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New Methods for Finite Element Model Updating Problems

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We consider two finite element model updating problems, which incorporate the measured modal data into the analytical finite element model, producing an adjusted model on the (mass) damping and stiffness, that closely matches the experimental modal data. We develop two efficient numerical algorithms for solving these problems. The new algorithms are direct methods that require $\mathcal{O}(nk^2)$ and $\mathcal{O}(nk^2+k^6)$ flops, respectively, and employ sparse matrix techniques when the analytic model is sparse. Here n is the dimension of the coefficient matrices defining the analytical model, and k is the number of measured eigenpairs.

Nomenclature

$\boldsymbol{b}$	=	see Eq. (21b); $k \times 1$
C	=	adjusted damping matrix, $n \times n$
C_a	=	damping matrix of the original finite element model (FEM), $n \times n$
D	=	see Eq. (13); $k \times k$
$\boldsymbol{e}_i$	=	ith standard vector of $\mathbb{R}^n$
G	=	see Eq. (21a); $k \times k$
I_k	=	$k \times k$ identity matrix
$\imath$	=	$\sqrt{-1}$
K	=	adjusted stiffness matrix, $n \times n$
K_a	=	stiffness matrix of the original FEM model, $n \times n$
k	=	number of measured eigenvalues or eigenvectors
M	=	adjusted mass matrix, $n \times n$
M_a	=	mass matrix of the original FEM model, $n \times n$
Q	=	orthogonal matrix, $n \times n$
R	=	upper triangular matrix, $k \times k$
$\mathrm{Re}(A)$	=	real part of A
$r_{i,j}$	=	ith component of $\boldsymbol{r}_j$
$\boldsymbol{r}_j$	=	jth column of R^{-1}, $k \times 1$
S	=	$R\Lambda R^{-1}$, $k \times k$
$\mathcal{S}_1(M_{11})$	=	see Eq. (36a); $k \times k$
$\mathcal{S}_2(M_{11})$	=	see Eq. (37a); $k \times 1$
$\mathbb{S}^{k\times k}$	=	set of all $k \times k$ symmetric matrices
$\mathcal{R}_1(D)$	=	see Eq. (36b); $k \times k$
$\mathcal{R}_2(D)$	=	see Eq. (37b); $k \times 1$
$\mathrm{tr}(\cdot)$	=	trace operator
$\mathrm{vec}(\cdot)$	=	vectorization operator
$\boldsymbol{x}$	=	solution of linear system, $k \times 1$
Γ_j	=	see Eq. (18); $k \times k$
δ	=	see Eq. (41)
Λ	=	diagonal block eigenvalue matrix, $k \times k$
λ	=	eigenvalues of quadratic eigenvalue problem
μ	=	weight factor
ν	=	weight factor
Φ	=	eigenvector matrix, $n \times k$
∇	=	gradient of a function
∇_A	=	gradient of a function with respective to the elements of A
$\circ$	=	Hadamard product
$\otimes$	=	Kronecker product
$(\cdot)^\top$	=	transpose
$\|\cdot\|_F$	=	Frobenius norm

I. Introduction

VIBRATING systems, such as automotives, bridges, highways, and buildings are usually described by distributed parameters. However, because of the lack of viable computational methods to handle distributed parameter systems, a finite element method is generally used to discretize such systems to an analytical finite element model (see Ref. 1, Chapter 2, for details), namely, a second-order differential equation

$$M_a\ddot{\boldsymbol{q}}(t) + C_a\dot{\boldsymbol{q}}(t) + K_a\boldsymbol{q}(t) = \boldsymbol{f}(t) \tag{1}$$

Here M_a, C_a, and $K_a \in \mathbb{R}^{n\times n}$ are all symmetric and represent the analytical mass, damping, and stiffness matrices, respectively (with M_a being symmetric positive definite, or $M_a > 0$), $\boldsymbol{q}(t)$ is the $n \times 1$ vector of positions, and $\boldsymbol{f}(t)$ is the $n \times 1$ vector of external force. It is known that solving the homogeneous equation (1) [i.e., $\boldsymbol{f}(t) \equiv 0$] corresponds to solving the quadratic eigenvalue problem (QEP)

$$Q_a(\lambda)\boldsymbol{x} = \left(\lambda^2 M_a + \lambda C_a + K_a\right)\boldsymbol{x} = 0 \tag{2}$$

by letting $\boldsymbol{q}(t) = e^{\lambda t}\boldsymbol{x}$. The scalar λ and the associated vector $\boldsymbol{x}$ in Eq. (2) are called, respectively, eigenvalues and eigenvectors of the quadratic pencil $Q_a(\lambda)$. Note that the QEP (2) has $2n$ finite eigenvalues because the leading M_a is nonsingular.

In the finite element model (2) for structural dynamics, the analytical mass and stiffness matrices are, in general, clearly defined by physical parameters and evaluated by static tests. However, the analytical damping matrix for precise dissipative effects is not well understood because it is a purely dynamics property that cannot be measured statically and must be determined by dynamic testing. This makes the process of modeling and experimental verification difficult. A common simplification is to assume proportional damping, which seems to be sufficient where damping levels are lower than 10% of critical.[2] Two new methods for damping matrix identification, which produce accurate representative damping matrices,[3] are developed. They serve to integrate the theory and practical application of damping matrix identification. Therefore, it is assumed in this paper that acceptable models of the analytical mass, damping, and stiffness matrices are available. It is our objective to incorporate

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the measured modal data into the finite element model, aiming to produce an adjusted finite element model on the mass, damping, and stiffness with modal properties that closely match the experimental modal data.

Finite element model updating (FEMU) problems have emerged in the 1990s as a significant subject to the design, construction, and maintenance of mechanical systems. Model updating, at its most ambitious form, attempts to correct errors in a finite element model. It uses measured data such as natural frequencies, damping ratios, mode shapes, and frequency response functions, which can usually be obtained by vibration test. In the past decade, a number of approaches to the FEMU problem are proposed (see Refs. 1 and 4 and references therein). For example, Baruch,[5] Baruch and Bar-Itzak,[6] Bermann,[7] Bermann and Nagy,[8] and Wei[9−11] proposed various updating methods to correct the analytical mass and stiffness matrices of undamped systems (i.e., $C_a = 0$). In Datta,[12] Datta et al.,[13] and Datta and Sarkissian,[14] studies are undertaken toward a nonsymmetric feedback design problem for second-order control system. That consideration eventually leads to a partial eigenstructure assignment problem for the QEP. A new symmetric feedback design for the QEP using symmetric eigenstructure assignment was recently developed in Ref. 15.

