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New Numerical Results for the Time-Fractional Phi-Four Equation Using a Novel Analytical Approach

Wei Gao ^{1,*}, Pundikala Veerasha ² , Doddabhadrappla Gowda Prakasha ³,
Haci Mehmet Baskonus ⁴  and Gulnur Yel ⁵ 

¹ School of Information Science and Technology, Yunnan Normal University, Kunming 650500, China

² Department of Mathematics, Karnatak University, Dharwad 580003, India; viru0913@gmail.com

³ Department of Mathematics, Faculty of Science, Davangere University, Shivagangothri, Davangere 577007, India; prakashadg@gmail.com

⁴ Department of Mathematics and Science Education, Faculty of Education, Harran University, Sanliurfa 63200, Turkey; hmbaskonus@gmail.com

⁵ Faculty of Educational Sciences, Final International University, Mersin 10, Kyrenia 99370, Turkey; gulnuryel33@gmail.com

* Correspondence: gaowei@ynnu.edu.cn

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Abstract: This manuscript investigates the fractional Phi-four equation by using q -homotopy analysis transform method (q -HATM) numerically. The Phi-four equation is obtained from one of the special cases of the Klein-Gordon model. Moreover, it is used to model the kink and anti-kink solitary wave interactions arising in nuclear particle physics and biological structures for the last several decades. The proposed technique is composed of Laplace transform and q -homotopy analysis techniques, and fractional derivative defined in the sense of Caputo. For the governing fractional-order model, the Banach's fixed point hypothesis is studied to establish the existence and uniqueness of the achieved solution. To illustrate and validate the effectiveness of the projected algorithm, we analyze the considered model in terms of arbitrary order with two distinct cases and also introduce corresponding numerical simulation. Moreover, the physical behaviors of the obtained solutions with respect to fractional-order are presented via various simulations.

Keywords: q -homotopy analysis method; Laplace transform; Phi-four equation; Caputo fractional derivative.

1. Introduction

Recently, many new fractional-order operator types have been introduced to the literature by some scholars, and established its fundamental properties [1–6]. Fractional theory is widely used to explain many properties of phenomena such as nanotechnology [7], optics [8], human diseases [9], chaos theory [10], and others [11–32]. As an example, the tumor-immune surveillance model has been investigated in [33]. Some important properties of dengue fever have been effectively investigated in [34,35]. The analytical as well as numerical solutions for the equations illustrating the above cited phenomena, have an important role in describing the behavior of nonlinear models ascends [36–40]. For the differential equation, symmetry is a transformation that keeps its family of solutions invariant, and further, its analysis can be applied to examine and illustrate various classes of differential equations. The study of fractional calculus associated to symmetry recently attracted many researchers from different disciplines in order to present their viewpoints while analyzing real-world problems.

The Klein-Gordon (KG) equation is derived by physicists Klein and Gordon to investigate the behavior of relativistic electrons. As a specific form of the KG equation, the Phi-four equation is defined to illustrate wave interactions [41,42] which is considered as in fractional order, namely, fractional Phi-four (FPF) equation

$$D^\mu v(x, t) = v_{xx}(x, t) + \lambda_1 v(x, t) + \lambda_2 v^3(x, t), \quad 1 < \mu \leq 2, \quad (1)$$

with

$$v(x, 0) = f_1(x) \text{ and } v_t(x, 0) = f_2(x).$$

Here, λ_1 and λ_2 are real-valued constants. Here, $v(x, t)$ is the normalized propagation of distance and retracted time t .

Many physicists and mathematicians recently proposed very accurate and more effective algorithms to find and examine the solution for equations describing nonlinear mechanisms arising in science and engineering. As an approximate method, the homotopy analysis method (HAM) has been introduced by L. Shijun [43,44], which is formed to describe deformation from zero to 1. Recently, it has been efficiently and beneficially employed to examine the nature of nonlinear problems without linearization and perturbation. However, HAM necessitates huge time and computer memory for computational work. Therefore, there is an essence of the consolidation of this scheme with familiar transform algorithms. The projected solution procedure is the mixture of q -HAM with LT [45]. Since the future scheme is a modified method of HAM, it does not kernel discretization, perturbation or linearization [46–56].

The considered nonlinear problem recently fascinated authors from various areas of science. Due to the numerous applications of the proposed model and the important role it plays in describing various nonlinear phenomena, several researchers found and investigated the solution numerically and also analytically; for instance, researchers in [57] find the new exact solution for the fractional case of considered nonlinear problem, and the spectral collocation method is considered in [58] to find the solution for PF equation. The considered nonlinear PF equation has attracted the attention of researchers, and, in order to present their viewpoint, many techniques are considered [59–65], presenting some interesting results. Recently many researchers from all over the world have investigated Fractional Calculus (FC) as an effective tool by comparing with other operators some important physical problems [66–79]. For instance, authors in [80] presented the reflection symmetry in fractional calculus, and proposed some interesting properties and corresponding consequences.

The main aim of this paper is to investigate the numerical solutions for the considered fractional Phi-Four equation using the novel technique. The proposed algorithm provides more liberty to consider the diverse class complex as well as nonlinear problems and the initial guess.

In this paper, we try to capture in the physical behaviors obtained numerical solutions with distinct arbitrary order and the parameters accessible by the projected algorithm. Moreover, in order to present accuracy and effectiveness, we present the numerical simulations. By the help of these results, we try to explain the diverse model exemplifying numerous phenomena arising in daily life. The rest of the paper is organized as follows: the basic properties of FC are presented in Section 2, the basic solution algorithm of the projected method for the considered problem is defined in Section 3, the convergence analysis of the considered scheme is illustrated in Section 4, the solution for FPF equation and its corresponding consequences in terms plots and also numerical simulation are respectively cited in Sections 5 and 6. Finally, some important findings of the paper are presented as a conclusion section.

2. Preliminaries

In this subsection of the paper, it is presented the basic definitions of fractional calculus.

