Reprinted from JOURNAL OF THE AERONAUTICAL SCIENCES

DECEMBER, 1957 VOLUME 24, NO. 12

New Methods in Heat Flow Analysis With Application to Flight Structures†

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SUMMARY

New methods are presented for the analysis of transient heat flow in complex structures, leading to drastic simplifications in the calculation and the possibility of including nonlinear and surface effects. These methods are in part a direct application of some general variational principles developed earlier for linear thermodynamics.[1–3] They are further developed in the particular case of purely thermal problems to include surface and boundary-layer heat transfer, nonlinear systems with temperature-dependent parameters, and radiation. The concepts of thermal potential, dissipation function, and generalized thermal force are introduced, leading to ordinary differential equations of the Lagrangian type for the thermal flow field. Because of the particular nature of heat flow phenomena, compared with dynamics, suitable procedures must be developed in order to formulate each problem in the simplest way. This is done by treating a number of examples. The concepts of penetration depth and transit time are introduced and discussed in connection with one-dimensional flow. Application of the general method to the heating of a slab, with temperature-dependent heat capacity, shows a substantial difference between the heating and cooling processes. An example of heat flow analysis of a supersonic wing structure by the present method is also given and requires only extremely simple calculations. The results are found to be in good agreement with those obtained by the classical and much more elaborate procedures.

(1) INTRODUCTION

THE ADVENT of supersonic flight has brought a new importance to heat conduction problems in engineering. The temperature ranges involved and the highly transient character of the phenomena require a new frame of reference and the development of methods more suited to the new problems.

It is our purpose here to initiate the development of a new approach to heat flow analysis. The expression *heat flow* is used in its broad sense and encompasses heat conduction and heat transfer, convection phenomena, and radiation. The substance of the method has been outlined in earlier publications on irreversible thermodynamics by this writer.[1, 2] The principles were given more specific formulation in reference 3 in the particular field of thermoelasticity and heat conduction. In those papers general methods are outlined by which the elasticity and the thermal conduction problem are treated in a unified way. The thermoelastic response to thermal excitation is considered to result from the application of generalized thermal forces, defined in the same way as the mechanical forces and leading to Lagrangian equations for the coupled elastic and thermal coordinates. The thermostatic and thermodynamic properties are completely defined by a thermoelastic potential and a generalized dissipation function.

The scope of the present study includes first a reformulation, in the special domain of thermal flow, of the previously established principles and methods. We make use of the concept of thermal force and of the dissipation function defined as previously except for a constant factor. We also introduce a thermal potential. We further extend these principles to include boundary-layer conduction, nonlinear phenomena with temperature dependent parameters, and radiation. In addition, several entirely different methods of applying the general principles are presented, by the treatment of specific examples, in an attempt to develop the art of solving complex problems of thermal flow analysis by procedures best suitable for each type of problem. It is, of course, not possible in such a short space to examine the innumerable variations and refinements of the method. We have therefore picked a limited number of examples and illustrations of a typical treatment. The heat flow in extremely com-

Presented at the Structures Session, Twenty-Fifth Annual Meeting, IAS, New York, January 28–31, 1957.
† This paper is based on Cornell Aeronautical Laboratory Report No. SA-987-S-3, May, 1956.
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plex structures is open to analysis by such methods without undue complications and with suitable flexibility with regard to the introduction of intuitive and experimental knowledge into the formulation.

While the mathematical basis of this new formulation is presented in the variational language, it is not a straightforward application of the variational calculus to the heat conduction equations. The physical phenomenon is considered to be represented by a vector field instead of a temperature field. This opens the way to new concepts such as generalized thermal force and to straightforward procedures for the interconnection of thermal networks.

From the standpoint of simplicity, compared to the classical treatment of heat conduction, the new approach bears similarity to the powerful simplification which was achieved in the vibration analysis of complex structures by the introduction of Lagrangian methods. We must bear in mind, however, that the steps are far from identical because of the essential physical difference between the nature of the phenomena of diffusion and dynamics. One of our purposes here is to show how special adaptation of the methods to heat flow phenomena lead to remarkable simplifications. An unexpected result is a simplification which leads to a nonlinear differential equation with a single unknown even in a physically linear problem. Another special feature of interest lies in the use of time-dependent approximate solutions, instead of the usual static distributions which are of standard use in the Rayleigh-Ritz solutions of dynamical problems. This also leads, of course, to a method of successive approximations, whereby a previous solution may be introduced in the following step and ordinary differential equations established for the correction. This constitutes, by the same token, a procedure for testing the accuracy of an approximate solution.

The basic principles in variational form as applied to a thermal flow field are introduced in Section (2). They are a direct consequence of the more general theories developed in references 1, 2, and 3. It is shown how boundary-layer heat transfer may be included.

The description of the flow field by means of generalized coordinates and the concept of thermal force are introduced in Section (3). By means of a thermal potential and a dissipation function, it is shown how the thermal flow problem may be formulated by differential equations of the Lagrangian type. The inclusion of surface and boundary-layer heat transfer is also discussed. Section (4) deals with linear systems and discusses thermal modes, normal coordinates, orthogonality, and general expressions for the thermal admittance in operational form. The basic mathematical and thermodynamic background for these concepts and their properties was established in references 1 and 2. Various ways of applying the general equations to thermal flow problems are discussed in a general vein in Section (5). Specific procedures for one-dimensional problems are developed in Section (6). One example uses normal coordinates and a modification of the latter for flows deviating only slightly from the steady state. Simpson's method of integration is used to write the equation directly in matrix form particularly suited to solution by methods of iteration, as in vibration analysis. A particularly simple way of analyzing heat conduction in a slab is illustrated in Section (7). This leads to a simple differential equation with one unknown. Comparison of the results with the exact series solution shows the method to be surprisingly accurate considering its simplicity.

The principles are further extended in Section (8) to include systems where dependence of the heat capacity and conductivity on the temperature leads to nonlinear equations. Temperature-dependent heat-transfer coefficients as well as surface radiation conditions are included in the formulation. This is illustrated in Section (9) by the example of heat conduction in a slab with temperature-dependent heat capacity. It is found that, because of the nonlinearity, the cooling proceeds quite differently from the heating.

As a final example, in Section (10) we treat the problem of a typical portion of a supersonic wing structure under aerodynamic heating. The example was solved previously by Pohle and Oliver.[6] The present results, obtained with very little effort, are compared with those calculated by the exact and considerably more elaborate procedure, and their accuracy is found to be quite satisfactory. Various methods of refinement of the solution of this particular problem are also discussed.

(2) Fundamental Variational Principles in Heat Conduction

Consider an isotropic body, with a thermal conductivity $k(x, y, z)$ and a heat capacity $c(x, y, z)$ per unit volume as functions of the coordinates. For our purpose it is convenient to introduce an excess temperature $\theta = T - T_0$ over an equilibrium temperature T_0. The temperature θ satisfies the equation of heat conduction,

$$(\partial/\partial x)\ [k(\partial\theta/\partial x)] + (\partial/\partial y)\ [k(\partial\theta/\partial y)] + (\partial/\partial z)\ [k(\partial\theta/\partial z)] = c(\partial\theta/\partial t) \quad (2.1)$$

We shall formulate a variational principle which is equivalent to this equation. To this effect we introduce a heat flow vector field $\boldsymbol{H}$ such that the rate of heat flow at every point is $\partial H/\partial t$ per unit area, normal to $\boldsymbol{H}$. Energy conservation is expressed by the relation

$$c\theta = -\text{div}\ \boldsymbol{H} \quad (2.2)$$

We define a *thermal potential*

$$V = (1/2) \iiint_\tau c\theta^2 d\tau \quad (2.3)$$

obtained by integration in the volume τ. We also define a variational invariant

$$\delta D = \iiint_\tau (1/k)\ (\partial \boldsymbol{H}/\partial t)\cdot\delta \boldsymbol{H} d\tau \quad (2.4)$$

We shall see that V plays a role analogous to a potential energy while δD is related to the concept of dissipation function. Except for a constant factor they are identical with the functions introduced by the writer in references 1, 2, and 3 in connection with linear irreversible thermodynamics. It was shown in these references that V may be considered as a generalization of the free energy for systems which are not in thermal equilibrium. The relation of δD to a dissipation function will appear more clearly in the next section dealing with generalized coordinates.

With these definitions we state the variational principle as

$$\delta V + \delta D = \iint_s \theta \boldsymbol{n} \cdot \delta \boldsymbol{H} dS \tag{2.5}$$

This equation is to be verified identically for all arbitrary variations $\delta \boldsymbol{H}$ of the heat flow field in the volume τ. The surface integral is extended to the boundary S of the domain with the unit normal $\boldsymbol{n}$ to the boundary taken positive inward. To show that Eq. (2.5) is equivalent to the heat conduction equation, we evaluate

$$\delta V = \iiint_\tau c\theta\delta\theta d\tau = -\iiint_\tau \theta\delta(\text{div } \boldsymbol{H}) d\tau \tag{2.6}$$

Integrating by parts, we find

$$\delta V = \iiint_\tau \delta\boldsymbol{H} \cdot \text{grad } \theta d\tau + \iint_s \theta \boldsymbol{n} \cdot \delta \boldsymbol{H} dS \tag{2.7}$$

Substituting Eqs. (2.4) and (2.7) in Eq. (2.5) and putting the coefficient of $\delta \boldsymbol{H}$ equal to zero yields

$$\text{grad } \theta + (1/k)\,(\partial \boldsymbol{H}/\partial t) = 0 \tag{2.8}$$

From Eqs. (2.8) and (2.2)

$$\text{div } (k \text{ grad } \theta) = c(\partial\theta/\partial t) \tag{2.9}$$

which is identical with Eq. (2.1). The validity of the variational principle (2.5) is thus established.

