question arises, however, whether the likelihood for Hopf bifurcations is not overly significant in the authors' problem formulation? Indeed, in the paper both P and Q are used as variables, so that Hopf bifurcations may easily occur. In the approach of Gao et al[A], where P is eliminated from the set of variables so that the resulting Jacobian matrix is almost symmetrical, the critical eigenvalues are always real.

The geometrical concept of a distance to the bifurcation point raises some interesting questions. If the distance is Euclidean, it implies that, when forming a sum of squares, real powers P and reactive powers Q are equally weighted. Is a heavier weighting of reactive powers not preferable when it comes to voltage stability studies? Certainly, weighting may affect the solution and may have some relation to my previous question regarding the nature of the resulting bifurcation point.

For an operating point still away from the point of collapse, distance is an integral concepts (the sum of infinitesimal distances). As alternatives, one may consider local indicators, mainly regarding the direction to the bifurcation point. These could be either the smallest eigenvalue with the corresponding eigenvectors or the smallest singular value with the corresponding singular vectors of the Jacobian matrix J in question (in short EIG and SVD). All three methods have in common the fact that their features are derived by considering a point approaching the one at bifurcation, as a limit. It is interesting to note that all three give the same result and are, perhaps, conceptually equivalent at the limit point of saddle bifurcation. For EIG and SVD this results from the simple fact that, *for a matrix A equal to the inverse of J that approaches singularity, the eigenvalue decomposition and the singular value decomposition are identical.* Indeed, as one eigenvalue of J approaches zero, A becomes a rank-one matrix, with the unique decomposition $A=\mu uv^T$ (if u and v are normalized to unit length). In this decomposition the scalar μ ($\to\infty$) is a measure of closeness to the bifurcation point; u and v are the critical eigen/singular vectors (there is now no distinction between the two!) of J or A.

Also noteworthy is the fact that, *even away from the bifurcation point, SVD gives exact local information* for both the direction u of largest voltage variation Δx and for the most sensitive direction v of the control vector $\Delta\lambda$, since the SVD for the relevant matrix A yields orthogonal modes. (The left and right eigenvectors of an eigenvalue decomposition are only mutually, rather than individually, orthogonal.) Therefore, u and v are simply the singular vectors corresponding to the largest singular value.

If several eigen/singular values of J become small simultaneously then, instead of single critical directions, the respective eigen/singular vectors define *subspaces* for maximal voltage variations and optimal control directions.

Comments and clarifications from the authors would be most appreciated.

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Ian Dobson: We thank Professors Pai and Semlyen for their discussions.

Professor Pai asks about the relation of the closest saddle node bifurcation to maximum loadability. The distance to a closest bifurcation is a minimum of the loadability given a slack bus or participation factors for increasing the generation as the load increases. It is possible to maximize the distance to a closest bifurcation by rescheduling the generation; that is, maximize over generation the minimum loading distance to bifurcation. This computation is suggested in [C1] and gives a maximum loadability of the system.

Concerning the closest bifurcation computations for an area: Since there is freedom in choosing how the system loading is parameterized, only the loads in a given area could be allowed to vary. The parameter space would then be a subspace of the full parameter space in which all loads vary. It is straightforward to compute the closest bifurcation in the subspace. This is one sense in which the computations can be done for an area.

Professor Pai correctly points out that some power system models will have Hopf bifurcations in addition to saddle node bifurcations. The method presented in the paper should apply to computing closest saddle node bifurcations in those power system models but the Hopf bifurcations would not necessarily be detected. The detection of the Hopf points depends on the algorithm used to detect the first bifurcation along a given ray of load increase. A continuation method could detect the first Hopf *or* saddle node bifurcation as can some direct methods [C2]. If a Hopf occurred first then the normal vector formula appropriate to the Hopf hypersurface would be used [12]. The method suggested in the paper to detect the first bifurcation along a given ray of load increase detects the first saddle node bifurcation but neglects the Hopf bifurcations.

In general, the power system operation and design to avoid instabilities must avoid both the saddle node and Hopf bifurcations and we agree that this is an interesting engineering problem. It is useful to compute the Hopf bifurcation assuming a given load increase and then use the Hopf normal vector formula to derive the sensitivities of the load power margin to Hopf bifurcation with respect to power system parameters and controls [C3]. Then the sensitivities may be used to change power system parameters to optimally increase the load power margin to Hopf bifurcation. Similar techniques could be used to "steer" the system away from a closest Hopf bifurcation once the closest Hopf bifurcation was computed using the iterative method.

Professor Semlyen questions the appropriateness of power system models in which Hopf bifurcations occur. First we note that the Hopf analysis is different from the saddle node analysis since it cannot be properly done without some knowledge of the details of the underlying dynamics [C3]. The present models (particularly of the load dynamics) have not been demonstrated to be suitable to make predictions about the real power system. Thus the current dynamic models may predict more or fewer Hopf bifurcations than actually occur. It is true that Hopf bifurcations can be eliminated by simplifying the power system model in such a way that the Jacobian is symmetric. However, even if the real system had no Hopf bifurcations, this does not necessarily justify the simplification of the model. On the contrary, one could argue that the power system model should be elaborated to model a well designed power system stabilizer to eliminate the Hopf bifurcations. It seems to us that the degree to which Hopf bifurcations are present at extreme power system loadings is unknown. Gao et al. [A] choose to keep P constant so that QV characteristics can be studied. They do not address the relation of their simplified model to more detailed models.

Professor Semlyen makes the pertinent observation that different weightings or scalings of the parameters will yield different answers for the closest bifurcation. In other words, different metrics yield different closest bifurcations. Thus there may be some arbitrariness in the closest bifurcation depending on the scaling chosen. We chose for P and Q what seemed to us the most straightforward choice of equal weighting. We would prefer the method to compute for us the relative importance of P and Q rather than assume the relative importance a priori. Other weightings are possible, but we do not know how to justify them. One sensible approach avoids this question by assuming constant power factor at each load and using either P to parameterize all the loads or Q to parameterize all the loads.

Professor Semlyen discusses the relation of eigenvalues, singular values and load power margins as voltage collapse indexes. Our opinion is that eigenvalues and singular values are relatively unsuitable voltage collapse indexes because they vary nonlinearly as the bifurcation is approached and it is hard to predict far from the bifurcation which eigenvalue or singular value will become zero at the bifurcation. Load power margin takes full account of the system nonlinearity (it is not based on a linearization valid only locally) and is readily understood. However, the singular value computations are well suited for computing the sensitivities of voltage magnitude at the stable equilibrium.

At a saddle node bifurcation the load power margin, an eigenvalue and a singular value are all zero. The coincidence of the eigenvectors and singular vectors corresponding to the zero eigenvalue and zero singular value is observed and proven in [14]. The inverse matrix A in Professor Semlyen's discussion is only asymptotically rank one if some suitable normalization of the matrix entries is used.

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