Definition 1. Fractional integral of $v(x, t) \in C_\delta (\delta \geq -1)$ with the order μ is defined in RL as

$$J^\mu v(x, t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \vartheta)^{\mu-1} v(x, \vartheta) d\vartheta, J^0 v(x, t) = v(x, t). \quad (2)$$

Definition 2. Fractional derivative of $f \in C_{-1}^n$ is given in the sense of Caputo as the following:

$$D_t^\mu v(x, t) = \begin{cases} \frac{d^\mu v(x, t)}{dt^\mu}, & \mu = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\mu)} \int_0^t (x, t - \vartheta)^{n-\mu-1} v^{(n)}(x, \vartheta) d\vartheta, & n-1 < \mu < n, n \in \mathbb{N}. \end{cases} \quad (3)$$

Definition 3. The Laplace transform (LT) of $D_t^\mu v(x, t)$ for a Caputo fractional derivative is introduced as below:

$$L[D_t^\mu v(x, t)] = s^\mu V(x, s) - \sum_{r=0}^{n-1} s^{\mu-r-1} v^{(r)}(x, 0^+), \quad (n-1 < \mu \leq n), \quad (4)$$

where $V(x, s)$ is LT of $v(x, t)$.

3. Solution Procedure for Fractional Phi-Four Equation

Here, we consider the nonlinear FPF equation to illustrate the basic solution algorithm of the projected method with associated initial conditions as follows:

$$D_t^\mu v(x, t) = \frac{\partial^2 v(x, t)}{\partial x^2} + \lambda_1 v(x, t) + \lambda_2 v^3(x, t), \quad 1 < \mu \leq 2, \quad t > 0, \quad (5)$$

and

$$v(x, 0) = g_1(x) \text{ and } v_t(x, 0) = g_2(x). \quad (6)$$

Here, $D_t^\mu v(x, t)$ signifies the fractional Caputo-derivative of $v(x, t)$. Here, $v(x, t)$ is a bounded function. On using the LT on Equation (5), and then by the help of Equation (6), it is given as following

$$L[v(x, t)] - \frac{1}{s}[g_1(x)] - \frac{1}{s^2}[g_2(x)] - \frac{1}{s^\mu} L\left\{\frac{\partial^2 v}{\partial x^2} + \lambda_1 v + \lambda_2 v^3\right\} = 0. \quad (7)$$

By the assist of Equation (7), the nonlinear operator is given as follows:

$$\mathcal{N}[\varphi(x, t; q)] = L[\varphi(x, t; q)] - \frac{1}{s}[g_1(x)] - \frac{1}{s^2}[g_2(x)] - \frac{1}{s^\mu} L\left\{\frac{\partial^2 \varphi(x, t; q)}{\partial x^2} + \lambda_1 \varphi(x, t; q) + \lambda_2 \varphi^3(x, t; q)\right\}, \quad (8)$$

Now, the homotopy structure is defined by

$$(1 - nq)L[\varphi(x, t; q) - v_0(x, t)] = \hbar q \mathcal{N}[\varphi(x, t; q)], \quad (9)$$

where $q \in [0, \frac{1}{n}] (n \geq 1)$ and $\hbar \neq 0$ are respectively the embedding and auxiliary parameter. Further, these parameters help us to control and adjust the convergence region of the obtained solution. For the proper choice of n and \hbar , the obtained solution quickly converges to the exact solution in an admissible domain. The following results are respectively true for $q = 0$ and $q = \frac{1}{n}$

$$\varphi(x, t; 0) = v_0(x, t), \quad \varphi\left(x, t; \frac{1}{n}\right) = v(x, t). \quad (10)$$

Near to q , we define $\varphi(x, t; q)$ in series form by help of the Taylor theorem and then one can obtain

$$\varphi(x, t; q) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t)q^m, \tag{11}$$

where

$$v_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \varphi(x, t; q)}{\partial q^m} \right|_{q=0}. \tag{12}$$

For the proper choice of $v_0(x, t)$, n and \hbar ; the series (11) converges at $q = \frac{1}{n}$. Then

$$v(x, t) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t) \left(\frac{1}{n}\right)^m. \tag{13}$$

Dividing by $m!$ after m -times differentiating Equation (9) with q and then putting $q = 0$, it simplifies to

$$L[v_m(x, t) - k_m v_{m-1}(x, t)] = \hbar \mathfrak{R}_m(\vec{v}_{m-1}), \tag{14}$$

where k_m and v_m are defined by

$$k_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \tag{15}$$

and

$$\vec{v}_m = \{v_0(x, t), v_1(x, t), \dots, v_m(x, t)\}. \tag{16}$$

On applying inverse LT on Equation (14), produces

$$v_m(x, t) = k_m v_{m-1}(x, t) + \hbar L^{-1}[\mathfrak{R}_m(\vec{v}_{m-1})], \tag{17}$$

where

$$\begin{aligned} \mathfrak{R}_m[\vec{v}_{m-1}] &= L[v_{m-1}(x, t)] - \left(1 - \frac{k_m}{n}\right) \left(\frac{1}{s} [g_1(x)] + \frac{1}{s^2} [g_2(x)]\right) \\ &\quad - \frac{1}{s^{\mu}} L \left\{ \frac{\partial^2 v_{m-1}}{\partial x^2} + \lambda_1 v_{m-1} + \lambda_2 \sum_{i=0}^{m-1} \left(\sum_{j=0}^i v_j v_{i-j} \right) v_{m-1-i} \right\}. \end{aligned} \tag{18}$$

With the help of Equations (17) and (18), we have

$$\begin{aligned} v_m(x, t) &= (k_m + \hbar) v_{m-1}(x, t) - \left(1 - \frac{k_m}{n}\right) L^{-1} \left\{ \frac{1}{s} [g_1(x)] + \frac{1}{s^2} [g_2(x)] \right\} \\ &\quad - \hbar L^{-1} \left[\frac{1}{s^{\mu}} L \left\{ \frac{\partial^2 v_{m-1}}{\partial x^2} + \lambda_1 v_{m-1} + \lambda_2 \sum_{i=0}^{m-1} \left(\sum_{j=0}^i v_j v_{i-j} \right) v_{m-1-i} \right\} \right]. \end{aligned} \tag{19}$$

Using Equation (19), we achieve the series of $v_m(x, t)$. Lastly, the series q -HATM solution is defined as

$$v(x, t) = \sum_{m=0}^{\infty} v_m(x, t). \tag{20}$$

4. Convergence Analysis of the Proposed Method

In this part of the paper, for the considered problem, we introduce the convergence analysis as follows.