The above derivation assumes a body of isotropic thermal conductivity. In the more general case of anisotropy the thermal conductivity is defined by a symmetric tensor

$$k_{ij} = k_{ji} \tag{2.10}$$

and the law of thermal conduction is written

$$\sum^{j} k_{ij}(\partial\theta/\partial x_j) = -(\partial H_i/\partial t) \tag{2.11}$$

Eq. (2.1) is replaced by

$$\sum^{ij} (\partial/\partial x_i)\,[k_{ij}(\partial\theta/\partial x_j)] = c(\partial\theta/\partial t) \tag{2.12}$$

Proceeding as above for the isotropic case, it can be shown that the variational principle (2.5) is also valid for the general anisotropic case, provided δD is replaced by

$$\delta D = \iiint_\tau \sum^{ij} \lambda_{ij}(\partial H_i/\partial t) \cdot \delta H_j d\tau \tag{2.13}$$

In this expression λ_{ij} is the inverse matrix of k_{ij} and represents the thermal resistivity

$$\lambda_{ij} = [k_{ij}]^{-1} \tag{2.14}$$

The law of thermal conduction is then

$$\partial\theta/\partial x_j = -\sum^{i} \lambda_{ij}(\partial H_i/\partial t) \tag{2.15}$$

The principles as stated above may be readily extended to include the effect of surface and boundary-layer heat transfer. This can be seen immediately since the heat-transfer layer is equivalent to a material with zero heat capacity. This will be developed more explicitly in the next section. The question of radiation loss will be discussed in Section (8) in connection with nonlinear problems.

From an intuitive viewpoint, it is interesting to point out the complete analogy of heat conduction and the seepage of a compressible viscous fluid through a porous solid. Such an analogy was discussed in reference 3. The mass flow rate corresponds to the rate of heat flow $\partial \boldsymbol{H}/\partial t$, the pressure to the temperature θ, and the increase of fluid mass per unit volume to the heat content. The fluid compressibility represents the heat capacity and the permeability is the equivalent of the thermal conductivity. The analogy holds for nonisotropic media and may be extended to nonlinear systems, as will be demonstrated in Section (8). It also indicates that such concepts of mechanics as generalized forces and coordinates may be introduced in thermal problems. It can be done directly without referring to the mechanical model, as will be shown in the following section by applying the variational principle demonstrated above.

(3) Generalized Coordinates and Thermal Force—Surface Heat Transfer

We now introduce the concept of generalized coordinates in conjunction with the variational principle. There are many ways in which these generalized coordinates may be defined. We may express the flow field $\boldsymbol{H}$ as a sum of fixed configurations $\boldsymbol{H}_i(x, y, z)$ with a variable amplitude of each configuration

$$\boldsymbol{H} = \sum^{i} q_i \boldsymbol{H}_i \tag{3.1}$$

The unknown amplitudes q_i are the generalized coordinates. However, these coordinates need not be related linearly to the field $\boldsymbol{H}$. More generally, we could write

$$\boldsymbol{H} = \boldsymbol{H}(q_1 \ldots q_n, x, y, z) \tag{3.2}$$

In this case the generalized coordinates are simply a set of n parameters defining the field configuration. As we shall see in the examples [Section (7)] it is advantageous sometimes even in linear problems to choose generalized coordinates which are related nonlinearly to the unknown thermal variables.

Consider now the variational relation (2.5) and let us apply it first to an isotropic medium.

The variations δH in this case are due entirely to the variations of q_i,

$$\delta H = \sum^{i} (\partial H/\partial q_i)\delta q_i \tag{3.3}$$

Also we may write

$$(\partial/\partial t)H = \sum^{i} (\partial H/\partial q_i)\dot{q}_i \tag{3.4}$$

Hence,

$$(\partial/\partial \dot{q}_i)\ (\partial H/\partial t) = \partial H/\partial q_i \tag{3.5}$$

The invariant $(\partial H/\partial t)\cdot\delta H$ in Eq. (2.4) may, therefore, be written

$$(\partial H/\partial t)\cdot\delta H = (\partial H/\partial t)\cdot \sum^{i} (\partial H/\partial q_i)\delta q_i = \sum^{i} (\partial H/\partial t)\ (\partial/\partial \dot{q}_i)\ (\partial H/\partial t)\delta q_i \tag{3.6}$$

or

$$(\partial H/\partial t)\cdot\delta H = \sum^{i} \delta q_i(\partial/\partial \dot{q}_i)\ [(1/2)\ (\partial H/\partial t)^2] \tag{3.7}$$

With these results the variational equation (2.5) becomes

$$\delta V + \sum^{i} \delta q_i(\partial D/\partial \dot{q}_i) = \iint_s \theta \boldsymbol{n}\cdot\delta \boldsymbol{H} dS \tag{3.8}$$

with

$$\partial D/\partial \dot{q}_i = \iiint_\tau (1/k) \times (\partial \boldsymbol{H}/\partial t)\cdot(\partial/\partial \dot{q}_i)\ (\partial \boldsymbol{H}/\partial t)d\tau \tag{3.9}$$

Hence, we may introduce the invariant

$$D = (1/2)\iiint (1/k)\ (\partial \boldsymbol{H}/\partial t)^2\ d\tau \tag{3.10}$$

Finally, we put

$$\iint_s \theta \boldsymbol{n}\cdot\delta \boldsymbol{H} dS = \sum^{i} Q_i\delta q_i \tag{3.11}$$

with

$$Q_i = \iint_s \theta \boldsymbol{n}\cdot(\partial \boldsymbol{H}/\partial q_i)dS \tag{3.12}$$

$$\delta V = \sum^{i} (\partial V/\partial q_i)\delta q_i \tag{3.13}$$

The variational principle, therefore, yields n equations for the field parameters q_i—namely,

$$(\partial V/\partial q_i) + (\partial D/\partial \dot{q}_i) = Q_i \tag{3.14}$$

These are the analog of the Lagrangian equations for a mechanical dissipative system with a potential energy V and a dissipation function D. The above equations have been derived for an isotropic medium. In the anisotropic case, it is easily verified that the same equations hold provided we define D as

$$D = (1/2)\iiint_\tau \sum^{ij} \lambda_{ij}(\partial H_i/\partial t)\ (\partial H_j/\partial t)d\tau \tag{3.15}$$

This expression for D as well as Eq. (3.10) shows its physical significance as it is related to the rate of entropy production.[2] We shall refer to D as a *dissipation function* also in purely thermal problems.

The generalized force Q_i will be referred to as the *thermal force.* It can be seen that Eq. (3.11) defines it in exactly the same way as a mechanical force—i.e., as the work done by a temperature θ on a virtual displacement $\delta \boldsymbol{H}$.

The above derivation does not require that the field $\boldsymbol{H}$ be completely defined by the coordinates q_i as in Eq. (3.2). It is possible to have the time appear explicitly in the description and write

$$\boldsymbol{H} = \boldsymbol{H}(q_1 \ldots q_n, x, y, z, t) \tag{3.16}$$

This will lead to the same differential equations (3.14) for the coordinates q_i but with the time variable appearing explicitly in the equations. This point is of importance in applications as it leads very often to considerable simplification to adopt a time-dependent description of the field.

Let us now consider the case of a surface heat-transfer effect and show how it may be included in the above formulation. We denote the temperature outside of the layer by θ_a and the temperature of the surface of the body by θ. With a heat-transfer coefficient K for the layer, the local rate of heat flow across the layer is

$$\partial H_n/\partial t = K(\theta_a - \theta) \tag{3.17}$$

The normal component of the heat flow *into the body* is denoted by H_n. From Eqs. (3.12) and (3.17) the thermal force at the surface of the body is

$$Q_i = \iint_s \theta(\partial H_n/\partial q_i)dS = -\iint_s (1/K)\ (\partial H_n/\partial t) \times (\partial H_n/\partial q_i)dS + \iint_s \theta_a(\partial H_n/\partial q_i)dS \tag{3.18}$$

Because of Eq. (3.5) we may write

$$\partial D_s/\partial \dot{q}_i = \iint_s (1/K)\ (\partial H_n/\partial t)\ (\partial H_n/\partial q_i)dS = \iint_s (1/K)\ (\partial H_n/\partial t)\ (\partial/\partial \dot{q}_i)\ (\partial H_n/\partial t)dS \tag{3.19}$$

With a surface dissipation function,

$$D_s = (1/2) \iint_s (1/K)\ (\partial H_n/\partial t)^2 dS \tag{3.20}$$

Hence, Eqs. (3.14) may again be applied to the total system including the surface layer provided we include D_s in the total dissipation function and write

$$D = (1/2) \iiint_\tau (1/k)\ (\partial \boldsymbol{H}/\partial t)^2 d\tau + (1/2) \iint_s (1/K)\ (\partial H_n/\partial t)^2 dS \tag{3.21}$$

We must also use a definition of the thermal force by means of the temperature outside the layer and write

$$Q_i = \iint_s \theta_a(\partial H_n/\partial q_i)dS \quad (3.22)$$

When the surface layer is a moving fluid boundary layer, θ_a is the so-called "adiabatic wall temperature."

It is of interest to point out that in Eq. (3.21) the boundary heat-transfer coefficient K may be time dependent. This is particularly useful in problems of heat transfer through boundary layers with variable fluid velocity.

(4) Thermal Admittance and Normal Coordinates

The case where Eqs. (3.14) are linear is of particular interest. This will happen, of course, if the physical problem is linear and if the generalized coordinates are related to the thermal field by a linear expression of the type (3.1). The thermal potential V and the dissipation function D are then positive definite quadratic forms

$$\left.\begin{aligned} V &= (1/2) \sum^{ij} a_{ij}q_iq_j \\ D &= (1/2) \sum^{ij} b_{ij}\dot{q}_i\dot{q}_j \end{aligned}\right\} \quad (4.1)$$

and Eqs. (3.14) read

$$\sum^{j} a_{ij}q_j + \sum^{j} b_{ij}\dot{q}_j = Q_i \quad (4.2)$$

with symmetric matrices, $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$.

We consider first the case where no thermal forces are applied ($Q_i = 0$). Solution of the homogeneous equations requires the solution of the characteristic determinant

$$\det\left|a_{ij} + pb_{ij}\right| = 0 \quad (4.3)$$

with p as unknown. It can be shown that the roots $-\lambda_s$ are real and never positive.[1] The characteristic solutions are

$$q_j^{(s)} = C^{(s)}\phi_j^{(s)}e^{-\lambda_s t} \quad (4.4)$$

where the $\phi_j^{(s)}$ are normalized modal distributions and $C^{(s)}$ arbitrary constants. We refer to these characteristic solutions as *thermal modes*. These modes satisfy orthogonality relations ($r \neq s$)

$$\sum^{ij} a_{ij}\phi_i^{(s)}\phi_j^{(r)} = \sum^{ij} b_{ij}\phi_i^{(s)}\phi_j^{(r)} = 0 \quad (4.5)$$

The proof of these relations is identical with that given in reference 1 in connection with relaxation phenomena.

Whether the roots λ_s are zero or infinite or multiple, there are always n such orthogonal modes in a system of n coordinates.

