NEW PROPERTIES OF MULTIPLE HARMONIC SUMS MODULO p AND p-ANALOGUES OF LESHCHINER'S SERIES

KH. HESSAMI PILEHROOD, T. HESSAMI PILEHROOD, AND R. TAURASO

ABSTRACT. In this paper we present some new binomial identities for multiple harmonic sums whose indices are the sequences $(\{1\}^a, c, \{1\}^b)$, $(\{2\}^a, c, \{2\}^b)$ and prove a number of congruences for these sums modulo a prime p. The congruences obtained allow us to find nice p-analogues of Leshchiner's series for zeta values and to refine a result due to M. Hoffman and J. Zhao about the set of generators of the multiple harmonic sums of weight 7 and 9 modulo p. As a further application we provide a new proof of Zagier's formula for $\zeta^*(\{2\}^a, 3, \{2\}^b)$ based on a finite identity for partial sums of the zeta-star series.

1. INTRODUCTION

In the last few years there has been a growing attention to the study of p-adic analogues of various binomial series related to multiple zeta values, which are nested generalizations of the classical Riemann zeta function $\zeta(s) = \sum_{n=1}^s 1/n^s$. The main reason of interest for such p-analogues is that they are related to divisibility properties of multiple harmonic sums which can be considered as elementary "bricks" for expressing complicated congruences. Before discussing this further we recall the precise definitions of such objects.

For $r \in \mathbb{N}$, $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in (\mathbb{Z}^*)^r$, and a non-negative integer n, the *alternating multiple harmonic sum* is defined by

$$H_n(s_1, s_2, \ldots, s_r) = \sum_{1 \le k_1 < k_2 < \cdots < k_r \le n} \prod_{i=1}^r \frac{\operatorname{sgn}(s_i)^{k_i}}{k_i^{|s_i|}}$$

and the *"odd" alternating multiple harmonic sum* is given by

$$\overline{H}_n(s_1, s_2, \ldots, s_r) = \sum_{0 \le k_1 < k_2 < \cdots < k_r < n} \prod_{i=1}^r \frac{\operatorname{sgn}(s_i)^{k_i}}{(2k_i+1)^{|s_i|}}.$$

If all $s_1, \ldots, s_r$ are positive, then $H_n(s_1, \ldots, s_r)$ and $\overline{H}_n(s_1, s_2, \ldots, s_r)$ are called the *multiple harmonic sum* (MHS) and *"odd" multiple harmonic sum,* respectively. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, it is also convenient to consider the *multiple star harmonic sum* (or non-strict MHS)

$$S_n(s_1, \ldots, s_r) = \sum_{1 \le k_1 \le \cdots \le k_r \le n} \frac{1}{k_1^{s_1} \cdots k_r^{s_r}}.$$

The integers $l(\mathbf{s}) := r$ and $w := |\mathbf{s}| := \sum_{i=1}^{r} |s_i|$ are called the length (or depth) and the weight of a multiple harmonic sum. By convention, we put $H_n(\mathbf{s}) = \overline{H}_n(\mathbf{s}) = 0$ if $n < r$, and $H_n(\emptyset) = \overline{H}_n(\emptyset) = S_n(\emptyset) = 1$. By $\{s_1, s_2, \ldots, s_j\}^m$ we denote the sequence of length mj with m repetitions of $(s_1, s_2, \ldots, s_j)$.

MHSs occur naturally in different areas of mathematics such as combinatorics, number theory, algebraic geometry, quantum groups and knot theory. For a long time, MHSs have been of interest to physicists [2, 30], who realized these sums as Mellin transforms of some functions occurring in quantum field theories. The limit cases of MHSs give rise to multiple zeta values (MZVs):

$$\zeta(s_1, s_2, \ldots, s_r) = \lim_{n \to \infty} H_n(s_1, s_2, \ldots, s_r),$$

$$\zeta^*(s_1, s_2, \ldots, s_r) = \lim_{n \to \infty} S_n(s_1, s_2, \ldots, s_r)$$

defined for $s_1, \ldots, s_{r-1} \geq 1$ and $s_r \geq 2$ to ensure convergence of the series.

The earliest results on MZVs are due to Euler who elaborated a method to reduce double sums of small weight to certain rational linear combinations of products of single sums. In particular, he proved the simple but non-trivial relation $\zeta(1, 2) = \zeta(3)$ and determined the explicit values of the zeta function at even integers:

$$\zeta(2m) = \frac{(-1)^m B_{2m}}{2(2m)!} (2\pi)^{2m},$$

where $B_k \in \mathbb{Q}$ are the Bernoulli numbers defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

It is easy to verify that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_{2m+1} = 0$ for $m \geq 1$. The denominators of the Bernoulli numbers B_{2m} are completely characterized by the Clausen–von Staudt Theorem [20, p. 233], which states that

$$\text{for} \quad m \in \mathbb{N}, \quad B_{2m} + \sum_{p-1|2m} \frac{1}{p} \quad \text{is an integer.}$$

Early results for the values modulo p of the multiple harmonic sums, when p is a prime greater than 3, go back to the classical Wolstenholme's Theorem [32]:

$$H_{p-1}(1) \equiv 0 \pmod{p^2}, \qquad H_{p-1}(2) \equiv 0 \pmod{p}.$$

Glaisher [8] in 1900, and Lehmer [21] in 1938, proved that even the multiple harmonic sums $H_{p-1}(m)$ modulo a higher power of a prime $p \geq m + 3$ are related to the Bernoulli numbers:

$$H_{p-1}(m) \equiv \begin{cases} \frac{m(m+1)}{2(m+2)} p^2 B_{p-m-2} \pmod{p^3} & \text{if} \quad m \quad \text{is odd,} \\ \frac{m}{m+1} p B_{p-m-1} \pmod{p^2} & \text{if} \quad m \quad \text{is even.} \end{cases}$$

The systematic study of MZVs began in the early 1990s with the works of Hoffman [16] and Zagier [33]. The set of the MZVs has a rich algebraic structure given by the shuffle and the stuffle (harmonic shuffle or quasi-shuffle) relations. These follow from the representation of multiple zeta values in terms of iterated integrals and harmonic sums, respectively. There are many conjectures concerning multiple zeta values, and despite some recent progress, lots of open questions still remain to be answered.

Let $\mathfrak{Z}_w$ denote the $\mathbb{Q}$-vector space spanned by the set of multiple zeta values $\zeta(s_1, \ldots, s_r)$ with $s_r \geq 2$ and the total weight $w = s_1 + \cdots + s_r$, and let $\mathfrak{Z}$ denote the $\mathbb{Q}$-vector space spanned by all multiple zeta values over $\mathbb{Q}$. A conjecture of Zagier [33] states that the dimension of the $\mathbb{Q}$-vector space $\mathfrak{Z}_w$ is given by the Perrin numbers d_w defined for $w \geq 3$ by the recurrence

$$d_w = d_{w-2} + d_{w-3}$$

with the initial conditions $d_0 = 1$, $d_1 = 0$, $d_2 = 1$. The upper bound $\dim \mathfrak{Z}_w \leq d_w$ was proved independently by Goncharov [9] and Terasoma [28].

It is easy to see that the Perrin number d_w is equal to the number of multiple zeta values $\zeta(s_1, \ldots, s_r)$ with $s_1 + \cdots + s_r = w$ and each $s_j \in \{2, 3\}$. While investigating the deep algebraic structure of $\mathfrak{Z}$, Hoffman [17] conjectured that the MZVs $\zeta(s_1, \ldots, s_r)$ of weight w with $s_j \in \{2, 3\}$ span the $\mathbb{Q}$-space $\mathfrak{Z}_w$. Very recently, this conjecture was proved using motivic ideas by Brown [4]. So the main problem which remains open is proving that the numbers $\zeta(s_1, \ldots, s_r)$ with $s_j \in \{2, 3\}$ are linearly independent over $\mathbb{Q}$.

According to Zagier's conjecture, a basis for $\mathfrak{Z}_w$ for $2 \leq w \leq 9$ should be given as follows (see [31]):

$$
\begin{aligned}
w = 2, \quad & d_2 = 1, \quad \zeta(2); \\
w = 3, \quad & d_3 = 1, \quad \zeta(3); \\
w = 4, \quad & d_4 = 1, \quad \zeta(2,2); \\
w = 5, \quad & d_5 = 2, \quad \zeta(2,3), \zeta(3,2); \\
w = 6, \quad & d_6 = 2, \quad \zeta(2,2,2), \zeta(3,3); \\
w = 7, \quad & d_7 = 3, \quad \zeta(2,2,3), \zeta(2,3,2), \zeta(3,2,2); \\
w = 8, \quad & d_8 = 4, \quad \zeta(2,2,2,2), \zeta(2,3,3), \zeta(3,2,3), \zeta(3,3,2); \\
w = 9, \quad & d_9 = 5, \quad \zeta(2,2,2,3), \zeta(2,2,3,2), \zeta(2,3,2,2), \zeta(3,2,2,2), \zeta(3,3,3).
\end{aligned}
$$

Brown [4] proved that part of this conjectural basis, namely the multiple zeta values of the form $\zeta(\{2\}^a, 3, \{2\}^b)$, can be expressed in terms of ordinary zeta values. The exact formulae were found and proved by Zagier [34]:

$$\zeta(\{2\}^a, 3, \{2\}^b) = 2 \sum_{r=1}^{K} (-1)^r \left(\binom{2r}{2a+2} - (1 - 2^{-2r}) \binom{2r}{2b+1} \right) \zeta(2r+1) H(K-r)$$

and
(1.1)

$$\zeta^*(\{2\}^a, 3, \{2\}^b) = -2 \sum_{r=1}^{K} \left(\binom{2r}{2a} - \delta_{r,a} - (1 - 2^{-2r}) \binom{2r}{2b+1} \right) \zeta(2r+1) H^*(K-r),$$

where $K = a + b + 1$ and

$$H(n) = \zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}, \quad H^*(n) = \zeta^*(\{2\}^n) = 2(1 - 2^{1-2n})\zeta(2n), \ n \geq 0.$$

The theory of multiple harmonic sums modulo p bears many similarities with the theory of multiple zeta values. In the early 2000s Zhao [35,37] began to generalize the Wolstenholme Theorem and Glaisher–Lehmer congruences to other multiple harmonic sums with special emphasis on the cases where the sums are divisible by a high power of a prime. Later, Hoffman [18] showed that the algebraic setup developed in order to study MZVs can be extended to deal with MHSs as well.

In [19], Hoffman described the possible relations modulo p between MHSs of given weight not exceeding 9.

For a positive integer w, let c_w denote the minimal number of harmonic sums of weight w which are needed to generate all MHSs of weight w modulo p for $p > w+1$. For the first few values of c_w, Hoffman [19] obtained the following table:

w	1	2	3	4	5	6	7	8	9
c_w	0	0	1	0	1	1	2	2	2

The value $c_9 = 2$ was calculated conditionally under the assumption of the congruence

$$(1.2) \qquad S_{p-1}(1,1,1,6) \equiv \frac{1}{54} B_{p-3}^3 + \frac{1889}{648} B_{p-9} \pmod{p},$$

which was first conjectured and verified for all primes $10 < p < 2000$ by Zhao [35, Prop. 3.2].