Theorem 1 (Convergence theorem). *Let $F : E \rightarrow E$ is mapping (nonlinear) with Banach space E . Presume that*

$$\|F(u) - F(v)\| \leq \alpha \|u - v\|, \quad \forall u, v \in E, \tag{21}$$

then there is a fixed point for F with the help of Banach's fixed point theory [81–83]. Then, for $u_0, v_0 \in E$ the sequence obtained by q -HATM solution converges to a specified point of F , and

$$\|v_m - v_p\| \leq \frac{\alpha^p}{1-\alpha} \|v_1 - v_0\|. \quad (22)$$

Proof. Let us consider a Banach space $(C[J], \|\cdot\|)$ with the norm defined as $\|g(t)\| = \max_{t \in J} |g(t)|$ for all continuous function on J . Now, we verify $\{v_p\}$ is Cauchy sequence in $(C[J], \|\cdot\|)$. For that, let

$$\begin{aligned} \|v_m - v_p\| &= \max_{t \in J} |v_m - v_p| \\ &= \max_{t \in J} \left| (k_m + \hbar)(v_{m-1} - v_{p-1}) - \hbar L^{-1} \left(\frac{1}{s^\mu} L \left[\left(\frac{\partial^2 v_{m-1}}{\partial x^2} - \frac{\partial^2 v_{p-1}}{\partial x^2} \right) + \lambda_1 (v_{m-1} - v_{p-1}) + \lambda_2 (v_{m-1}^3 - v_{p-1}^3) \right] \right) \right| \\ &\leq \max_{t \in J} \left\{ (k_m + \hbar) |v_{m-1} - v_{p-1}| - \hbar L^{-1} \left(\frac{1}{s^\mu} L \left[\left| \frac{\partial^2 v_{m-1}}{\partial x^2} - \frac{\partial^2 v_{p-1}}{\partial x^2} \right| \right. \right. \right. \\ &\quad \left. \left. \left. + \lambda_1 |v_{m-1} - v_{p-1}| + \lambda_2 |v_{m-1}^3 - v_{p-1}^3| \right] \right) \right\}. \end{aligned}$$

For LT , with the help of the convolution theorem, we get

$$\begin{aligned} \|v_m - v_p\| &\leq \max_{t \in J} [(k_m + \hbar) |v_{m-1} - v_{p-1}| \\ &\quad - \hbar \int_0^t \left(\left| \frac{\partial^2 v_{m-1}}{\partial x^2} - \frac{\partial^2 v_{p-1}}{\partial x^2} \right| + \lambda_1 |v_{m-1} - v_{p-1}| + \lambda_2 |v_{m-1}^3 - v_{p-1}^3| \right) \frac{(t-\xi)^\mu}{\Gamma(\mu+1)} d\xi \\ &\leq \max_{t \in J} [(k_m + \hbar) |v_{m-1} - v_{p-1}| - \hbar \int_0^t (\delta^2 |v_{m-1} - v_{p-1}| + \lambda_1 |v_{m-1} - v_{p-1}| \\ &\quad + \lambda_2 |v_{m-1} - v_{p-1}| (P^2 + Q^2 + PQ)) \frac{(t-\xi)^\mu}{\Gamma(\mu+1)} d\xi]. \end{aligned}$$

Then, the above inequality reduces to

$$\begin{aligned} \|v_m - v_p\| &\leq \max_{t \in J} [(k_m + \hbar) |v_{m-1} - v_{p-1}| + \hbar (\delta^2 |v_{m-1} - v_{p-1}| \\ &\quad + \lambda_1 |v_{m-1} - v_{p-1}| + \lambda_2 |v_{m-1} - v_{p-1}| (P^2 + Q^2 + PQ)) \mathcal{T}], \\ \|v_m - v_p\| &\leq \alpha \|v_{m-1} - v_{p-1}\|. \end{aligned}$$

Setting $m = p + 1$, it yields

$$\|v_{p+1} - v_p\| \leq \alpha \|v_p - v_{p-1}\| \leq \alpha^2 \|v_{p-1} - v_{p-2}\| \leq \dots \leq \alpha^p \|v_1 - v_0\|.$$

On using triangular inequality, we have

$$\begin{aligned} \|v_m - v_p\| &\leq \|v_{p+1} - v_p\| + \|v_{p+2} - v_{p+1}\| + \dots + \|v_m - v_{m-1}\| \\ &\leq [\alpha^p + \alpha^{p+1} + \dots + \alpha^{m-1}] \|v_1 - v_0\| \\ &\leq \alpha^p [1 + \alpha + \alpha^2 + \dots + \alpha^{m-p-1}] \|v_1 - v_0\| \\ &\leq \alpha^p \left[\frac{1 - \alpha^{m-p-1}}{1 - \alpha} \right] \|v_1 - v_0\|. \end{aligned}$$

As $0 < \alpha < 1$, so $1 - \alpha^{m-p-1} < 1$, then we have

$$\|v_m - v_p\| \leq \frac{\alpha^p}{1-\alpha} \|v_1 - v_0\|.$$

But $\|v_1 - v_0\| < \infty$, consequently as $m \rightarrow \infty$ than $\|v_m - v_p\| \rightarrow 0$, which proves $\{v_p\}$ is Cauchy sequence in $C[J]$. It completes our required proof. \square

Theorem 2 (Uniqueness theorem). For Equation (5), the q -HATM solution is unique wherever $0 < \alpha < 1$, where $\alpha = (k_m + \hbar) + \hbar(\delta^2 + \lambda_1 + \lambda_2(P^2 + Q^2 + PQ))\mathcal{T}$.

