NEW POINT ESTIMATES FOR PROBABILITY MOMENTS

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ABSTRACT: There are many areas of structural safety and structural dynamics in which it is often desirable to compute the first few statistical moments of a function of random variables. The usual approximation is by the Taylor expansion method. This approach requires the computation of derivatives. In order to avoid the computation of derivatives, point estimates for probability moments have been proposed. However, the accuracy is quite low, and sometimes, the estimating points may be outside the region in which the random variable is defined. In the present paper, new point estimates for probability moments are proposed, in which increasing the number of estimating points is easier because the estimating points are independent of the random variable in its original space and the use of high-order moments of the random variables is not required. By using this approximation, the practicability and accuracy of point estimates can be much improved.

INTRODUCTION

The uncertainties treated in classical stochastic dynamic analysis are usually restricted to the excitation (Ibrahim 1987; Singh and Lee 1993). However, it has recently been recognized that the model parameters such as material properties are often poorly known. The inclusion of their uncertainties has therefore become an increasingly important problem in many areas of dynamics. In mathematics, this problem can, in the final analysis, be summarized as the computation of the first few statistical moments of a function of random variables, which are expressed as the following:

$$\mu_g = \int G(\mathbf{X})f(\mathbf{X})\, d\mathbf{X} \tag{1}$$

$$M_{kg} = \int (G(\mathbf{X}) - \mu_g)^k f(\mathbf{X})\, d\mathbf{X} \quad \text{for} \quad k \geq 2 \tag{2}$$

where $G(\mathbf{X})$ is a function of the random variables $\mathbf{X}$; μ_g is the mean value of $G(\mathbf{X})$; M_{kg} is the kth central moment of $G(\mathbf{X})$; and $f(\mathbf{X})$ is the joint probability density function of $\mathbf{X}$.

Because $G(\mathbf{X})$ is generally a complicated and implicit function, the computation of (1) and (2) by direct integration is almost impossible. The usual approximation is by the Taylor expansion method (Ibrahim 1987; Singh and Lee 1993). This approach requires the computation of derivatives (Rosenblueth 1975), which are generally difficult to obtain. Rosenblueth (1975) has given expressions that do not involve derivatives for estimating the first few moments of a function of random variables. The method uses a weighted sum of the function evaluated at a finite number of points, which are chosen to satisfy the following equation:

$$\sum_{j=1}^{m} P_j(x_j - \mu_x)^k = M_{kx} \tag{3}$$

where x_j, $j = 1, \ldots, m$ are estimating points; and P_j, $j = 1, \ldots, m$ are the corresponding weights.

Rosenblueth (1975) gives expressions for a two point estimate. Gorman (1980) derived expressions for a three point estimate as follows:

$$x_- = \mu_x - \frac{\sigma_x}{2}(\theta - \alpha_{3x}); \quad P_- = \frac{1}{2}\left(\frac{1 + \alpha_{3x}/\theta}{\alpha_{4x} - \alpha_{3x}^2}\right) \tag{4a,b}$$

$$x_0 = \mu_x; \quad P_0 = 1 - \frac{1}{\alpha_{4x} - \alpha_{3x}^2} \tag{5a,b}$$

$$x_+ = \mu_x + \frac{\sigma_x}{2}(\theta + \alpha_{3x}); \quad P_+ = \frac{1}{2}\left(\frac{1 - \alpha_{3x}/\theta}{\alpha_{4x} - \alpha_{3x}^2}\right) \tag{6a,b}$$

where

$$\theta = (4\alpha_{4x} - 3\alpha_{3x}^2)^{1/2} \tag{7}$$

x_-, x_0, and x_+ are the estimating points; P_-, P_0, and P_+ are the corresponding weights; and α_{3x} and α_{4x} are the 3rd and 4th dimensionless central moment, i.e., the skewness and kurtosis, respectively.

For a function $y = y(x)$, the kth central moment of y can be calculated by

$$\mu_y = P_- y(x_-) + P_0 y(x_0) + P_+ y(x_+) \tag{8}$$

$$M_{ky} = P_-(y(x_-) - \mu_y)^k + P_0(y(x_0) - \mu_y)^k + P_+(y(x_+) - \mu_y)^k \tag{9}$$

Eqs. (4)–(9) have been applied to system reliability analysis and response uncertainty evaluation (Gorman 1980; Ono et al. 1985; Zhao et al. 1999), and it has been found that the method has the following weaknesses:

1. The accuracy in general cases is quite low (Gorman 1980), especially in cases where the parameter uncertainties and the nonlinearity in the performance function are both large, or when the method is used to evaluate high-order moments (Zhao et al. 1999).
2. For some random variables, such as variables having a lognormal or exponential distribution, if the standard deviation is relatively large, then x_- given by (4) will be outside of the region in which the random variable is defined, and so the computation will be impossible.