The purpose of the present paper is to study multiple harmonic sums of the form

$$S_n(\{2\}^a, c, \{2\}^b), \quad S_n(\{1\}^a, c, \{1\}^b), \qquad c = 1, 2, 3, \dots.$$

We prove some new binomial identities for these sums and apply them to obtain congruences modulo a prime p for

$$(1.3) \qquad S_{p-1}(\{2\}^a, 1, \{2\}^b), \quad S_{p-1}(\{2\}^a, 3, \{2\}^b), \quad S_{p-1}(\{1\}^a, 2, \{1\}^b).$$

This allows us to give a new proof of Zagier's formula (1.1) for $\zeta^*(\{2\}^a, 3, \{2\}^b)$ based on a finite identity for $S_n(\{2\}^a, 3, \{2\}^b)$ and to formulate its finite p-analogue. In addition, those congruences of the finite sums (1.3) sharpen Hoffman's and Zhao's results on MHSs of small weights. Indeed, we show that in weight 7 the number of generators c_7 equals 1, and in weight 9 we prove congruence (1.2), which implies the equality $c_9 = 2$ unconditionally. From [19, Section 10] and Corollaries 4.3 and 4.4 below, the set of generators modulo p for multiple harmonic sums $S_{p-1}(\mathbf{s})$ of weight $w = |\mathbf{s}| \le 9$ when $p > w + 1$ in terms of Bernoulli numbers is as follows:

$w = 1, \quad c_1 = 0, \quad 0;$
$w = 2, \quad c_2 = 0, \quad 0;$
$w = 3, \quad c_3 = 1, \quad S_{p-1}(1,2) \equiv B_{p-3};$
$w = 4, \quad c_4 = 0, \quad 0;$
$w = 5, \quad c_5 = 1, \quad S_{p-1}(1,4) \equiv B_{p-5};$
$w = 6, \quad c_6 = 1, \quad S_{p-1}(1,1,4) \equiv -\frac{1}{6} B_{p-3}^2;$
$w = 7, \quad c_7 = 1, \quad S_{p-1}(1,6) \equiv B_{p-7};$
$w = 8, \quad c_8 = 2, \quad S_{p-1}(1,1,6), \ S_{p-1}(1,4)S_{p-1}(1,2) \equiv B_{p-5}B_{p-3};$
$w = 9, \quad c_9 = 2, \quad S_{p-1}(1,8) \equiv B_{p-9}, \ S_{p-1}(1,2)S_{p-1}(1,1,4) \equiv -\frac{1}{6} B_{p-3}^3.$

Hoffman [19] conjectured that all multiple harmonic sums $S_{p-1}(\mathbf{s})$ (or $H_{p-1}(\mathbf{s})$) can be written modulo p as sums of products of "height one sums" $S_{p-1}(1, \dots, 1, 2h)$ (or $H_{p-1}(1, \dots, 1, 2h)$) with rational coefficients. Note that this conjecture has been confirmed only for sums of weight w with $w \le 9$. Zhao [36] continued computations of sets of generators for $w = 10, 11, 12$. Using reduction, stuffle and duality relations he found that the MHSs of weight 10 can be generated by the set

$$B_{p-3}B_{p-7} \equiv S_{p-1}(1,2)S_{p-1}(1,6), \quad B_{p-5}^2 \equiv S_{p-1}^2(1,4),$$

$$S_{p-1}(1,1,8), \ S_{p-1}(1,1,1,6), \ S_{p-1}(2,2,1,4,1),$$

which according to Hoffman's conjecture still contains the extra term $S_{p-1}(2,2,1,4,1)$. Zhao [38] kindly communicated to us that by using our Theorems 4.1 and 4.2 (see below) it is possible to reduce the set of 8 generators for MHSs of weight 11 found in [36] to the set of 5 elements:

$$B_{p-11} \equiv S_{p-1}(1,10),\ B_{p-5}B_{p-3}^2,\ B_{p-3}S_{p-1}(1,1,6),\ S_{p-1}(1,1,1,8),\ S_{p-1}(5,3,2,1).$$

In contrast to the corresponding conjecture for MZVs, not all "height one sums" are claimed to be linearly independent modulo p, because they satisfy the duality relation [19, Theorem 5.2]:

$$S_{p-1}(\{1\}^{k-1}, h) \equiv (-1)^{k+h} S_{p-1}(\{1\}^{h-1}, k) \pmod{p}, \quad p > \max(h, k).$$

In [36, Remark 2.3], Zhao conjectured that weight-8 generators $S_{p-1}(1,1,6)$ and $B_{p-3}B_{p-5}$ should be linearly independent modulo p over $\mathbb{Q}(10)$, where

$$\mathbb{Q}(w) := \{a/b \in \mathbb{Q} : a/b \text{ is reduced and if a prime } q|b, \text{ then } q \leq w\}.$$

Zhao [36] also formulated a general conjecture on linear independence modulo p of products of Bernoulli numbers

$$B_{p-i_1} B_{p-i_2} \cdots B_{p-i_r}$$

over $\mathbb{Q}(w+2)$, where $i_1, \ldots, i_r$ are odd indices greater than 1 with $i_1 + \cdots + i_r = w$, which, in particular, implies the linear independence of weight-9 generators B_{p-3}^3 and B_{p-9} modulo p over $\mathbb{Q}(11)$.

As an application of our results to ordinary zeta values, we consider generalizations of Apéry's famous series,

$$(1.4) \qquad \zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}, \qquad \zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}},$$

used in his irrationality proof of $\zeta(2)$ and $\zeta(3)$. Among different extensions [1, 3, 11, 13, 22] of series (1.4), we focus on those obtained by Leshchiner [22] for values of the Riemann zeta function $\zeta(s)$ and Dirichlet beta function $\beta(s) = \sum_{n=0}^{\infty} (-1)^n / (2n+1)^s$. Note that the value $\beta(2m+1)$ as well as $\zeta(2m+2)$ can be expressed in terms of π,

$$\beta(2m+1) = \frac{(-1)^m E_{2m}}{2^{2m+2}(2m)!} \pi^{2m+1},$$

where m is a non-negative integer and $E_{2m} \in \mathbb{Q}$ are Euler numbers.

Leshchiner established the following four expansions in terms of multiple harmonic sums that generalize series (1.4):

$$(1.5) \qquad \left(1 - \frac{1}{2^{2m+1}}\right)\zeta(2m+2) = \frac{3}{2} \sum_{k=1}^{\infty} \frac{(-1)^m H_{k-1}(\{2\}^m)}{k^2 \binom{2k}{k}} + 2 \sum_{j=1}^{m} \sum_{k=1}^{\infty} \frac{(-1)^{m-j} H_{k-1}(\{2\}^{m-j})}{k^{2j+2} \binom{2k}{k}},$$

$$(1.6) \qquad (-1)^{m-1} \cdot \zeta(2m+3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^k H_{k-1}(\{2\}^m)}{k^3 \binom{2k}{k}} + 2 \sum_{j=1}^{m} \sum_{k=1}^{\infty} \frac{(-1)^{k-j} H_{k-1}(\{2\}^{m-j})}{k^{2j+3} \binom{2k}{k}},$$

$$(1.7) \quad \left(1 - \frac{1}{2^{2m+2}}\right)\zeta(2m+2) = \frac{5}{4}\sum_{k=0}^{\infty}\frac{(-1)^{k+m}\binom{2k}{k}\overline{H}_k(\{2\}^m)}{16^k(2k+1)^2}$$

$$+ \sum_{j=1}^{m}\sum_{k=0}^{\infty}\frac{(-1)^{k+m-j}\binom{2k}{k}\overline{H}_k(\{2\}^{m-j})}{16^k(2k+1)^{2j+2}},$$

$$(1.8)\quad \beta(2m+1) = \frac{3}{4}\sum_{k=0}^{\infty}\frac{(-1)^m\binom{2k}{k}\overline{H}_k(\{2\}^m)}{16^k(2k+1)} + \sum_{j=1}^{m}\sum_{k=0}^{\infty}\frac{(-1)^{m-j}\binom{2k}{k}\overline{H}_k(\{2\}^{m-j})}{16^k(2k+1)^{2j+1}}.$$

Indeed, if we put $m = 0$ in (1.5), (1.6), we get Apéry's series (1.4).

Recently, the authors [14, 15, 26], showed that the series (1.5), (1.6) for $m = 0, 1$ and the series (1.7), (1.8) for $m = 0$ admit very nice p-analogues. Indeed, if we truncate the series (1.5), (1.6) with $m = 0$ or 1 up to $p - 1$ and the series (1.7), (1.8) with $m = 0$ up to $(p - 3)/2$, where p is an odd prime, we get congruences for the finite sums modulo powers of p expressible in terms of Bernoulli numbers. In this paper, we extend these results to any non-negative integer m. This is done by showing that the finite p-analogues of the series (1.5)–(1.8) are related with MHSs of the form

$$H_{p-1}(\{2\}^m, 3), \quad H_{p-1}(\{2\}^m, 1), \quad \overline{H}_{\frac{p-1}{2}}(\{2\}^m, 3), \quad \overline{H}_{\frac{p-1}{2}}(\{2\}^m, 1).$$

2. Identities for multiple harmonic sums

We start with a list of binomial identities that we will need later. Note that a generalization of (2.5) appears in [10, (4.20)].

Lemma 2.1. *For any positive integers m, n and a non-negative integer l we have*

$$(2.1)\qquad \sum_{k=l+1}^{n}(-1)^{k-1}\binom{mn}{n-k} = (-1)^l\binom{mn-1}{n-l-1},$$

$$(2.2)\qquad 2\sum_{k=l+1}^{n}\frac{k\binom{n}{k}}{\binom{n+k}{k}} = \frac{n\binom{n-1}{l}}{\binom{n+l}{l}},$$

$$(2.3)\qquad 2\sum_{k=1}^{n}\frac{\binom{n}{k}}{k\binom{n+k}{k}} = \sum_{k=1}^{n}\frac{1}{k}.$$

If $l \geq 1$, then

$$(2.4)\qquad \sum_{k=l}^{n}\frac{\binom{k}{l}}{k^2\binom{k+l}{l}} = \frac{\binom{n}{l}}{l^2\binom{n+l}{l}}.$$

If $n \geq 2$, then

$$(2.5)\qquad \sum_{k=1}^{n}\frac{(-1)^k k^2\binom{n}{k}}{\binom{n+k}{k}} = 0.$$

Proof. For the proof of our identities it is useful to consider the usual binomial coefficient $\binom{r}{k}$ in a more general setting, that is, to allow an arbitrary integer to appear in the lower index of $\binom{r}{k}$. For this purpose we set

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots 1} & \text{if integer} \quad k \geq 0, \\ 0 & \text{if integer} \quad k < 0. \end{cases}$$

To prove the first identity we first observe that

$$(2.6) \qquad (-1)^{k-1}\binom{mn}{n-k} = G(n,k+1) - G(n,k)$$

for positive integers n, k with

$$G(n,k) = (-1)^k\binom{mn-1}{n-k}.$$

Then summing both sides of equation (2.6) over k from $l+1$ to n we obtain

$$\sum_{k=l+1}^{n}(-1)^{k-1}\binom{mn}{n-k} = G(n,n+1) - G(n,l+1) = -G(n,l+1) = (-1)^l\binom{mn-1}{n-l-1}.$$

Similarly, for the proof of the second identity we have

$$(2.7) \qquad \frac{2k\binom{n}{k}}{\binom{n+k}{k}} = G(n,k+1) - G(n,k)$$

for positive integers n, k with

$$G(n,k) = -\frac{(n+k)\binom{n}{k}}{\binom{n+k}{k}}.$$

Summing both sides of equation (2.7) over k from $l+1$ to n we easily obtain the result.

To prove identity (2.3) we set

$$f(m,k) := \frac{\binom{m}{k}}{k\binom{m+k}{k}} \qquad \text{and} \qquad G(m,k) := -\frac{\binom{m+1}{k}}{2(m+1)\binom{m+k}{k}}, \quad m \geq 0,\ k \geq 1.$$

Then it is easy to see that

$$(2.8) \qquad f(m+1,k) - f(m,k) = G(m,k+1) - G(m,k).$$

Summing both sides of equation (2.8) over m from 0 to $n-1$ we have

$$f(n,k) = \sum_{m=0}^{n-1}(G(m,k+1) - G(m,k)).$$

Now summing the above equation once again over k from 1 to n we obtain

$$\sum_{k=1}^{n}f(n,k) = \sum_{m=0}^{n-1}\sum_{k=1}^{n}(G(m,k+1) - G(m,k)) = \sum_{m=0}^{n-1}(G(m,n+1) - G(m,1))$$

$$= -\sum_{m=0}^{n-1}G(m,1) = \frac{1}{2}\sum_{m=0}^{n-1}\frac{1}{m+1},$$

as required.