The FEMU problem for damped systems was first proposed by Friswell et al.[2] They considered the mass matrix to be exact and updated the damping and stiffness matrices by using the measured modal data as a reference. Following the basic idea of Refs. 5 and 6, they minimized the difference between the analytical and updated damping/stiffness matrices, subject to the constraints that the eigenmatrix equation is satisfied and the damping/stiffness matrices are symmetric. That is, the FEMU problem proposed by Ref. 2 can be formulated by the following constrained optimization problem.

Problem FEMU-I. Find $n \times n$ real matrices C and K to minimize the objective function

$$J = \tfrac{1}{2}\nu \left\| M_a^{-\frac{1}{2}}(C - C_a)M_a^{-\frac{1}{2}} \right\|_F^2 + \tfrac{1}{2}\left\| M_a^{-\frac{1}{2}}(K - K_a)M_a^{-\frac{1}{2}} \right\|_F^2 \quad (3a)$$

subject to

$$M_a \Phi \Lambda^2 + C\Phi\Lambda + K\Phi = 0 \quad (3b)$$

$$C^\top = C, \qquad K^\top = K \quad (3c)$$

Here M_a, C_a, and K_a are, respectively, the analytical mass, damping, and stiffness matrices; $\nu > 0$ is a weighting parameter; and C and K are, respectively, the updated damping and stiffness matrices. The measured eigenvalue matrix Λ and the associated eigenvector matrix Φ satisfy

$$\Lambda = \mathrm{diag}\left(\lambda_1^{[2]}, \ldots, \lambda_\ell^{[2]}, \lambda_{2\ell+1}, \ldots, \lambda_k\right) \in \mathbb{R}^{k\times k} \quad (4a)$$

with $k \ll n$ and

$$\lambda_j^{[2]} = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}, \qquad \beta_j \neq 0$$

for $j = 1, \ldots, \ell$, and

$$\Phi = [\varphi_{1R}, \varphi_{1I}, \ldots, \varphi_{\ell R}, \varphi_{\ell I}; \varphi_{2\ell+1}, \ldots, \varphi_k] \in \mathbb{R}^{n\times k} \quad (4b)$$

Throughout this paper, we assume that Λ in Eq. (4a) has only simple eigenvalues and Φ in Eq. (4b) is of full column rank.

In a finite element model, the mass is usually well defined by physical parameters. However, we shall consider a more general and interesting problem that the analytical mass, damping, and stiffness matrices are all allowed to be updated. The second FEMU problem can be formulated by the constrained optimization problem:

Problem FEMU-II. Determine the $n \times n$ real matrices M, C, and K to minimize the objective function

$$J = \tfrac{1}{2}\mu \left\| M_a^{-\frac{1}{2}}(M - M_a)M_a^{-\frac{1}{2}} \right\|_F^2 + \tfrac{1}{2}\nu \left\| M_a^{-\frac{1}{2}}(C - C_a)M_a^{-\frac{1}{2}} \right\|_F^2$$
$$+ \tfrac{1}{2}\left\| M_a^{-\frac{1}{2}}(K - K_a)M_a^{-\frac{1}{2}} \right\|_F^2 \quad (5a)$$

subject to

$$M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0 \quad (5b)$$

$$M^\top = M, \qquad C^\top = C, \qquad K^\top = K \quad (5c)$$

Here M_a, C_a, and K_a are, respectively, the analytical mass, damping, and stiffness matrices; μ, $\nu > 0$ are weighting parameters; and M, C, and K are, respectively, the updated mass, damping, and stiffness matrices. The measured eigenvalue matrix Λ and the associated eigenvector matrix Φ are defined in Eqs. (4a) and (4b), respectively.

For problem FEMU-I, Friswell et al.[2] and Pilkey[3] proposed an updating method by using the Lagrange multiplier method to solve Eq. (3). The solutions C and K are given by

$$C = C_a - (2/\nu)M_a \,\mathrm{Re}\left(\Gamma_\Lambda \Lambda \Phi^\top + \Phi\Lambda\Gamma_\Lambda^\top\right)M_a \quad (6)$$

$$K = K_a - 2M_a \,\mathrm{Re}\left(\Gamma_\Lambda \Phi^\top + \Phi\Gamma_\Lambda^\top\right)M_a \quad (7)$$

where $\Gamma_\Lambda \in \mathbb{C}^{n\times k}$ solves the linear equation

$$2M_a \,\mathrm{Re}\left(\Gamma_\Lambda \Phi^\top + \Phi\Gamma_\Lambda^\top\right)M_a\Phi + (2/\nu)M_a \,\mathrm{Re}\left(\Gamma_\Lambda\Lambda\Phi^\top\right.$$
$$\left. + \Phi\Lambda\Gamma_\Lambda^\top\right)M_a\Phi\Lambda = M_a\Phi\Lambda^2 + C_a\Phi\Lambda + K_a\Phi \quad (8)$$

There are two weaknesses for the method. First, the solution Γ_Λ in Eq. (8) is, in general, complex, whereas the updated matrices C and K are expected to be real symmetric. Second, the dimension n of coefficient matrices in the finite element model (2) is usually quite large. It is impractical to solve the large and dense linear system (8), which requires $\mathcal{O}(n^3k^3)$ flops.

In Sec. III, we develop an efficient algorithm for solving problem FEMU-I in Eq. (3). The new algorithm is a direct method, which avoids the Lagrange multiplier method in Refs. 2 and 3 requiring only $\mathcal{O}(n^2k)$ flops. In practice, M_a, C_a, and K_a are usually sparse with $\mathcal{O}(n)$ nonzero entries, and the computational cost is then reduced to $\mathcal{O}(nk^2)$ flops.

For problem FEMU-II, a dual optimization approach with a Newton-type method has been developed in Ref. 16 to solve Eq. (5), for $M > 0$ and $K \geq 0$ (positive semidefinite). It is an iterative algorithm that solves an $nk \times nk$ linear system by the conjugate gradient method in each iteration.

