New models for current distributions and scalar potential formulations of magnetic field problems

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Given volume distributions of stationary or quasistationary electric current are replaced by equivalent distributions of fictitious magnetization and, eventually, of surface current, on the basis of the Ampèrian model for magnetized media. The fictitious magnetization is subsequently replaced by the equivalent distribution of fictitious magnetic charge. Consequently, the magnetic field due to given volume currents is determined from that produced by the corresponding charges and surface currents. This modeling method is also presented for generalized distributions of current. A scalar potential is introduced to describe the field in the models constructed. This scalar potential is a single-valued function of position when the current distribution is modeled by using a fictitious magnetization and only an equivalent charge distribution within the region considered. The modeling procedure is flexible and the models proposed yield an easier physical interpretation and a substantially reduced amount of computation with respect to vector or combined vector and scalar potential methods used so far. A few illustrative examples are given. This paper relates to stationary or quasistationary magnetic fields, but the modeling technique and the scalar potential presented are applicable to problems relative to any physical fields governed by the same equations.

I. INTRODUCTION

The stationary or quasistationary macroscopic magnetic field in regions with known distributions of volume current density $\mathbf{J}$ is described by the classical equations[1]

$$\text{curl}\, \mathbf{H} = \mathbf{J}\,, \tag{1}$$

$$\text{div}\, \mathbf{B} = 0\,, \tag{2}$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})\,, \tag{3}$$

where $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{M}$ are the field intensity, the magnetic induction, and the magnetization vector, respectively, and μ_0 is the permeability of free space. The conditions across the surfaces of discontinuity of the field quantities are

$$\hat{\mathbf{n}}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s\,, \tag{4}$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0\,, \tag{5}$$

with $\hat{\mathbf{n}}_{12}$ being the normal unit vector oriented from side 1 to side 2 of each surface, and $\mathbf{J}_s$ the local surface current density.

The magnetic field can be determined everywhere (including the regions with $\mathbf{J} \neq 0$) by means of a magnetic vector potential [from Eq. (2), $\mathbf{B} = \text{curl}\, \mathbf{A}$]. Wherever $\mathbf{J} = 0$, the field intensity is irrotational and can be derived from a magnetic scalar potential (from $\text{curl}\, \mathbf{H} = 0$, $\mathbf{H} = -\text{grad}\, \Phi_m$). The scalar potential is easier to compute from the corresponding partial differential or integral equations than the vector potential, especially in three-dimensional configurations, since the vector potential has, in general, three components. On the other hand, it is easier to visualize a scalar potential than a vector potential, the surfaces of constant scalar potential being normal to its gradient. However, there are two main difficulties related to the usage of this scalar potential. First, it is a multivalued function of position in the presence of current distributions and, second, it is useless wherever $\mathbf{J} \neq 0$. Such a scalar potential has been used for calculating the field produced by filamentary currents in various magnetic systems (see Ref. 2, for instance) and for problems of pronounced skin effect.[3] Magnetic scalar potentials and their multipole expansions were also constructed for specified regions outside bounded volume or surface current distributions.[4,5] It should be noted that the lowest nonvanishing term in the expansion of the magnetic field outside a localized quasistationary current distribution corresponds to an "effective" magnetization, which is directly related to the current density.[6]

In general, the field intensity in Eqs. (1)–(3), in the presence of given current distributions, can be decomposed in two components: one which satisfies the inhomogeneous Eq. (1) and another one which is irrotational, and can therefore be derived from a scalar potential. Numerous authors consider the former component as being just the solenoidal field produced by the given, localized current distribution in a homogeneous, unbounded region (and calculated, for example, by applying the Biot–Savart formula); the latter component is now due to the magnetization within the magnetic materials. This type of a decomposition is used, for instance, in Refs. 7 and 8, where two different scalar potentials are considered, one for regions without currents and the other one for regions with currents, the boundary conditions at interfaces being expressed in terms of the previously computed solenoidal component of the field intensity. Blewett[9] used a decomposition of the magnetic field intensity in which the component satisfying Eq. (1) is chosen to be zero outside the outer surfaces of the current coils and with a zero tangential component on these surfaces, and gave a formula for calculating this field component for coils of rectangular cross section; subsequently, the scalar potentials inside and outside the coils were adjusted to represent a continuous

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function within the whole region. This procedure was applied to stationary field computations,[10,11] as well as to the computation of eddy-current effects in a system with moving solid conductors.[12] Many other appropriate expressions for the component which satisfies the inhomogeneous Eq. (1) (often called electric vector potential) and the corresponding scalar potential have been used for the solution of magnetic field problems related to a large variety of electromagnetic devices. It should be pointed out that, in the case of eddy-current problems, the induced current density is unknown and, therefore, the electric vector potential is also unknown.[13,14] This vector potential can be chosen to be zero in nonconducting regions, such that only the scalar potential needs to be computed there,[14] but the general three-dimensional eddy-current problem cannot be reduced to the computation of a single scalar function.[15]

In this paper we only consider known stationary or quasistationary current distributions, such as those within the coils of various electromagnetic devices, or of magnetic systems in particle accelerators, magnetohydrodynamic energy converters, tokamak configurations, etc. First, we model volume current distributions by using distributions of fictitious magnetization, volume and surface magnetic charge, and, eventually, surface current. For a large class of practical coils, the fictitious magnetization can easily be chosen to have a zero volume divergence, so that the corresponding volume charge is equal to zero. Now the calculation of the field due to volume currents is reduced to that of the field determined by surface charges and, eventually, surface currents. Second, we define a scalar potential associated to the models developed and formulate the field problems in terms of this potential. It is shown how to model given current distributions in order to obtain a scalar potential which is a single-valued function of position.