For proving identity (2.4), it is easy to see that

$$(2.9) \qquad \frac{l^2\binom{k}{l}}{k^2\binom{k+l}{l}} = G(k+1,l) - G(k,l)$$

for positive integers k, l with

$$G(k,l) = \frac{(k^2-l^2)\binom{k}{l}}{k^2\binom{k+l}{l}}.$$

Then summing both sides of equation (2.9) over k from l to n we get

$$l^2 \sum_{k=l}^{n} \frac{\binom{k}{l}}{k^2 \binom{k+l}{l}} = G(n+1,l) - G(l,l) = G(n+1,l) = \frac{\binom{n}{l}}{\binom{n+l}{l}}.$$

To prove (2.5) we note that for integers $n \geq 2$, $k \geq 0$,

(2.10)
$$\frac{(-1)^k k^2 \binom{n}{k}}{\binom{n+k}{k}} = G(n, k+1) - G(n, k),$$

with

$$G(n,k) = \frac{(-1)^{k-1} k(k-1)(n+k)\binom{n}{k}}{2(n-1)\binom{n+k}{k}}.$$

Then summing both sides of equation (2.10) over k from 1 to n we obtain

$$\sum_{k=1}^{n} \frac{(-1)^k k^2 \binom{n}{k}}{\binom{n+k}{k}} = G(n, n+1) - G(n, 1) = 0,$$

and the lemma is proved.   □

Lemma 2.2. *Let k, m, n be positive integers, $A_{n,k}^{(m)} = (-1)^k \binom{mn}{n-k} c_n^{(m)}$ where $c_n^{(m)}$ is an arbitrary sequence independent of k, and a be a non-negative integer. Then for each $c \in \mathbb{N}$,*

(2.11)

$$\frac{1}{n^c} \sum_{k=1}^{n} \frac{H_{k-1}(\mathbf{b}) A_{n,k}^{(m)}}{k^a} = \sum_{k=1}^{n} \frac{H_{k-1}(\mathbf{b}) A_{n,k}^{(m)}}{k^{a+c}} + \sum_{\substack{j+|\mathbf{s}|=a+c \\ j \geq 0, s_1 > a}} m^{l(\mathbf{s})} \sum_{k=1}^{n} \frac{H_{k-1}(\mathbf{b}, \mathbf{s}) A_{n,k}^{(m)}}{k^j},$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{N}^r$ for $r \geq 1$, and $|\mathbf{s}| = \sum_{i=1}^{r} s_i$, $l(\mathbf{s}) = r$.

Proof. We first note that by (2.1),

(2.12)
$$\frac{m}{l} \sum_{k=l+1}^{n} A_{n,k}^{(m)} = \left(\frac{1}{n} - \frac{1}{l}\right) A_{n,l}^{(m)}.$$

We prove (2.11) by induction on c. By (2.12), we have that

$$m \sum_{k=1}^{n} H_{k-1}(\mathbf{b}, a+1) A_{n,k}^{(m)} = m \sum_{l=1}^{n} \frac{H_{l-1}(\mathbf{b})}{l^{a+1}} \sum_{k=l+1}^{n} A_{n,k}^{(m)}$$

$$= \sum_{l=1}^{n} \frac{H_{l-1}(\mathbf{b}) A_{n,l}^{(m)}}{l^a} \left(\frac{1}{n} - \frac{1}{l}\right)$$

$$= \frac{1}{n} \sum_{l=1}^{n} \frac{H_{l-1}(\mathbf{b}) A_{n,l}^{(m)}}{l^a} - \sum_{l=1}^{n} \frac{H_{l-1}(\mathbf{b}) A_{n,l}^{(m)}}{l^{a+1}},$$

which is (2.11) for $c = 1$. Now let $c > 1$. Then by induction, we have

$$\frac{1}{n^c}\sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b})A_{n,k}^{(m)}}{k^a} = \frac{1}{n^{c-1}}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b})A_{n,k}^{(m)}}{k^a}\right)$$

$$= \frac{1}{n^{c-1}}\sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b})A_{n,k}^{(m)}}{k^{a+1}} + \frac{m}{n^{c-1}}\sum_{k=1}^{n}H_{k-1}(\mathbf{b}, a+1)A_{n,k}^{(m)}$$

$$= \sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b})A_{n,k}^{(m)}}{k^{a+c}} + \sum_{\substack{j+|\mathbf{s}|=a+c \\ j\geq 0, s_1 > a+1}} m^{l(\mathbf{s})}\sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b}, \mathbf{s})A_{n,k}^{(m)}}{k^j}$$

$$+ m\sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b}, a+1)A_{n,k}^{(m)}}{k^{c-1}} + m\sum_{\substack{j+|\mathbf{s}|=c-1 \\ j\geq 0, s_1 > 0}} m^{l(\mathbf{s})}\sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b}, a+1, \mathbf{s})A_{n,k}^{(m)}}{k^j}$$

$$= \sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b})A_{n,k}^{(m)}}{k^{a+c}} + \sum_{\substack{j+|\mathbf{s}|=a+c \\ j\geq 0, s_1 > a+1}} m^{l(\mathbf{s})}\sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b}, \mathbf{s})A_{n,k}^{(m)}}{k^j}$$

$$+ \sum_{\substack{j+|\mathbf{s}|=a+c \\ j\geq 0, s_1 = a+1}} m^{l(\mathbf{s})}\sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{b}, \mathbf{s})A_{n,k}^{(m)}}{k^j}.$$

$\square$

Theorem 2.3. *For a positive integer n and non-negative integers a, b, and $c \geq 2$,*

$$S_n(\{2\}^a, c, \{2\}^b) = 2\sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k^{2a+2b+c}\binom{n+k}{k}}$$

(2.13)

$$+ 4\sum_{\substack{i+j+|\mathbf{s}|=c \\ i\geq 1, j\geq 2, |\mathbf{s}|\geq 0}} 2^{l(\mathbf{s})}\sum_{k=1}^{n}\frac{H_{k-1}(2a+i, \mathbf{s})(-1)^{k-1}\binom{n}{k}}{k^{2b+j}\binom{n+k}{k}},$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{N}^r$ for $r \geq 0$, and $|\mathbf{s}| = \sum_{i=1}^{r} s_i$, $l(\mathbf{s}) = r$.

Proof. The proof is by induction on n. For $n = 1$ we have $S_1(\{2\}^a, c, \{2\}^b) = 1$, and the formula is true. For $n > 1$ we proceed as follows. If $c = 2$, then we should prove that

(2.14)
$$S_n(\{2\}^m) = 2\sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k^{2m}\binom{n+k}{k}},$$

where $m = a + b + 1$. First note that

$$S_n(\{2\}^m) = \sum_{1\leq k_1 \leq k_2 \leq \cdots \leq k_m \leq n}\frac{1}{k_1^2 k_2^2 \cdots k_m^2} = \sum_{l=0}^{m}\frac{1}{n^{2(m-l)}}S_{n-1}(\{2\}^l).$$

Then by the induction hypothesis, we have that

$$S_n(\{2\}^m) = 2\sum_{l=0}^{n} \frac{1}{n^{2(m-l)}} \sum_{k=1}^{n-1} \frac{(-1)^{k-1}\binom{n-1}{k}}{k^{2l}\binom{n+k-1}{k}} = \frac{2}{n^{2m}} \sum_{k=1}^{n-1} \frac{(-1)^{k-1}\binom{n-1}{k}}{\binom{n+k-1}{k}} \sum_{l=0}^{m} \frac{n^{2l}}{k^{2l}}$$

$$= \frac{2}{n^{2m+2}} \sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}(n^{2m+2}-k^{2m+2})}{k^{2m}\binom{n+k}{k}}$$

$$= 2\sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k^{2m}\binom{n+k}{k}} - \frac{2}{n^{2m+2}} \sum_{k=1}^{n} \frac{(-1)^{k-1}k^2\binom{n}{k}}{\binom{n+k}{k}}$$

and formula (2.14) easily follows by the equation (2.5).

To prove (2.13) for $c > 2$ we note that for $n > 1$,

$$S_n(\{2\}^a, c, \{2\}^b) = \sum_{1 \le k_1 \le \cdots \le k_a \le k_{a+1} \le k_{a+2} \le \cdots \le k_{a+b+1} \le n} \frac{1}{k_1^2 \cdots k_a^2 k_{a+1}^c k_{a+2}^2 \cdots k_{a+b+1}^2}$$

$$= \sum_{l=0}^{b} \frac{1}{n^{2(b-l)}} S_{n-1}(\{2\}^a, c, \{2\}^l) + \frac{1}{n^{2b+c}} S_n(\{2\}^a).$$

Then by the induction hypothesis and formula (2.14), we have

$$S_n(\{2\}^a, c, \{2\}^b) = 2\sum_{l=0}^{b} \frac{1}{n^{2(b-l)}} \sum_{k=1}^{n-1} \frac{(-1)^{k-1}\binom{n-1}{k}}{k^{2(a+l)+c}\binom{n+k-1}{k}}$$

$$+ 4\sum_{l=0}^{b} \frac{1}{n^{2(b-l)}} \sum_{\substack{i+j+|\mathbf{s}|=c \\ i\ge 1, j\ge 2, |\mathbf{s}|\ge 0}} 2^{l(\mathbf{s})} \sum_{k=1}^{n-1} \frac{H_{k-1}(2a+i,\mathbf{s})(-1)^{k-1}\binom{n-1}{k}}{k^{2l+j}\binom{n-1+k}{k}}$$

$$+ \frac{2}{n^{2b+c}} \sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k^{2a}\binom{n+k}{k}}.$$

Changing the order of summation and summing the inner sum

$$\frac{\binom{n-1}{k}}{\binom{n-1+k}{k}} \sum_{l=0}^{b} \frac{n^{2l}}{k^{2l}} = \frac{n^{2b+2}-k^{2b+2}}{(n^2-k^2)k^{2b}} \frac{\binom{n-1}{k}}{\binom{n-1+k}{k}} = \left(\frac{n^{2b}}{k^{2b}} - \frac{k^2}{n^2}\right) \frac{\binom{n}{k}}{\binom{n+k}{k}},$$

we obtain

$$S_n(\{2\}^a, c, \{2\}^b) = 2\sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k^{2a+2b+c}\binom{n+k}{k}}$$

$$+ 4\sum_{\substack{i+j+|\mathbf{s}|=c \\ i\ge 1, j\ge 2, |\mathbf{s}|\ge 0}} 2^{l(\mathbf{s})} \sum_{k=1}^{n} \frac{H_{k-1}(2a+i,\mathbf{s})(-1)^{k-1}\binom{n}{k}}{k^{2b+j}\binom{n+k}{k}}$$

$$+ \frac{2}{n^{2b+c}} \sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k^{2a}\binom{n+k}{k}} - \frac{2}{n^{2b+2}} \sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k^{2a+c-2}\binom{n+k}{k}}$$

$$- \frac{4}{n^{2b+2}} \sum_{\substack{i+j+|\mathbf{s}|=c \\ i\ge 1, j\ge 2, |\mathbf{s}|\ge 0}} 2^{l(\mathbf{s})} \sum_{k=1}^{n} \frac{H_{k-1}(2a+i,\mathbf{s})(-1)^{k-1}\binom{n}{k}}{k^{j-2}\binom{n+k}{k}}.$$

The final result follows as soon as we show that

$$\frac{1}{n^{c-2}}\sum_{k=1}^{n}\frac{A_{n,k}^{(2)}}{k^{2a}} = \sum_{k=1}^{n}\frac{A_{n,k}^{(2)}}{k^{2a+c-2}} + 2\sum_{\substack{i+j+|\mathbf{s}|=c\\i\geq 1,j\geq 2,|\mathbf{s}|\geq 0}}2^{l(\mathbf{s})}\sum_{k=1}^{n}\frac{H_{k-1}(2a+i,\mathbf{s})A_{n,k}^{(2)}}{k^{j-2}},$$

where $A_{n,k}^{(2)} = (-1)^{k-1}\binom{n}{k}/\binom{n+k}{k}$. This identity holds by Lemma 2.2 because

$$2\sum_{\substack{i+j+|\mathbf{s}|=c\\i\geq 1,j\geq 2,|\mathbf{s}|\geq 0}}2^{l(\mathbf{s})}\sum_{k=1}^{n}\frac{H_{k-1}(2a+i,\mathbf{s})A_{n,k}^{(2)}}{k^{j-2}} = \sum_{\substack{j+|\mathbf{s}|=2a+c-2\\j\geq 0,s_1>2a}}2^{l(\mathbf{s})}\sum_{k=1}^{n}\frac{H_{k-1}(\mathbf{s})A_{n,k}^{(2)}}{k^j}.$$

□

Corollary 2.4. *For a positive integer n and non-negative integers a,b,*

$$S_n(\{2\}^a) = 2\sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k^{2a}\binom{n+k}{k}},$$

(2.15)

$$S_n(\{2\}^a,3,\{2\}^b) = 2\sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k^{2(a+b)+3}\binom{n+k}{k}} + 4\sum_{k=1}^{n}\frac{H_{k-1}(2a+1)(-1)^{k-1}\binom{n}{k}}{k^{2b+2}\binom{n+k}{k}}.$$