Because the procedure of point estimates is very simple and does not require the computation of derivatives, if the two weaknesses described above can be eliminated, the uncertainty analysis in structural dynamics will become simpler. The object of this paper is to propose new point estimates for probability moments that remove these two weaknesses.

NEW POINT ESTIMATES FOR PROBABILITY MOMENTS

Basic Ideas

The most basic way to improve the accuracy of the current point estimates is to increase the number of estimating points.

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In general, for an m point estimate, supposing that an estimating point is fixed at $x_0 = \mu_x$, $2m - 1$ equations expressed in (3) will be needed and the first $2(m - 1)$ moments of x will be required. However, the high-order moments of an arbitrary random variable are not always easy to obtain, and it is generally hard to accept the use of moments higher than 4th order in engineering. Even if the moments higher than 4th order could be used, it would be almost impossible to obtain the general expressions for $x_1, x_2, \ldots, x_m$ and $P_1, P_2, \ldots, P_m$ for an arbitrary random variable. Furthermore, because the moment M_{kx} is dependent on the distribution of the random variable, it is difficult to avoid that the estimating point may move outside the region on which the random variable is defined. Due to the above problems, the three point estimate seems to be the limit, unless a different route can be found.

In order to avoid the aforementioned problems, the estimating points will be obtained, in the present paper, in the standard normal space. This is because any set of random variables can easily be transformed into a set of standard normal random variables through the Rosenblatt transformation (Hohenbichler and Rackwitz 1981)

$$\mathbf{U} = T(\mathbf{X}) \tag{10}$$

Using the inverse Rosenblatt transformation, (1) and (2) can be rewritten as

$$\mu_g = \int G[T^{-1}(\mathbf{U})]\phi(\mathbf{U})\, d\mathbf{U} \tag{11}$$

$$M_{kg} = \int (G[T^{-1}(\mathbf{U})] - \mu_g)^k \phi(\mathbf{U})\, d\mathbf{U} \quad \text{for} \quad k > 2 \tag{12}$$

where ϕ is the probability density function of standard normal random variables.

Because the estimating points and their corresponding weights for a standard normal random variable can be directly obtained by utilizing those of Hermite integration, the solution of the $2m - 1$ equations is unnecessary. Furthermore, because no central moment of the original random variables is required when obtaining the estimating points in the standard normal space, and the normal random variable is defined within the whole range $(-\infty, \infty)$, the problem of an estimating point going outside of the region on which the random variable is defined is avoided.

After obtaining the estimating points $u_1, u_2, \ldots, u_m$ and their corresponding weights $P_1, P_2, \ldots, P_m$, the kth central moment of a function $y = y(x)$ can be calculated as

$$\mu_y = \sum_{j=1}^{m} P_j y[(T^{-1}(u_j)] \tag{13}$$

$$M_{ky} = \sum_{j=1}^{m} P_j (y[(T^{-1}(u_j)] - \mu_y)^k \tag{14}$$

Here $T^{-1}(u_j)$ is the inverse Rosenblatt transformation. Note that the general expression for the function $G[T^{-1}(\mathbf{U})]$ in (13) and (14) is not necessary, and that the inverse Rosenblatt transformation is only required at the estimating points.

Estimating Points and Corresponding Weights in Standard Normal Space

Substituting the characteristics of the normal random variables into (3), one obtains

$$\int u^k \exp\left(-\frac{1}{2}u^2\right) du = \sqrt{2\pi} \sum_{j=1}^{m} P_j u_j^k \tag{15}$$

The left-hand side of (15) is a Hermite integration with weight function $\exp(-u^2/2)$. The estimating points u_i and their corresponding weights P can readily be obtained as

$$u_i = \sqrt{2}x_i; \quad P_i = \frac{w_i}{\sqrt{\pi}} \tag{16a,b}$$

where x_i and w_i are the abscissas and weights for Hermite integration with weight function $\exp(-x^2)$ (Abramowitz and Stegun 1972).