Proof. For fractional PF equation cited in Equation (5), the solution is illustrated as following

$$v(x, t) = \sum_{m=0}^{\infty} v_m(x, t)$$

where

$$v_m(x, t) = (k_m + \hbar)v_{m-1}(x, t) - \left(1 - \frac{k_m}{n}\right)L^{-1}\left\{\frac{1}{s} [f_1(x)] + \frac{1}{s^2} [f_2(x)]\right\} - \hbar L^{-1}\left[\frac{1}{s^\mu} L\left\{\frac{\partial^2 v_{m-1}}{\partial x^2} + \lambda_1 v_{m-1} + \lambda_2 \sum_{i=0}^{m-1} \left(\sum_{j=0}^i v_j v_{i-j}\right) v_{m-1-i}\right\}\right].$$

Suppose v and v^* be the two different solutions for Equation (5) such that $|v| \leq P$ and $|v^*| \leq Q$, then using the above relation, it is obtained as following

$$|v - v^*| = \left| (k_m + \hbar)(v - v^*) - \hbar L^{-1}\left\{\frac{1}{s^\mu} L\left(\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v^*}{\partial x^2}\right) + \lambda_1(v - v^*) + \lambda_2(v^3 - v^{*3})\right)\right\}\right|. \tag{23}$$

For Laplace transform, we have, by the help of the convolution theorem,

$$\begin{aligned} |v - v^*| &= (k_m + \hbar)|v - v^*| + \hbar \int_0^t \left(\left| \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v^*}{\partial x^2} \right| + |\lambda_1(v - v^*)| + |\lambda_2(v^3 - v^{*3})| \right) \frac{(t-\xi)^\mu}{\Gamma(\mu+1)} d\xi, \\ &\leq (k_m + \hbar)|v - v^*| + \hbar \int_0^t \left(\frac{\partial^2}{\partial x^2} |v - v^*| + \lambda_1 |v - v^*| + \lambda_2 |v - v^*|(v^2 + v^{*2} + vv^*) \right) \frac{(t-\xi)^\mu}{\Gamma(\mu+1)} d\xi, \\ &\leq (k_m + \hbar)|v - v^*| + \hbar \int_0^t (\delta^2 |v - v^*| + \lambda_1 |v - v^*| + \lambda_2 |v - v^*|(P^2 + Q^2 + PQ)) \frac{(t-\xi)^\mu}{\Gamma(\mu+1)} d\xi, \end{aligned}$$

where $\delta^2 = \frac{\partial^2}{\partial x^2}$. Using integral mean value, we have

$$\begin{aligned} |v - v^*| &\leq (k_m + \hbar)|v - v^*| + \hbar(\delta^2 |v - v^*| + \lambda_1 |v - v^*| \\ &+ \lambda_2 |v - v^*|(P^2 + Q^2 + PQ))\mathcal{T} |v - v^*| \leq \alpha |v - v^*| (1 - \alpha) |v - v^*| \leq 0. \end{aligned}$$

Since $0 < \alpha < 1$, therefore $|v - v^*| = 0$, which gives $v = v^*$. This completes the required proof. \square

5. Solution for Fractional PF Equation

In order to validate the applicability and efficiency of the future technique, here we consider the two distinct examples.

Example 1. Consider the time-fractional Phi-four equation [84–86]

$$D_t^\mu v(x, t) = \frac{\partial^2 v(x, t)}{\partial x^2} - \lambda_1 v(x, t) - \lambda_2 v^3(x, t), \tag{24}$$

subjected to the initial conditions

$$v(x, 0) = \sqrt{\frac{-\lambda_1^2}{\lambda_2}} \tanh\left(\lambda_1 \sqrt{\frac{1}{2(\eta^2-1)}} x\right) \text{ and } v_t(x, 0) = -\lambda_1 \eta \sqrt{\frac{-\lambda_1^2}{2\lambda_2(\eta^2-1)}} \operatorname{sech}^2\left(\lambda_1 \sqrt{\frac{1}{2(\eta^2-1)}} x\right). \tag{25}$$

By using the proposed algorithm and Equation (25), the Equation (24) becomes

$$\begin{aligned}
 v_m(x, t) = & (k_m + \hbar)v_{m-1}(x, t) - \left(1 - \frac{k_m}{n}\right)L^{-1}\left\{\frac{1}{s}\left(\sqrt{\frac{-\lambda_1^2}{\lambda_2}} \tan h\left(\lambda_1 \sqrt{\frac{1}{2(\eta^2-1)}}x\right)\right)\right. \\
 & \left. - \frac{1}{s^2}\left(\lambda_1\eta \sqrt{\frac{-\lambda_1^2}{2\lambda_2(\eta^2-1)}} \sec h^2\left(\lambda_1 \sqrt{\frac{1}{2(\eta^2-1)}}x\right)\right)\right\} - \hbar L^{-1}\left[\frac{1}{s^\mu}L\left\{\frac{\partial^2 v_{m-1}}{\partial x^2} - \lambda_1 v\right.\right. \\
 & \left. \left. - \lambda_2 \sum_{i=0}^{m-1}\left(\sum_{j=0}^i v_j v_{i-j}\right)v_{m-1-i}\right]\right].
 \end{aligned}
 \tag{26}$$

We get the iterative terms of $v_m(x, t)$ by solving Equation (26), and as follows