Theorem 2.5. *For a positive integer n and non-negative integers a,b,c,*
(2.16)

$$S_n(\{1\}^a,c,\{1\}^b) = \sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k^{a+b+c}} + \sum_{\substack{i+j+|\mathbf{s}|=c\\i\geq 1,j\geq 1,|\mathbf{s}|\geq 0}}\sum_{k=1}^{n}\frac{H_{k-1}(a+i,\mathbf{s})(-1)^{k-1}\binom{n}{k}}{k^{b+j}}.$$

Proof. Note that the case $c=1$ is well known (see [6, Corollary 3]). It also follows from the polynomial identity

$$\sum_{1\leq k_1\leq\cdots\leq k_m\leq n}\frac{(1+x)^{k_1}-1}{k_1\cdots k_m} = \sum_{k=1}^{n}\binom{n}{k}\frac{x^k}{k^m}$$

which appeared in [27, Lemma 5.5]. For $c>1$ the identity can be proved by induction on n. By a similar argument as in the proof of Theorem 2.3, we have

$$S_n(\{1\}^a,c,\{1\}^b) = \sum_{l=0}^{b}\frac{1}{n^{b-l}}S_{n-1}(\{1\}^a,c,\{1\}^l) + \frac{1}{n^{b+c}}S_n(\{1\}^a)$$

$$= \frac{1}{n^{b+c}}\sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k^a} + \sum_{l=0}^{b}\frac{1}{n^{b-l}}\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\binom{n-1}{k}}{k^{a+c+l}}$$

$$+ \sum_{l=0}^{b}\frac{1}{n^{b-l}}\sum_{\substack{i+j+|\mathbf{s}|=c\\i\geq 1,j\geq 1,|\mathbf{s}|\geq 0}}\sum_{k=1}^{n-1}\frac{H_{k-1}(a+i,\mathbf{s})(-1)^{k-1}\binom{n-1}{k}}{k^{l+j}}.$$

Changing the order of summation and summing the inner sum

$$\binom{n-1}{k}\sum_{l=0}^{b}\frac{n^l}{k^l} = \left(\frac{n^b}{k^b}-\frac{k}{n}\right)\binom{n}{k}$$

we obtain

$$S_n(\{1\}^a, c, \{1\}^b) = \sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k^{a+b+c}} + \sum_{\substack{i+j+|\mathbf{s}|=c \\ i\geq 1, j\geq 1, |\mathbf{s}|\geq 0}} \sum_{k=1}^{n} \frac{H_{k-1}(a+i, \mathbf{s})(-1)^{k-1}\binom{n}{k}}{k^{b+j}}$$

$$+ \frac{1}{n^{b+c}} \sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k^a} - \frac{1}{n^{b+1}} \sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k^{a+c-1}}$$

$$- \frac{1}{n^{b+1}} \sum_{\substack{i+j+|\mathbf{s}|=c \\ i\geq 1, j\geq 1, |\mathbf{s}|\geq 0}} \sum_{k=1}^{n} \frac{H_{k-1}(a+i, \mathbf{s})(-1)^{k-1}\binom{n}{k}}{k^{j-1}}.$$

Now the result easily follows from Lemma 2.2 with $A_{n,k}^{(1)} = (-1)^{k-1}\binom{n}{k}$ and the equality

$$\sum_{\substack{i+j+|\mathbf{s}|=c \\ i\geq 1, j\geq 1, |\mathbf{s}|\geq 0}} \sum_{k=1}^{n} \frac{H_{k-1}(a+i, \mathbf{s})(-1)^{k-1}\binom{n}{k}}{k^{j-1}} = \sum_{\substack{j+|\mathbf{s}|=a+c-1 \\ j\geq 0, s_1 > a}} \sum_{k=1}^{n} \frac{H_{k-1}(\mathbf{s})(-1)^{k-1}\binom{n}{k}}{k^{j}}.$$

$\square$

Theorem 2.6. *Let a, b be integers satisfying $a \geq 1$, $b \geq 0$. Then for any positive integer n,*

$$(2.17) \qquad S_n(1, \{2\}^b) = 2 \sum_{k=1}^{n} \frac{\binom{n}{k}}{k^{2b+1}\binom{n+k}{k}},$$

$$(2.18) \qquad S_n(\{2\}^a, 1, \{2\}^b) = 2 \sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k^{2(a+b)+1}\binom{n+k}{k}} - 4 \sum_{k=1}^{n} \frac{H_{k-1}(-2a)\binom{n}{k}}{k^{2b+1}\binom{n+k}{k}}.$$

Proof. To prove the first identity we proceed by induction on b. For $b = 0$ its validity follows from (2.3). Assume the formula holds for $b > 0$. Then we easily obtain from (2.4),

$$S_n(1, \{2\}^{b+1}) = \sum_{k=1}^{n} \frac{S_k(1, \{2\}^b)}{k^2} = 2 \sum_{k=1}^{n} \frac{1}{k^2} \sum_{l=1}^{k} \frac{\binom{k}{l}}{l^{2m+1}\binom{k+l}{l}}$$

$$= 2 \sum_{l=1}^{n} \frac{1}{l^{2m+1}} \sum_{k=l}^{n} \frac{\binom{k}{l}}{k^2\binom{k+l}{l}} = 2 \sum_{l=1}^{n} \frac{\binom{n}{l}}{l^{2m+3}\binom{n+l}{l}},$$

as required. To prove the second identity of our theorem we proceed by induction on n. Obviously, it is valid for $n = 1$. For $n > 1$ we use the equality

$$S_n(\{2\}^a, 1, \{2\}^b) = \sum_{l=0}^{b} \frac{1}{n^{2(b-l)}} S_{n-1}(\{2\}^a, 1, \{2\}^l) + \frac{1}{n^{2b+1}} S_n(\{2\}^a)$$

and apply the same arguments as in the proof of Theorem 2.3. Then with the help of formula (2.2) we may easily deduce the result. $\square$

Remark 2.7. Note that the identities for the finite multiple harmonic sums found in Theorems 2.3 and 2.6 can be used for evaluating multiple zeta-star values. For

example, letting n tend to infinity in (2.15), (2.17), (2.18) we get expressions for $\zeta^*(\{2\}^a, 3, \{2\}^b)$, $\zeta^*(\{2\}^a, 1, \{2\}^b)$ in terms of alternating Euler sums:

$$(2.19) \qquad \zeta^*(\{2\}^a, 3, \{2\}^b) = 2\overline{\zeta}(2a+2b+3) + 4\sum_{k=1}^{\infty} \frac{(-1)^{k-1} H_{k-1}(2a+1)}{k^{2b+2}},$$

$$(2.20) \qquad \zeta^*(\{2\}^a, 1, \{2\}^b) = 2\overline{\zeta}(2a+2b+1) - 4\sum_{k=1}^{\infty} \frac{H_{k-1}(-2a)}{k^{2b+1}}, \quad a,b \geq 1,$$

$$(2.21) \qquad \zeta^*(1, \{2\}^b) = 2\zeta(2b+1), \qquad b \geq 1,$$

where

$$\overline{\zeta}(s) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} = (1 - 2^{1-s})\zeta(s),$$

with $\overline{\zeta}(1) = \log 2$, is the alternating zeta function. Formula (2.21) was proved previously by several authors (see [23, p. 292, Ex. b], [39], [29]). Evaluations for length 2 Euler sums of odd weight appearing on the right-hand sides of (2.19) and (2.20) in terms of zeta values are well known (see, for example, [7, Theorem 7.2]). Therefore by [7, Theorem 7.2], we obtain

$$\zeta^*(\{2\}^a, 3, \{2\}^b) = 4\sum_{r=1}^{K} \left(\binom{2r}{2b+1}\left(1 - \frac{1}{2^{2r}}\right) + \delta_{r,a} - \binom{2r}{2a} \right) \zeta(2r+1)\overline{\zeta}(2K-2r),$$

where $\overline{\zeta}(0) := 1$, $K = a+b+1$, which gives another proof of Zagier's formula (1.1) for the zeta-star value $\zeta^*(\{2\}^a, 3, \{2\}^b)$, and
(2.22)

$$\zeta^*(\{2\}^a, 1, \{2\}^b) = 4\sum_{r=1}^{a+b} \left(\binom{2r}{2b} - \binom{2r}{2a-1}\left(1 - \frac{1}{2^{2r}}\right) \right) \zeta(2r+1)\overline{\zeta}(2a+2b-2r)$$

for $a,b \geq 1$. Note that formula (2.22) was also proved in [25, Theorem 1.6] by another method.

3. Auxiliary results on congruences

In this section we collect several congruences that will be required later in this paper.

(i) ([35, Theorem 1.6]) for positive integers a, r and for any prime $p > ar + 2$,

$$H_{p-1}(\{a\}^r) \equiv \begin{cases} (-1)^r \frac{a(ar+1)}{2(ar+2)} p^2 B_{p-ar-2} \pmod{p^3} & \text{if } ar \text{ is odd}, \\ (-1)^{r-1} \frac{a}{ar+1} p B_{p-ar-1} \pmod{p^2} & \text{if } ar \text{ is even}; \end{cases}$$

(ii) ([35, Theorems 3.1, 3.2]) for positive integers a_1, a_2 and for any prime $p \geq a_1 + a_2$,

$$H_{p-1}(a_1, a_2) \equiv \frac{(-1)^{a_2}}{a_1+a_2}\binom{a_1+a_2}{a_1} B_{p-a_1-a_2} \pmod{p};$$

moreover, if $a_1 + a_2$ is even, then for any prime $p > a_1 + a_2 + 1$,

$$H_{p-1}(a_1, a_2) \equiv p\left[(-1)^{a_1} a_2 \binom{a_1+a_2+1}{a_1} - (-1)^{a_1} a_1 \binom{a_1+a_2+1}{a_2} - a_1 - a_2 \right]$$
$$\times \frac{B_{p-a_1-a_2-1}}{2(a_1+a_2+1)} \pmod{p^2};$$

(iii) ([35, Theorem 3.5]) if $a_1, a_2, a_3 \in \mathbb{N}$ and $w := a_1 + a_2 + a_3$ is odd, then for any prime $p > w$, we have

$$H_{p-1}(a_1, a_2, a_3) \equiv \left[(-1)^{a_1} \binom{w}{a_1} - (-1)^{a_3} \binom{w}{a_3} \right] \frac{B_{p-w}}{2w} \pmod{p};$$

(iv) ([24, Theorem 5.2]) for a positive integer a and for any prime $p \geq a + 2$, we have

$$H_{\frac{p-1}{2}}(a) \equiv \begin{cases} -2q_p(2) + pq_p^2(2) - p^2 \left(\frac{2}{3} q_p^3(2) + \frac{7}{12} B_{p-3} \right) \pmod{p^3} & \text{if } a = 1, \\ \frac{a(2^{a+1}-1)}{2(a+1)} p B_{p-a-1} \pmod{p^2} & \text{if } a \text{ is even}, \\ -\frac{2^a - 2}{a} B_{p-a} \pmod{p} & \text{if } a > 1 \text{ is odd}, \end{cases}$$

where $q_p(2) = (2^{p-1} - 1)/p$ is the so-called *Fermat quotient*;

(v) ([15, Lemma 1]) if a, b are positive integers and $a + b$ is odd, then for any prime $p > a + b$,

$$H_{\frac{p-1}{2}}(a, b) \equiv \frac{B_{p-a-b}}{2(a+b)} \left((-1)^b \binom{a+b}{a} + 2^{a+b} - 2 \right) \pmod{p};$$

(vi) by (i) and (iv), for any positive integer a and any prime $p \geq a + 2$, we have

$$H_{p-1}(-a) = 2^{1-a} H_{\frac{p-1}{2}}(a) - H_{p-1}(a)$$

$$\equiv \begin{cases} -2q_p(2) + pq_p^2(2) - p^2 \left(\frac{2}{3} q_p^3(2) + \frac{1}{4} B_{p-3} \right) \pmod{p^3} & \text{if } a = 1, \\ \frac{a(1-2^{-a})}{a+1} p B_{p-a-1} \pmod{p^2} & \text{if } a \text{ is even}, \\ -\frac{2(1-2^{1-a})}{a} B_{p-a} \pmod{p} & \text{if } a > 1 \text{ is odd}; \end{cases}$$

(vii) ([27, Theorem 3.1]) for positive integers a, b of distinct parity and a prime $p \geq a + b + 1$,

$$H_{p-1}(-a, b) \equiv H_{p-1}(a, -b) \equiv \frac{1 - 2^{1-a-b}}{a+b} B_{p-a-b} \pmod{p},$$

$$H_{p-1}(-a, -b) \equiv \frac{2^{1-a-b} - 1}{a+b} (-1)^b \binom{a+b}{b} B_{p-a-b} \pmod{p};$$

(viii) ([27, Theorem 4.1]) if $a, b, c \in \mathbb{N}$ and $w := a + b + c$ is odd, then for any prime $p > w$,

$$2H_{p-1}(a, -b, -c) \equiv H_{p-1}(c + b, a) + H_{p-1}(-c, -b - a)$$
$$- H_{p-1}(-c) H_{p-1}(-b, a) \pmod{p},$$

$$2H_{p-1}(-a, b, -c) \equiv H_{p-1}(-c) H_{p-1}(b, -a) - H_{p-1}(-c, b) H_{p-1}(-a)$$
$$+ H_{p-1}(-c - b, -a) + H_{p-1}(-c, -b - a) \pmod{p}.$$

In [5, Lemma 6.2] it was proved that for positive integers a, b and a prime $p > 2a + 2b + 1$,

$$H_{p-1}(-2a, -2b) \equiv 0 \pmod{p}.$$

However, we will need a stronger version of this congruence.