II. MODELS FOR GIVEN VOLUME CURRENT DISTRIBUTIONS

It is well known[16–18] that under static or stationary conditions, from the point of view of the macroscopic field produced in free space, as well as of the forces and torques exerted in external fields, a magnetized or electrically polarized volume element is equivalent either to an elementary duplet of charges (electric dipole) or to an elementary current loop (magnetic dipole). On the basis of this equivalence, a distribution of magnetization or electric polarization can be macroscopically modeled with the aid of an equivalent distribution of charge or an equivalent distribution of current (Ampèrian model). In the case of a given volume distribution of magnetization $\mathbf{M}$, for instance, the equivalent volume and surface fictitious charge distributions in free space are, respectively,

$$\rho_m = -\mu_0 \operatorname{div} \mathbf{M} , \tag{6}$$

$$\rho_{sm} = -\mu_0 \hat{\mathbf{n}}_{12} \cdot (\mathbf{M}_2 - \mathbf{M}_1) . \tag{7}$$

In this model, the field intensity $\mathbf{H}$ produced by the charge distribution in Eqs. (6) and (7) in free space is identical to that due to the given magnetization, and the magnetic induction $\mathbf{B}$ is determined from Eq. (3). The Ampèrian model for the same case consists of the following volume and surface fictitious current distributions in free space, respectively:

$$\mathbf{J}_m = \operatorname{curl} \mathbf{M} , \tag{8}$$

$$\mathbf{J}_{sm} = \hat{\mathbf{n}}_{12} \times (\mathbf{M}_2 - \mathbf{M}_1) . \tag{9}$$

In this model, the magnetic induction $\mathbf{B}$ produced by the current distribution in Eqs. (8) and (9) in free space is identical to that due to the given magnetization, and the field intensity $\mathbf{H}$ is obtained from Eq. (3). Similar expressions correspond to the charge model and to the current model for a given volume distribution of electric polarization. These classical models have been widely used for calculating electric and magnetic fields and forces in the presence of electrically polarized or magnetized media, as well as those due to point, line, and surface charges, or line and surface currents.[16]

Consider now a volume distribution of stationary or quasistationary current of known density $\mathbf{J}$, in a nonmagnetic material region of permeability μ_0. The modeling method presented in this paper is based on treating the given volume distribution of current as if it would represent an Ampèrian current distribution corresponding to a fictitious magnetization $\mathbf{M}_c$, such that [see Eqs. (8) and (9)]

$$\operatorname{curl} \mathbf{M}_c = \mathbf{J} . \tag{10}$$

The vector field $\mathbf{M}_c$ is not uniquely determined by this equation alone. Since we do not impose any *a priori* conditions for its divergence, simple expressions for $\mathbf{M}_c$ can be found extremely easily for practical distributions of current density (see Sec. III). For a given current distribution we may construct more than one model; the fictitious magnetization $\mathbf{M}_c$ may be chosen different from zero outside the regions with $\mathbf{J} \neq 0$, but its curl must be zero wherever $\mathbf{J} = 0$. In the following we assume that $\mathbf{M}_c$ has no line or point singularities anywhere, i.e., its closed line and closed surface integrals tend to zero when the path of integration shrinks to a point and the surface of integration shrinks to a line segment or to a point.

In the Ampèrian model, the effect of the magnetization $\mathbf{M}_c$ is the same as that of a distribution of volume current [see Eq. (8)], whose density is just the given $\mathbf{J}$ [in Eq. (10)], and of a distribution of surface current [see Eq. (9)],

$$\hat{\mathbf{n}}_{12} \times (\mathbf{M}_{c_2} - \mathbf{M}_{c_1}) \equiv -\mathbf{J}_{sc} . \tag{11}$$

Consequently, for calculating the magnetic induction, the given volume distribution $\mathbf{J}$ can be replaced by the fictitious distributions of magnetization $\mathbf{M}_c$ and of surface current of density $\mathbf{J}_{sc}$ on all the surfaces of discontinuity of the tangential component of $\mathbf{M}_c$. Thus,

$$\mathbf{B} = \mu_0 \mathbf{H} = \mathbf{B}_{M_c} + \mathbf{B}_{J_{sc}} , \tag{12}$$

where $\mathbf{B}_{M_c}$ is due to $\mathbf{M}_c$ and $\mathbf{B}_{J_{sc}}$ to $\mathbf{J}_{sc}$.

On the other hand, the field intensity $\mathbf{H}_{M_c}$ produced by the magnetization $\mathbf{M}_c$ is that determined in free space by the following distributions of volume charge density and of surface charge density on all the surfaces of discontinuity of the normal component of $\mathbf{M}_c$, respectively [see Eqs. (6) and (7)]:

$$\rho_c = -\mu_0 \operatorname{div} \mathbf{M}_c , \tag{13}$$

$$\rho_{sc} = -\mu_0 \hat{\mathbf{n}}_{12} \cdot (\mathbf{M}_{c_2} - \mathbf{M}_{c_1}) , \tag{14}$$

and the magnetic induction $\mathbf{B}_{M_c}$ is given by

$$\mathbf{B}_{M_c} = \mu_0 (\mathbf{H}_{M_c} + \mathbf{M}_c) . \quad (15)$$

From Eqs. (12) and (15), the field intensity can be expressed in the form

$$\mathbf{H} = \mathbf{M}_c + \mathbf{H}_{M_c} + \mathbf{H}_{J_{sc}} , \quad (16)$$

where $\mathbf{H}_{J_{sc}} = \mathbf{B}_{J_{sc}}/\mu_0$.

Therefore, in a region of permeability μ_0, the field intensity due to a distribution of volume current density $\mathbf{J}$ can be obtained by the superposition of $\mathbf{M}_c$ and the field intensity due to the charge distributions ρ_c and ρ_{sc}, and to the surface current distribution $\mathbf{J}_{sc}$.

An elementary formula can be written for the field in Eq. (16) produced by a given, localized volume current distribution in an unbounded, homogeneous three-dimensional space,

$$\mathbf{H}(\mathbf{r}) = \mathbf{M}_c(\mathbf{r}) + \frac{1}{4\pi\mu_0} \times \left(\int \frac{\rho_c(\mathbf{r}')\mathbf{R}}{R^3} dv' + \int \frac{\rho_{sc}(\mathbf{r}')\mathbf{R}}{R^3} dS' \right) + \frac{1}{4\pi} \int \frac{\mathbf{J}_{sc}(\mathbf{r}')\times\mathbf{R}}{R^3} dS' , \quad (17)$$

where dv' and dS' are the elements of volume and surface, respectively, $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors defining the field point and the source point, respectively, with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and the surface integrals are performed over the surfaces of discontinuity of $\mathbf{M}_c$.

A few special cases are particularly important for field computations. If $\mathbf{M}_c$ is chosen such that $\mathbf{J}_{sc}$ in Eq. (11) is zero everywhere, the given volume current distribution is modeled in terms of $\mathbf{M}_c$ and charge distributions only; Eq. (17) does not contain the last integral and the difference $\mathbf{H} - \mathbf{M}_c$ can now be derived from a single-valued scalar potential produced by ρ_c and ρ_{sc} (see Sec. IV). If ρ_c in Eq. (13) is zero everywhere, the volume current distribution is modeled in terms of $\mathbf{M}_c$ and only surface distributions of charge and current, and the field in Eq. (17) is expressed only in terms of surface integrals; when $\mathbf{M}_c$ is chosen to be zero outside the regions with $\mathbf{J}\neq 0$, then these surface integrals are taken only over the boundary of that region. It should be noted that the field in Eq. (17) is indentical to that given by the classical Biot–Savart volume integral formula, and by choosing various vector fields $\mathbf{M}_c$, corresponding to the same volume current distributions (see Sec. III), one obtains various vector integral identities.