$$\begin{aligned}
 v_0(x, t) = & \sqrt{\frac{-\lambda_1^2}{\lambda_2}} \tan h\left(\lambda_1 \sqrt{\frac{1}{2(\eta^2-1)}}x\right) - \lambda_1 \eta t \sqrt{\frac{-\lambda_1^2}{2\lambda_2(\eta^2-1)}} \sec h^2\left(\lambda_1 \sqrt{\frac{1}{2(\eta^2-1)}}x\right) \\
 v_1(x, t) = & \frac{\lambda_1^2 \eta^2 \hbar t^\mu}{8(-1+\eta^2)\Gamma[\mu+1]} \sec h^6\left(\frac{\lambda_1 x \sqrt{\frac{1}{-1+\eta^2}}}{\sqrt{2}}\right) \left(-3 \sqrt{2} \lambda_1 t \eta \sqrt{\frac{\lambda_1^2}{\lambda_2 - \eta^2 \lambda_2}}\right. \\
 & \left. + 2 \sqrt{2} \lambda_1^3 t^3 \eta \sqrt{\frac{\lambda_1^2}{\lambda_2 - \eta^2 \lambda_2}}\right. \\
 & \left. - 2 \sqrt{2} \lambda_1 t \eta \sqrt{\frac{\lambda_1^2}{\lambda_2 - \eta^2 \lambda_2}} \cos h\left(\sqrt{2} \lambda_1 x \sqrt{\frac{1}{-1+\eta^2}}\right)\right. \\
 & \left. + \sqrt{2} \lambda_1 t \eta \sqrt{\frac{\lambda_1^2}{\lambda_2 - \eta^2 \lambda_2}} \cos h\left(2 \sqrt{2} \lambda_1 x \sqrt{\frac{1}{-1+\eta^2}}\right)\right. \\
 & \left. + 2 \sqrt{-\frac{\lambda_1^2}{\lambda_2}} \sin h\left(\sqrt{2} \lambda_1 x \sqrt{\frac{1}{-1+\eta^2}}\right) + 6 t^2 \left(-\frac{\lambda_1^2}{\lambda_2}\right)^{3/2} \lambda_2 \sin h\left(\sqrt{2} \lambda_1 x \sqrt{\frac{1}{-1+\eta^2}}\right)\right. \\
 & \left. + \sqrt{-\frac{\lambda_1^2}{\lambda_2}} \sin h\left(2 \sqrt{2} \lambda_1 x \sqrt{\frac{1}{-1+\eta^2}}\right)\right),
 \end{aligned}$$

Similarly, the remaining part of the series can be calculated (See Figures 1–3 and Table 1). Now, the series form of the q -HATM solution is as follows:

$$v(x, t) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t).$$

Table 1. Numerical simulation conducted for FPF equation defined in Example1 with different x and t at $\lambda_1 = 1, \lambda_2 = -1, \eta = 3, n = 1, \mu = 2$ and $\hbar = -1$.

x/t	$ v_{Exact} - v_{q-HATM}^{(3)} $				
	0.01	0.02	0.03	0.04	0.05
-5	9.18570×10^{-8}	7.38629×10^{-7}	2.50566×10^{-6}	5.96979×10^{-6}	1.17195×10^{-5}
-3	3.68683×10^{-8}	3.07718×10^{-7}	1.08179×10^{-6}	2.66706×10^{-6}	5.41063×10^{-6}
-1	2.15263×10^{-7}	1.70951×10^{-6}	5.72612×10^{-6}	1.34678×10^{-5}	2.60945×10^{-5}
1	2.18306×10^{-7}	1.75819×10^{-6}	5.97254×10^{-6}	1.42464×10^{-5}	2.79946×10^{-5}
3	3.36893×10^{-8}	2.56849×10^{-7}	8.24228×10^{-7}	1.85290×10^{-6}	3.42245×10^{-6}
5	9.09190×10^{-8}	7.23617×10^{-7}	2.42963×10^{-6}		1.11322×10^{-5}

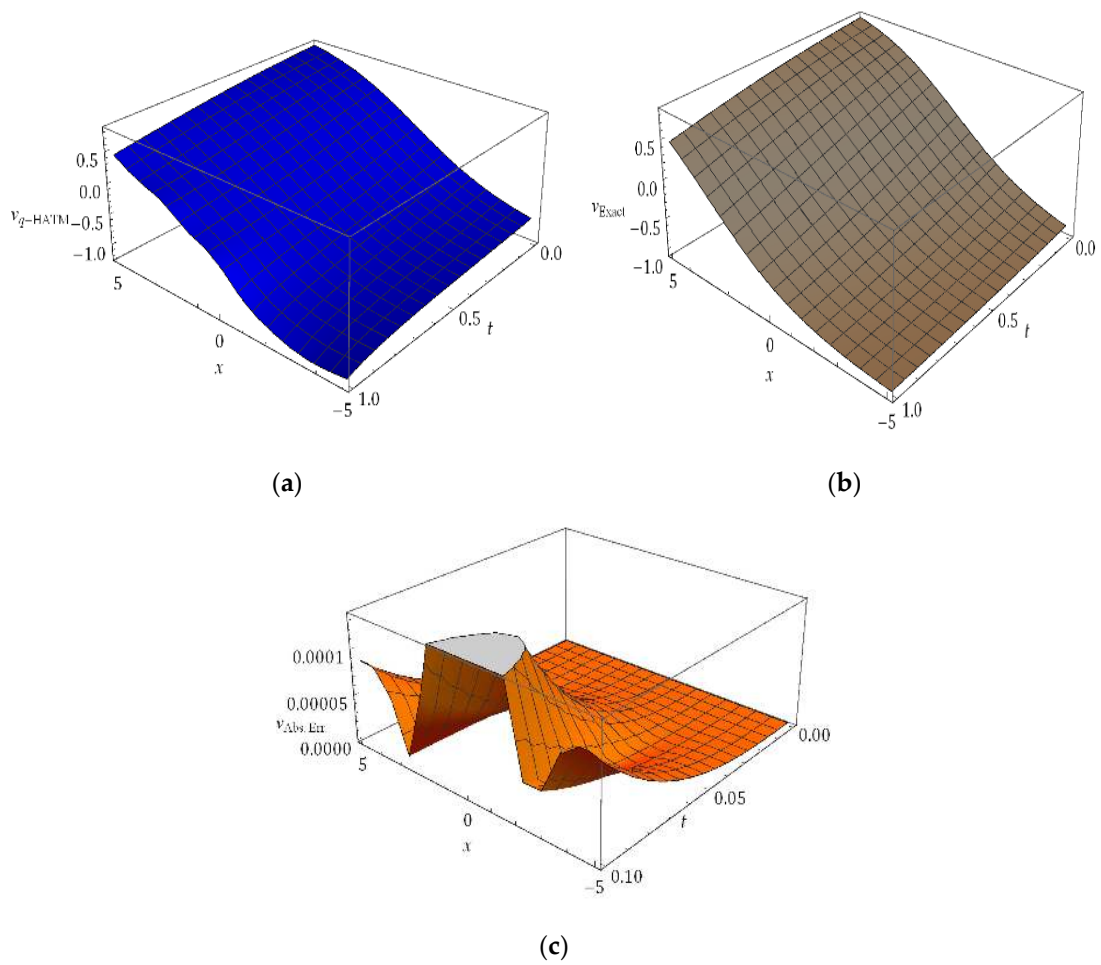


Figure 1. 3D graphs of (a) Obtained solution, (b) Exact solution, (c) Absolute error = $|v_{Exact} - v_{App}|$ for Example 1 at $n = 1$, $\hbar = -1$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\eta = 3$ and $\mu = 2$.