In Sec. IV, we develop an efficient algorithm to solve problem FEMU-II in Eq. (5), dropping the positive semidefiniteness requirement on K. The new algorithm is a direct method that computes a symmetric positive-definite M when the weighting parameter μ in Eq. (5a) is larger than δ [given by Eq. (41)]. The computation cost is $\mathcal{O}(n^2k)$ flops and reduces to $\mathcal{O}(nk^2)$ flops when M_a, C_a, and K_a are sparse with $\mathcal{O}(n)$ nonzero entries.

II. Solving a Partially Described Inverse QEP

For a given matrix pair $(\Lambda, \Phi) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k} (k \leq n)$, where Λ and Φ are defined by Eqs. (4a) and (4b), respectively, we now consider the partially described inverse quadratic eigenvalue problem (PD-IQEP):

Find a general form of symmetric matrices M, C, and K, with M being positive definite that satisfies the equation

$$M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0 \quad (9a)$$

$$M^\top = M > 0, \qquad C^\top = C, \qquad K^\top = K \quad (9b)$$

A general solution to the PD-IQEP is given in Ref. 15 as follows:

Theorem II.1. Let Φ have the QR factorization

$$\Phi = Q\begin{bmatrix} R \\ 0 \end{bmatrix} \equiv [Q_1, Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix} \quad (10)$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal with $Q_1 \in \mathbb{R}^{n\times k}$ and $R \in \mathbb{R}^{k\times k}$ is nonsingular, and let $S = R\Lambda R^{-1}$. Then the general solution to the

PD-IQEP defined by Eqs. (9a) and (9b) is given by

$$M = Q\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} Q^\top, \qquad C = Q\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} Q^\top$$

$$K = Q\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} Q^\top \tag{11}$$

Here the $n \times n$ symmetric positive-definite matrix

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

the $(n-k)\times(n-k)$ symmetric submatrices C_{22} and K_{22}, and the $(n-k)\times k$ submatrix $C_{21} = C_{12}^\top$ can be arbitrarily chosen. The symmetric submatrices C_{11} and K_{11} and the submatrices K_{21} and K_{12} satisfy

$$C_{11} = -\left(M_{11}S + S^\top M_{11} + R^{-\top} D R^{-1}\right) \tag{12a}$$

$$K_{11} = S^\top M_{11} S + R^{-\top} D \Lambda R^{-1} \tag{12b}$$

$$K_{21} = K_{12}^\top = -\left(M_{21}S^2 + C_{21}S\right) \tag{12c}$$

with

$$D = \operatorname{diag}\left(\begin{bmatrix} \xi_1 & \eta_1 \\ \eta_1 & -\xi_1 \end{bmatrix}, \ldots, \begin{bmatrix} \xi_\ell & \eta_\ell \\ \eta_\ell & -\xi_\ell \end{bmatrix}, \xi_{2\ell+1}, \ldots, \xi_k\right) \tag{13}$$

and ξ_i and η_i being arbitrary real numbers.

In the rest of this paper, we will utilize this result to develop two efficient algorithms for solving problems FEMU-I and FEMU-II described in Sec. I.

III. Solving Problem FEMU-I

To solve problem FEMU-I, we first solve two optimization problems. Let D and R be given in Eqs. (13) and (10), respectively. We denote

$$R^{-1} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k] = \begin{bmatrix} r_{11} & \cdots & r_{1k} \\ & \ddots & \vdots \\ 0 & & r_{kk} \end{bmatrix} \tag{14}$$

Problem I. Given $A = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k]$, $B = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k] \in \mathbb{R}^{k\times k}$, and $\nu > 0$, let

$$\boldsymbol{x} = (\xi_1, \eta_1, \ldots, \xi_\ell, \eta_\ell, \xi_{2\ell+1}, \ldots, \xi_k)^\top \tag{15}$$

be constructed from the matrix D in Eq. (13). Find $\boldsymbol{x}^*$ to minimize

$$f(\boldsymbol{x}) = \nu\|A + R^{-\top} D R^{-1}\|_F^2 + \|B - R^{-\top}\Lambda^\top D R^{-1}\|_F^2 = \sum_{j=1}^{k} f_j(\boldsymbol{x}) \tag{16a}$$

where

$$f_j(\boldsymbol{x}) = \nu\left\|\boldsymbol{a}_j + R^{-\top} D \boldsymbol{r}_j\right\|_2^2 + \left\|\boldsymbol{b}_j - R^{-\top}\Lambda^\top D \boldsymbol{r}_j\right\|_2^2 \tag{16b}$$

Solution. Note that

$$\begin{bmatrix} \xi & \eta \\ \eta & -\xi \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u & v \\ -v & u \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

The vector $D\boldsymbol{r}_j$ in Eq. (16b) can be rewritten as

$$D\boldsymbol{r}_j = \Gamma_j \boldsymbol{x} \qquad j = 1, \ldots, k \tag{17}$$

where

$$\Gamma_j = \operatorname{diag}\left(\begin{bmatrix} r_{1j} & r_{2j} \\ -r_{2j} & r_{1j} \end{bmatrix}, \ldots, \begin{bmatrix} r_{2\ell-1,j} & r_{2\ell,j} \\ -r_{2\ell,j} & r_{2\ell-1,j} \end{bmatrix}, r_{2\ell+1,j}, \ldots, r_{k,j}\right) \in \mathbb{R}^{k\times k} \tag{18}$$

Substituting Eq. (17) into Eq. (16b), and then differentiating $f_j(\boldsymbol{x})$, we have

$$\begin{aligned} \nabla f_j(\boldsymbol{x}) &= \left(\frac{\partial f_j}{\partial x_1}, \ldots, \frac{\partial f_j}{\partial x_k}\right)^\top \\ &= 2\nu\left(R^{-\top}\Gamma_j\right)^\top\left(\boldsymbol{a}_j + R^{-\top}\Gamma_j\boldsymbol{x}\right) - 2\left(R^{-\top}\Lambda^\top\Gamma_j\right)^\top \\ &\quad \times \left(\boldsymbol{b}_j - R^{-\top}\Lambda^\top\Gamma_j\boldsymbol{x}\right) \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \nabla f(\boldsymbol{x}) &= \sum_{j=1}^{k} \nabla f_j(\boldsymbol{x}) \\ &= 2\sum_{j=1}^{k}\left[\nu\left(R^{-\top}\Gamma_j\right)^\top \boldsymbol{a}_j + \nu\Gamma_j^\top (R^\top R)^{-1}\Gamma_j\boldsymbol{x} - \Gamma_j^\top \Lambda R^{-1}\boldsymbol{b}_j \right. \\ &\quad \left. + \Gamma_j^\top\Lambda(R^\top R)^{-1}\Lambda^\top\Gamma_j\boldsymbol{x}\right] \end{aligned} \tag{19}$$