For regions with a linear, isotropic, and homogeneous magnetic material, the above results remain valid if μ_0 is replaced by the permeability μ of the medium.

The field in the presence of inhomogeneous (but isotropic) magnetic materials, with the permeability μ varying within the regions occupied by the distributions of $\mathbf{J}$ and $\mathbf{M}_c$, can be determined as shown in Sec. IV. The models for volume current distributions are constructed by keeping the local values of permeability, with the same surface current distribution $\mathbf{J}_{sc}$, as in Eq. (11), and with the following distributions of fictitious charge:

$$\rho_c = -\operatorname{div}(\mu\mathbf{M}_c) , \quad (18)$$

$$\rho_{sc} = -\hat{\mathbf{n}}_{12}\cdot(\mu_2\mathbf{M}_{c_2} - \mu_1\mathbf{M}_{c_1}) . \quad (19)$$

The results of the analysis presented in this section may be formalized by stating the following modeling *theorem*: Let $\mathbf{J}(\mathbf{r})$ be the piecewise continuous density of a given volume distribution of stationary or quasistationary current, and let $\mathbf{M}_c(\mathbf{r})$ be any piecewise continuous fictitious magnetization which satisfies the equation curl $\mathbf{M}_c = \mathbf{J}$ and has no point or line singularities. Let $\rho_c = -\operatorname{div}(\mu\mathbf{M}_c)$, $\rho_{sc} = -\hat{\mathbf{n}}_{12}\cdot(\mu_2\mathbf{M}_{c_2} - \mu_1\mathbf{M}_{c_1})$, and $\mathbf{J}_{sc} = -\hat{\mathbf{n}}_{12}\times(\mathbf{M}_{c_2} - \mathbf{M}_{c_1})$ be, respectively, densities of fictitious volume charge, surface charge, and surface current, where μ is the permeability and $\hat{\mathbf{n}}_{12}$ the unit normal on the surfaces of discontinuity of $\mathbf{M}_c$ and μ. Then, the magnetic field intensity due to $\mathbf{J}$ is equal to the sum of $\mathbf{M}_c$ and the field intensity due to the distributions of ρ_c, ρ_{sc}, and $\mathbf{J}_{sc}$.

The same modeling procedure is applicable to current sheets of a given surface density $\mathbf{J}_s$, by means of a fictitious magnetization $\mathbf{M}_c$ which satisfies the equations curl $\mathbf{M}_c = 0$ and $\hat{\mathbf{n}}_{12}\times(\mathbf{M}_{c_2} - \mathbf{M}_{c_1}) = \mathbf{J}_s$ [instead of Eq. (10)], and the above theorem can be correspondingly extended. The case of a generalized current distribution, including line currents, is presented in the Appendix.

As we shall illustrate in a few examples, the volume current distributions in practical configurations can be modeled in terms of $\mathbf{M}_c$ and only a distribution of surface charge. The amount of computation necessary to determine the field on the basis of such a model is substantially reduced, since the computation is reduced to that of the field due to surface charge distributions.

The price paid for applying the method proposed in this paper is that of the construction of an appropriate model for a given problem, i.e., the choice of an appropriate distribution of fictitious magnetization $\mathbf{M}_c$. A vector field which satisfies Eq. (10) and supplementary conditions can be easily determined in particular situations.[9,12,14] The only condition for the fictitious magnetization $\mathbf{M}_c$ used in our modeling procedure is to satisfy Eq. (10), without any other restrictions inside or outside the region occupied by the given current distribution. Therefore, for each configuration we have a few possible models to choose from, in order to optimize the field calculations. In the case of practical coils, for instance, for each of their straight or circular current tubes of constant cross-sectional area, with the current density $\mathbf{J}$ being constant over the cross section, one can always choose $\mathbf{M}_c$ inside the tube as having only one component, in a direction which is perpendicular to $\mathbf{J}$, and depending linearly on the distance along the direction of $\mathbf{M}_c\times\mathbf{J}$; the volume divergence of such an $\mathbf{M}_c$ is equal to zero.

It should be noted that, in a similar manner, a given volume distribution of static or quasistatic electric charge can be modeled in terms of fictitious distributions of electric polarization, volume and surface current, and surface charge.

III. ILLUSTRATIVE EXAMPLES

A. Infinitely long, straight conductors

A straight conductor of an arbitrary cross section is shown in Fig. 1, with the volume current density oriented along the positive z axis, $\mathbf{J} = \hat{\mathbf{z}}J$.

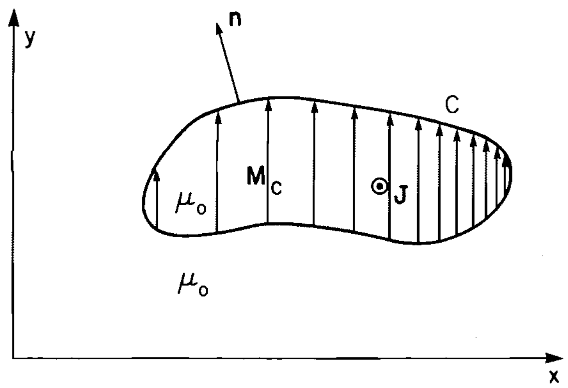

FIG. 1. Cross section of cylindrical conductor carrying current.