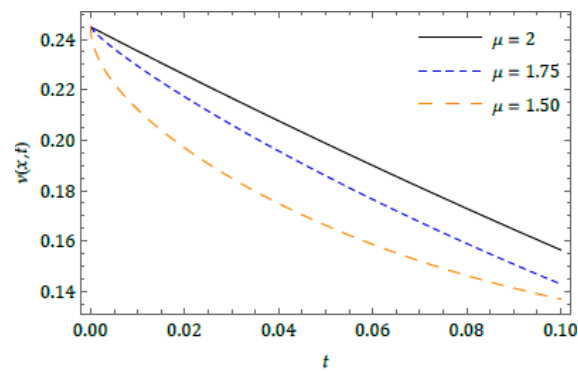


Figure 2. 2D graph of q -homotopy analysis transform method (q -HATM) solution at $n = 1$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\eta = 3$, $x = 1$ and $\hbar = -1$ with distinct μ for Example 1.

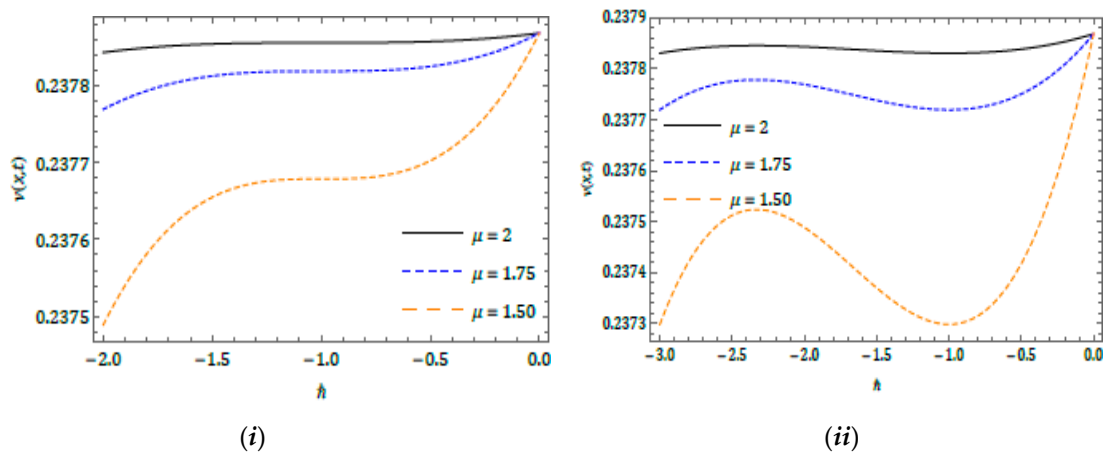


Figure 3. 2D graphs of \hbar -curves for $v(x,t)$ of Example 1 at (i) $n = 1$, (ii) $n = 2$ with different μ with $\lambda_1 = 1$, $\lambda_2 = -1$, $\eta = 3$, $x = 1$ and $t = 0.01$.

For $n = 1$, $\mu = 2$ and $\hbar = -1$, the q -HATM solution converges to analytical solution

$$v(x,t) = \sqrt{\frac{-\lambda_1^2}{\lambda_2}} \tanh\left(\lambda_1 \sqrt{\frac{1}{2(\eta^2 - 1)}}(x - \eta t)\right).$$

Example 2. Consider the FPF equation at $\lambda_1 = -1$ and $\lambda_2 = 1$ [57]

$$D_t^\mu v(x,t) = \frac{\partial^2 v(x,t)}{\partial x^2} + v(x,t) - v^3(x,t), \tag{27}$$

subjected to the initial conditions

$$v(x, 0) = \tanh\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) \text{ and } v_t(x, 0) = \tanh\left(\sqrt{\frac{1}{2(1-k^2)}}x\right). \tag{28}$$

By using the proposed algorithm and Equation (28), the Equation (27) becomes

$$v_m(x,t) = (k_m + \hbar)v_{m-1}(x,t) - \left(1 - \frac{k_m}{n}\right)L^{-1}\left\{\left(\frac{1}{s} + \frac{1}{s^2}\right)\tanh\left(\sqrt{\frac{1}{2(1-k^2)}}x\right)\right\} - \hbar L^{-1}\left[\frac{1}{s^\mu}L\left\{\frac{\partial^2 v_{m-1}}{\partial x^2} + v - \sum_{i=0}^{m-1}\left(\sum_{j=0}^i v_j v_{i-j}\right)v_{m-1-i}\right\}\right]. \tag{29}$$

We evaluate the iterative terms of $v_m(x, t)$ with the assistance of Equation (29), and as follows