Setting $\nabla f(\boldsymbol{x}) = 0$, we derive the linear equation for $\boldsymbol{x}$:

$$G\boldsymbol{x} = \boldsymbol{b} \tag{20}$$

where

$$G = \sum_{j=1}^{k}\left[\nu\Gamma_j^\top (R^\top R)^{-1}\Gamma_j + \Gamma_j^\top\Lambda(R^\top R)^{-1}\Lambda^\top\Gamma_j\right] \tag{21a}$$

$$\boldsymbol{b} = \sum_{j=1}^{k}\left(\Gamma_j^\top \Lambda R^{-1}\boldsymbol{b}_j - \nu\Gamma_j^\top R^{-1}\boldsymbol{a}_j\right) \tag{21b}$$

Because the function $f(\boldsymbol{x})$ in Eq. (16a) must have an optimum, the linear system of Eq. (20) is consistent, and therefore $\boldsymbol{x} = \boldsymbol{x}^*$ is solvable.

Problem II. Given $E, F \in \mathbb{R}^{(n-k)\times k}$, $\nu > 0$, and $S = R\Lambda R^{-1}$ as in Theorem II.1; minimize

$$g(X) = \nu\|E - X\|_F^2 + \|F + XS\|_F^2 \tag{22}$$

for $X = [x_{ij}] \in \mathbb{R}^{(n-k)\times k}$.

Solution. Differentiating Eq. (22) yields

$$\begin{aligned} \frac{\partial g}{\partial x_{ij}} &= -2\nu\operatorname{tr}\left[(E - X)^\top \boldsymbol{e}_i\boldsymbol{e}_j^\top\right] + 2\operatorname{tr}\left[(F + XS)^\top \boldsymbol{e}_i\boldsymbol{e}_j^\top S\right] \\ &= -2\nu\boldsymbol{e}_i^\top (E - X)\boldsymbol{e}_j + \boldsymbol{e}_i^\top (F + XS)S^\top \boldsymbol{e}_j \end{aligned}$$

and so we have

$$\nabla g(X) = 2[-\nu E + \nu X + FS^\top + XSS^\top] \tag{23}$$

By solving $\nabla g(X) = 0$, we get

$$X = (\nu E - FS^\top)(\nu I + SS^\top)^{-1} \tag{24}$$

We now return to problem FEMU-I. Let

$$C_a := M_a^{-\frac{1}{2}} C_a M_a^{-\frac{1}{2}}, \qquad K_a := M_a^{-\frac{1}{2}} K_a M_a^{-\frac{1}{2}} \tag{25a}$$

$$C := M_a^{-\frac{1}{2}} C M_a^{-\frac{1}{2}}, \qquad K := M_a^{-\frac{1}{2}} K M_a^{-\frac{1}{2}} \tag{25b}$$

$$\Phi := M_a^{\frac{1}{2}}\Phi, \qquad M := M_a^{-\frac{1}{2}} M_a M_a^{-\frac{1}{2}} = I \tag{25c}$$

Then it follows from Eqs. (11–13) and $Q = [Q_1, Q_2]$ that problem FEMU-I becomes

$$\min\left\{\frac{1}{2}\nu\left\|Q^\top C_a Q - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{bmatrix}\right\|_F^2 + \frac{1}{2}\left\|Q^\top K_a Q - \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix}\right\|_F^2\right\} = \frac{1}{2}[f(D) + 2g(C_{21}) + h(C_{22}, K_{22})] \tag{26}$$

where

$$f(D) = \nu\|A + R^{-\top}DR^{-1}\|_F^2 + \|B - R^{-\top}\Lambda^\top DR^{-1}\|_F^2$$

$$g(C_{21}) = \nu\|E - C_{21}\|_F^2 + \|F + C_{21}S\|_F^2$$

$$h(C_{22}, K_{22}) = \nu\left\|C_{22} - Q_2^\top C_a Q_2\right\|_F^2 + \left\|K_{22} - Q_2^\top K_a Q_2\right\|_F^2$$

with

$$A = Q_1^\top C_a Q_1 + S + S^\top, \qquad B = Q_1^\top K_a Q_1 - S^\top S \tag{27a}$$

$$E = Q_2^\top C_a Q_1, \qquad F = Q_2^\top K_a Q_1 \tag{27b}$$

Clearly, Eq. (26) achieves its minimal value if and only if

$$\min f(D), \qquad \min g(C_{21}), \qquad \min h(C_{22}, K_{22})$$

are achieved. Obviously, $h(C_{22}, K_{22})$ is minimized if and only if

$$C_{22} = Q_2^\top C_a Q_2, \qquad K_{22} = Q_2^\top K_a Q_2 \tag{28}$$

The optimization problems $\min f(D)$ and $\min g(C_{21})$ can be solved via problems I and II, with the matrices A, B, E, and F defined by Eq. (27).

In summary, we have the following algorithm:

Algorithm I. For a given $\nu > 0$, an analytical quadratic pencil $Q_a(\lambda) = \lambda^2 M_a + \lambda C_a + K_a$ and a matrix pair $(\Lambda, \Phi) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ as defined in Eq. (4), we seek the symmetric solutions C and K to problem FEMU-I:

1) Set

$$C_a := M_a^{-\frac{1}{2}} C_a M_a^{-\frac{1}{2}}, \qquad K_a := M_a^{-\frac{1}{2}} K_a M_a^{-\frac{1}{2}}, \qquad \Phi := M_a^{\frac{1}{2}}\Phi$$

2) Compute the QR-factorization of Φ:

$$\Phi = [Q_1, Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix}, \qquad S = R\Lambda R^{-1}$$

3) Compute $C_{22} = Q_2^\top C_a Q_2$ and $K_{22} = Q_2^\top K_a Q_2$.