$\mathbf{M}_c$ in Eq. (10) can be chosen as

$$\mathbf{M}_c(x,y) = \left| \begin{array}{ll} \hat{\mathbf{y}} \int_0^x J(x,y)\,dx & \text{inside the conductor}, \\ 0 & \text{outside the conductor}, \end{array} \right. \quad (20a)$$

where $\hat{\mathbf{y}}$ is the unit vector along the positive y axis. In general, the corresponding volume charge density in Eq. (13) is not zero. If J depends only on one coordinate, say x, then the divergence of $\mathbf{M}_c$ in Eq. (20a) is zero, and the magnetic field produced can be expressed in terms of $\mathbf{M}_c$ and only of line integrals along the conductor contour C. In the case of a constant current density, $\mathbf{J} = \text{const}$,

$$\mathbf{M}_c(x,y) = \left| \begin{array}{ll} \hat{\mathbf{y}} Jx & \text{inside the conductor}, \\ 0 & \text{outside the conductor}. \end{array} \right. \quad (20b)$$

Equations (13), (14), and (11) give

$$\begin{aligned} &\rho_c = 0, \\ &\rho_{sc} = \mu_0 \hat{\mathbf{n}} \cdot \mathbf{M}_c = \mu_0 Jxn_y \quad \text{on } C, \\ &\mathbf{J}_{sc} = \hat{\mathbf{n}} \times \mathbf{M}_c = \hat{\mathbf{z}} Jxn_x \quad \text{on } C, \end{aligned} \quad (21)$$

where n_x and n_y are the direction cosines of the outward normal $\hat{\mathbf{n}}$. The field intensity in Eq. (17) is

$$\mathbf{H}(x,y) = \mathbf{M}_c(x,y) + \frac{J}{2\pi} \times \left(\oint_C \frac{x' n'_y \mathbf{R}}{R^2} dl' + \hat{\mathbf{z}} \times \oint_C \frac{x' n'_x \mathbf{R}}{R^2} dl' \right), \quad (22)$$

with

$$\mathbf{R} = \hat{\mathbf{x}}(x - x') + \hat{\mathbf{y}}(y - y'), \quad (23)$$

$\hat{\mathbf{x}}$ being the unit vector along the positive x axis. The contour integrals in expression (22) require a substantially reduced amount of computation as compared to the integral over the conductor cross section in the Biot–Savart formula. It should be noted that for a straight conductor of a general polygonal cross section, with $J = \text{const}$, choosing M_c as in (20b) yields surface densities ρ_{sc} and $\mathbf{J}_{sc}$ in (21) which are either constant or depending linearly on the distance along the cross-sectional sides. For such a conductor the integrations in Eq. (22) can be analytically performed.

In the case of an infinitely long, straight conductor of rectangular cross section, carrying a current of constant density, $\mathbf{J} = \hat{\mathbf{z}} J$, we can use the models presented in Fig. 2. If we choose $\mathbf{M}_c$, as given in Eq. (20b), we obtain the model in Fig. 2(a), with only surface distributions of charge and current,

$$\begin{aligned} &\rho_{sc} = \pm \mu_0 Jx \quad \text{for} \quad x \in (0,a), \quad y = \pm b/2, \\ &\mathbf{J}_{sc} = \hat{\mathbf{z}} Ja \quad \text{for} \quad x = a, \quad y \in (-b/2, b/2). \end{aligned} \quad (24)$$

If we choose

$$\mathbf{M}_c(x,y) = \left| \begin{array}{ll} \hat{\mathbf{y}} Jx & \text{inside the conductor}, \\ \hat{\mathbf{y}} Ja & \text{for } x \in (a,\infty), \quad y \in (-b/2, b/2), \\ 0 & \text{elsewhere}, \end{array} \right. \quad (25)$$

we obtain the model in Fig. 2(b), with a surface distribution of charge only,

$$\rho_{sc} = \left| \begin{array}{ll} \pm \mu_0 Jx & \text{for } x \in (0,a), \quad y = \pm b/2, \\ \pm \mu_0 Ja & \text{for } x \in (a,\infty), \quad y = \pm b/2. \end{array} \right. \quad (26)$$

For this model, the only integral in formula (17) is that in ρ_{sc} and the calculation of the field is reduced to the evaluation of single integrals,

$$\mathbf{H}(x,y) = \mathbf{M}_c(x,y) + \frac{J}{2\pi} \left(\int_0^a \frac{x' \mathbf{R}}{R^2} \Big|_{y' = -b/2}^{y' = b/2} dx' + a \int_a^\infty \frac{\mathbf{R}}{R^2} \Big|_{y' = -b/2}^{y' = b/2} dx' \right), \quad (27)$$

with $\mathbf{M}_c = 0$ outside the semi-infinite slab shown in Fig. 2(b).

The same type of simple models can be constructed in the direction of the negative x axis, and also with the fictitious magnetization $\mathbf{M}_c$ along the x axis. It should be remarked that the distribution of $\mathbf{M}_c$ in Fig. 2(b) can be ended

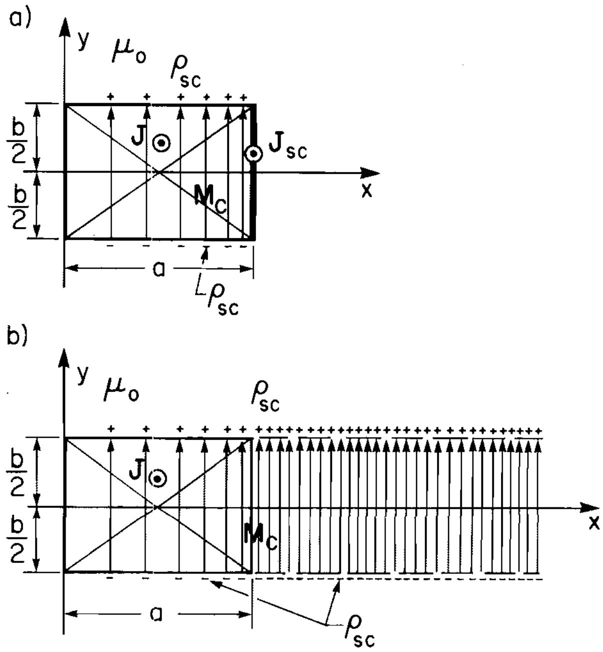

FIG. 2. Conductor of rectangular cross section carrying uniformly distributed current: (a) model with surface current and charge; (b) model with surface charge only.

at any $x=a'>a$ by placing a current sheet of density $\mathbf{J}_{sc}=\hat{\mathbf{z}}\,Ja$ [as in Eq. (24)] at $x=a'$, $y\in(-b/2,b/2)$. Similar models can be used for infinitely long, straight current sheets.

B. Toroidal coils

Consider a toroidal conductor of an arbitrary cross section, as shown in Fig. 3. In circular cylindrical coodinates (r,ϕ,z), with the z axis chosen as the axis of symmetry, the current density has only one component in the ϕ direction, $\mathbf{J}=\hat{\boldsymbol{\phi}}\,J$.