For a five point estimate in standard normal space,

$$u_0 = 0; \quad P_0 = 8/15 \tag{17a,b}$$

$$u_{1+} = -u_{1-} = 1.3556262; \quad P_1 = 0.2220759 \tag{17c,d}$$

$$u_{2+} = -u_{2-} = 2.8569700; \quad P_2 = 1.12574 \times 10^{-2} \tag{17e,f}$$

For a seven point estimate in standard normal space,

$$u_0 = 0; \quad P_0 = 16/35 \tag{18a,b}$$

$$u_{1+} = -u_{1-} = 1.1544054; \quad P_1 = 0.2401233 \tag{18c,d}$$

$$u_{2+} = -u_{2-} = 2.3667594; \quad P_2 = 3.07571 \times 10^{-2} \tag{18e,f}$$

$$u_{3+} = -u_{3-} = 3.7504397; \quad P_3 = 5.48269 \times 10^{-4} \tag{18g,h}$$

Point Estimates for Function of *N* Variables

The procedure described above can be generalized to a function of many variables $Z = G(\mathbf{X})$, where $\mathbf{X} = x_1, x_2, \ldots, x_n$. The joint probability density is assumed to be concentrated at points in the k^n hyperquadrants of the space defined by the n random variables, where k is the number of estimating points used in the point estimates for functions of single random variables. The computation becomes excessive when n is large.

When the random variables are mutually independent, Rosenblueth (1975) approximates $G(\mathbf{X})$ by the following function:

$$Z = G'(\mathbf{X}) = G_\mu \prod_{i=1}^{n} \left(\frac{Z_i}{G_\mu}\right) \tag{19}$$

where G_μ is the function evaluated at the variable means; and Z_i are the functions computed as though x_i were the only random variable, with the other variables set equal to their mean values.

Because the probability moments of Z_i are estimated using two points, only $2n + 1$ points are required in the approximation. Gorman (1980) improved the approximation by evaluating the probability moments of Z_i using his three point estimate, in which again only $2n + 1$ points are required because the variable mean point is common to all of the n variables. It will be pointed out that there are significant errors in these approximations.

In the present paper the function $G(\mathbf{X})$ is approximated by the following function:

$$G'(\mathbf{X}) = \sum_{i=1}^{n} (G_i - G_\mu) + G_\mu \tag{20}$$

where

$$G_i = G[T^{-1}(\mathbf{U}_i)] \tag{21}$$

$\mathbf{U}_i$ means u_i is the only random variable, with the other variables set equal to mean values transformed into standard normal space; G_i is a function of only u_i.

Because $\mathbf{U} = T(\mathbf{X})$ are mutually independent, and so G_i are also independent from each other, one can obtain the following equations

$$\mu_G = \sum_{i=1}^{n} (\mu_i - G_\mu) + G_\mu \tag{22}$$

$$\sigma_G^2 = \sum_{i=1}^{n} \sigma_i^2 \tag{23}$$

$$\alpha_{3G}\sigma_G^3 = \sum_{i=1}^{n} \alpha_{3i}\sigma_i^3 \tag{24}$$

$$\alpha_{4G}\sigma_G^4 = \sum_{i=1}^{n} \alpha_{4i}\sigma_i^4 + 6 \sum_{i=1}^{n-1} \sum_{j>1}^{n} \sigma_i^2\sigma_j^2 \tag{25}$$

where μ_i, α_i, α_{3i}, α_{4i} are the probability moments of G_i, which can be obtained by using the point estimate described in the previous section. Only $5n$ and $7n$ points are required when using five point and seven point estimates, respectively, for the probability moments of G_i. Note that since $\mathbf{U} = T(\mathbf{X})$ are mutually independent, the assumption that $\mathbf{X}$ should be mutually independent is not required in (22)–(25).

NUMERICAL EXAMPLES AND INVESTIGATIONS

First Example

The first example considers the following function of a standard exponential random variable:

$$y = -\ln(x) \tag{26}$$

where $x > 0$; $\mu_x = 1$; $\sigma_x = 1$; $\alpha_{3x} = 2$; $\alpha_{4x} = 9$; and y is a random variable with standard extreme value distribution with $\mu_y = 0.577216$, $\sigma_y = 1.28255$, $\alpha_{3y} = 1.139548$, and $\alpha_{4y} = 5.4$.