4) Solve $G\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{x} = (\xi_1, \eta_1, \ldots, \xi_\ell, \eta_\ell, \xi_{2\ell+1}, \ldots, \xi_k)^\top$, where

$$G = \sum_{j=1}^{k} \Gamma_j^\top[\nu(R^\top R)^{-1} + \Lambda(R^\top R)^{-1}\Lambda^\top]\Gamma_j$$

$$\boldsymbol{b} = \sum_{j=1}^{k}\Gamma_j^\top\left(\Lambda R^{-1}\boldsymbol{v}_j - \nu R^{-1}\boldsymbol{u}_j\right)$$

$$\Gamma_j = \mathrm{diag}\left(\begin{bmatrix} r_{1,j} & r_{2,j} \\ -r_{2,j} & r_{1,j}\end{bmatrix}, \ldots, \begin{bmatrix} r_{2\ell-1,j} & r_{2\ell,j} \\ -r_{2\ell,j} & r_{2\ell-1,j}\end{bmatrix}, r_{2\ell+1,j}, \ldots, r_{k,j}\right)$$

$$[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k] = Q_1^\top C_a Q_1 + S + S^\top$$

$$[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k] = Q_1^\top K_a Q_1 - S^\top S$$

$$(r_{1,j}, \ldots, r_{k,j})^\top = R^{-1}\boldsymbol{e}_j$$

5) Form D as in Eq. (13), and compute

$$C_{11} = -(S + S^\top + R^{-\top}DR^{-1}), \qquad K_{11} = S^\top S + R^{-\top}D\Lambda R^{-1}$$

$$C_{21} = Q_2^\top\left(\nu C_a Q_1 - K_a Q_1 S^\top\right)(\nu I + SS^\top)^{-1}, \qquad K_{21} = -C_{21}S$$

6) Compute

$$C = M_a^{\frac{1}{2}} Q \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{bmatrix} Q^\top M_a^{\frac{1}{2}}$$

$$K = M_a^{\frac{1}{2}} Q \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix} Q^\top M_a^{\frac{1}{2}}$$

where $Q = [Q_1, Q_2]$.

Note that the linear system in step 4 is solvable because the cost function has a global minimizer.

Remark III.1. a) In a finite element model, the analytical matrices M_a, C_a, and K_a are usually very large and sparse. Matrix M_a is, in general, diagonal or banded and therefore easily invertible. In practice, the number of measured eigenpairs is much less than the dimension of the finite element model, that is, $k \ll n$. The orthogonal matrix $Q = [Q_1, Q_2]$ in step 2 of Algorithm I can be computed and stored in the form of a diagonal matrix plus a low rank updating by Householder transformations. Suppose the multiplication of the sparse matrix C_a or K_a to a vector needs $\mathcal{O}(n)$ flops. Then, the computational cost of Algorithm I is $\mathcal{O}(nk^2)$ flops. Obviously, if the analytical matrices are all dense, then the computational cost of Algorithm I will increase to $\mathcal{O}(n^2k)$ flops.

b) Using Algorithm I to solve problem FEMU-I in Eq. (3) is different from using Eqs. (6–8). The latter needs to solve a large (and possibly dense) $nk \times nk$ linear system in Eq. (8), which is impractical when n is very large.

IV. Solving Problem FEMU-II

According to Eq. (25), for simplicity and without loss of generality, we can assume that $M_a = I$ in the rest of this section. Applying Eqs. (11–13), we can easily derive that problem FEMU-II in Eq. (5) is equivalent to the following unconstrained optimization problem:

$$\min\left\{\frac{1}{2}\mu\left\|Q^\top M_a Q - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix}\right\|_F^2 + \frac{1}{2}\nu\left\|Q^\top C_a Q - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{bmatrix}\right\|_F^2 + \frac{1}{2}\left\|Q^\top K_a Q - \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix}\right\|_F^2\right\}$$

$$= f(M_{11}, D) + 2g(M_{21}, C_{21}) + h(M_{22}, C_{22}, K_{22}) \tag{29}$$

where

$$f(M_{11}, D) = \tfrac{1}{2}\mu\|M_{11} - I_k\|_F^2 + \tfrac{1}{2}\nu\left\|Q_1^\top C_a Q_1 + M_{11}S + S^\top M_{11} + R^{-\top}DR^{-1}\right\|_F^2 + \tfrac{1}{2}\left\|Q_1^\top K_a Q_1 - S^\top M_{11}S - R^{-\top}D\Lambda R^{-1}\right\|_F^2$$

$$g(M_{21}, C_{21}) = \tfrac{1}{2}\mu\|M_{21}\|_F^2 + \tfrac{1}{2}\nu\left\|Q_2^\top C_a Q_1 - C_{21}\right\|_F^2 + \tfrac{1}{2}\left\|Q_2^\top K_a Q_1 + M_{21}S^2 + C_{21}S\right\|_F^2$$

$$h(M_{22}, C_{22}, K_{22}) = \tfrac{1}{2}\mu\|I_{n-k} - M_{22}\|_F^2 + \tfrac{1}{2}\nu\left\|Q_2^\top C_a Q_2 - C_{22}\right\|_F^2 + \tfrac{1}{2}\left\|Q_2^\top K_a Q_2 - K_{22}\right\|_F^2$$

Here, $M_{11} \in \mathbb{S}^{k\times k}$, M_{22}, C_{22}, $K_{22} \in \mathbb{S}^{(n-k)\times(n-k)}$, M_{21}, $C_{21} \in \mathbb{R}^{(n-k)\times k}$, and D is defined by Eq. (13).

Clearly, Eq. (29) holds if and only if

$$\min f(M_{11}, D), \qquad \min g(M_{21}, C_{21}), \qquad \min h(M_{22}, C_{22}, K_{22}) \tag{30}$$

are achieved. It is easy to see that $h(M_{22}, C_{22}, K_{22})$ achieves its minimal value if and only if

$$M_{22} = I_{n-k}, \qquad C_{22} = Q_2^\top C_a Q_2, \qquad K_{22} = Q_2^\top K_a Q_2 \tag{31}$$

Differentiating $g(M_{21}, C_{21})$ with respect to M_{21} and C_{21}, we obtain

$$\nabla_{M_{21}} g = M_{21}[\mu I + S^2(S^2)^\top] + C_{21} S(S^2)^\top + K_{21}^a (S^2)^\top$$

$$\nabla_{C_{21}} g = M_{21} S^2 S^\top + C_{21}(\nu I + SS^\top) + K_{21}^a S^\top - \nu C_{21}^a$$

where $C_{21}^a = Q_2^\top C_a Q_1$, $K_{21}^a = Q_2^\top K_a Q_1$. Thus, it follows that $g(M_{21}, C_{21})$ achieves its minimal value if and only if

$$M_{21}[\mu I + S^2(S^2)^\top] + C_{21} S(S^2)^\top + K_{21}^a (S^2)^\top = 0$$

$$M_{21} S^2 S^\top + C_{21}(\nu I + SS^\top) + K_{21}^a S^\top - \nu C_{21}^a = 0 \tag{32}$$