$\mathbf{M}_c$ in Eq. (10) can be chosen, for example, as

$$\mathbf{M}_c(z,r)=\begin{cases}-\hat{\mathbf{z}}\displaystyle\int_{r_0}^{r}J(z,r)\,dr & \text{inside the coil},\\ 0 & \text{outside the coil},\end{cases}\tag{28}$$

where r_0 is an arbitrary distance from the z axis. If J depends only on the r coordinate, then the volume charge density in Eq. (13) is equal to zero, and models with only surface distributions of charge and current can be constructed. Taking into account Eqs. (14) and (11), the field in Eq. (17) can be expressed in terms of integrals over the cross-sectional contour, whose integrands contain complete elliptic integrals.

In the particular case of a toroidal coil of rectangular cross section, with $J=\text{const}$, we can use the models shown in Fig. 4. Choosing

$$\mathbf{M}_c=\begin{cases}-\hat{\mathbf{z}}\,J(r-a) & \text{inside the coil},\\ 0 & \text{outside the coil},\end{cases}\tag{29}$$

yields the model in Fig. 4(a), with surface distribution of charge and current,

$$\rho_{sc}=\mp\mu_0 J(r-a)\quad\text{for}\quad z=\pm c,\quad r\in(a,b),$$
$$\mathbf{J}_{sc}=\hat{\boldsymbol{\phi}}\,J(b-a)\quad\text{for}\quad z\in(-c,c),\quad r=b.\tag{30}$$

Choosing

$$\mathbf{M}_c=\begin{cases}-\hat{\mathbf{z}}\,J(r-a) & \text{inside the coil},\\ -\hat{\mathbf{z}}\,J(b-a) & \text{for } z\in(-c,c),\ r\in(b,\infty),\\ 0 & \text{elsewhere},\end{cases}\tag{31}$$

yields the model in Fig. 4(b), with a surface distribution of charge only,

$$\rho_{sc}=\begin{cases}\mp\mu_0 J(r-a) & \text{for } z=\pm c,\quad r\in(a,b),\\ \mp\mu_0 J(b-a) & \text{for } z=\pm c,\quad r\in(b,\infty).\end{cases}\tag{32}$$

Other equivalent models for this particular volume current distribution can be obtained by choosing

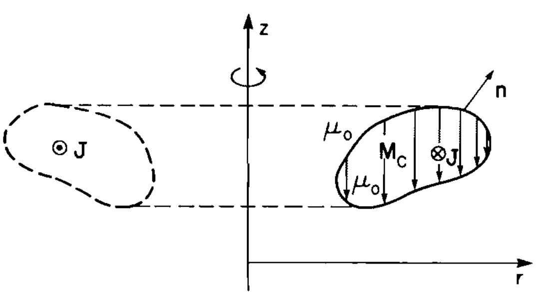

FIG. 3. Azimuthal section of toroidal conductor.

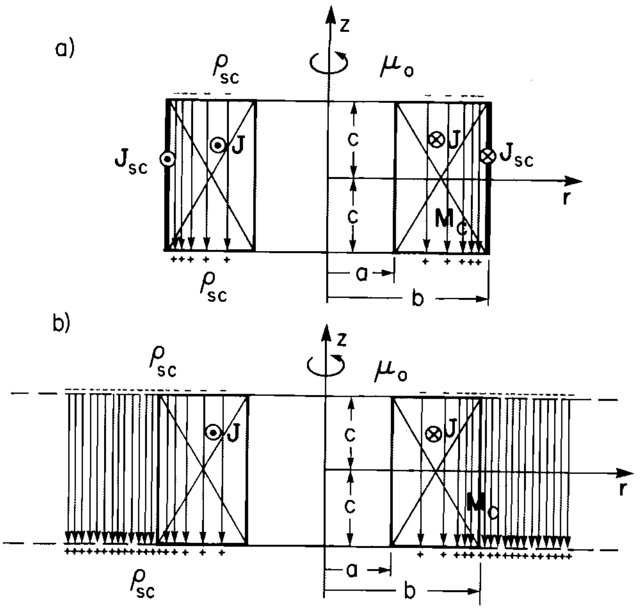

FIG. 4. Toroidal coil of rectangular cross section carrying uniformly distributed current: (a) model with surface current and charge; (b) model with surface charge only.

$$\mathbf{M}_c=\begin{cases}\hat{\mathbf{z}}\,J(b-r) & \text{inside the coil},\\ 0 & \text{outside the coil},\end{cases}\tag{33}$$

which yields

$$\rho_{sc}=\pm\mu_0 J(b-r)\quad\text{for } z=\pm c,\quad r\in(a,b),$$
$$\mathbf{J}_{sc}=\hat{\boldsymbol{\phi}}\,J(b-a)\quad\text{for } z\in(-c,c),\quad r=a,\tag{34}$$

or by choosing[19]

$$\mathbf{M}_c=\begin{cases}\hat{\mathbf{z}}\,J(b-r) & \text{inside the coil},\\ \hat{\mathbf{z}}\,J(b-a) & \text{for } z\in(-c,c),\ r\in(0,a),\\ 0 & \text{elsewhere},\end{cases}\tag{35}$$

which yields only

$$\rho_{sc}=\begin{cases}\pm\mu_0 J(b-r) & \text{for } z=\pm c,\quad r\in(a,b),\\ \pm\mu_0 J(b-a) & \text{for } z=\pm c,\quad r\in(0,a).\end{cases}\tag{36}$$

As in the case of a straight conductor of rectangular cross section, mathematically equivalent models can be obtained from the model presented in Fig. 4(b), for instance, if the distribution of fictitious magnetization $\mathbf{M}_c$ is ended at any $r=b'>b$, by placing a current sheet of density $\mathbf{J}_{sc}=\hat{\boldsymbol{\phi}}\,J(b-a)$ [as in Eq. (30)] at $z\in(-c,c)$, $r=b'$. In Sec. V we shall use this modeling procedure for formulating boundary-value problems in terms of a single-valued scalar potential in the presence of volume current distributions. Similar models can be constructed for axisymmetric current sheets.

It should be remarked that when the entire current distribution in a given region is modeled by using a fictitious magnetization $\mathbf{M}_c$ and a charge distribution only, the closed line integral of $\mathbf{H}-\mathbf{M}_c$ is equal to zero,

$$\oint(\mathbf{H}-\mathbf{M}_c)\cdot d\mathbf{l}=0,\tag{37}$$

for any path of integration inside that region, in a similar manner as in the case of an electrostatic field.