$$\begin{aligned}
 v_0(x, t) &= (1 + t) \tan h\left(\sqrt{\frac{1}{2(1-k^2)}}x\right), \\
 v_1(x, t) &= \frac{\hbar t^\mu}{\Gamma[\mu+1]} \tan h\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) \left(-1 - t - \frac{(1+t)\sec h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right)}{k^2-1} + (1+t)^3 \tan h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right)\right), \\
 v_2(x, t) &= \frac{\hbar(n+\hbar)t^\mu}{\Gamma[\mu+1]} \tan h\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) \left(-1 - t - \frac{(1+t)\sec h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right)}{k^2-1} + (1+t)^3 \tan h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right)\right) \\
 &+ \frac{\hbar^2 t^{2\mu}}{\Gamma[2\mu+1]} \tan h\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) \left(1 + t + \frac{(1+t)}{-1+k^2} \sec h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) + \frac{2(1+t)(k^2(1+t)^2-t(2+t))}{(-1+k^2)^2}\right. \\
 &\times \sec h^4\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) - \frac{(1+t)(k^2(1+t)^2-t(2+t))(-2+\cos h\left(\sqrt{2}\sqrt{\frac{1}{1-k^2}}x\right))}{(-1+k^2)^2} \sec h^4\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) \\
 &- (1+t)^3 \tan h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) + \frac{2\sec h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right)}{2-2k^2} (-1 - t - \frac{(1+t)\text{Sech}\left[\sqrt{\frac{1}{2-2k^2}}x\right]^2}{-1+k^2}) \\
 &+ (1+t)^3 \tan h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) + 3(1+t)^2 \tan h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right) (-1 - t \\
 &\left. - \frac{(1+t)\sec h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right)}{-1+k^2} + (1+t)^3 \tan h^2\left(\sqrt{\frac{1}{2(1-k^2)}}x\right)\right),
 \end{aligned}$$

Now, the series form of q -HATM solution is as follows:

$$v(x, t) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t).$$

For $n = 1$, $\mu = 2$ and $\hbar = -1$, the q -HATM solution converges to analytical solution $v(x, t) = \tan h\left(\sqrt{\frac{1}{2(1-k^2)}}(x - kt)\right)$ (See Figures 4–6 and Table 2).

Table 2. Numerical simulation conducted for fractional Phi-four (FPF) equation defined in Example2 with different x and t at $k = 0.1$, $\mu = 2$, $n = 1$ and $\hbar = -1$.

x/t	$ v_{Exact} - v_{q-HATM}^{(3)} $				
	0.01	0.02	0.03	0.04	0.05
-5	1.96030×10^{-5}	9.96170×10^{-3}	1.99430×10^{-2}	2.99243×10^{-2}	3.99056×10^{-2}
-3	3.17168×10^{-4}	9.36662×10^{-3}	1.90505×10^{-2}	2.87343×10^{-2}	3.84183×10^{-2}
-1	4.33502×10^{-3}	1.33124×10^{-3}	6.99789×10^{-3}	1.26649×10^{-2}	1.83324×10^{-2}
1	1.65573×10^{-2}	2.31142×10^{-2}	2.96714×10^{-2}	3.62290×10^{-2}	4.27869×10^{-2}
3	1.97624×10^{-2}	2.95239×10^{-2}	3.92856×10^{-2}	4.90472×10^{-2}	5.88089×10^{-2}
5	1.99868×10^{-2}	2.99728×10^{-2}	3.99588×10^{-2}	4.99447×10^{-2}	5.99307×10^{-2}

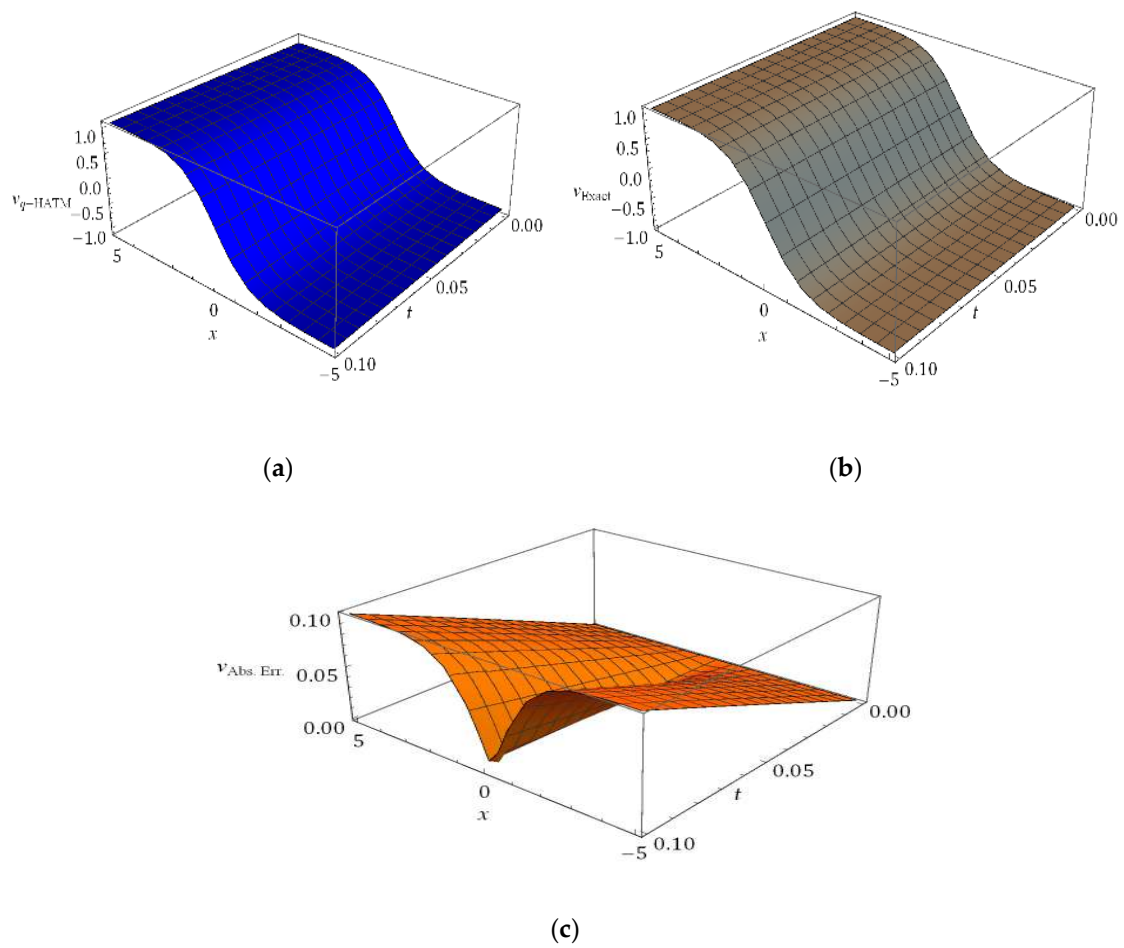


Figure 4. 3Dsurfaces of (a) q -HATM solution, (b) Exact solution, (c) Absolute error = $|v_{Exa.} - v_{App.}|$ for Example2 at $n = 1$, $\hbar = -1$, $k = 0.1$ and $\mu = 2$.