Substituting the moments of x into (4)–(7), one finds that x_- gives a value outside of the region in which x is defined. This means that Rosenblueth's two point and Gorman's three point estimates can not be applied in such an example. Using the point estimates in standard normal space proposed in the present paper, the first four moments of y can easily be evaluated without the problem described above. The results obtained by using the seven point estimate are $\mu_y = 0.577214$, $\sigma_y = 1.282547$, $\alpha_{3y} = 1.139665$, and $\alpha_{4y} = 5.399765$. They give good approximations of the exact results in this example.

Second Example

The second example considers the following function (Rosenblueth 1975):

$$y = x^k \tag{27}$$

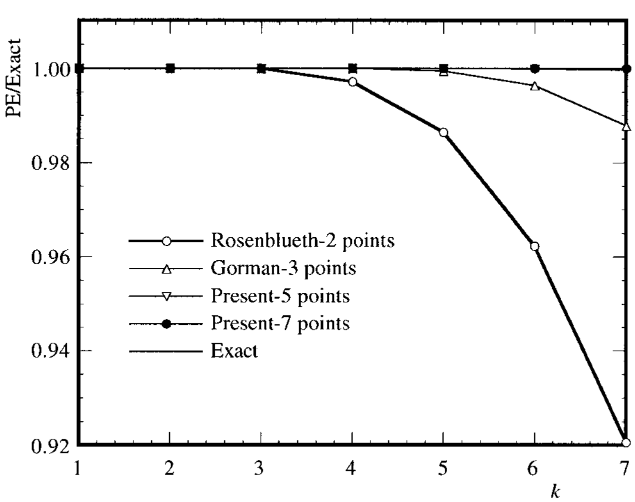

FIG. 1. Point Estimate Results for Mean Values in Example 2

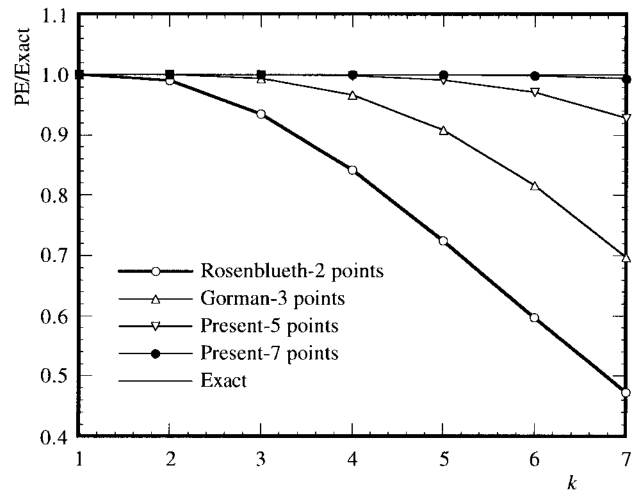

FIG. 2. Point Estimate Results for Standard Deviation in Example 2

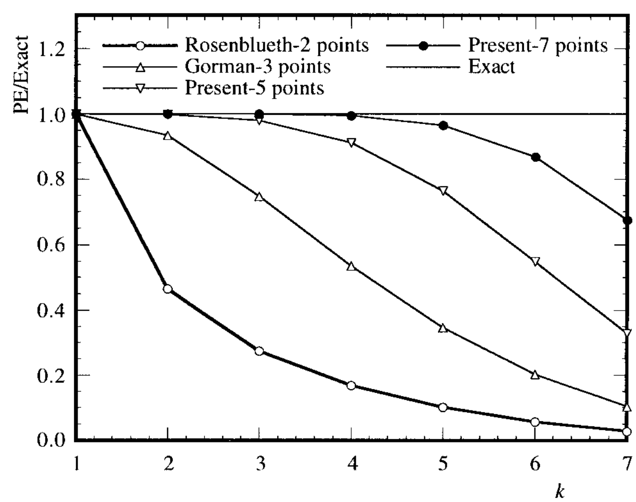

FIG. 3. Point Estimate Results for Skewness in Example 2

where x is a lognormal random variable with mean value $\mu = 1$ and coefficient of variation $V = 0.2$. The exact first four moments are given by Gorman (1980).

Plots of the ratio of the approximate mean and coefficient of variation to the exact values for values of k from 1 to 7 are shown in Figs. 1 and 2, respectively, for Rosenblueth's two point, Gorman's three point, and the present five and seven point estimates. Figs. 1 and 2 show that the present five and seven point estimates give great improvement over Rosenblueth's two point and Gorman's three point estimates, and the seven point estimate agrees with the exact ones very well.