IV. SCALAR POTENTIAL FOR MAGNETIC FIELD PROBLEMS IN THE PRESENCE OF GIVEN CURRENT DISTRIBUTIONS

In the following we introduce a scalar potential based directly on the modeling method presented in Sec. II, for which the boundary conditions are readily available, without being necessary to perform supplementary computations. In addition, for a large class of practical boundary-value problems, the given current distributions can be modeled such that the corresponding scalar potential is a single-valued function of position within the entire problem region. Field analysis by using this scalar potential is much simpler than that based on various methods developed so far.

Using the "del" operator, Eqs. (1) and (10) yield

$$\nabla\times(\mathbf{H}-\mathbf{M}_c)=0\,, \tag{38}$$

and, hence, $\mathbf{H}-\mathbf{M}_c$ can be derived from a scalar potential Φ_c,

$$\mathbf{H}-\mathbf{M}_c=-\nabla\Phi_c\,. \tag{39}$$

Once the model for the current distribution is constructed, the fictitious magnetization $\mathbf{M}_c$ is known and the field intensity is given by

$$\mathbf{H}=\mathbf{M}_c-\nabla\Phi_c\,. \tag{40}$$

From Eqs. (2), (3), and (40) one obtains the equation satisfied by Φ_c. For an isotropic medium, for example, Eq. (3) can be written as

$$\mathbf{B}=\mu\mathbf{H}+\mu_0\mathbf{M}_p\,, \tag{41}$$

where μ is the permeability of the medium and $\mathbf{M}_p$ the permanent magnetization, eventually present at the point considered. The corresponding equation in Φ_c is

$$\nabla^2\Phi_c+(\nabla\mu/\mu)\cdot\nabla\Phi_c=-(1/\mu)(\rho_c+\rho_{M_p})\,, \tag{42}$$

where ρ_c is given by Eq. (18) and $\rho_{M_p}=-\mu_0\nabla\cdot\mathbf{M}_p$. In the case of a linear, isotropic, and homogeneous medium, without permanent magnetization, Φ_c satisfies the Poisson equation

$$\nabla^2\Phi_c=-\rho_c/\mu\,, \tag{43}$$

with $\rho_c/\mu=-\nabla\cdot\mathbf{M}_c$. If $\mathbf{M}_c$ in Eq. (10) is chosen such that its volume divergence is equal to zero, then the scalar potential Φ_c satisfies the Laplace equation within linear, isotropic, and homogeneous materials,

$$\nabla^2\Phi_c=0\,. \tag{44}$$

The boundary conditions and the conditions at the surfaces of discontinuity of the quantities **H**, **B**, and $\mathbf{M}_c$ can be expressed in terms of Φ_c and its first spatial derivatives, by taking into account the model adopted for the current distribution. Assuming $\mathbf{M}_p=0$, Eqs. (4) and (5), with (40), (41), (11), and (19), yield

$$\hat{\mathbf{n}}_{12}\times(\nabla\Phi_{c_1}-\nabla\Phi_{c_2})=\mathbf{J}_s+\mathbf{J}_{sc}\,, \tag{45}$$

$$\mu_1\frac{\partial\Phi_{c_1}}{\partial n_{12}}-\mu_2\frac{\partial\Phi_{c_2}}{\partial n_{12}}=\rho_{sc}\,, \tag{46}$$

which give the behavior of the tangential and normal components of $\nabla\Phi_c$, respectively. Equation (45) shows that, wherever $\mathbf{J}_s+\mathbf{J}_{sc}=0$, and therefore across any surface for models constructed by using fictitious charge distributions only, the tangential component of $\nabla\Phi_c$ is continuous, which is equivalent to the continuity of the scalar potential itself,

$$\Phi_{c_1}=\Phi_{c_2}\,. \tag{47}$$

For $\mu_1=\mu_2\equiv\mu$, Eq. (46) becomes

$$\frac{\partial\Phi_{c_1}}{\partial n_{12}}-\frac{\partial\Phi_{c_2}}{\partial n_{12}}=\frac{\rho_{sc}}{\mu}\,, \tag{48}$$

with $\rho_{sc}/\mu=-\hat{\mathbf{n}}_{12}\cdot(\mathbf{M}_{c_2}-\mathbf{M}_{c_1})$.

The solution of Eq. (43) for the special case of a linear, homogeneous, unbounded region, along with Eq. (40), yields the expression in Eq. (17) (with $\mu\equiv\mu_0$).

Once the model for a current distribution given in a region is adopted, i.e., once the distribution of fictitious magnetization $\mathbf{M}_c$ is chosen, all the conditions necessary for determining the scalar potential Φ_c are known. If the surface current density $\mathbf{J}_s+\mathbf{J}_{sc}$ is not zero inside the region, or if there are line currents which are not modeled by equivalent charge distributions (see the Appendix), then the scalar potential Φ_c is a multivalued function of position. It can be made single valued by using appropriate "cuts" which include the surfaces of existing current sheets, as well as open surfaces bounded by the filamentary currents. Due to the flexibility of the modeling technique presented in this paper, for magnetic field problems related to a large class of practical systems, models can always be constructed to contain only fictitious charge distributions within the region considered. We can easily see that, if the whole distribution of stationary or quasistationary current inside a region is modeled by using a distribution of fictitious magnetization and only charge distributions, then the scalar potential Φ_c is a single-valued function of position in that region. The truth of this statement is evident if one remarks that, for such a model, the field $\mathbf{H}-\mathbf{M}_c=-\nabla\Phi_c$ is of the same nature as the electrostatic field produced by electric charge distributions. A generalized presentation of this statement in the form of a theorem, including the modeling of surface and filamentary currents, and its proof is given in the Appendix.

V. EXAMPLE OF BOUNDARY-VALUE PROBLEM FORMULATION

Consider a toroidal magnetic core, assumed to be of an ideal ferromagnetic material ($\mu\to\infty$), with two toroidal coils of rectangular cross section, as shown in Fig. 5. The axes of the two coils are parallel to the axis of the magnetic core but, in general, the three axes do not coincide. The currents carried by the two coils are uniformly distributed over their axial cross sections, and are equal and opposite in direction,

$$J_1S_1=J_2S_2\equiv I\,. \tag{49}$$

where S_1 and S_2 are the cross-sectional areas of the coils.