Lemma 3.1. *Let* a, b *be positive integers and a prime* $p > 2a + 2b + 1$. *Then the following congruence holds modulo* p^2 :

$$H_{p-1}(-2a, -2b) \equiv \left(\frac{(a-b)(1 - 2^{-2a-2b})}{(2a+1)(2b+1)} \binom{2a+2b}{2a} - \frac{a+b}{2a+2b+1} \right) pB_{p-2a-2b-1}.$$

Proof. Clearly (see [27, Lemma 7.1]),

$$H_{p-1}(-2a, -2b) = \sum_{1 \leq j < k \leq p-1} \frac{(-1)^{j+k}}{j^{2a}k^{2b}} = \sum_{1 \leq k < j \leq p-1} \frac{(-1)^{p-j+p-k}}{(p-j)^{2a}(p-k)^{2b}}$$

$$\equiv \sum_{1 \leq k < j \leq p-1} \frac{(-1)^{k+j}}{k^{2b}j^{2a}} \left(1 + \frac{2bp}{k} \right) \left(1 + \frac{2ap}{j} \right) \equiv H_{p-1}(-2b, -2a)$$

$$+ 2bpH_{p-1}(-2b-1, -2a) + 2apH_{p-1}(-2b, -2a-1) \pmod{p^2}.$$

Therefore by (vii), we obtain

$$H_{p-1}(-2a, -2b) \equiv H_{p-1}(-2b, -2a)$$

(3.1)
$$+ \frac{2p(a-b)(1 - 2^{-2a-2b})}{(2a+1)(2b+1)} \binom{2a+2b}{2a} B_{p-2a-2b-1} \pmod{p^2}.$$

On the other hand, we have

$$H_{p-1}(-2a, -2b) + H_{p-1}(-2b, -2a)$$
$$\equiv H_{p-1}(-2a)H_{p-1}(-2b) - H_{p-1}(2a+2b) \pmod{p^2},$$

and therefore by (vi) and (i),

$$H_{p-1}(-2a, -2b) + H_{p-1}(-2b, -2a) \equiv -H_{p-1}(2a+2b)$$

(3.2)
$$\equiv -\frac{(2a+2b)p}{2a+2b+1} B_{p-2a-2b-1} \pmod{p^2}.$$

Now from (3.1) and (3.2) we obtain the required congruence. $\qquad\square$

Lemma 3.2. *For positive integers* a, b *and a prime* $p > 2a + 2b + 1$, *the following congruence holds modulo* p :

$$H_{\frac{p-1}{2}}(-2a, -2b-1) \equiv \frac{2^{2a+2b} - 1}{2a+2b+1} \left(\frac{1}{2^{2a+2b+1}} \binom{2a+2b+1}{2a} + 1 \right) B_{p-2a-2b-1}.$$

Proof. It is easily seen that

$$H_{p-1}(-2a, -2b-1) - H_{\frac{p-1}{2}}(-2a, -2b-1) = \sum_{k=\frac{p+1}{2}}^{p-1} \frac{(-1)^k H_{k-1}(-2a)}{k^{2b+1}}$$

(3.3)
$$= \sum_{k=1}^{(p-1)/2} \frac{(-1)^{p-k} H_{p-k-1}(-2a)}{(p-k)^{2b+1}} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^k H_{p-k-1}(-2a)}{k^{2b+1}} \pmod{p}.$$

For the harmonic number $H_{p-k-1}(-2a)$, by (vi) we have

$$H_{p-k-1}(-2a) = \sum_{j=1}^{p-k-1} \frac{(-1)^j}{j^{2a}} = \sum_{j=k+1}^{p-1} \frac{(-1)^{p-j}}{(p-j)^{2a}} \equiv -\sum_{j=k+1}^{p-1} \frac{(-1)^j}{j^{2a}}$$
$$= -(H_{p-1}(-2a) - H_k(-2a)) \equiv H_k(-2a) \pmod{p}.$$

Substituting the above congruence in (3.3) we obtain

$$H_{p-1}(-2a, -2b-1) - H_{\frac{p-1}{2}}(-2a, -2b-1) \equiv \sum_{k=1}^{(p-1)/2} \frac{(-1)^k H_k(-2a)}{k^{2b+1}}$$

$$= H_{\frac{p-1}{2}}(-2a, -2b-1) + H_{\frac{p-1}{2}}(2a+2b+1) \pmod{p},$$

which by (iv) and (vii) implies the required result. $\qquad\square$

4. New congruences for multiple harmonic sums

In this section we state and prove our main results on multiple harmonic sums.

Theorem 4.1. *Let a, b be non-negative integers and p be a prime such that $p > 2a + 2b + 3$. Then*

$$S_{p-1}(\{2\}^a, 3, \{2\}^b) \equiv \frac{b-a}{(a+1)(b+1)}\binom{2a+2b+2}{2a+1} B_{p-2a-2b-3} \pmod{p},$$

$$H_{p-1}(\{2\}^a, 3, \{2\}^b) \equiv \frac{(-1)^{a+b}(a-b)}{(a+1)(b+1)}\binom{2a+2b+2}{2a+1} B_{p-2a-2b-3} \pmod{p}.$$

Proof. From Theorem 2.3 we have

$$S_{p-1}(\{2\}^a, 3, \{2\}^b) = 2\sum_{k=1}^{p-1} \frac{(-1)^{k-1}\binom{p-1}{k}}{k^{2a+2b+3}\binom{p-1+k}{k}} + 4\sum_{k=1}^{p-1} \frac{H_{k-1}(2a+1)(-1)^{k-1}\binom{p-1}{k}}{k^{2b+2}\binom{p-1+k}{k}}.$$

Since

$$\frac{\binom{p-1}{k}}{\binom{p-1+k}{k}} = \frac{(p-1)(p-2)\cdots(p-k)}{p(p+1)\cdots(p+k-1)} = \frac{(-1)^k k}{p}\prod_{j=1}^{k}\left(1-\frac{p}{j}\right)\prod_{j=1}^{k-1}\left(1+\frac{p}{j}\right)^{-1}$$

$$(4.1) \qquad \equiv \frac{(-1)^k k}{p}(1 - pH_k(1))(1 - pH_{k-1}(1))$$

$$\equiv \frac{(-1)^k k}{p}\left(1 - 2pH_{k-1}(1) - \frac{p}{k}\right) \pmod{p},$$

we obtain

$$S_{p-1}(\{2\}^a, 3, \{2\}^b) \equiv -\frac{2}{p}\sum_{k=1}^{p-1} \frac{1}{k^{2a+2b+2}}\left(1 - 2pH_{k-1}(1) - \frac{p}{k}\right)$$

$$-\frac{4}{p}\sum_{k=1}^{p-1} \frac{H_{k-1}(2a+1)}{k^{2b+1}}\left(1 - 2pH_{k-1}(1) - \frac{p}{k}\right)$$

$$= -\frac{2}{p}H_{p-1}(2a+2b+2) + 4H_{p-1}(1, 2a+2b+2)$$

$$-\frac{4}{p}H_{p-1}(2a+1, 2b+1) + 8H_{p-1}(2a+1, 1, 2b+1)$$

$$+ 8H_{p-1}(1, 2a+1, 2b+1) + 8H_{p-1}(2a+2, 2b+1)$$

$$+ 4H_{p-1}(2a+1, 2b+2) \pmod{p}.$$

Now by (i)–(iii), we easily obtain the required congruence for $S_{p-1}(\{2\}^a, 3, \{2\}^b)$. To prove the corresponding congruence for the H-sum, we apply the following

identity which relates S-version and H-version multiple harmonic sums (see [18, Theorem 6.8]):

$$(4.2) \qquad (-1)^{l(\mathbf{s})} S_n(\overline{\mathbf{s}}) = \sum_{\bigsqcup_{i=1}^l \mathbf{s}_i = \mathbf{s}} (-1)^l \prod_{i=1}^l H_n(\mathbf{s}_i),$$

where $\bigsqcup_{i=1}^l \mathbf{s}_i$ is the concatenation of $\mathbf{s}_1$ to $\mathbf{s}_l$ and the vector $\overline{\mathbf{s}} = (s_r, \dots, s_1)$ is obtained from $\mathbf{s} = (s_1, \dots, s_r)$ by reversing its coordinates. We substitute $\mathbf{s} = (\{2\}^b, 3, \{2\}^a)$ and $n = p-1$ in (4.2) and note that if $l \geq 2$, then one of $\mathbf{s}_i = (\{2\}^c)$ for some positive c and therefore $H_{p-1}(\mathbf{s}_i) \equiv 0 \pmod{p}$. Thus

$$(4.3) \qquad (-1)^{a+b+1} S_{p-1}(\{2\}^a, 3, \{2\}^b) \equiv -H_{p-1}(\{2\}^b, 3, \{2\}^a) \pmod{p}.$$

On the other hand, we have (see [19, Theorem 4.5])

$$(4.4) \qquad H_{p-1}(\{2\}^a, 3, \{2\}^b) \equiv -H_{p-1}(\{2\}^b, 3, \{2\}^a) \pmod{p}.$$

Now from (4.3) and (4.4) we get the required congruence and the proof is complete. $\qquad\square$

Theorem 4.2. *Let a, b be non-negative integers and a prime $p > 2a + 2b + 1$. Then*

$$S_{p-1}(\{2\}^a, 1, \{2\}^b) \equiv \frac{4(b-a)(1 - 4^{-a-b})}{(2a+1)(2b+1)} \binom{2a+2b}{2a} B_{p-2a-2b-1} \pmod{p},$$

$$H_{p-1}(\{2\}^a, 1, \{2\}^b) \equiv \frac{4(-1)^{a+b}(a-b)(1 - 4^{-a-b})}{(2a+1)(2b+1)} \binom{2a+2b}{2a} B_{p-2a-2b-1} \pmod{p}.$$

Proof. We begin by considering the case $a \geq 1$. Then from identity (2.18) with $n = p-1$ and congruence (4.1) we have

$$S_{p-1}(\{2\}^a, 1, \{2\}^b) \equiv -\frac{2}{p} \sum_{k=1}^{p-1} \frac{1}{k^{2a+2b}} \left(1 - 2pH_{k-1}(1) - \frac{p}{k}\right)$$

$$-\frac{4}{p} \sum_{k=1}^{p-1} \frac{H_{k-1}(-2a)(-1)^k}{k^{2b}} \left(1 - 2pH_{k-1}(1) - \frac{p}{k}\right) \pmod{p}.$$

By the obvious equality

$$(4.5) \quad H_n(a)H_n(b) = H_n(a, b) + H_n(b, a) + H_n\big(\mathrm{sgn}(ab)(|a| + |b|)\big), \quad a, b \in \mathbb{Z} \setminus \{0\},$$

we obtain

$$S_{p-1}(\{2\}^a, 1, \{2\}^b) \equiv -\frac{2}{p} H_{p-1}(2a + 2b) + 4H_{p-1}(1, 2a + 2b) - \frac{4}{p} H_{p-1}(-2a, -2b)$$

$$+ 8H_{p-1}(-2a, 1, -2b) + 8H_{p-1}(1, -2a, -2b)$$

$$+ 8H_{p-1}(-2a - 1, -2b) + 4H_{p-1}(-2a, -2b - 1) \pmod{p}.$$

Applying (viii), (4.5), (vi) and simplifying, we deduce that

$$S_{p-1}(\{2\}^a, 1, \{2\}^b) \equiv -\frac{2}{p} H_{p-1}(2a + 2b) - \frac{4}{p} H_{p-1}(-2a, -2b) \pmod{p}.$$