Let

$$S_\nu = S^\top (\nu I + SS^\top)^{-1} S \tag{33}$$

and assume that $\mu I + S^2(I - S_\nu)(S^2)^\top$ is nonsingular, then Eq. (32) gives rise to

$$M_{21} = \left[K_{21}^a (S_\nu - I) - \nu C_{21}^a (\nu I + SS^\top)^{-1} S\right] \times (S^2)^\top \left[\mu I + S^2 (I - S_\nu)(S^2)^\top\right]^{-1} \tag{34}$$

$$C_{21} = \left[\nu C_{21}^a - M_{21} S^2 S^\top - K_{21}^a S^\top\right](\nu I + SS^\top)^{-1} \tag{35}$$

Next, we consider $\min f(M_{11}, D)$. Differentiating $f(M_{11}, D)$ with respect to the elements of M_{11} (which is symmetric), we have

$$\nabla_{M_{11}} f = (2E - I) \circ [\mathcal{S}_1(M_{11}) + \mathcal{R}_1(D) - B_1]$$

where $\circ$ stands for the Hadamard product (i.e., the componentwise product), E is the matrix of all 1s, and

$$\mathcal{S}_1(M_{11}) = \mu M_{11} + \nu\left(SS^\top M_{11} + M_{11} SS^\top\right) + \nu\left(S M_{11} S + S^\top M_{11} S^\top\right) + SS^\top M_{11} SS^\top \tag{36a}$$

$$\mathcal{R}_1(D) = \nu(S R^{-\top} D R^{-1} + R^{-\top} D R^{-1} S^\top) + S R^{-\top} \Lambda^\top D R^{-1} S^\top \tag{36b}$$

$$B_1 = \mu I - \nu\left(S C_{11}^a + C_{11}^a S^\top\right) + S K_{11}^a S^\top \tag{36c}$$

with $C_{11}^a = Q_1^\top C_a Q_1$, $K_{11}^a = Q_1^\top K_a Q_1$.

On the other hand, similar to Eq. (19), we have

$$\nabla_D f = \mathcal{S}_2(M_{11}) + \mathcal{R}_2(D) - \boldsymbol{b}_2$$

where

$$\mathcal{S}_2(M_{11}) = \sum_{j=1}^{k} \Gamma_j^\top \left[\nu R^{-1}\left(M_{11} S + S^\top M_{11}\right) + \Lambda R^{-1} S^\top M_{11} S\right] \boldsymbol{e}_j \tag{37a}$$

$$\mathcal{R}_2(D) = \sum_{j=1}^{k} \Gamma_j^\top (\nu R^{-1} R^{-\top} + \Lambda R^{-1} R^{-\top} \Lambda^\top) \Gamma_j \boldsymbol{x} \tag{37b}$$

$$\boldsymbol{b}_2 = \sum_{j=1}^{k} \Gamma_j^\top \left(\Lambda R^{-1} K_{11}^a - \nu R^{-1} C_{11}^a\right) \boldsymbol{e}_j \tag{37c}$$

in which $\boldsymbol{x}$ and Γ_j are defined by Eqs. (15) and (18), respectively.

Consequently, it follows that $f(M_{11}, D)$ achieves its minimal value if and only if

$$\mathcal{S}_1(M_{11}) + \mathcal{R}_1(D) = B_1, \qquad \mathcal{S}_2(M_{11}) + \mathcal{R}_2(D) = \boldsymbol{b}_2 \tag{38}$$

One of the simplest methods to solve Eq. (38) is to apply the Kronecker product and vec operator to rewrite Eq. (38) as

$$\begin{bmatrix} E_1 & F \\ F^\top & E_2 \end{bmatrix} \begin{bmatrix} \mathrm{vec}(M_{11}) \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(B_1) \\ \boldsymbol{b}_2 \end{bmatrix} \tag{39}$$

with

$$E_1 = \mu I_{k^2} + \nu\left[I_k \otimes (SS^\top) + (SS^\top) \otimes I_k\right] + \nu(S^\top \otimes S + S \otimes S^\top) + (SS^\top) \otimes (SS^\top)$$

$$E_2 = \sum_{j=1}^{k} \Gamma_j^\top (\nu R^{-1} R^{-\top} + \Lambda R^{-1} R^{-\top} \Lambda^\top) \Gamma_j \tag{40}$$

$$F = \left\{\nu\left[I_k \otimes (S R^{-\top}) + S \otimes R^{-\top}\right] + S \otimes (S R^{-\top} \Lambda^\top)\right\} \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_k \end{bmatrix}$$

Gaussian elimination with partial pivoting can then be applied.

Overall we have the following:

Algorithm II. For given $\mu, \nu > 0$, an analytical quadratic pencil $Q_a(\lambda) = \lambda^2 M_a + \lambda C_a + K_a$, and a matrix pair $(\Lambda, \Phi) \in \mathbb{R}^{k \times k} \times \mathbb{R}^{n \times k}$ as defined in Eq. (4), we seek the symmetric solutions M, C, and K to problem FEMU-II:

1) Set

$$C_a := M_a^{-\frac{1}{2}} C_a M_a^{-\frac{1}{2}}, \qquad K_a := M_a^{-\frac{1}{2}} K_a M_a^{-\frac{1}{2}}, \qquad \Phi := M_a^{\frac{1}{2}} \Phi$$

2) Compute the QR factorization of Φ:

$$\Phi = [Q_1, Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}, \qquad S = R \Lambda R^{-1}$$

3) Compute $M_{22} = I_{n-k}$, $C_{22} = Q_2^\top C_a Q_2$, and $K_{22} = Q_2^\top K_a Q_2$.

4) Solve Eq. (39) for $\mathrm{vec}(M_{11})$ and $\boldsymbol{x}$, and compute M_{21} and C_{21} by Eqs. (34) and (35), respectively.