With (49), the boundary condition for the magnetic field problem inside the magnetic circuit cavity is that of a zero tangential component of the field intensity,

$$H_{\tan}=0\ \text{for}\left|\begin{array}{l} z=0 \text{ and } z=h\,,\quad r\in(R_1,R_2)\,; \\ z\in(o,h)\,,\quad r=R_1 \text{ and } r=R_2\,. \end{array}\right. \tag{50}$$

The volume current distribution of each of the two coils can be modeled as shown in example B, Sec. III, by ending

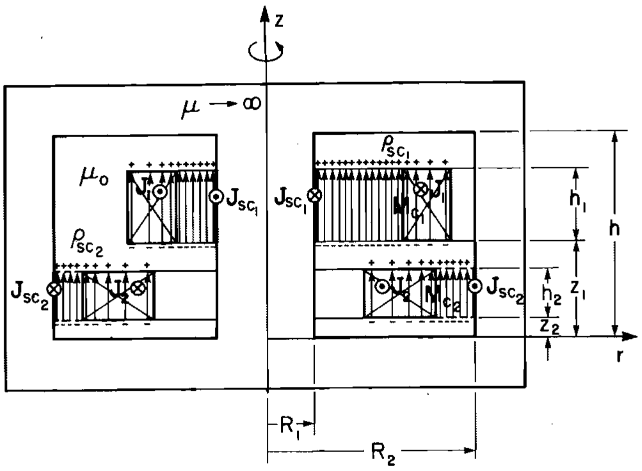

FIG. 5. Axial section of a toroidal magnetic core with two coils.

the distributions of fictitious magnetization $\mathbf{M}_{c_1}$ and $\mathbf{M}_{c_2}$ either at $r = R_1$ or at $r = R_2$, such that the model contains a distribution of magnetization and surface charge only inside the cavity. The magnetizations $\mathbf{M}_{c_1}$ and $\mathbf{M}_{c_2}$, and the corresponding surface charge densities ρ_{sc_1} and ρ_{sc_2}, shown in Fig. 5, have elementary expressions [see Eqs. (35), (36) and (31), (32)]; the surface density of the fictitious current in Eq. (11) is

$$\mathbf{J}_{sc_1} = \hat{\boldsymbol{\phi}}\, I/h_1 \quad \text{for } z \in (z_1, z_1 + h_1)\,, \quad r = R_1\,,$$
$$\mathbf{J}_{sc_2} = -\hat{\boldsymbol{\phi}}\, I/h_2 \quad \text{for } z \in (z_2, z_2 + h_2)\,, \quad r = R_2\,. \tag{51}$$

The field problem can be formulated in terms of a single-valued scalar potential Φ_c, as defined by Eq. (39). Inside the cavity region, assumed to have a constant permeability μ_0, Φ_c satisfies the Laplace Eq. (44) everywhere, except the points on the surfaces S where the surface charge density $\rho_{sc} \neq 0$. The scalar potential equation at all the points within the problem region can be written in the form of a generalized Poisson equation,[20]

$$\nabla^2 \Phi_c = -(1/\mu_0)\, \rho_{sc}\, \delta_{(S)}\,, \tag{52}$$

where $\delta_{(S)}$ is a generalized function which satisfies the condition in Eq. (A2) (see the Appendix). The conditions on the region boundary Σ are obtained from Eqs. (A12), (39), (45), (50), and (51), and can be expressed, for instance, as follows:

$$\Phi_c\Big|_{\substack{z\in(0,\,z_1)\\ r=R_1}} = \Phi_c\Big|_{\substack{z=0\\ r\in(R_1,R_2)}} = \Phi_c\Big|_{\substack{z\in(0,\,z_2)\\ r=R_2}} = 0\,,$$
$$\Phi_c\Big|_{\substack{z\in(z_2,\,z_2+h_2)\\ r=R_2}} = \frac{I}{h_2}(z - z_2)\,,$$
$$\Phi_c\Big|_{\substack{z\in(z_2+h_2,\,h)\\ r=R_2}} = \Phi_c\Big|_{\substack{z=h\\ r\in(R_1,R_2)}} = \Phi_c\Big|_{\substack{z\in(z_1+h_1,\,h)\\ r=R_1}} = I\,,$$
$$\Phi_c\Big|_{\substack{z\in(z_1,\,z_1+h_1)\\ r=R_1}} = \frac{I}{h_1}(z - z_1)\,. \tag{53}$$

Therefore, the boundary-value problem for Φ_c is an interior Dirichlet problem, with a known surface charge distribution in a homogeneous region. Its solution can be obtained by applying, for intance, the Green function method,

$$\Phi_c(\mathbf{r}) = \frac{1}{\mu_0}\int_S \rho_{sc}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\,dS' - \oint_\Sigma \Phi_c(\mathbf{r}')\,\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\,dS'\,, \tag{54}$$

where $\partial/\partial n'$ denotes the derivative along the outward normal and $G(\mathbf{r},\mathbf{r}')$ is the Dirichlet Green function for the Laplacian, relative to the region $z\in(0,h)$, $r\in(R_1,R_2)$, which can be expressed in terms of Bessel functions.[21]

The component of Φ_c given by the integral over the region boundary [last term in Eq. (54)] represents the contribution of the fictitious current sheets on the cavity walls, in the model adopted. This component is axisymmetric, has an analytical expression, and is independent of the position of the axes and of the radial dimensions of the two coils. In fact, it is identical to the classical magnetic scalar potential corresponding to the magnetic field intensity which would be produced in the region considered by two current sheets alone, of densities given in Eq. (51). The component of Φ_c given by the first term in the right-hand side of Eq. (54) is identical to the electrostatic potential which would be produced in the region considered by a charge distribution of density ρ_{sc} if the entire boundary were kept at zero potential. The integral which gives this component can be numerically evaluated. When the coils and the magnetic core are coaxial, this component also has an analytical expression.

A similar formulation and solution can be used for a system with more than two coils, if the algebraic sum of their currents is equal to zero. The solution presented is much simpler and more useful for calculating local field quantities, as well as global quantities (stored energy, inductances, forces), than those corresponding to various multivalued scalar potential or vector potential formulations.

Similar models can be constructed for even more general systems, where the coil axes are not parallel to the magnetic core axis. For such a system, the models on the boundary present not only a surface current distribution, but also a surface charge distribution. In the case of a magnetic circuit whose cavity is arbitrarily shaped, when the Dirichlet Green function is not available in an analytical form, the scalar potential Φ_c can be determined by means of boundary scalar integral equations, formulated on the basis of the modeling technique presented. The method elaborated in this paper requires an amount of computation which is substantially reduced with respect to that required by other methods developed so far.

VI. CONCLUSIONS

A new modeling method for given current distributions and the corresponding scalar potential formulation of magnetic field problems have been presented. The method is simple and flexible, in the sense that for a given problem one can construct the most appropriate models in order to reduce the necessary amount of computation. A general theorem shows the conditions required for the model to obtain the associated scalar potential as a single-valued function of position.