Now the required congruence follows from (i) and Lemma 3.1. The case $a = 0$ is handled in the same way with the help of identity (2.17) and congruences (vi), (vii). The corresponding congruence for $H_{p-1}(\{2\}^a, 1, \{2\}^b)$ easily follows from relation (4.2). $\qquad\square$

Corollary 4.3. *Let p be a prime greater than* 7. *Then*

$$(4.6) \qquad S_{p-1}(1,1,1,4) \equiv S_{p-1}(1,2,2,2) \equiv \frac{27}{16} B_{p-7} \pmod{p},$$

$$(4.7) \qquad S_{p-1}(1,1,1,6) \equiv \frac{1}{54} B_{p-3}^3 + \frac{1889}{648} B_{p-9} \pmod{p}.$$

Proof. The first congruence in (4.6) follows from [19, Theorem 7.3] and the second one is a consequence of Theorem 4.2 with $a = 0$, $b = 3$. To prove congruence (4.7) we set $a = 0$, $b = 4$ in Theorem 4.2 to obtain

$$(4.8) \qquad S_{p-1}(1,\{2\}^4) \equiv \frac{85}{48} B_{p-9} \pmod{p}.$$

On the other hand, using [19, Theorem 6.4] we can rewrite the length 5 sum $S_{p-1}(1,\{2\}^4)$ in terms of length 4 sums and $S_{p-1}(1,8)$ as

$$2S_{p-1}(1,\{2\}^4) \equiv S_{p-1}(1,2,2,4) + S_{p-1}(1,2,4,2) + S_{p-1}(1,4,2,2)$$
$$+ S_{p-1}(3,2,2,2) - \frac{255}{18} S_{p-1}(1,8) \pmod{p}.$$

Now applying [19, Theorem 7.5] and expressing all length 4 sums in the congruence above in terms of the three quantities $S_{p-1}(1,1,1,6)$, $S_{p-1}(1,8) \equiv B_{p-9}$, and $S_{p-1}(1,2)S_{p-1}(1,1,4) \equiv -\frac{1}{6} B_{p-3}^3 \pmod{p}$, we obtain
(4.9)
$$2S_{p-1}(1,\{2\}^4) \equiv 3S_{p-1}(1,1,1,6) + \frac{1}{3} S_{p-1}(1,2)S_{p-1}(1,1,4) - \frac{281}{54} S_{p-1}(1,8) \pmod{p}.$$

Comparing congruences (4.8) and (4.9) we conclude that

$$S_{p-1}(1,1,1,6) \equiv -\frac{1}{9} S_{p-1}(1,2)S_{p-1}(1,1,4) + \frac{1889}{648} S_{p-1}(1,8)$$

$$\equiv \frac{1}{54} B_{p-3}^3 + \frac{1889}{648} B_{p-9} \pmod{p}.$$

$\square$

From Corollary 4.3, [19, Theorems 7.3, 7.5], and [36, Section 2] we obtain the following description of multiple harmonic sums of weight 7 and 9.

Corollary 4.4. *Let p be a prime greater than* 7. *Then*
   (i) *all multiple harmonic sums $S_{p-1}(\mathbf{s})$, $H_{p-1}(\mathbf{s})$, of weight $|\mathbf{s}| = 7$ belong to $\mathbb{Q}(2)B_{p-7}$ modulo p;*
   (ii) *all multiple harmonic sums $S_{p-1}(\mathbf{s})$, $H_{p-1}(\mathbf{s})$ of weight $|\mathbf{s}| = 9$ can be written modulo p as linear combinations of the two quantities B_{p-9} and B_{p-3}^3 with coefficients in $\mathbb{Q}(3)$.*

In [35, Theorem 3.16], Zhao proved that

$$S_{p-1}(\{1\}^a, 2, \{1\}^b) \equiv H_{p-1}(\{1\}^a, 2, \{1\}^b) \equiv 0 \pmod{p}$$

if $a + b$ is even and a prime $p > a + b + 3$. In the next theorem we prove a deeper result on the above sums.

Theorem 4.5. *For non-negative integers a, b and a prime $p > a + b + 3$,*

$$S_{p-1}(\{1\}^a, 2, \{1\}^b) \equiv \begin{cases} \frac{(-1)^b}{a+b+2}\binom{a+b+2}{a+1}B_{p-a-b-2} & \pmod{p} & \text{if } a+b \text{ is odd,} \\ \frac{pB_{p-a-b-3}}{2(a+b+3)}\left(1 + (-1)^a\binom{a+b+3}{a+2}\right) \pmod{p^2} & \text{if } a+b \text{ is even;} \end{cases}$$

$$H_{p-1}(\{1\}^a, 2, \{1\}^b) \equiv \begin{cases} \frac{(-1)^b}{a+b+2}\binom{a+b+2}{a+1}B_{p-a-b-2} & \pmod{p} & \text{if } a+b \text{ is odd,} \\ \frac{pB_{p-a-b-3}}{2(a+b+3)}\left(1 + (-1)^a\binom{a+b+3}{b+2}\right) \pmod{p^2} & \text{if } a+b \text{ is even.} \end{cases}$$

Proof. From Theorem 2.5 for $c = 2$ we have

$$S_n(\{1\}^a, 2, \{1\}^b) = \sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k^{a+b+2}} + \sum_{k=1}^{n}\frac{H_{k-1}(a+1)(-1)^{k-1}\binom{n}{k}}{k^{b+1}}.$$

Setting $n = p - 1$ and observing that

$$\binom{p-1}{k} = \frac{(p-1)(p-2)\cdots(p-k)}{k!} \equiv (-1)^k(1 - pH_k(1)) \pmod{p^2},$$

we have

$$S_{p-1}(\{1\}^a, 2, \{1\}^b) \equiv -\sum_{k=1}^{p-1}\frac{1 - pH_k(1)}{k^{a+b+2}} - \sum_{k=1}^{p-1}\frac{H_{k-1}(a+1)(1 - pH_k(1))}{k^{b+1}}$$

(4.10)
$$= -H_{p-1}(a+b+2) + pH_{p-1}(1, a+b+2) + pH_{p-1}(a+b+3)$$
$$- H_{p-1}(a+1, b+1) + pH_{p-1}(a+1, b+2) + pH_{p-1}(a+1, 1, b+1)$$
$$+ pH_{p-1}(1, a+1, b+1) + pH_{p-1}(a+2, b+1) \pmod{p^2}.$$

If $a + b$ is odd, then by (ii) we obtain

$$S_{p-1}(\{1\}^a, 2, \{1\}^b) \equiv -H_{p-1}(a+1, b+1) \equiv \frac{(-1)^b}{a+b+2}\binom{a+b+2}{a+1}B_{p-a-b-2} \pmod{p}.$$

If $a + b$ is even, then from (4.10) and (i)–(iii) after simplifying we get the second congruence for $S_{p-1}(\{1\}^a, 2, \{1\}^b)$.

Considering equality (4.2) with $n = p - 1$, $\mathbf{s} = (\{1\}^a, 2, \{1\}^b)$ modulo p we find

$$(-1)^{a+b+1}S_{p-1}(\{1\}^b, 2, \{1\}^a) \equiv -H_{p-1}(\{1\}^a, 2, \{1\}^b) \pmod{p}$$

and therefore

(4.11) $\quad H_{p-1}(\{1\}^a, 2, \{1\}^b) \equiv \begin{cases} -S_{p-1}(\{1\}^b, 2, \{1\}^a) & \pmod{p} & \text{if } a+b \text{ is odd,} \\ 0 & \pmod{p} & \text{if } a+b \text{ is even.} \end{cases}$

Similarly, from identity (4.2) considered modulo p^2 we obtain

$$(-1)^{a+b}S_{p-1}(\{1\}^b, 2, \{1\}^a)$$

(4.12)
$$\equiv H_{p-1}(\{1\}^a, 2, \{1\}^b) - \prod_{\substack{\mathbf{s}_1 \sqcup \mathbf{s}_2 = \\ (\{1\}^a, 2, \{1\}^b)}} H_{p-1}(\mathbf{s}_1)H_{p-1}(\mathbf{s}_2) \pmod{p^2}.$$

It is clear that the product $H_{p-1}(\mathbf{s}_1)H_{p-1}(\mathbf{s}_2)$ has one of the forms

$$H_{p-1}(\{1\}^m)H_{p-1}(\{1\}^{a-m}, 2, \{1\}^b), \qquad 1 \le m \le a,$$

or

$$H_{p-1}(\{1\}^a, 2, \{1\}^l) H_{p-1}(\{1\}^{b-l}), \qquad 0 \le l \le b.$$

If $a + b$ is even, then by (i) and (4.11), it easily follows that both products are congruent to zero modulo p^2. Thus in this case from (4.12) we obtain

$$S_{p-1}(\{1\}^b, 2, \{1\}^a) \equiv H_{p-1}(\{1\}^a, 2, \{1\}^b) \pmod{p^2},$$

and the proof is complete. $\qquad\square$

Remark 4.6. It is worth mentioning that there is a notion of duality for multiple harmonic sums: given $\mathbf{s} = (s_1, s_2, \ldots, s_r)$ we define the power set $P(\mathbf{s})$ to be the partial sum sequence $(s_1, s_1 + s_2, \ldots, s_1 + \cdots + s_{r-1})$ as a subset of $\{1, 2, \ldots, |\mathbf{s}| - 1\}$. Then $\mathbf{s}^*$ is the composition of weight $|\mathbf{s}|$ corresponding to the complement subset of $P(\mathbf{s})$ in $\{1, 2, \ldots, |\mathbf{s}| - 1\}$, namely,

$$\mathbf{s}^* = P^{-1}(\{1, 2, \ldots, |\mathbf{s}| - 1\} \setminus P(\mathbf{s})).$$

It is easy to verify that $(\mathbf{s}^*)^* = \mathbf{s}$. Moreover, by [18, Theorem 6.7] we have that

$$S_{p-1}(\mathbf{s}) \equiv -S_{p-1}(\mathbf{s}^*) \pmod{p}$$

for any prime p. In particular, for any positive integers a, b it follows that

$$S_{p-1}(\{2\}^a, 1, \{2\}^b) \equiv -S_{p-1}(1, \{2\}^{a-1}, 3, \{2\}^{b-1}, 1) \pmod{p},$$
$$S_{p-1}(\{2\}^a, 3, \{2\}^b) \equiv -S_{p-1}(1, \{2\}^a, 1, \{2\}^b, 1) \pmod{p},$$
$$S_{p-1}(\{1\}^a, 2, \{1\}^b) \equiv -S_{p-1}(a + 1, b + 1) \pmod{p},$$

and by Theorem 4.1 and Theorem 4.2 we are able to find $S_{p-1}(1, \{2\}^a, 1, \{2\}^b, 1)$ and $S_{p-1}(1, \{2\}^{a-1}, 3, \{2\}^{b-1}, 1)$ modulo a prime p in terms of Bernoulli numbers.

5. Congruences for "odd" multiple harmonic sums

In this section we prove some congruences involving "odd" multiple harmonic sums. These results will be needed in the next sections.

Lemma 5.1. *Let $\mathbf{s} \in \mathbb{N}^r$. Then for any prime $p > 3$,*

$$\overline{H}_{\frac{p-1}{2}}(\mathbf{s}) \equiv \frac{(-1)^{|\mathbf{s}|}}{2^{|\mathbf{s}|}} H_{\frac{p-1}{2}}(\overline{\mathbf{s}}) \pmod{p}.$$

Proof. Changing the order of summation in the definition of $\overline{H}_n(\mathbf{s})$, with $n = \frac{p-1}{2}$, we obtain

$$
\begin{aligned}
\overline{H}_n(s_1, s_2, \ldots, s_r) &= \sum_{0 \le k_1 < \cdots < k_r < n} \frac{1}{(2k_1 + 1)^{s_1} \cdots (2k_r + 1)^{s_r}} \\
&= \sum_{n > k_1 > \cdots > k_r \ge 0} \frac{1}{(2(n - 1 - k_1) + 1)^{s_1} \cdots (2(n - 1 - k_r) + 1)^{s_r}} \\
&\equiv \sum_{0 \le k_r < \cdots < k_1 \le n-1} \frac{(-1)^{s_1 + \cdots + s_r}}{2^{s_1 + \cdots + s_r} (k_1 + 1)^{s_1} \cdots (k_r + 1)^{s_r}} \\
&= \frac{(-1)^{|\mathbf{s}|}}{2^{|\mathbf{s}|}} H_n(s_r, \ldots, s_1) \pmod{p}.
\end{aligned}
$$

$\qquad\square$

Theorem 5.2. *For any positive integer m and any prime $p > 2m + 1$, we have*

$$(5.1) \qquad \overline{H}_{\frac{p-1}{2}}(2m) \equiv \frac{m}{4^m(2m+1)} \, p \, B_{p-2m-1} \pmod{p^2},$$

$$(5.2) \qquad \overline{H}_{\frac{p-1}{2}}(\{2\}^m) \equiv \frac{(-1)^{m-1}}{4^m(2m+1)} \, p \, B_{p-2m-1} \pmod{p^2}.$$

Proof. Let $n = (p-1)/2$. It is easy to deduce (see [15, Lemma 2]) that

$$\overline{H}_n(2m) = H_{2n}(2m) - \frac{H_n(2m)}{2^{2m}},$$

which by (i) and (iv) implies the first congruence.