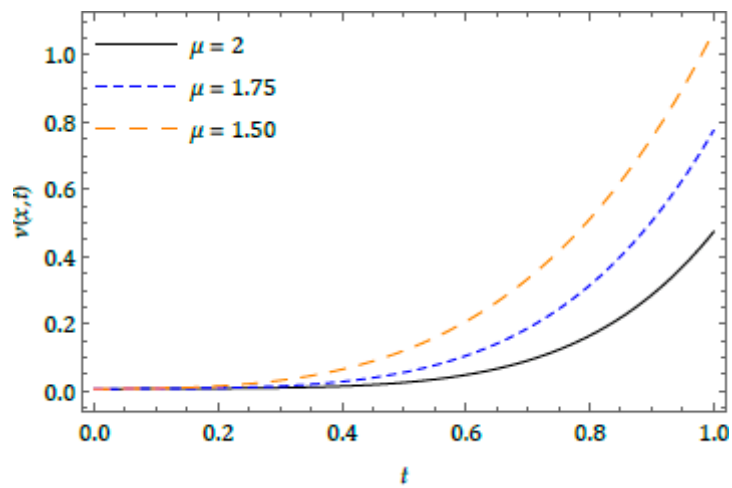


Figure 5. 2D graph of q -HATM solution at $n = 1$, $\hbar = -1$, $x = 0.01$ and $k = 0.1$ with different μ for Example2.

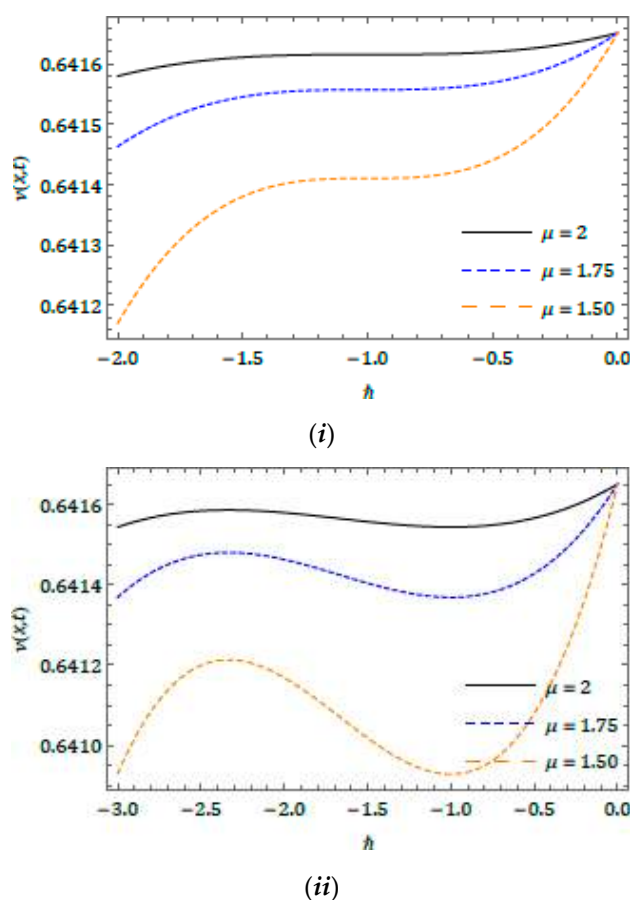


Figure 6. 2D graphs of for of Example2 at (i) $n = 1$, (ii) $n = 2$ with distinct μ with $k = 0.1$, $x = 1$ and $t = 0.01$.

6. Numerical Results and Discussion

Here, we have demonstrated the numerical simulations of the Phi-four equation having fractional-order by using q -HATM. The fourth order solution (i.e., up to $v_4(x, t)$) we consider to illustrate the behavior of the obtained solution in terms of plots and tables. In Tables 1 and 2, the numerical studies have been conducted to ensure the exactness of the projected algorithm, and further from the tables, it is confirmed that the considered model conspicuously depends on the time. The surfaces of the q -HATM solution, exact solution and absolute error are captured in Figure 1 for the FPF equation considered in Example 1. To show the projected problem remarkably depends on arbitrary order and the response of q -HATM solution for different μ , the plots have been drowned and are cited in Figures 2 and 5 for Examples 1 and 2., respectively. Similarly, nature obtained and exact solutions in association with absolute error for Example 2, are presented in Figure 4. The \hbar -curves have been drowned to analyze the behavior of the achieved solution related homotopy parameter \hbar with different μ for both examples and have been respectively captured in Figures 3 and 6. These curves can help us to adjust and regulate the region of convergence for the q -HATM solution. For unsuitable \hbar , the q -HATM solution swiftly converges to exact solution. The demonstrated plots help us to better understand the nature of the fractional Phi-four equation when temporal-spatial variables vary in comparison with arbitrary order.

7. Conclusions

In the present framework, q -HATM has been employed to find the numerical solutions for the fractional Phi-four equation. With the help of Banach's fixed point theory, the convergence analysis

projecting the nonlinear problem has been presented. To present the efficiency as well as the applicability of the projected algorithm, we have considered two distinct cases. The present study confirms that the projected nonlinear problem is remarkable with the time instant as well as the time history and these can be effectively exemplified by employing the concept of fractional calculus. The governing model plays a vibrant role while analyzing many physical phenomena, and thus, for future work, this can be examined by using recently proposed and nurtured effective and accurate methods [87,88]. With the help of the obtained results, we can capture more interesting consequences. Finally, it has been observed that the considered scheme is highly effective, more accurate and extremely methodical, and this can be employed to exemplify the various classes of nonlinear models that exist in science and technology. The results obtained demonstrate that the considered method is more effective and easy to employ to scrutinize the behaviors of fractional differential equations with multi-dimensions arising in associated areas of science and technology.

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