These mathematical properties are identical with those of a dynamic system with inertia and elasticity. The well-known numerical methods used in vibration analysis, such as iteration, for instance, may be applied to the evaluation of the thermal modes.

We should note that, in general, a thermal system will contain heat flow configurations with conservative fields—i.e., such that div $H = 0$. The thermal potential V is independent of the particular generalized coordinates corresponding to these fields, while D depends on these coordinates. These fields also constitute thermal modes of infinite relaxation time—i.e., of zero roots λ_s. The case is analogous to that of ignorable coordinates in dynamics, such as, for instance, the coordinate corresponding to the free motion of a solid.

Once the thermal modes are known, the basic equations (4.2) may be expressed in terms of normal coördinates defined by the relations

$$q_i = \sum^{s} \phi_i^{(s)}\xi_s \quad (4.6)$$

With these coordinates all equations are uncoupled since V and D reduce to sums of squares. Using properly normalized modal distributions, Eqs. (4.2) become

$$\lambda_s\xi_s + \dot{\xi}_s = \Xi_s \quad (4.7)$$

The generalized forces are

$$\Xi_s = \sum^{j} \phi_j^{(s)}Q_j \quad (4.8)$$

The operational solution of Eq. (4.7) reads

$$\xi_s = \Xi_s/(p + \lambda_s) \quad (4.9)$$

with $$p = d/dt$$

Substituting Eqs. (4.8) and (4.9) into Eq. (4.6) we obtain the operational solution of the original Eqs. (4.2) as

$$q_i = \sum^{j} A_{ij}Q_j \quad (4.10)$$

with an operational matrix

$$A_{ij} = A_{ji} = \sum^{s} [C_{ij}^{(s)}/(p + \lambda_s)] \quad (4.11)$$

which constitutes the thermal admittance of the system. The coefficients $C_{ij}^{(s)}$ are

$$C_{ij}^{(s)} = \phi_i^{(s)}\phi_j^{(s)} \quad (4.12)$$

If there are multiple roots, we may collect all the terms corresponding to the same root into one single term so that the summation in Eq. (4.11) for C_{ij}^s may be taken to extend to all distinct roots only.

In a thermal system there are no infinite values of λ_s—i.e., all relaxation constants are finite. However, as pointed out above there may be zero roots λ_s corresponding to steady heat flow through the system. This may be expressed by separating the zero root term in the admittance,

$$A_{ij} = \sum^{s} [C_{ij}^{(s)}/(p + \lambda_s)] + (C_0/p) \quad (4.13)$$

The significance of these operators is brought out by considering the case where all thermal forces vanish except Q_j. Assuming the time dependence such that Q_j corresponds to a sudden rise in temperature at $t = 0$ expressed by a Heaviside step function, we write

$$Q_j = 1(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad (4.14)$$

The coordinate q_i is then

$$q_i = \left[\sum^{s} \frac{C_{ij}^{(s)}}{p + \lambda_s} + \frac{C_0}{p}\right] 1(t) \qquad (4.15)$$

From the significance of these operators[4] we obtain

$$q_i = \sum^{s} (C_{ij}^{(s)}/\lambda_s)\,(1 - e^{-\lambda_s t}) + C_0 t \qquad (4.16)$$

The bracketed term corresponds to the establishment of an equilibrium temperature distribution, while the term proportional to t represents the steady-state heat flow through the system after the equilibrium temperatures are established.

(5) Some General Properties and Procedures

There are certain general properties of the temperatures and thermal flow fields which are worthy of a closer scrutiny because of their bearing on basic methods of solution of the equations.

It was pointed out above that, in general, a thermal system will contain a divergence-free field of flow corresponding to ignorable coordinates. Let us examine this point more closely by separating the flow field into a divergence free field $\boldsymbol{H}_f$ and a remainder $\boldsymbol{H}_\theta$.

$$\boldsymbol{H} = \boldsymbol{H}_f + \boldsymbol{H}_\theta \qquad (5.1)$$

The field $\boldsymbol{H}_\theta$ is associated with temperature changes while $\boldsymbol{H}_f$ is not. Each field may be represented by its own set of generalized coordinates q_i and f_i

$$\boldsymbol{H}_f = \sum^{i} \boldsymbol{F}_i f_i, \quad \boldsymbol{H}_\theta = \sum^{i} \boldsymbol{\Theta}_i q_i \qquad (5.2)$$

The f_i's represent the ignorable coordinates. By definition,

$$\operatorname{div} \boldsymbol{H}_f = \operatorname{div} \boldsymbol{F}_i = 0 \qquad (5.3)$$

Since the temperature is

$$\theta = -(1/c) \operatorname{div} \boldsymbol{H} = -(1/c) \sum^{i} q_i \operatorname{div} \boldsymbol{\Theta}_i \qquad (5.4)$$

the thermal potential V is independent of the f_i coordinates. The dissipation function including the surface heat transfer is

$$D = \frac{1}{2}\iiint_\tau \frac{1}{k}\left(\frac{\partial \boldsymbol{H}}{\partial t}\right)^2 d\tau + \frac{1}{2}\iint_s \frac{1}{K}\left(\frac{\partial H_n}{\partial t}\right)^2 dS \qquad (5.5)$$

or

$$D = (1/2) \sum^{ij} b_{ij}'' \dot{f}_i \dot{f}_j + (1/2) \sum^{ij} b_{ij}' \dot{f}_i \dot{q}_j + (1/2) \sum^{ij} b_{ij} \dot{q}_i \dot{q}_j \qquad (5.6)$$

with

$$\left.\begin{aligned} b_{ij}'' &= \iiint_\tau \frac{1}{k} \boldsymbol{F}_i \cdot \boldsymbol{F}_j d\tau + \iint_s \frac{1}{K} F_{in} F_{jn} dS \\ b_{ij}' &= \iiint_\tau \frac{1}{k} \boldsymbol{F}_i \cdot \boldsymbol{\Theta}_j d\tau + \iint_s \frac{1}{K} F_{in} \Theta_{jn} dS \\ b_{ij} &= \iiint_\tau \frac{1}{k} \boldsymbol{\Theta}_i \cdot \boldsymbol{\Theta}_j d\tau + \iint_s \frac{1}{K} \Theta_{in} \Theta_{jn} dS \end{aligned}\right\} \qquad (5.7)$$

The subscript n indicates normal components of the vectors. Now we may choose the fields $\boldsymbol{F}_i$ and $\boldsymbol{\Theta}_i$ in such a way that

$$b_{ij}' = \iiint_\tau (1/k) \boldsymbol{F}_i \cdot \boldsymbol{\Theta}_j d\tau + \iint_s (1/K) F_{in} \Theta_{jn} dS = 0 \qquad (5.8)$$

i.e., such that the $\boldsymbol{\Theta}_i$'s are orthogonal to the divergence-free field $\boldsymbol{F}_i$. The thermal potential V is independent of f_i. We have

$$V = (1/2) \iiint_\tau (1/c)\,(\operatorname{div} \boldsymbol{H})^2 d\tau \qquad (5.9)$$

or

$$V = (1/2) \sum^{ij} a_{ij} q_i q_j \qquad (5.10)$$

with

$$a_{ij} = \iiint_\tau (1/c)\,(\operatorname{div} \boldsymbol{\Theta}_i)\,(\operatorname{div} \boldsymbol{\Theta}_j) d\tau \qquad (5.11)$$

Under these conditions the equations for q_i are independent of the coordinates f_i, and we may write

$$\sum^{j} a_{ij} q_j + \sum^{j} b_{ij} \dot{q}_j = Q_i \qquad (5.12)$$

The question arises of how to establish such a coordinate system. This may be done by introducing scalar field distributions ψ_i associated with a thermal flow field by

$$\boldsymbol{\Theta}_i = k \operatorname{grad} \psi_i \qquad (5.13)^*$$

Then by Green's formula, and taking into account Eq. (5.3), we find

$$\iiint_\tau \frac{1}{k} \boldsymbol{F}_i \cdot \boldsymbol{\Theta}_j d\tau = \iint_s \psi_j F_{in} dS \qquad (5.14)$$

where n is the normal to the boundary taken positive outward. Condition (5.8) becomes

$$\iint_s F_{in}[\psi_j + (1/K)\Theta_{jn}] dS = 0 \qquad (5.15)$$

This is satisfied if the scalar ψ_i satisfies the following condition at the boundary:

$$\psi_j + (k/K) \operatorname{grad}_n \psi_j = 0 \qquad (5.16)$$

Because of Eq. (5.3) we may also substitute a constant on the right-hand side of this condition instead of zero. With such a coordinate system we have to solve the differential equations (5.12) for q_i, each of which is associated with a particular temperature configuration. The equations for f_i are irrelevant as far as temperature is concerned and are only of importance if we wish to calculate the heat flow. The equations for f_i are

$$\sum^{j} b_{ij}'' \dot{f}_j = Q_i'' \qquad (5.17)$$

where Q_i'' is the thermal force associated with f_i. The solution of this system is trivial. As we have pointed

* If k is discontinuous, we must choose ψ_i in such a way that the normal component $k \operatorname{grad}_n \psi_i$ is continuous across the surface of discontinuity of k, as well as the corresponding temperature.

out, these ignorable coordinates are the analog of those representing the free body motion in the vibration analysis of an elastic structure. They are eliminated here by a similar condition of orthogonality.

The differential system (5.12) with a reduced number of unknowns possesses the same properties as the general system with its own characteristic values, all different from zero. If the system is represented by its normal coordinates, the matrices a_{ij} and b_{ij} are diagonalized. Two normal coordinates ξ_r and ξ_s correspond to fields

$$H_\theta^{(r)} = \mathbf{\Theta}^{(r)}\xi_r, \quad H_\theta^{(s)} = \mathbf{\Theta}^{(s)}\xi_s \tag{5.18}$$

The vanishing of the nondiagonal term a_{rs} means

$$a_{rs} = \iiint_\tau (1/c)\,(\text{div }\mathbf{\Theta}^{(r)})\,(\text{div }\mathbf{\Theta}^{(s)})d\tau = 0 \tag{5.19}$$

The temperature fields associated with the modes ξ_r and ξ_s are

$$\theta_r = -(1/c)\text{ div }\mathbf{\Theta}^{(r)}, \quad \theta_s = -(1/c)\text{ div }\mathbf{\Theta}^{(s)} \tag{5.20}$$

In terms of the temperature fields the orthogonality condition (5.19) becomes

$$\iiint_\tau c\theta_r\theta_s d\tau = 0 \tag{5.21}$$

This condition is valid for nonisotropic media. The orthogonality condition in terms of the flow fields is obtained by putting the nondiagonal coefficients b_{rs} equal to zero. We find

$$\iiint_\tau (1/k)\mathbf{\Theta}^{(r)}\cdot\mathbf{\Theta}^{(s)}d\tau + \iint_s (1/K)\Theta_n^{(r)}\Theta_n^{(s)}dS = 0 \tag{5.22}$$

For an anisotropic medium it is easily verified by using Eq. (3.15) that the orthogonality condition becomes

$$\iiint_\tau \lambda_{ij}\Theta_i^{(r)}\Theta_j^{(s)}d\tau + \iint_s (1/K)\Theta_n^{(r)}\Theta_n^{(s)}dS = 0 \tag{5.23}$$

where $\Theta_i^{(r)}$ and $\Theta_j^{(s)}$ are the Cartesian components of the flow field.