5) Compute

$$C_{11} = -\left(M_{11} S + S^\top M_{11} + R^{-\top} D R^{-1}\right)$$

$$K_{11} = S^\top M_{11} S + R^{-\top} D \Lambda R^{-1}, \qquad K_{21} = -\left(M_{21} S^2 + C_{21} S\right)$$

where D is formed from $\boldsymbol{x}$ by Eq. (13).

6) Compute

$$M = M_a^{\frac{1}{2}} Q \begin{bmatrix} M_{11} & M_{21}^\top \\ M_{21} & M_{22} \end{bmatrix} Q^\top M_a^{\frac{1}{2}}$$

$$C = M_a^{\frac{1}{2}} Q \begin{bmatrix} C_{11} & C_{21}^\top \\ C_{21} & C_{22} \end{bmatrix} Q^\top M_a^{\frac{1}{2}}$$

$$K = M_a^{\frac{1}{2}} Q \begin{bmatrix} K_{11} & K_{21}^\top \\ K_{21} & K_{22} \end{bmatrix} Q^\top M_a^{\frac{1}{2}}$$

where $Q = [Q_1, Q_2]$.

Note that linear system Eq. (39) in step 4 is solvable because the cost function has global minimizer.

Remark IV.1. a) Similar to Remark III.1a and from Eq. (39), the computational cost of Algorithm II is $\mathcal{O}(nk^2 + k^6)$ flops, provided M_a, C_a, and K_a are sparse with $\mathcal{O}(n)$ nonzero entries. The computational cost is increased to $\mathcal{O}(n^2 k + k^6)$ flops when the analytical matrices are all dense. 2) Note that the updated mass matrix M generated by Algorithm II might not be positive definite. However, if

$\mu > \delta$, then M is positive definite, where

$$\delta = \min\left\{\nu\left\|M_a^{-\frac{1}{2}}(C-C_a)M_a^{-\frac{1}{2}}\right\|_F^2 + \left\|M_a^{-\frac{1}{2}}(K-K_a)M_a^{-\frac{1}{2}}\right\|_F^2 : M_a\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0 \quad \text{with} \quad C, K \quad \text{symmetric}\right\} \tag{41}$$

In fact, if (M, C, K) is a solution to problem FEMU-II in Eq. (5), then it is easily seen that

$$J = \tfrac{1}{2}\mu\left\|M_a^{-\frac{1}{2}}(M-M_a)M_a^{-\frac{1}{2}}\right\|_F^2 + \tfrac{1}{2}\nu\left\|M_a^{-\frac{1}{2}}(C-C_a)M_a^{-\frac{1}{2}}\right\|_F^2 + \tfrac{1}{2}\left\|M_a^{-\frac{1}{2}}(K-K_a)M_a^{-\frac{1}{2}}\right\|_F^2 \le \tfrac{1}{2}\delta$$

Hence, we have

$$\|E\|_2 \le \|E\|_F = \left\|M_a^{-\frac{1}{2}}(M-M_a)M_a^{-\frac{1}{2}}\right\|_F \le \delta/\mu < 1$$

where $E = M_a^{-1/2}MM_a^{-1/2} - I$. By the perturbation theorem of symmetric matrices, we have

$$\lambda_{\min}\left(M_a^{-\frac{1}{2}}MM_a^{-\frac{1}{2}}\right) \ge 1 - \|E\|_2 > 0$$

where $\lambda_{\min}(M_a^{-1/2}MM_a^{-1/2})$ denotes the minimal eigenvalue of $M_a^{-1/2}MM_a^{-1/2}$. Thus, we show that M is symmetric positive definite provided that $\mu > \delta$.

In practice, the analytical mass matrix M_a is quite accurate. So, the weighting μ in Eq. (5a) should be chosen sufficiently large. Thus, the condition $\mu > \delta$ can be easily satisfied so that the updated mass matrix M is symmetric positive definite.

V. Numerical Results

A set of pseudosimulation data was provided by the Boeing Company for testing. After a model reduction technique, we get three symmetric analytical matrices M_a, C_a, and K_a with dimension 42 and M_a being positive definite. The 2-norms of M_a, C_a, and K_a are 3.9057×10^8, 1.2250×10^8, and 2.0326×10^8, respectively.

Test 1. Because $M_a > 0$, the quadratic pencil $Q_a(\lambda) = \lambda^2 M_a + \lambda C_a + K_a$ has 84 finite eigenvalues. We first compute all 84 eigenpairs of $Q_a(\lambda)$ by solving a generalized eigenvalue problem of a linearization of $Q_a(\lambda)$. Then the measured eigenpairs $(\Lambda_a, \Phi_a) \in \mathbb{R}^{14\times 14} \times \mathbb{R}^{42\times 14}$ are chosen from those 84 computed eigenpairs of $Q_a(\lambda)$ so that eigenvalues of Λ_a are nearest to the original. Actually, the relative residual is estimated by

$$\frac{\left\|M_a\Phi_a\Lambda_a^2 + C_a\Phi_a\Lambda_a + K_a\Phi_a\right\|_F}{\left\|M_a\Phi_a\Lambda_a^2\right\|_F + \|C_a\Phi_a\Lambda_a\|_F + \|K_a\Phi_a\|_F} = 4.0671 \times 10^{-10}$$

Intuitively, the optimal solutions C and K for problem FEMU-I should be very close to C_a and K_a, respectively. We use Algorithm I to solve problem FEMU-I with $\nu = 1$; the relative errors of the updated matrices are estimated by

$$\|C - C_a\|_{F_a}\big/\kappa_1 \simeq 10^{-10}, \qquad \|K - K_a\|_{F_a}\big/\kappa_1 \simeq 10^{-10}$$

where $\|\cdot\|_{F_a} = \|M_a^{-1/2}(\cdot)M_a^{-1/2}\|_F$ and $\kappa_1 = \max\{\|C_a\|_{F_a}, \|K_a\|_{F_a}\}$. The relative residual of (Λ_a, Φ_a) is estimated by

$$\frac{\left\|M_a\Phi_a\Lambda_a^2 + C\Phi_a\Lambda_a + K\Phi_a\right\|_F}{\left\|M_a\Phi_a\Lambda_a^2\right\|_F + \|C\Phi_a\Lambda_a\|_F + \|K\Phi_a\|_F} = 5.4135 \times 10^{-14}$$