A few elementary examples illustrate the general modeling procedure, as well as specific techniques for solving magnetic field boundary-value problems in terms of a single-valued scalar potential. The models used allow an easier physical interpretation of the results, since the field due to volume current distributions is determined from that due to surface charge distributions. At the same time, these examples illustrate the efficiency of the proposed method for computational purposes. The type of models elaborated for the examples in Sec. III can also be used for all practical current distributions which can be decomposed in straight current tubes of finite length and current tubes in the form of a portion of a toroid. Closed analytical formulas can be obtained for the field due to these basic elements of practical coils.

The modeling method and the scalar potential formulation presented in this paper can be readily extended to systems with anisotropic or nonlinear materials. Although the presentation has been made for stationary or quasistationary magnetic fields, the results obtained in this paper are applicable to other vector fields described by the same equations.

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APPENDIX: MODELING OF GENERALIZED CURRENT DISTRIBUTIONS

In the presence of volume, surface, and filamentary currents in a region D, the generalized distribution of volume current density can be written in the form[20]

$$\mathbf{j}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) + \mathbf{J}_s(\mathbf{r})\delta_{(S)} + \hat{\mathbf{t}}_c(\mathbf{r}) i_c \delta_{(C)} , \quad \text{(A1)}$$

where: $\mathbf{J}$ is the ordinary volume density of current, defined for $\mathbf{r}\notin S$ and $\mathbf{r}\notin C$, $\mathbf{J}_s$ is the surface density of current given on piecewise smooth surfaces S inside D, i_c is the intensity of current along piecewise smooth curves C inside D, $\hat{\mathbf{t}}_c$ is the unit vector tangential to C, along the direction of the current i_c, $\delta_{(S)}$ and $\delta_{(C)}$ are generalized functions[22] which satisfy the relationships

$$\int_D f(\mathbf{r})\delta_{(S)}\, dv = \int_S f(\mathbf{r})\, dS , \quad \text{(A2)}$$

$$\int_D f(\mathbf{r})\delta_{(C)}\, dv = \int_C f(\mathbf{r})\, dl , \quad \text{(A3)}$$

for any function $f(\mathbf{r})$ given in D and continuous on S and C.

Equation (1) can be written in a generalized form as

$$\mathit{curl}\ \mathbf{H} = \mathbf{j} , \quad \text{(A4)}$$

where the generalized curl operator is[20]

$$\mathit{curl}\ \mathbf{H} = \text{curl}\ \mathbf{H} + \text{curl}\ s\ \mathbf{H}\delta_{(S)} + \text{curl}\ l\, \mathbf{H}\delta_{(C)} , \quad \text{(A5)}$$

with curl s and curl l being the surface curl and the line curl, respectively,

$$\text{curl}\ s\ \mathbf{H} \equiv \hat{\mathbf{n}}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) , \quad \text{(A6)}$$

$$\text{curl}\ l\, \mathbf{H} \equiv \hat{\mathbf{t}}_c \lim_{C_l \to P_c} \oint_{C_l} \mathbf{H} \cdot \mathbf{dl} ; \quad \text{(A7)}$$

the line curl is defined at the points P_c of a curve C which has the property that the circulation of the vector field along any closed path C_l, enclosing once the curve, is different from zero in the limit when C_l shrinks to P_c, with the unit vector $\hat{\mathbf{t}}_c$, tangential to C, associated with the direction of integration according to the right-handed screw rule.

In order to model the whole distribution of given current (A1) by means of a fictitious magnetization $\mathbf{M}_c$ and distributions of charge only, the distribution of $\mathbf{M}_c$ must satisfy the generalized equation

$$\mathit{curl}\ \mathbf{M}_c = \mathbf{j} , \quad \text{(A8)}$$

with no restrictive conditions imposed on the divergence of $\mathbf{M}_c$. The volume and surface currents in Eq. (A1) can be modeled in terms of an ordinary volume distribution of magnetization, as shown in Sec. II, with curl $\mathbf{M}_c = \mathbf{J}$ and curl s $\mathbf{M}_c = \mathbf{J}_s$. Each filamentary current loop C, carrying a current i_c, represented in the last term of Eq. (A1), can be modeled by a distribution of surface magnetization $\mathbf{M}_{sc}$ over an arbitrary open surface S_c, bounded by the loop,

$$\mathbf{M}_{sc} = \hat{\mathbf{n}}_{S_c} i_c , \quad \text{(A9)}$$

where $\hat{\mathbf{n}}_{S_c}$ is the unit vector normal to S_c, associated with the direction of i_c according to the right-handed screw rule. The volume magnetization corresponding to $\mathbf{M}_{sc}$ is

$$\mathbf{M}_{sc}\delta_{(S_c)} = \hat{\mathbf{n}}_{S_c} i_c \delta_{(S_c)} , \quad \text{(A10)}$$

and the surface density of (positive) charge of the equivalent double layer is $\mu i_c \delta_{(S_c)}$. The line curl of the magnetization in Eq. (A10) is just $\hat{\mathbf{t}}_c i_c$, which corresponds to the last term in Eq. (A1).

We can now generalize the statement in Sec. IV, in the form of the following *theorem*: If the volume, surface, and line distributions of stationary or quasistationary current inside a region are modeled by means of a distribution of fictitious magnetization whose generalized curl is equal to the generalized volume current density, then the difference between the magnetic field intensity and this fictitious magnetization can be derived from a scalar potential which is a single-valued function of position in that region. To prove this theorem we show that the line integral of $\mathbf{H} - \mathbf{M}_c = -\nabla\Phi_c$ along any closed path Γ, in the region considered, is equal to zero. Indeed, applying the generalized Stokes' theorem[20] and using Eqs. (A4) and (A8) yield

$$\oint_\Gamma (\mathbf{H} - \mathbf{M}_c) \cdot \mathbf{dl} = \int_{S_\Gamma} (\mathit{curl}\ \mathbf{H} - \mathit{curl}\ \mathbf{M}_c) \cdot \mathbf{dS} = 0 . \quad \text{(A11)}$$

Thus, the scalar potential difference between any two points P and P_0 in the region is independent of the path of integration between the two points,

$$\Phi_{c_P} - \Phi_{c_{P_0}} = -\int_{P_0}^{P} (\mathbf{H} - \mathbf{M}_c) \cdot \mathbf{dl} , \quad \text{(A12)}$$

and, therefore, Φ_c is a single-valued function of position.

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