In solving Eqs. (5.12) for the coordinates we may first calculate the normal coordinates. The normal coordinates may be conveniently obtained by an iteration procedure applied to the homogeneous system—i.e., putting $Q_i = 0$. The homogeneous equations in matrix form are

$$[a_{ij}]\,[q_j] = [b_{ij}]\,[\dot{q}_j] \tag{5.24}$$

For an exponential solution of the type $e^{-\lambda t}[q_j]$ we write

$$(1/\lambda)\,[q_i] = [a_{ij}]^{-1}[b_{ij}]\,[q_j] \tag{5.25}$$

The iteration procedure is the same as the well-known one used in vibration analysis.[4] Substituting a column q_j on the right-hand side, we obtain a new column which is again substituted. This process converges toward the thermal mode of smallest value of the relaxation constant λ. Use of the orthogonality condition yields successively the modes of higher λ.

We notice that if we use the thermal modes as the coordinates, the steady-state solution is represented by a superposition of a complete set of modal distributions.

In many problems with slowly varying temperatures the deviation from the steady state may be small and only a small number of relaxation modes with slow decay are excited. In such a case it is preferable to proceed as follows. Assume we have calculated the normal coordinates of the system. The equations then are

$$\lambda_r\xi_r + \dot{\xi}_r = \Xi_r \tag{5.26}$$

If the forces were applied very slowly, the solution for ξ_r would be

$$\zeta_r = (1/\lambda_r)\,\Xi_r \tag{5.27}$$

We may separate the solution into an instantaneous steady state ζ_r and a correction ξ_r—i.e., replace ξ_r by $\xi_r + \zeta_r$. This yields

$$\lambda_r\xi_r + \dot{\xi}_r = -\dot{\zeta}_r = -(1/\lambda_r)\dot{\Xi}_r \tag{5.28}$$

The force $\dot{\Xi}_r$ is found by calculating the virtual work of the time derivative of the boundary temperature $\dot{\theta}$ on each relaxation mode. It is seen that if the temperature varies slowly, only a small number of modes are excited. This is particularly advantageous if we evaluate the modes by iteration since only a few modes of lowest value of λ_s need be calculated. The steady-state solution ζ_r is not evaluated by normal coordinates, but the temperature field may be calculated directly. This is best illustrated by the example in the following section.

Before closing this discussion, attention should be called to an important feature. With the exception of normal coordinates the methods discussed above are not restricted to component field configurations $\mathbf{\Theta}_i$ which are constant. We may introduce variable fields such as

$$H_\theta = \mathbf{\Theta}_0(x, y, z, t) + \sum^i \mathbf{\Theta}_i(x, y, z, t)q_i \tag{5.29}$$

This is extremely important since it opens the way to a method of successive approximation where $\mathbf{\Theta}_0$ is an approximate solution to which q_i terms are added as a correction. In this case the differential equations for q_i may or may not have time-dependent coefficients.

Another important point, already mentioned before, is the possibility of including boundary heat-transfer coefficients K which are time dependent. Such is the case, for instance, in aerodynamic heating.

Methods outlined here all lead to linear equations. They are not the only ones available. In some problems it is preferable to choose the generalized coordinates in a way different from the above. This will be illustrated in Section (7).

(6) Application to Some One-Dimensional Problems

We shall now illustrate the methods of thermal flow analysis, outlined in the previous section, by formulating them more precisely on some problems of one-dimensional flow. Other methods relative to one-dimensional flow will be introduced in the following section.

In the present approach the unknown variables represent the amplitudes of certain given field configurations. In reference 3 we have discussed the solution of a plate problem by means of normal coordinates. This was the problem of a plate brought suddenly to a temperature θ_0 on one face and kept at temperature $\theta = 0$ on the other. Here we shall discuss the case where the plate is thermally insulated on the other face. The temperature is represented as a Fourier series,

$$h = c\theta = \sum_0^\infty q_n \sin\,[n + (1/2)]\,(\pi y/l) \quad (6.1)^*$$

and the corresponding heat flow is derived by integration. Since $H = 0$ at $y = l$, we write

$$H = \sum_0^\infty \frac{q_n l}{\pi[n + (1/2)]} \cos\left(n + \frac{1}{2}\right)\frac{\pi y}{l} \quad (6.2)$$

This corresponds to distributions of the scalar ψ_n

$$\psi_n = \sin\,[n + (1/2)]\,(\pi y/l) \quad (6.3)$$

satisfying condition (5.16).

The thermal potential and dissipation function are

$$\left.\begin{aligned} V &= \frac{1}{2c}\int_0^l h^2 dy = \frac{l}{4c}\sum_0^\infty q_n^2 \\ D &= \frac{1}{2k}\int_0^l \left(\frac{\partial H}{\partial t}\right)^2 dy = \frac{l^3}{4k\pi^2}\sum_0^\infty \frac{\dot{q}_n^2}{[n + (1/2)]^2} \end{aligned}\right\} \quad (6.4)$$

These forms reduce to sums of squares because the component field configurations satisfy the orthogonality relations (5.19) and (5.22). The thermal force Q_n associated with each coordinate is derived from the variation δH due to δq_n at $y = 0$,

$$Q_n \delta q_n = \theta_0 \delta H = \frac{\theta_0 l}{\pi[n + (1/2)]}\delta q_n \quad (6.5)$$

Hence, the differential equation (3.14) in the present case takes the form

$$\frac{l}{2c} q_n + \frac{l^3}{2k\pi^2[n + (1/2)]^2}\dot{q}_n = \frac{\theta_0 l}{\pi[n + (1/2)]} \quad (6.6)$$

These are uncoupled equations as should be since we have chosen normal coordinates. The equations are valid, of course, for any arbitrary variation of θ_0 with time. In our particular example, we have

$$\theta_0(t) = \theta_0 1(t) \quad (6.7)$$

where $1(t)$ is the Heaviside step function,

$$1(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

In more general cases the choice of normal coordinates as above will have the inconvenience of the uniformly distributed temperature at equilibrium being represented by a slowly convergent Fourier series. It may be preferable therefore to proceed as already suggested in the general discussion of Section (5). The temperature is represented by a term $\theta_0(t)$ independent of y, corresponding to the instantaneous steady state and a residue θ_1,

$$\theta = \theta_0(t) + \theta_1 \quad (6.8)$$

We apply the procedure leading to Eq. (5.28). The residue θ_1 is represented by normal coordinates q_n satisfying the same Eq. (6.6) except that the generalized force is now due to $\dot{\theta}_0$ and is divided by the characteristic value

$$\lambda_n = (k\pi^2/cl^2)\,[n + (1/2)]^2 \quad (6.9)$$

The equations are

$$\frac{l}{2c} q_n + \frac{l}{2c\lambda_n}\dot{q}_n = -\frac{l}{\lambda_n \pi[n + (1/2)]}\dot{\theta}_0 \quad (6.10)$$

In the present case, where θ_0 is given by Eq. (6.7), the solution is

$$q_n = -\frac{2c\,\theta_0}{\pi[n + (1/2)]} e^{-\lambda_n t} \quad (6.11)$$

The temperature θ_1 associated with this coordinate is given by Eq. (6.1)

$$c\theta_1 = -\sum_0^\infty \frac{2c\,\theta_0}{\pi[n + (1/2)]} e^{-\lambda_n t} \sin\left(n + \frac{1}{2}\right)\frac{\pi y}{l} \quad (6.12)$$

The total temperature field is

$$\theta = \theta_0 + \theta_1 = \theta_0 - \sum_0^\infty \{2\theta_0/\pi[n + (1/2)]\} \times e^{-\lambda_n t} \sin\,[n + (1/2)]\,(\pi y/l) \quad (6.13)$$

which is the classical solution for this problem.

It is, of course, possible to use any type of function, such as trigonometric functions, which are not necessarily orthogonal, and polynomials. For instance, we could use a polynomial

$$\theta = \sum_0^\infty q_n y^n \quad (6.14)$$

This might be particularly useful for a heterogeneous material, such as a slab made up of layers of different materials.

A convenient method for numerical work is to take advantage of the accuracy of Simpson's rule in performing the integrations by adapting it to the present case. The slab is divided into an even number, $2N$,

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* These sine functions are chosen because they are the natural thermal modes of the system. It can be inferred from results of the previous section that they constitute a complete orthogonal set of functions.

of intervals and the temperatures at the point of subdivision are $\theta_i (i = 0, 1, \ldots 2N)$. In pairs of intervals the temperature is represented by a parabola passing through the three ordinates of the two intervals. The heat flow is obtained by integration of the parabolic area.* By integrating[4] in the successive intervals, the heat flow vectors H_i at the various points of subdivision are given by

$$\frac{1}{c\Delta y}\begin{bmatrix} H_1 - H_0 \\ H_2 - H_1 \\ - \\ - \\ H_{2N} - H_{2N-1} \end{bmatrix} = \begin{bmatrix} -5/12 & -2/3 & 1/12 & 0 & 0 & - \\ 1/12 & -2/3 & -5/12 & 0 & 0 & - \\ 0 & 0 & -5/12 & -2/3 & 1/12 & - \\ 0 & 0 & 1/12 & -2/3 & -5/12 & - \\ - & - & - & - & - & - \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \cdot \\ \cdot \\ \theta_{2N} \end{bmatrix} \tag{6.15}$$

by addition of successive lines

$$\frac{1}{c\Delta y}\begin{bmatrix} H_1 - H_0 \\ H_2 - H_0 \\ - \\ - \\ H_{2N} - H_0 \end{bmatrix} = \begin{bmatrix} -5/12 & -2/3 & 1/12 & 0 & 0 & - \\ -1/3 & -4/3 & -1/3 & 0 & 0 & - \\ -1/3 & -4/3 & -3/4 & -2/3 & 1/12 & - \\ -1/3 & -4/3 & -2/3 & -4/3 & -1/3 & - \\ - & - & - & - & - & - \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \cdot \\ \cdot \\ \theta_{2N} \end{bmatrix} \tag{6.16}$$