We use Algorithm II to solve problem FEMU-II with $\mu = \nu = 1$; the relative errors of updated matrices are estimated by

$$\|M - M_a\|_{F_a}\big/\kappa_2 \simeq 10^{-10}, \qquad \|C - C_a\|_{F_a}\big/\kappa_2 \simeq 10^{-10}$$

$$\|K - K_a\|_{F_a}\big/\kappa_2 \simeq 10^{-10}$$

where $\kappa_2 = \max\{\|M_a\|_{F_a}, \|C_a\|_{F_a}, \|K_a\|_{F_a}\}$. The relative residual of (Λ_a, Φ_a) is estimated by

$$\frac{\left\|M\Phi_a\Lambda_a^2 + C\Phi_a\Lambda_a + K\Phi_a\right\|_F}{\left\|M\Phi_a\Lambda_a^2\right\|_F + \|C\Phi_a\Lambda_a\|_F + \|K\Phi_a\|_F} = 1.0821 \times 10^{-13}$$

Test 2. Consider the given measured eigenvalues

$$\{\lambda_{mj}\}_{j=1}^{14} = \{-0.60939 \pm 37.365\iota, -0.73496 \pm 36.707\iota, -2.8832 \pm 31.992\iota, -1.8907 \pm 61.437\iota, -1.9112 \pm 54.181\iota, -2.2785 \pm 39.639\iota, -5.0387, -4.3416\} \tag{42}$$

The eigenpairs of the experimental model are used to create the experimental modal data. It is assumed that only the fundamental mode characteristics are experimentally determined, and only $s(s \le 42)$ components of eigenvector are measured. Suppose now we are given the measured mode shapes $\boldsymbol{v}_j \in \mathbb{R}^s$, $j = 1, \ldots, 14$. The measured eigenvectors φ_j is estimated by

$$\varphi_j = D_j\tilde{D}_j^{\dagger}\boldsymbol{v}_j, \qquad j = 1, \ldots 14 \tag{43}$$

where D_j is defined by $D_j = [\lambda_{mj}^2 M_a + \lambda_{mj}C_a + K_a]^{-1}B_a$ with the control influence matrix $B_a \in \mathbb{R}^{n\times t}(t \le s)$. The matrix $\tilde{D}_j$ consists of the first s rows of D_j, and the superscript $\dagger$ denotes the pseudo inverse. We first construct the eigenmatrix pair (Λ, Φ) associated with Eqs. (42) and (43) as in Eq. (4). Then we use Algorithm I to compute the updated matrices C and K with $\nu = 0.1$, 1.0, and 10, respectively. The numerical results are shown in Table 1.

Here, δ is given in Eq. (41), and the relative residual is defined by

$$r_1 = \frac{\left\|M_a\Phi\Lambda^2 + C\Phi\Lambda + K\Phi\right\|_F}{\left\|M_a\Phi\Lambda^2\right\|_F + \|C\Phi\Lambda\|_F + \|K\Phi\|_F}$$

We use Algorithm II to compute the new updated matrices M, C, and K with $\nu = 1$ and $\mu \in (10^7, 10^{11})$. In Fig. 1, we plot the minimal eigenvalue of the symmetric matrix M vs μ. We see that the minimal eigenvalue of the mass matrix M becomes negative when μ less then 2×10^8.

We now fix $\nu = 1$ and use Algorithm II to compute the updated matrices M, C, and K with $\mu = 5.0 \times 10^8, 5.0 \times 10^9$, and 5.0×10^{10}, respectively. The numerical results are shown in Table 2.

Here, $\kappa_2 = \max\{\|M_a\|_{F_a}, \|C_a\|_{F_a}, \|K_a\|_{F_a}\}$, and the relative residual is defined by

$$r_2 = \frac{\|M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi\|_F}{\|M\Phi\Lambda^2\|_F + \|C\Phi\Lambda\|_F + \|K\Phi\|_F}$$

From the accurate relative residuals in Tables 1 and 2, we see that the new proposed methods have high efficiency and reliability.

Table 1 Relative residuals and optimal values

	ν		
Values	0.1	1.0	10
r_1	1.4725×10^{-14}	1.4826×10^{-14}	1.4859×10^{-14}
δ	6.5057×10^9	6.5960×10^9	7.9249×10^9

Table 2 Relative residuals and relative errors of updated matrices

	μ		
Values	5.0×10^8	5.0×10^9	5.0×10^{10}
r_2	1.8392×10^{-14}	1.2380×10^{-14}	1.3865×10^{-14}
$\|M - M_a\|_{F_a}/\kappa_2$	3.6414×10^{-6}	7.8126×10^{-7}	1.2492×10^{-7}
$\|C - C_a\|_{F_a}/\kappa_2$	4.2456×10^{-2}	4.2461×10^{-2}	4.2485×10^{-2}
$\|K - K_a\|_{F_a}/\kappa_2$	3.3737×10^{-1}	3.5835×10^{-1}	3.6718×10^{-1}

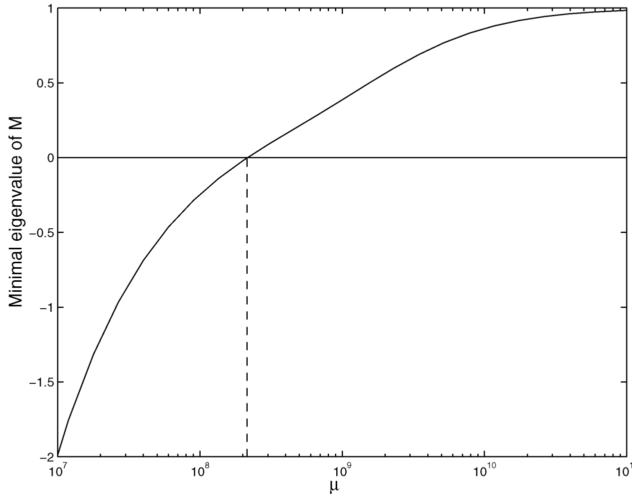

Fig. 1 Minimal eigenvalue of symmetric matrix *M* vs μ.

VI. Conclusions

In this paper, we have developed two efficient numerical algorithms for finite element model updating problems. The new algorithms compute symmetric updated (mass) damping and stiffness that closely match the experimental modal data. The updated mass is symmetric positive definite when the weighting parameter μ is chosen sufficiently large. The new algorithms are direct methods that are highly efficient and reliable, according to our numerical experiments. The algorithms produce encouraging results and interesting insight in a simple pseudo test suit provided by the Boeing Company.

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