From [35, Section 2.4, Lemma 2.12] it follows that the "odd" multiple harmonic sum $\overline{H}_n$ as well as H_n satisfies the following stuffle relation:

$$(5.3) \qquad m\overline{H}_n(\{2\}^m) = \sum_{k=1}^m (-1)^{k-1} \overline{H}_n(2k) \cdot \overline{H}_n(\{2\}^{m-k}).$$

Now the congruence (5.2) easily follows from the above identity and (5.1) by induction on m. Indeed, for $m = 1$ it follows from (5.1), and if $m > 1$, then by (5.3) and induction hypothesis, we have

$$m\overline{H}_n(\{2\}^m) \equiv (-1)^{m-1} \overline{H}_n(2m) \pmod{p^2},$$

and the theorem is proved. $\qquad\qquad\square$

Theorem 5.3. *Let a, b be non-negative integers and a prime $p > 2a + 2b + 3$. Then*

$$S_{\frac{p-1}{2}}(\{2\}^a, 3, \{2\}^b) \equiv -\frac{B_{p-2a-2b-3}}{b+1}\binom{2a+2b+2}{2a+1} \pmod{p},$$

$$H_{\frac{p-1}{2}}(\{2\}^a, 3, \{2\}^b) \equiv (-1)^{a+b+1} \frac{B_{p-2a-2b-3}}{a+1}\binom{2a+2b+2}{2a+1} \pmod{p}.$$

Proof. Setting $n = \frac{p-1}{2}$ in identity (2.15) and observing that

$$(5.4) \qquad \frac{\binom{n}{k}}{\binom{n+k}{k}} = (-1)^k \frac{\left(\frac{1}{2} - \frac{p}{2}\right)\left(\frac{3}{2} - \frac{p}{2}\right)\cdots\left(\frac{2k-1}{2} - \frac{p}{2}\right)}{\left(\frac{1}{2} + \frac{p}{2}\right)\left(\frac{3}{2} + \frac{p}{2}\right)\cdots\left(\frac{2k-1}{2} + \frac{p}{2}\right)} \equiv (-1)^k \pmod{p},$$

we obtain

$$S_{\frac{p-1}{2}}(\{2\}^a, 3, \{2\}^b) \equiv -2H_{\frac{p-1}{2}}(2a+2b+3) - 4H_{\frac{p-1}{2}}(2a+1, 2b+2) \pmod{p}.$$

Now the required congruence easily follows by (iv) and (v). From Lemma 5.1 and Theorem 5.2 it follows that

$$H_{\frac{p-1}{2}}(\{2\}^m) \equiv 2^{2m} \, \overline{H}_{\frac{p-1}{2}}(\{2\}^m) \equiv 0 \pmod{p}.$$

Therefore applying identity (4.2) with $n = \frac{p-1}{2}$ and $\mathbf{s} = (\{2\}^a, 3, \{2\}^b)$ we obtain

$$(-1)^{a+b+1} S_{\frac{p-1}{2}}(\{2\}^b, 3, \{2\}^a) \equiv -H_{\frac{p-1}{2}}(\{2\}^a, 3, \{2\}^b) \pmod{p}$$

and the theorem is proved. $\qquad\qquad\square$

Theorem 5.4. *Let a, b be non-negative integers, not both zero, and a prime $p > 2a + 2b + 1$. Then*

$$S_{\frac{p-1}{2}}(\{2\}^a, 1, \{2\}^b) \equiv \frac{2^{1-2a-2b} - 2}{2b + 1}\binom{2a + 2b}{2a}B_{p-2a-2b-1} \pmod{p},$$

$$H_{\frac{p-1}{2}}(\{2\}^a, 1, \{2\}^b) \equiv \frac{(-1)^{a+b}(2^{1-2a-2b} - 2)}{2a + 1}\binom{2a + 2b}{2a}B_{p-2a-2b-1} \pmod{p}.$$

Proof. If $a \geq 1$, then from identity (2.18) with $n = (p-1)/2$ and congruence (5.4) we find

$$S_{\frac{p-1}{2}}(\{2\}^a, 1, \{2\}^b) \equiv -2H_{\frac{p-1}{2}}(2a + 2b + 1) - 4H_{\frac{p-1}{2}}(-2a, -2b - 1) \pmod{p}.$$

Now the required result easily follows by (iv) and Lemma 3.2. If $a = 0$, then from (2.17) we have

$$(5.5) \qquad S_{\frac{p-1}{2}}(1, \{2\}^b) \equiv 2H_{\frac{p-1}{2}}(-2b - 1) \pmod{p}.$$

Moreover,

$$H_{\frac{p-1}{2}}(-2b - 1) = \sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^{2b+1}} = \sum_{k=(p+1)/2}^{p-1}\frac{(-1)^{p-k}}{(p-k)^{2b+1}}$$

$$\equiv \sum_{k=(p+1)/2}^{p-1}\frac{(-1)^k}{k^{2b+1}} = H_{p-1}(-2b - 1) - H_{\frac{p-1}{2}}(-2b - 1) \pmod{p}.$$

Therefore, since $b > 0$, by (vi) it follows that

$$(5.6) \qquad H_{\frac{p-1}{2}}(-2b - 1) \equiv \frac{1}{2}H_{p-1}(-2b - 1) \equiv \frac{2^{-2b} - 1}{2b + 1}B_{p-2b-1} \pmod{p}.$$

From (5.5) and (5.6) we conclude the truth of the first congruence for $a = 0$. From identity (4.2) we easily obtain

$$(-1)^{a+b}S_{\frac{p-1}{2}}(\{2\}^b, 1, \{2\}^a) \equiv H_{\frac{p-1}{2}}(\{2\}^a, 1, \{2\}^b) \pmod{p},$$

and the proof is complete. $\qquad\qquad\square$

Lemma 5.5. *Let r, a be positive integers. Then for any prime $p > r + 2$, we have*

$$(5.7) \begin{aligned} H_{\frac{p-1}{2}}(r) &\equiv H_{\lfloor\frac{p}{4}\rfloor}(r) + (-2)^r\sum_{k=0}^{a}\binom{r - 1 + k}{k}H_{\frac{p-1}{2}}(r + k)p^k \\ &\quad - (-1)^r\sum_{k=0}^{a}\binom{r - 1 + k}{k}\frac{H_{\lfloor\frac{p}{4}\rfloor}(r + k)}{2^k}p^k \pmod{p^{a+1}}. \end{aligned}$$

If r is a non-negative integer, then for any prime $p > 2r + 3$, we have

$$\overline{H}_{\frac{p-1}{2}}(-2r - 1) \equiv \frac{(-1)^{\frac{p+1}{2}}}{4^{2r+1}}H_{\frac{p-1}{2}}(2r + 1) \pmod{p^2}.$$

Proof. Let $n = (p-1)/2$, $m = \lfloor p/4 \rfloor$. Then the first congruence follows easily from the equality

$$H_n(r) = H_m(r) + 2^r\left(\sum_{k=1}^{n}\frac{1}{(p - k)^r} - \sum_{k=1}^{m}\frac{1}{(p - 2k)^r}\right).$$

To prove the second one, we have

$$(-1)^n \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)^{2r+1}} = \sum_{k=1}^{n} \frac{(-1)^k}{(p-2k)^{2r+1}}$$

$$= 2\sum_{k=1}^{m} \frac{1}{(p-4k)^{2r+1}} - \sum_{k=1}^{n} \frac{1}{(p-2k)^{2r+1}}$$

$$= 2\sum_{k=1}^{m} \frac{1}{(4k)^{2r+1}(p/(4k)-1)^{2r+1}} - \sum_{k=1}^{n} \frac{1}{(2k)^{2r+1}(p/(2k)-1)^{2r+1}}$$

$$\equiv -\frac{1}{2^{4r+1}}\left(H_m(2r+1) + \frac{(2r+1)p}{4} H_m(2r+2) \right)$$

$$+ \frac{1}{2^{2r+1}} H_n(2r+1) \pmod{p^2}.$$

From (5.7) with $a = 1$ and r replaced by $2r + 1$, we get

$$2H_m(2r+1) + \frac{(2r+1)p}{2} H_m(2r+2) \equiv (1 + 2^{2r+1})H_n(2r+1) \pmod{p^2},$$

which after substitution into the right-hand side of the above congruence implies the result. $\qquad\square$

6. Results on p-analogues of Leshchiner's series

In this section, we consider finite p-analogues of zeta and beta values arising from the truncation of Leshchiner's series (1.5)–(1.8) and defined by the finite sums

$$\zeta_p(2m+2) := \frac{3}{2} \sum_{k=1}^{p-1} \frac{(-1)^m H_{k-1}(\{2\}^m)}{k^2 \binom{2k}{k}} + 2\sum_{j=1}^{m}\sum_{k=1}^{p-1} \frac{(-1)^{m-j} H_{k-1}(\{2\}^{m-j})}{k^{2j+2} \binom{2k}{k}},$$

$$\zeta_p(2m+3) := \frac{5}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k+m-1} H_{k-1}(\{2\}^m)}{k^3 \binom{2k}{k}}$$

$$+ 2\sum_{j=1}^{m}\sum_{k=1}^{p-1} \frac{(-1)^{k+m-j-1} H_{k-1}(\{2\}^{m-j})}{k^{2j+3} \binom{2k}{k}},$$

$$\overline{\zeta}_p(2m+2) := \frac{5}{4} \sum_{k=0}^{(p-3)/2} \frac{(-1)^{k+m}\binom{2k}{k}\overline{H}_k(\{2\}^m)}{16^k(2k+1)^2}$$

$$+ \sum_{j=1}^{m}\sum_{k=0}^{(p-3)/2} \frac{(-1)^{k+m-j}\binom{2k}{k}\overline{H}_k(\{2\}^{m-j})}{16^k(2k+1)^{2j+2}},$$

$$\beta_p(2m+1) := \frac{3}{4} \sum_{k=0}^{(p-3)/2} \frac{(-1)^m\binom{2k}{k}\overline{H}_k(\{2\}^m)}{16^k(2k+1)}$$

$$+ \sum_{j=1}^{m}\sum_{k=0}^{(p-3)/2} \frac{(-1)^{m-j}\binom{2k}{k}\overline{H}_k(\{2\}^{m-j})}{16^k(2k+1)^{2j+1}}.$$

Applications of our results on congruences obtained in Sections 4 and 5 allowed us to establish the following congruences for the finite p-analogues of Leshchiner's series.

Theorem 6.1. *Let m be a non-negative integer and p be a prime such that $p > 2m + 3$. Then*

$$2p\,\zeta_p(2m+2) \equiv -\frac{4m(1-2^{-2m})}{2m+1}\,B_{p-2m-1} \pmod{p},$$

$$\zeta_p(2m+3) \equiv -\frac{(m+1)(2m+1)}{2m+3}\,B_{p-2m-3} \pmod{p}.$$

Note that for $\zeta_p(2)$, $\zeta_p(3)$, $\zeta_p(4)$, stronger congruences were proved by the authors in [14, 26]. For $\zeta_p(5)$, we recover the congruence $\zeta_p(5) \equiv -6/5\,B_{p-5} \pmod{p}$ proved in [14].