By putting

$$q_{-1} = H_0/c\Delta y$$

this may also be written

$$\frac{1}{c\Delta y}\begin{bmatrix} H_0 \\ \cdot \\ \cdot \\ \cdot \\ H_{2N} \end{bmatrix} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} q_{-1} \\ \theta_0 \\ \cdot \\ \cdot \\ \theta_{2N} \end{bmatrix} \tag{6.17}$$

with

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & - \\ 1 & -5/12 & -2/3 & 1/12 & 0 & 0 & - \\ 1 & -1/3 & -4/3 & -1/3 & 0 & 0 & - \\ 1 & -1/3 & -4/3 & -3/4 & -2/3 & 1/12 & - \\ 1 & -1/3 & -4/3 & -2/3 & -4/3 & -1/3 & - \\ - & - & - & - & - & - & - \end{bmatrix} \tag{6.18}$$

The dissipation function is evaluated by Simpson's method expressed in matrix form.

$$D = (1/2k) \int_0^l (\partial H/\partial t)^2 dy = (\Delta y/2k)\ [\dot{H}_i]'\ [S]\ [\dot{H}_i] \tag{6.19}$$

where

$$[\dot{H}_i] = \begin{bmatrix} \dot{H}_0 \\ \cdot \\ \cdot \\ \cdot \\ \dot{H}_{2N} \end{bmatrix}$$

$$[\dot{H}_i]' = [\dot{H}_0 \ldots \dot{H}_{2N}] \tag{6.20}$$

and $[S]$ is the diagonal matrix corresponding to Simpson's rule

$$[S] = \frac{1}{3}\begin{bmatrix} 1 & & & & & & & \\ & 4 & & & & 0 & & \\ & & 2 & & & & & \\ & & & 4 & & & & \\ & & & & 2 & & & \\ & & & & & \cdot & & \\ & 0 & & & & & \cdot & \\ & & & & & & 4 & \\ & & & & & & & 1 \end{bmatrix} \tag{6.21}$$

* This integration of a continuous distribution of temperature in order to calculate the coefficients of the differential equation is the reason for the considerably increased accuracy of the present method over the usual finite difference methods of solution.

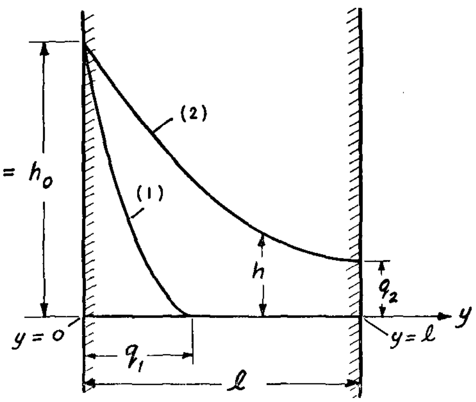

FIG. 1. Distribution of $h = C\theta$ for first phase (1), and second phase (2), in the heating of a slab.

Substituting in the expression for D

$$D = (c^2/2k)\ (\Delta y)^3\ [\dot{q}_i]'\ [B]\ [\dot{q}_i] \tag{6.22}$$

with the matrix

$$\begin{bmatrix} q_{-1} \\ \theta_0 \\ \cdot \\ \cdot \\ \cdot \\ \theta_{2N} \end{bmatrix} = [q_i] \tag{6.23}$$

We have introduced the transposed $[\dot{q}_i]'$ of $[\dot{q}_i]$ and

$$[B] = [A]'\ [S]\ [A] \tag{6.24}$$

where $[A]'$ is the transposed of $[A]$. The thermal potential is again by Simpson's rule

$$V = (c/2) \int_0^l \theta^2 dy = (c/2)\Delta y[\theta_i]'\ [S]\ [\theta_i] \tag{6.25}$$

We may consider the $2N + 2$ parameters $q_{-1}, \theta_0 \ldots \theta_{2N}$ as the generalized coordinates and proceed as in the general case. We must evaluate the corresponding generalized forces. Let us assume that the temperature θ_{2N} is given while the surface at $i = 0$ is insulated. This means $H_0 = 0$. In this case there are $2N + 1$ coordinates $\theta_0\theta_1 \ldots \theta_{2N}$ and the matrix $[A]$ is reduced to that in relation (6.16). The generalized forces are evaluated by the expressions

$$\theta_{2N}\delta H_{2N} = Q_i\delta\theta_i \tag{6.26}$$

The differential equations are

$$[S]\ [\theta_i] + (c/k)\ (\Delta y)^2\ [B]\ [\dot{\theta}_i] = [Q_i] \tag{6.27}$$

Since $[S]$ is diagonal, the thermal modes may be obtained immediately by iteration.

This method is quite general. If we are dealing with a composite slab, the procedure is identical.

The case where there is a coefficient of heat transfer K at the surface of the slab is easily taken care of by adding a term to the dissipation function in accordance with Eq. (3.21).

The additional term is

$$D = (1/2K)\dot{H}_{2N}^2 \tag{6.28}$$

The thermal potential remains unchanged.

(7) Heating of a Slab With Constant Parameters—the Concepts of Penetration Depth and Transit Time

In the previous section we have outlined procedures for the analysis of one-dimensional problems leading to linear differential equations for a sequence of unknown parameters representing the amplitudes of component flow fields. The flexibility of the present method is well illustrated by the possibility of using an entirely different approach where only one unknown parameter is used to represent the unknown field. Although the problem is physically linear, the parameter is found to satisfy a nonlinear differential equation of the first order. The nonlinearity in this case is compensated by the simplicity of the field representation. This approach also leads to two very useful concepts in practical applications—i.e., *penetration depth* and *transit time.*

Consider a plate of thickness l with constant values of the thermal conductivity k and heat capacity c (see Fig. 1).

One face at $y = 0$ is heated suddenly to the temperature θ_0 at $t = 0$. The other face at $y = l$ is thermally insulated so that no heat flows across it.

We shall assume the temperature distribution to be represented by a parabola and consider two phases in the phenomenon. In the first phase, the temperature has not yet begun to rise at the opposite wall and everything occurs as if the wall thickness were infinite. During this phase the heat content distribution h, which is proportional to the temperature, is approximated by

$$\begin{aligned} h &= h_0\,[1 - (y/q_1)]^2 \quad \text{for} \quad y < q_1 \\ h &= 0 \qquad\qquad\qquad\quad \text{for} \quad y > q_1 \end{aligned} \tag{7.1}$$

with $h = c\theta, \quad h_0 = c\theta_0$

The parameter q_1 is the generalized coordinate and may be called the *penetration depth.* The problem is to establish the value of q_1 as a function of time.

The thermal potential is

$$V = (c/2) \int_0^l \theta^2 dy = (1/2c) \int_0^{q_1} h^2 dy \tag{7.2}$$

hence,

$$V = (1/10)\ (h_0^2/c)q_1 \tag{7.3}$$

In order to determine the dissipation function we must introduce the heat flow vector H. This is obtained from h by integrating (2.2)—i.e., from

$$-dH/dy = h \tag{7.4}$$

Taking into account the condition $H = 0$ at $y = q_1$ we find

$$\left.\begin{aligned} H &= (1/3)h_0q_1 - h_0y + h_0(y^2/q_1) - (h_0/3)\ (y^3/q_1^2) \\ \dot{H} &= (1/3)h_0\dot{q}_1 - h_0(y^2/q_1^2)\dot{q}_1 + (2/3)h_0(y^3/q_1^3)\dot{q}_1 \end{aligned}\right\} \tag{7.5}$$

The dissipation function is

$$D = (1/2k)\int_0^{q_1} (\dot{H})^2 dy = (1/2)\cdot(13/315)\cdot(h_0^2/k)q_1\dot{q}_1^2 \quad (7.6)$$

Finally, we must evaluate the generalized force. This is obtained by considering the virtual displacement δH at $y = 0$. From Eq. (7.5) we have

$$\delta H = (1/3)h_0\delta q_1 \quad (7.7)$$

By definition,

$$Q_1\delta q_1 = \theta_0\delta H = (1/3)h_0\theta_0\delta q_1 \quad (7.8)$$

Hence, $$Q_1 = (1/3)h_0\theta_0 = (1/3c)h_0^2 \quad (7.9)$$

The differential equation for q_1 is

$$(\partial V/\partial q_1) + (\partial D/\partial \dot{q}_1) = Q_1 \quad (7.10)$$

or $$(1/10)\,(h_0^2/c) + (13/315)\,(h_0^2/k)q_1\dot{q}_1 = (1/3)h_0^2 \quad (7.11)$$

This is a first-order differential equation for q_1, which is easily integrated. With the initial condition $q_1 = 0$ for $t = 0$ we find

$$q_1 = 3.36\sqrt{(k/c)t} \quad (7.12)$$

In the first phase the penetration depth is proportional to $\sqrt{t}$. It will be noted that we have treated a linear problem by a nonlinear differential equation. The nonlinearity in this case is of a purely geometrical nature and is introduced by the fact that the generalized coordinate is related nonlinearly to the physical variable to be determined. The general disadvantage of nonlinearity is, in this case, largely outweighed by the remarkable simplicity of the solution.

From Eqs. (7.1) and (7.12) we derive the temperature distribution during the first phase

$$\theta = \theta_0\left[1 - \frac{y}{3.36\sqrt{k(t/c)}}\right]^2 \quad (7.13)$$

This phase ends when the temperature begins to rise at the boundary—i.e., at a time when $q_1 = l$ or

$$t_1 = 0.0885\, cl^2/k \quad (7.14)$$

We shall call this the *transit time*. With this transit time we may also write

$$\left.\begin{aligned} q_1 &= l\sqrt{t/t_1} \\ \theta/\theta_0 &= \left[1 - (y/l)\sqrt{t_1/t}\,\right]^2 \end{aligned}\right\} \quad (7.15)$$

In the second phase the heat content $c\theta$ at $y = l$ is used as generalized coordinate q_2 and the heat content distribution is again approximated by a parabola

$$h = (h_0 - q_2)\,[1 - (y/l)]^2 + q_2 \quad (7.16)$$

The thermal potential is

$$V = (1/2c)\int_0^l h^2 dy = (1/2c)h_0^2 l + (4l/15c)\,(q_2 - h_0)^2 + (2lh_0/3c)\,(q_2 - h_0) \quad (7.17)$$

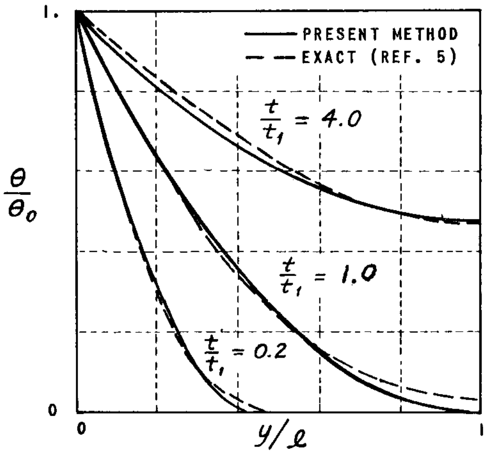

FIG. 2. Temperature distribution at various times in a slab.