Plots of the ratio of the approximate skewness to the exact values for values of k from 1 to 7 are shown in Fig. 3 for Rosenblueth's two point, Gorman's three point, and the five and seven point estimates. From Fig. 3, it can be seen that the present five and seven point estimates give much improvement over Rosenblueth's two point and Gorman's three point estimates, but still give significant error when k is large. In this case, more estimating points are needed to obtain more accurate results.

Third Example

The third example considers the following function of multirandom variables:

$$y = x_1^2 x_2^2 + 2x_3^4 \tag{28}$$

TABLE 1. Comparison between Results of Point Estimates and Exact Results for Example 3

Moment (1)	V (2)	Gorman (3)	Present		Exact (6)
			5 points (4)	7 points (5)	
μ_y	0.1	3.1439	3.1430	3.1430	3.1431
	0.2	3.6254	3.6106	3.6106	3.6122
	0.3	4.6194	4.5338	4.5342	4.5423
	0.4	6.5159	6.1863	6.1927	6.2184
σ_y	0.1	0.9392	0.9277	0.9277	0.9296
	0.2	2.5017	2.4368	2.4409	2.4539
	0.3	5.8079	5.7113	5.8567	5.9008
	0.4	13.580	13.326	14.997	15.349
α_{3y}	0.1	1.1547	1.1428	1.1453	1.1537
	0.2	2.1811	3.0120	3.2858	3.2762
	0.3	3.2885	5.5551	8.6631	9.7069
	0.4	5.0529	7.5452	18.825	38.779
α_{4y}	0.1	4.1434	5.5440	5.6479	5.6397
	0.2	7.6257	18.147	27.284	28.876
	0.3	15.563	43.157	161.12	385.28
	0.4	39.643	65.915	561.31	15610

where x_1, x_2, and x_3 are independent lognormal random variables with $\mu_{x1} = \mu_{x2} = \mu_{x3} = 1$ and $V_{x1} = V_{x2} = V_{x3} = V$.

The approximate mean and standard deviation for values of V from 0.1 to 0.4 are listed in Table 1, for Gorman's approximation and the present approximation with five and seven point estimates for a single variable. Table 1 shows that the results obtained by all the approximations are close to the exact results for the mean value μ_y. For the standard deviation σ_y, there are significant errors in Gorman's approximation, while the results obtained by using the seven point estimate are in good agreement with the exact results. For the skewness α_{3y} and kurtosis α_{4y}, results obtained by Gorman's approximation are seen to be far from the exact results, while those obtained by using the present approximation are relatively close to the exact results. The values of α_{3y} and α_{4y} obtained by using the seven point estimate agree with the exact results quite well for V equal to 0.1 and 0.2. For V = 0.3 and 0.4, more estimating points are needed to obtain a good accuracy.

CONCLUSIONS

1. New point estimates in standard normal space were proposed for evaluating probability moments of a function of random variables. With these new point estimates, increasing the number of estimating points is easier because the estimating points are independent of the random variable in the original space and the use of high-order moments of the random variables is not required.
2. By using the present five and seven point estimates, the accuracy of the approximation can be greatly improved. However, for approximation of high-order moments, more estimating points will be required in order to improve accuracy.
3. An approximation function for n random variables was given, in which $5n$ points and $7n$ points were used. They generally give more accurate approximations than the current methods.

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APPENDIX II. NOTATION

The following symbols are used in this paper:

$f(\mathbf{X})$ = joint probability density function of $\mathbf{X}$;
$G(\mathbf{X})$ = performance function of structures;
G_i = functions computed as though u_i were the only random variable, with other variables set equal to mean values transformed into standard normal space;
G_μ = function G evaluated at variable means;
M_k = kth central moment;
n = number of random variables;
P_0 = weight corresponding to estimating point u_0;
P_i = weight corresponding to estimating point u_i and u_{i+};
$\mathbf{U}$ = random variables in standard normal space;
u_0 = central estimating point in standard normal space;
u_{i-} = ith estimating point little than u_0;
u_{i+} = ith estimating point larger than u_0;
$\mathbf{X}$ = random variables in original space;
α_3 = 3rd dimensionless central moment, known as skewness;
α_4 = 4th dimensionless central moment, known as kurtosis;
μ_g = mean value;
σ_g = standard deviation; and
$\phi(x)$ = standard normal density distribution with argument x.