The heat flow vector is found by integrating Eq. (7.4) with the boundary condition $H = 0$ at $y = l$. We find

$$-H = (h_0 - q_2)\,[y - (y^2/l) + (y^3/3l^2) - (l/3)] + q_2(y - l) \quad (7.18)$$

$$\dot{H} = \dot{q}_2[-(y^2/l) + (y^3/3l^2) + (2/3)l] \quad (7.19)$$

Hence the dissipation function is

$$D = (1/2k)\int_0^l \dot{H}^2 dy = (34/315)\,(l^3/k)\dot{q}_2^2 \quad (7.20)$$

We must finally evaluate the generalized force Q_2 by considering the value of δH at $y = 0$. We write

$$Q_2\delta q_2 = \theta_0\delta H = (2/3)\theta_0 l\delta q_2 \quad (7.21)$$

Hence, $$Q_2 = (2/3)\theta_0 l = (2/3)\,(h_0/c)l \quad (7.22)$$

The differential equation for q_2 is

$$(\partial V/\partial q_2) + (\partial D/\partial \dot{q}_2) = Q_2 \quad (7.23)$$

or explicitly, after simplification and introduction of the transit time t_1,

$$q_2 + 4.67t_1\dot{q}_2 = h_0 \quad (7.24)$$

The solution with initial conditions $q_2 = 0$ at $t = t_1$ is

$$(q_2/h_0) = 1 - \exp\left\{-0.214[(t/t_1) - 1]\right\} \quad (7.25)$$

With this value of q_2/h_0 the temperature distribution in the second phase is given by

$$\theta/\theta_0 = [1 - (q_2/h_0)]\,[1 - (y/l)]^2 + (q_2/h_0) \quad (7.26)$$

This expression, along with Eq. (7.13), gives the complete time history of the temperature. The distributions θ/θ_0 thus obtained are plotted in Fig. 2 as functions of y/l for three values of the time ratio

$$t/t_1 = 0.2,\ 1.0,\ 4.0$$

The dotted line shows the results obtained by the classical Fourier series expansion and is taken from reference 5. Agreement of the present method with the exact solution is seen to be quite satisfactory.

It is of interest to evaluate the order of magnitude of the transit time t_1 [see Eq. (7.14)]. The value of the diffusivity $\kappa = k/c$ is $\kappa = 0.12$ cm.²/sec. for steel. Considering a steel plate 1 in. thick, $l = 2.5$ cm. and $t_1 = 4.6$ sec. For a dural plate of same thickness $\kappa = 0.86$ cm.²/sec., $t_1 = 0.62$ sec.

This concept of transit time is useful in many ways. It gives immediately the order of magnitude of specific transient effects in complex structures and indicates which features of the thermal flow may be neglected in comparison with other time constants.

The solution obtained here for the sudden rise in temperature at the wall involves very simple functions. It is, therefore, very simple to express in closed form the temperature field originating when the wall temperature is an arbitrary function of time using Duhamel's integral.[4] This will be applied in the example of Section (10).

(8) Extension to Nonlinear Systems With Temperature Dependent Parameters and Surface Radiation

In a medium where the heat capacity $c(x, y, z, \theta)$ and thermal conductivity $k(x, y, z, \theta)$ are functions of the temperature θ, the thermal conduction is governed by a nonlinear equation identical in form with Eq. (2.1). We will show that the variational principle as expressed by Eq. (2.5) is still valid in this case, provided we define the thermal potential in an appropriate way. We introduce

$$h(x, y, z, \theta) = \int_0^\theta c\,d\theta \tag{8.1}$$

the total heat acquired by the unit volume. Further, we define a density function as

$$F(x, y, z, h) = \int_0^h \theta\,dh = \int_0^\theta c\theta\,d\theta \tag{8.2}$$

Finally, the thermal potential is defined for the volume τ as

$$V = \iiint_\tau F\,d\tau \tag{8.3}$$

Comparing with the definition in the linear case we see that F replaces the quantity $(1/2)c\theta^2$ which is obtained from Eq. (8.2) if c is independent of the temperature.

With this definition we evaluate the variational quantity

$$\delta V = \iiint (\partial F/\partial h)\delta h\,d\tau \tag{8.4}$$

Conservation of energy requires

$$h = -\text{div}\,\boldsymbol{H} \tag{8.5}$$

Then, from Eq. (8.2), $\quad \partial F/\partial h = \theta \qquad (8.6)$

Hence,

$$\delta V = -\iiint_\tau \theta\,\text{div}\,(\delta \boldsymbol{H})d\tau \tag{8.7}$$

which is formally identical with Eq. (2.6). The variation δD is defined in the same way as for the linear case. We may then proceed exactly as in the linear case substituting δV in Eq. (2.5) and deriving the differential equation of heat conduction from the variational principle.

For the anisotropic nonlinear case the definition of V is the same as for the isotropic case and δD remains formally identical with the linear definition. The only difference lies in the fact that the thermal resistivity matrix λ_{ij} is now a function of the temperature θ.

The analogy of heat conduction to fluid flow in a porous material holds for the nonlinear case. The compressibility of the fluid and the permeability are functions of the pressure. The mechanical potential V is then given by the same Eq. (8.3), where F is defined as

$$F = \int_0^h p\,dh \tag{8.8}$$

The incremental fluid pressure is p and h is the incremental mass of fluid acquired by the pores per unit volume of bulk material. It is related to the mass flow field $\boldsymbol{H}$ by relation (8.5). The definition of the dissipation function is identical with Eq. (3.10) and is proportional to the energy dissipated through friction.

It should be noted that an equivalent formulation of the variational principle is obtained in the isotropic case when the heat conductivity is a function only of the temperature. By a well-known transformation we introduce the variable

$$u = \int_0^\theta k(\theta)d\theta \tag{8.9}$$

We may write

$$k\,(\partial\theta/\partial x) = (\partial u/\partial x), \quad \text{etc.} \tag{8.10}$$

$$c(\partial\theta/\partial t) = (c/k)\,(\partial u/\partial t) \tag{8.11}$$

Eq. (2.1) then reads

$$\nabla^2 u = (c/k)\,(\partial u/\partial t) \tag{8.12}$$

The problem is then mathematically identical with the case of a temperature distribution u in a medium of thermal conductivity equal to unity and a heat capacity c/k function of u.

The extension of the variational principle and the associated differential equations (3.14) for the generalized coordinates is completely general. It includes the case where the surface heat-transfer coefficient K is a function of both the outside temperature and the wall temperature. Moreover, it may also be a given function of time. This feature leads to an exact treatment of problems which involve boundary-layer heat transfer under variable flow conditions and including radiation losses in the high temperature range. This is in addition to the inclusion of temperature dependence of heat capacity and thermal conductivity of the materials in cases where they vary widely in the considered temperature range.

Surface radiation effects may be incorporated in the general expression for the dissipation function. We may visualize a heat flow bifurcation at the surface. The heat flowing into the surface per unit area may be written $H_b - H_r$ when H_b is the flow through the boundary layer and H_r the heat flow lost by radiation. The radiation is equivalent to an additional branch of the thermal flow network. The radiation loss may be written

$$\dot{H}_r = \epsilon\sigma[(T+\theta)^4 - T^4] \qquad (8.13)$$

where $T + \theta$ is the absolute temperature of the surface, ϵ the emissivity, and σ the Stefan constant. This is equivalent to

$$\dot{H}_r = K_r\theta \qquad (8.14)$$

with a temperature-dependent heat-transfer coefficient

$$K_r = (\epsilon\sigma/\theta)\ [(T+\theta)^4 - T^4] \qquad (8.15)$$

If we wish to include radiation, the dissipation function (3.21) is replaced by

$$D = \frac{1}{2}\iiint_\tau \frac{1}{k}\left(\frac{\partial H}{\partial t}\right)^2 d\tau + \frac{1}{2}\iint_s \frac{1}{K_b}(\dot{H}_b)^2 dS + \frac{1}{2}\iint_s \frac{1}{K_r}(\dot{H}_r)^2 dS \qquad (8.16)$$

where K_b is the boundary-layer heat-transfer coefficient. Note that the use of a reference temperature T is purely a matter of convenience. We could very well have put $T = 0$, in which case θ becomes the absolute temperature.

(9) Heating and Cooling of a Slab With Temperature Dependent Parameters

As an application of the method to a nonlinear problem let us consider a homogeneous slab with heat capacity and conductivity dependent on the temperature. We have shown that the problem may be reduced to one of constant conductivity. Therefore, we shall consider the case where the heat capacity alone is a function of the temperature. This being a nonlinear problem, different results will be obtained for heating and cooling.

We first assume that the face at $y = 0$ is brought suddenly to a higher temperature θ_0 and that the heat capacity varies by a factor of two within the range of temperatures considered.

$$c = c_0[1 + (\theta/\theta_0)] \qquad (9.1)$$

In Section (7) we have seen that there is a first phase where the temperature is approximately the same as in an infinitely thick slab. Using again the parabolic approximation we put

$$\theta/\theta_0 = [1 - (y/q_1)]^2 \qquad (9.2)$$

where q_1 is the depth of penetration of the temperature rise. Following the general procedure of Section (8) for the nonlinear case we evaluate

$$h = \int_0^\theta c d\theta = c_0\theta + (c_0/2)\ (\theta^2/\theta_0) \qquad (9.3)$$

$$F = \int_0^\theta c\theta d\theta = (1/2)c_0\theta^2 + (c_0/3)\ (\theta^3/\theta_0) \qquad (9.4)$$

The thermal potential is

$$V = \int_0^{q_1} F dy = (31/210)c_0\theta_0^2 q_1 \qquad (9.5)$$

The heat flow H is obtained from

$$H = \int_y^{q_1} h dy \qquad (9.6)$$

Putting $\qquad \zeta = 1 - (y/q_1) \qquad (9.7)$

we find

$$H = q_1 \int_0^s h d\zeta = q_1 c_0\theta_0[(1/3)\zeta^3 + (1/10)\zeta^5] \qquad (9.8)$$

Since $\qquad \dot{\zeta} = (y/q_1^2)\dot{q}_1 = -(1-\zeta)\ (\dot{q}_1/q_1) \qquad (9.9)$

the dissipation function is

$$D = \frac{1}{2k}\int_0^{q_1} \dot{H}^2 dy = \frac{1}{2k} q_1 \int_0^\zeta \dot{H}^2 d\zeta = \frac{0.0648}{2K} c_0^2\theta_0^2 q_1\dot{q}_1^2 \qquad (9.10)$$

The thermal force is obtained from δH at $y = 0$

$$Q_1\delta q_1 = \theta_0\delta H = (13/30)c_0\theta_0^2\delta q_1 \qquad (9.11)$$

Hence, $\qquad Q_1 = (13/30)c_0\theta_0^2 \qquad (9.12)$

The differential equation for q_1 is

$$(31/210)c_0\theta_0^2 + (0.0648/k)c_0^2\theta_0^2 q_1\dot{q}_1 = (13/30)c_0\theta_0^2 \qquad (9.13)$$

The solution with initial conditions $q_1 = 0$ at $t = 0$ is

$$q_1 = 2.97\sqrt{kt/c_0} \qquad (9.14)$$

The transit time is found by putting $q_1 = l$,

$$t_1 = 0.106\ c_0(l/k) \qquad (9.15)$$

We notice that the transit time for the present case where the heat capacity varies in the ratio of two to one is obtained by putting $c = 1.28c_0$ in Eq. (7.14) for the linear case. This is appreciably lower than the average value $c = 1.5c_0$, indicating that the heat propagation is controlled more by the value of c in the region of lower temperature.

It is of interest for comparison to consider the problem of cooling instead of heating. In this case the face of the wall at $y = 0$ is cooled suddenly to a temperature $\theta = -\theta_0$ at time $t = 0$. We assume again a variation of the heat capacity from $c = c_0$ to $c = 2c_0$ between the extremes of temperature. This is expressed by the formula

$$c = 2c_0[1 + (1/2)\ (\theta/\theta_0)] \qquad (9.16)$$

Because the problem is nonlinear the cooling problem

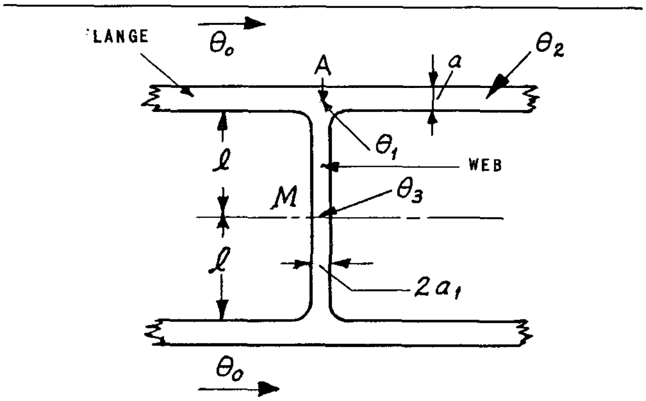

FIG. 3. Cross section of wing structure.

is different from that of heating *even if it occurs between the same temperature limits.*

Proceeding exactly as above, we find

$$V = (16/105)c_0\theta_0^2 q_1 \qquad (9.17)$$

$$H = 2c_0\theta_0 q_1[(1/20)\zeta^5 - (1/3)\zeta^3] \qquad (9.18)$$

$$D = 0.0634\,(c_0^2/k)\theta_0^2 q_1\dot{q}_1^2 \qquad (9.19)$$

The thermal force is found by forming δH at $y = 0$ or $\zeta = 1$;

$$Q_1\delta q_1 = -\theta_0\delta H = (17/30)c_0\theta_0^2\delta q_1 \qquad (9.20)$$

Hence the equation

$$2q_1\dot{q}_1 = 6.5(k/c_0) \qquad (9.21)$$

or

$$a_1 = 2.55\sqrt{kt/c_0} \qquad (9.22)$$

The transit time t_1—i.e., the time at which the opposite wall begins to cool off— is

$$t_1 = 0.154(c_0/k)l^2 \qquad (9.23)$$

If we compare this with Eq. (9.15) for the heating problem, we notice that it is quite a bit larger. In other words, the cooling takes about 40 per cent longer. Further comparison with the value (7.14) for the linear problem shows that the heat capacity behaves as if it had a constant value 1.74 c_0 which is higher than the average value 1.5 c_0. Therefore, the cooling occurs as if the heat capacity were higher than the average value while in the heating it is lower. In either case, the effective heat capacity tends to be closer to its value at the initial wall temperature.*

(10) Application to Supersonic Wing Structures

We consider the supersonic wing structure whose cross section is shown in Fig. 3. It is heated on both sides by air brought suddenly at a temperature θ_0 above the reference level $\theta = 0$ at time $t = 0$. The heat is transmitted through the boundary layer to the flange and the web. We propose to calculate the temperature field in this structure as a function of time.

* In general, we may conclude that the effective heat capacity will correspond to a temperature about midway between the average and the initial values.

In order to shorten the present calculation, certain simplifying assumptions are introduced. The simplifications are by no means essential to the method. However, they do lead to answers which are of acceptable accuracy in our problem.

The present numerical solution must be considered as a first approximation which, although quite satisfactory to our purpose, may be improved to include any of the more complex features of the actual problems as pointed out in more detail below.

We assume the heat flow to be one-dimensional—i.e., the temperature is uniform across the thickness in both web and flange. This assumption is equivalent to stating that the transit time across the thickness is a small fraction of the total time history of the system. The transit time for a steel plate of 1/2 in. thickness is of the order of 1 sec. The boundary-layer heat-transfer coefficient is taken to be a constant K. We consider the case where the wing is heated equally from both sides so that the problem is symmetric and no heat flow occurs at point M middle of the web. The procedure is easily adapted to the case of unsymmetric heating. The *equivalent one-dimensional system* is shown in Fig. 4.

A first step is to calculate the temperature history for the flange alone in the absence of any web. If a denotes the flange thickness and c its heat capacity per unit volume, the temperature of the flange θ_2 obeys the differential equation

$$ac\,\dot{\theta}_2 = K(\theta_0 - \theta_2) \qquad (10.1)$$

The solution with the initial condition $\theta_2 = 0$ is

$$\theta_2 = \theta_0[1 - e^{-t/\tau}] \qquad (10.2)$$

We have introduced the relaxation time of the flange boundary-layer system

$$\tau = ac/K \qquad (10.3)$$

In the numerical case considered below this relaxation time is found to be $\tau = 86$ sec. We now introduce the influence of the web. In the one-dimensional system of Fig. 4, the web of actual thickness $2a_1$ is represented by a plate of half the thickness, a_1, attached end-on to

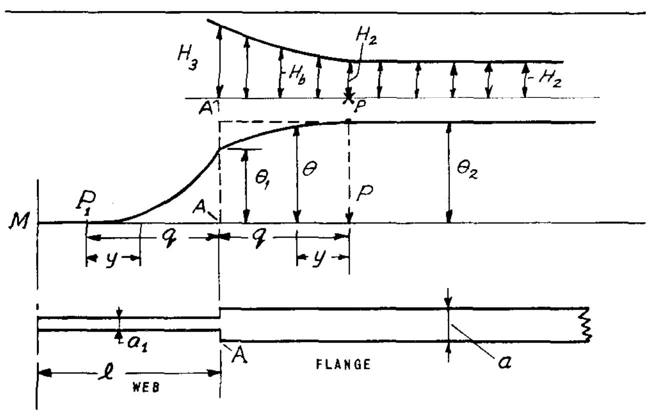

FIG. 4. Equivalent one-dimensional system with approximate temperature distribution θ and heat inflow H_b through the boundary layer.

a plate representing the flange. In the first phase of the thermal history the heat has not yet had time to reach the mid-web M. This phase occurs when the elapsed time t is smaller than the transit time t_1 associated with the half length l of the web. This transit time according to Eq. (7.14) is

$$t_1 = 0.0885(l^2/\kappa) \tag{10.4}$$

where κ is the diffusivity of the web. In the numerical example below the half length of the steel web is 4.69 in. and the transit time of the half web is $t_1 = 105$ sec.

In the first phase we take advantage of a general feature exhibited by the approximate solution developed in Section (7) for the penetration of heat in a slab.

We shall assume that in the web the penetration of heat is of a similar nature and that the temperature distribution in the web begins to rise at a moving point P_1 whose distance from the joint A of web and flange is determined by Eq. (7.12) for the penetration depth—i.e.,

$$P_1A = q = 3.36\sqrt{\kappa t} \tag{10.5}$$

Furthermore, we assume that the distribution of temperature in the web between the moving point P_1 and the joint A is parabolic. This temperature in the web is

$$\theta = \theta_1(y^2/q^2) \tag{10.6}$$

where θ_1 is the temperature at point A and y is counted from P_1 as origin. Similarly, we assume that the influence of the web on the temperature in the flange is to produce a drop below the temperature θ_2 calculated above for the isolated flange, and that this drop penetrates to a point P whose distance from A is determined by the same Eq. (10.5) for the penetration depth—i.e., $PA = q$. The temperature distribution in the flange between P and A is again assumed parabolic. We write for this temperature

$$\theta = \theta_2 + (\theta_1 - \theta_2)\,(y^2/q^2) \tag{10.7}$$

where y is counted from P as origin. Finally, we also assume a parabolic distribution for the total heat H_b per unit length which has flowed through the boundary layer between points A and P. We write

$$H_b = H_2 + (H_3 - H_2)\,(y^2/q^2) \tag{10.8}$$

The quantity H_2 represents the heat which has flowed into the flange without the web—i.e., outside of the region AP. It is given by

$$H_2 = ac\theta_2 \tag{10.9}$$

It is a known function of time through Eq. (10.2) for θ_2. The unknown quantities in the above expressions are H_3, which is the value of H_b at the joint A, and θ_1. However, these two quantities are related by the law of conservation of total heat. We state that the total heat, which has flowed through the boundary layer between A and P_1, is equal to the amount of heat stored in the web and flange between P_1 and P. This is expressed by

$$\int_P^A H_b dy = ac\int_P^A \theta dy + a_1 c\int_{P_1}^A \theta dy \tag{10.10}$$

Substituting the values (10.6) and (10.7) for θ and (10.8) for H_b gives H_3 as a function of θ_1,

$$H_3 = ac\beta\theta_1 \tag{10.11}$$

with $\quad \beta = 1 + \gamma, \quad \gamma = a_1/a$

We are thus left with a single unknown, the temperature θ_1 of the joint. A differential equation for this quantity may be found by applying the general principles developed in the previous sections. The dissipation function of the boundary layer is

$$D = (1/2K)\int_0^q \dot{H}_b^2 dy = (1/2K)\dot{H}_2^2 q + (1/3) \times$$
$$(\dot{H}_2/K)\,(\dot{H}_3 - \dot{H}_2)q + (1/10K)(\dot{H}_3 - \dot{H}_2)^2 q \tag{10.12}$$

Substituting Eq. (10.9) and Eq. (10.11) yields

$$D = (a^2c^2/K)q[(1/2)\dot{\theta}_2^2 + (1/3)\dot{\theta}_2(\beta\dot{\theta}_1 - \dot{\theta}_2) +$$
$$(1/10)\,(\beta\dot{\theta}_1 - \dot{\theta}_2)^2] \tag{10.13}$$

The generalized force Θ_1 associated with the coordinate θ_1 is found by evaluating the virtual work of the temperature on both sides of the boundary layer for a virtual change of heat flow. From Eqs. (10.8) and (10.11) we derive

$$\delta H_b = (y^2/q^2)\delta H_3 = ac\beta(y^2/q^2)\delta\theta_1 \tag{10.14}$$

The thermal force Θ_1 is defined by

$$\Theta_1\delta\theta_1 = \int_P^A (\theta_0 - \theta)\delta H_b dy \tag{10.15}$$

where θ is given by Eq. (10.7). Hence,

$$\Theta_1 = (1/3)ac\beta q(\theta_0 - \theta_2) - (1/5)ac\beta q(\theta_1 - \theta_2) \tag{10.16}$$

The differential equation for θ_1 is

$$\partial D/\partial\dot{\theta}_1 = \Theta_1 \tag{10.17}$$

or

$$(ac/3K)\dot{\theta}_2 + (1/5K)ac(\beta\dot{\theta}_1 - \dot{\theta}_2) =$$
$$(1/3)\,(\theta_0 - \theta_2) - (1/5)\,(\theta_1 - \theta_2) \tag{10.18}$$

There are cancellations of terms due to Eq. (10.1). Moreover, by putting $\theta_0 - \theta_1 = z$ and introducing τ from Eq. (10.3), Eq. (10.18) reduces to

$$z + \beta\tau\dot{z} = \gamma\tau\dot{\theta}_2 \tag{10.19}$$

With the initial condition $z = 0$ at $t = 0$ and the value (10.2) for θ_2, the solution of this equation is

$$z = \theta_0[e^{-t/\beta\tau} - e^{-t/\tau}] \tag{10.20}$$

This may be written

$$\theta_1 = \theta_0(1 - e^{-t/\beta\tau}) \tag{10.21}$$

At this point it is of interest to introduce numerical values. The dimensions are taken to be

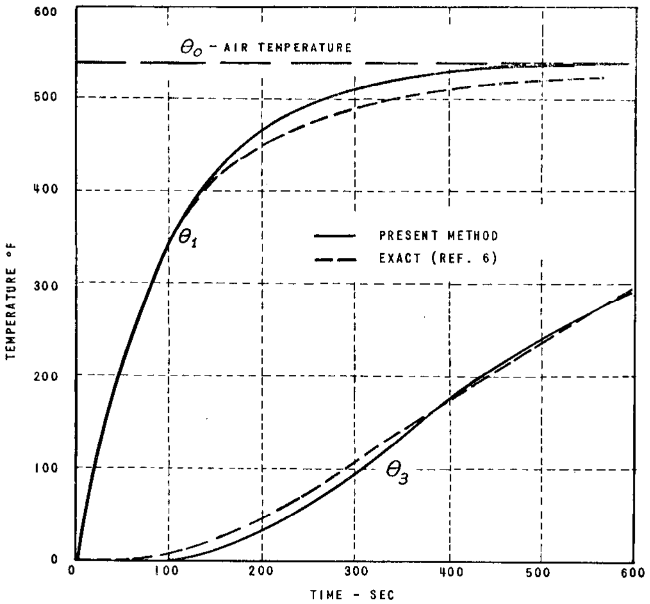

FIG. 5. Temperature θ_1 at the joint A and θ_3 at the midweb M.

l = 4.69 in., half length of web
a = 0.375 in., thickness of flange
$2a_1$ = 0.1 in., thickness of web

The material is steel, for which

$\kappa = k/c = 0.0186$ in.²/sec. diffusivity
$c = 68.6$ B.t.u./ft.³ °F. heat capacity per cu.ft.

The heat-transfer coefficient of the boundary layer is assumed to be

$$K = 90 \text{ B.t.u./hour ft.}^2 \text{ °F.}$$

With these values the relaxation time of the flange-boundary-layer system is

$$\tau = ac/\kappa = 86 \text{ sec.}$$

The air is assumed to rise at $t = 0$ to a temperature $\theta_0 = 540$°F. above the initial level. We introduce the value $\beta = 1 + (a_1/a) = 1.133$. The numerical value of the joint temperature θ_1 as a function of time calculated from (10.21) is plotted in Fig. 5. The present numerical values are the same as in the example treated by Pohle and Oliver (see reference 6),* who solved the heat conduction problem in the one-dimensional model by the exact method using the series and Laplace transform solutions of the corresponding partial differential equations. The procedure is quite elaborate. The exact value of θ_1 taken from reference 6 is also plotted for comparison in Fig. 5. It is seen to be in excellent agreement with our approximation.

The result also points to the fact, which could have been surmised, that the influence on the temperature θ_1 of the boundary condition at the mid-web M is not preponderant in this case. The assumption introduced for the first phase ($t < t_1$) is practically valid for the complete time history of θ_1. We may therefore use the approximate value (10.21) of θ_1 to compute the time history of the temperature at any point in the web by applying the Duhamel integral to the approximate solution developed in Section (7). For instance, let us calculate the temperature at the mid-web M in that manner. According to Eqs. (7.25) and (7.26) a sudden unit rise of temperature at A at time $t = 0$ produces no rise of temperature at the mid-web M until $t > t_1$. After that at time $t = t_1 + \Delta t$ the temperature is

$$\theta = 1 - e^{-\delta\Delta t} \qquad (10.22)$$

where $\delta = 0.214/t_1$. The transit time t_1 for the half web as given by Eq. (7.14) is

$$t_1 = 0.0885(l^2/\kappa) = 105 \text{ sec.}$$

By Duhamel's integral the temperature θ_3 at M is at time $t_1 + \Delta t$

$$\theta_3 = \int_0^{\Delta t} [1 - e^{-\delta(\Delta t - t')}]\dot{\theta}_1 dt' \qquad (10.23)$$

Introducing the value (10.21) for θ_1 we find

$$\theta_3 = \theta_0[1 - 1.25e^{-\delta\Delta t} + 0.25e^{-\Delta t/\beta\tau}] \qquad (10.24)$$

This is plotted for times $t = t_1 + \Delta t$ in Fig. 5 and compared with the exact value from reference 6. Again the two plots are in excellent agreement.

Before terminating this example we shall briefly outline how the present method may be used to improve the accuracy of the solution and to include more complex features and refinements.

A first step in this direction is to use the temperature history of the web calculated by expressions such as Eq. (10.23) instead of the parabolic distribution assumed above. The whole procedure is then repeated leading to an improved differential equation for θ_1. A further improvement would take into account the calculated temperatures in the flange. This will lead, in general, to a differential equation with variable coefficients.

Another improvement would be to get rid of the assumption of one-dimensional flow. For instance, we would start with the isolated flange problem and introduce instead of θ_2 two unknown temperatures, one on the side of the boundary layer, the other on the inside, with a parabolic temperature distribution across the thickness. We may then proceed further along similar lines. Two-dimensional flow in the web may also be introduced by a parabolic correction term for the temperature across the thickness along with a correction term for the nonuniformity of the heat flow near the joint.

In the present example we have not mentioned the influence of the distance between webs. This effect may, of course, be taken into account by separating the problem into a third phase corresponding to the transit time associated with the distance between the webs.

Antisymmetric heat flow results in the temperature θ_3 being kept zero. A parabolic approximation may be introduced in the second phase. It will tend exponen-

* The transient heat flow for the same geometry was also investigated by Hoff,[7] Parkes,[8] and Schuh.[9]

tially to the straight line distribution corresponding to steady-state heat flow along the web.

Leakage from radiation and convection is also easily corrected for by introducing additional coordinates corresponding to the approximate leakage flow distribution.

Nonlinear problems, as we have seen by the example of Section (9), may be handled by the present methods. For instance, if in the present example we wish to take into account the dependence of heat capacity and conduction on temperature, we could first assume that points P and P_1 move at speeds determined by approximate nonlinear solutions obtained as in Section (9). Nonlinear leakage due to high temperature radiation may also be introduced.

In many cases the heat-transfer coefficients of the boundary layer will be a function of both the time and temperature. This again imposes no restriction on the application of the method, the difference being simply that the differential equations in such a case will generally have nonconstant coefficients.

It may happen in such structures as we have considered here that we must take into account the effect of a thermal resistance between the flange and the web. In such a case we should introduce the temperatures θ_1 and θ_4 of each side of the resistance. Proceeding as above leads to one differential equation for these two unknowns. Another equation is obtained from the relation between the temperature drop through the resistance and the total rate of heat flow. We write

$$K_1(\theta_1 - \theta_4) = (1/3)c(d/dt)\,(\theta_4 q) \qquad (10.25)$$

where K_1 is the heat-transfer coefficient of the resistance.

Finally we wish to call attention to the generality of the method of successive approximation illustrated by the present example. We have just calculated the temperature θ_2 in the flange without the web and then introduced the influence of the web. In general it will be possible to proceed in a similar way by first calculating the temperature fields in simple parts of the structure and introduce gradually various complexities and refinements. Each step will involve the solution of relatively simple first-order ordinary differential equations.

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