Theorem 6.2. *Let m be a non-negative integer and p be a prime such that $p > 2m + 3$. Then*

$$\beta_p(2m+1) \equiv \frac{(-1)^{(p+1)/2}m(4^m-1)}{16^m(2m+1)}\,B_{p-2m-1} \pmod{p},$$

$$\overline{\zeta}_p(2m+2) \equiv -\frac{2m^2+3m+2}{2^{2m+3}(2m+3)}\,p\,B_{p-2m-3} \pmod{p^2}.$$

Note that for $\beta_p(1)$ and $\overline{\zeta}_p(2)$, sharper congruences were proved in [15].

7. FINITE IDENTITIES AND THEIR FORMAL PROOFS

By revisiting the combinatorial proof (due to D. Zagier) presented in Section 5 in [22], it is possible to obtain finite versions of identities (1.5)–(1.8). Here we derive these identities in a formal unified way based on applications of appropriate WZ pairs.

Lemma 7.1. *Let m be a non-negative integer. Then for every positive integer n, we have*

(7.1)
$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^{2m+2}} - \frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^{n+k+m}H_{k-1}(\{2\}^m)}{k^2\binom{n}{k}\binom{n+k}{k}}$$
$$= \frac{3}{2}\sum_{k=1}^{n}\frac{(-1)^m H_{k-1}(\{2\}^m)}{k^2\binom{2k}{k}} + 2\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{(-1)^{m-j}H_{k-1}(\{2\}^{m-j})}{k^{2j+2}\binom{2k}{k}},$$

(7.2)
$$\sum_{k=1}^{n}\frac{1}{k^{2m+3}} - \frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^{k+m}H_{k-1}(\{2\}^m)}{k^3\binom{n}{k}\binom{n+k}{k}}$$
$$= \frac{5}{2}\sum_{k=1}^{n}\frac{(-1)^{k+m-1}H_{k-1}(\{2\}^m)}{k^3\binom{2k}{k}} + 2\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{(-1)^{k+m-j-1}H_{k-1}(\{2\}^{m-j})}{k^{2j+3}\binom{2k}{k}}.$$

Proof. For non-negative integers n, k, we consider a pair of functions

$$F_1(n,k) = \frac{(-1)^{n+k}(n-k-1)!(1+a)_k(1-a)_k}{2(n+k+1)!}, \qquad n \geq k+1,$$

$$G_1(n,k) = \frac{(-1)^{n+k}(n-k)!(n+1)(1+a)_k(1-a)_k}{((n+1)^2-a^2)(n+k+1)!}, \qquad n \geq k,$$

where $(c)_k$ is the shifted factorial defined by $(c)_k = c(c+1)\cdots(c+k-1)$, $k \geq 1$, and $(c)_0 = 1$. It is easy to check that (F_1, G_1) is a WZ pair, i.e., it satisfies the relation

$$F_1(n+1, k) - F_1(n, k) = G_1(n, k+1) - G_1(n, k).$$

Summing the above equality first over k and then over n, we get the following summation formula (see [14, Section 4]):

(7.3)
$$\sum_{n=1}^{N} G_1(n-1, 0) = \sum_{n=1}^{N}(G_1(n-1, n-1) + F_1(n, n-1)) - \sum_{k=1}^{N} F_1(N, k-1), \quad N \in \mathbb{N}.$$

Substituting (F_1, G_1) in (7.3) we obtain

(7.4)
$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^2 - a^2} = \frac{1}{2}\sum_{n=1}^{N} \frac{1}{n^2\binom{2n}{n}} \frac{3n^2 + a^2}{n^2 - a^2} \prod_{m=1}^{n-1}\left(1 - \frac{a^2}{m^2}\right)$$
$$+ \frac{1}{2}\sum_{k=1}^{N} \frac{(-1)^{N+k}}{k^2\binom{N}{k}\binom{N+k}{k}} \prod_{m=1}^{k-1}\left(1 - \frac{a^2}{m^2}\right).$$

Expanding both sides of (7.4) in powers of a^2 and comparing coefficients of a^{2m} we derive the finite identity (7.1). Similarly, considering the WZ pair

$$F_2(n, k) = \frac{(-1)^n}{k+1} F_1(n, k), \qquad G_2(n, k) = \frac{(-1)^n}{n+1} G_1(n, k)$$

and applying the same argument as above we obtain the second identity. $\square$

Lemma 7.2. *For any non-negative integers m, n we have*

(7.5)
$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^{2m+1}} - \frac{1}{4}\sum_{k=0}^{n} \frac{(-1)^{n+k+m}\binom{2k}{k}\overline{H}_k(\{2\}^m)}{16^k(2k+1)\binom{n+k+1}{2k+1}}$$
$$= \frac{3}{4}\sum_{k=0}^{n} \frac{(-1)^m\binom{2k}{k}\overline{H}_k(\{2\}^m)}{16^k(2k+1)} + \sum_{j=1}^{m}\sum_{k=0}^{n} \frac{(-1)^{m-j}\binom{2k}{k}\overline{H}_k(\{2\}^{m-j})}{16^k(2k+1)^{2j+1}},$$

(7.6)
$$\sum_{k=0}^{n} \frac{1}{(2k+1)^{2m+2}} + \frac{1}{4}\sum_{k=0}^{n} \frac{(-1)^{k+m}\binom{2k}{k}\overline{H}_k(\{2\}^m)}{16^k(2k+1)^2\binom{n+k+1}{2k+1}}$$
$$= \frac{5}{4}\sum_{k=0}^{n} \frac{(-1)^{k+m}\binom{2k}{k}\overline{H}_k(\{2\}^m)}{16^k(2k+1)^2} + \sum_{j=1}^{m}\sum_{k=0}^{n} \frac{(-1)^{k+m-j}\binom{2k}{k}\overline{H}_k(\{2\}^{m-j})}{16^k(2k+1)^{2j+2}}.$$

Proof. As in the proof of Lemma 7.1, applying summation formula (7.3) to the WZ pairs

$$F_1(n, k) = \frac{(-1)^{n+k}(n-k-1)!}{4(n+k)!}\left(\frac{1+a}{2}\right)_k\left(\frac{1-a}{2}\right)_k, \quad n \geq k+1,$$

$$G_1(n, k) = \frac{(-1)^{n+k}(n-k)!(2n+1)}{(n+k)!((2n+1)^2 - a^2)}\left(\frac{1+a}{2}\right)_k\left(\frac{1-a}{2}\right)_k, \quad n \geq k,$$

and

$$F_2(n,k) = \frac{(-1)^n}{2k+1} F_1(n,k), \qquad G_2(n,k) = \frac{(-1)^n}{2n+1} G_1(n,k),$$

we get the required identities. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

8. Proofs of Theorems 6.1 and 6.2

Proof of Theorem 6.1. Setting $n = p - 1$ in (7.1), we get

$$\zeta_p(2m+2) = H_{p-1}(-2m-2) - \frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{k+m}H_{k-1}(\{2\}^m)}{k^2\binom{p-1}{k}\binom{p-1+k}{k}}.$$

Since

$$\text{(8.1)}\qquad \frac{(-1)^k}{k^2\binom{p-1}{k}\binom{p-1+k}{k}} = \frac{1}{pk}\prod_{m=1}^{k}\left(1-\frac{p}{m}\right)^{-1}\prod_{m=1}^{k-1}\left(1+\frac{p}{m}\right)^{-1}$$

$$\equiv \frac{1}{pk} + \frac{1}{k^2} + \frac{pH_k(2)}{k} \pmod{p^2},$$

we obtain

$$\zeta_p(2m+2) \equiv -H_{p-1}(-2m-2) - \frac{(-1)^m}{2p}H_{p-1}(\{2\}^m,1)$$

$$- \frac{(-1)^m}{2}H_{p-1}(\{2\}^{m+1}) \pmod{p},$$

which by (i), (vi), and Theorem 4.2 implies the required congruence.

Similarly, setting $n = p - 1$ in (7.2) and applying (8.1), (i) and Theorem 4.1 we get

$$\zeta_p(2m+3) = H_{p-1}(2m+3) - \frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{k+m}H_{k-1}(\{2\}^m)}{k^3\binom{p-1}{k}\binom{p-1+k}{k}}$$

$$\equiv H_{p-1}(2m+3) + \frac{(-1)^{m-1}}{2}\left(\frac{1}{p}H_{p-1}(\{2\}^{m+1}) + H_{p-1}(\{2\}^m,3)\right)$$

$$\equiv -\frac{B_{p-2m-3}}{2m+3} + \frac{(-1)^{m-1}}{2}H_{p-1}(\{2\}^m,3)$$

$$\equiv -\frac{(m+1)(2m+1)}{2m+3}B_{p-2m-3} \pmod{p}.$$

Proof of Theorem 6.2. Setting $n = (p-3)/2$ in (7.5), we get

$$\beta_p(2m+1) = \overline{H}_{\frac{p-1}{2}}(-2m-1) + \frac{(-1)^{(p-1)/2+m}}{4}\sum_{k=0}^{(p-3)/2}\frac{(-1)^k\binom{2k}{k}\overline{H}_k(\{2\}^m)}{16^k(2k+1)\binom{(p-1)/2+k}{2k+1}}.$$

Taking into account that
$$\text{(8.2)}$$
$$\frac{\binom{2k}{k}}{(-16)^k\binom{(p-1)/2+k}{2k+1}} = \frac{2(2k+1)}{p-1-2k}\prod_{j=0}^{k-1}\left(1-\frac{p^2}{(2j+1)^2}\right)^{-1} \equiv -2 - \frac{2p}{2k+1} \pmod{p^2},$$

we obtain by Theorem 5.2,

$$\beta_p(2m+1) \equiv \overline{H}_{\frac{p-1}{2}}(-2m-1) + \frac{(-1)^{(p+1)/2+m}}{2} \sum_{k=0}^{(p-3)/2} \frac{\overline{H}_k(\{2\}^m)}{2k+1}\left(1 + \frac{p}{2k+1}\right)$$

$$\equiv \overline{H}_{\frac{p-1}{2}}(-2m-1) + \frac{(-1)^{(p+1)/2+m}}{2} \overline{H}_{\frac{p-1}{2}}(\{2\}^m, 1) \pmod{p^2},$$

which implies the first congruence of the theorem by Lemmas 5.5 and 5.1, and Theorem 5.4.

Similarly, from (7.6) by (8.2), Theorem 5.2, Lemma 5.1 and Theorem 5.3, we have

$$\overline{\zeta}_p(2m+2) \equiv \overline{H}_{\frac{p-1}{2}}(2m+2) + \frac{(-1)^{m-1}}{2}\left(\overline{H}_{\frac{p-1}{2}}(\{2\}^{m+1}) + p\overline{H}_{\frac{p-1}{2}}(\{2\}^m, 3)\right)$$

$$\equiv \frac{2m+1}{2^{2m+3}(2m+3)} p B_{p-2m-3} + \frac{(-1)^{m-1}}{2} p \overline{H}_{\frac{p-1}{2}}(\{2\}^m, 3)$$

$$\equiv -\frac{2m^2+3m+2}{2^{2m+3}(2m+3)} p B_{p-2m-3} \pmod{p^2}.$$

Acknowledgements

References

[1] David H. Bailey, Jonathan M. Borwein, and David M. Bradley, *Experimental determination of Apéry-like identities for $\zeta(2n+2)$*, Experiment. Math. **15** (2006), no. 3, 281–289. MR2264467 (2007f:11144)

[2] J. Blümlen, S. Kurth, *Harmonic sums and Mellin transforms up to two-loop order*, Phys. Rev. D**60** (1999) 014018, 31pp.; arXiv: hep-ph/9810241.

[3] Jonathan Borwein and David Bradley, *Empirically determined Apéry-like formulae for $\zeta(4n+3)$*, Experiment. Math. **6** (1997), no. 3, 181–194. MR1481588 (98m:11142)

[4] Francis Brown, *Mixed Tate motives over $\mathbb{Z}$*, Ann. of Math. (2) **175** (2012), no. 2, 949–976, DOI 10.4007/annals.2012.175.2.10. MR2993755

[5] Marc Chamberland and Karl Dilcher, *Divisibility properties of a class of binomial sums*, J. Number Theory **120** (2006), no. 2, 349–371, DOI 10.1016/j.jnt.2006.01.009. MR2257551 (2007g:11020)

[6] Karl Dilcher, *Some q-series identities related to divisor functions*, Discrete Math. **145** (1995), no. 1-3, 83–93, DOI 10.1016/0012-365X(95)00092-B. MR1356587 (96i:11020)

[7] Philippe Flajolet and Bruno Salvy, *Euler sums and contour integral representations*, Experiment. Math. **7** (1998), no. 1, 15–35. MR1618286 (99c:11110)

[8] J. W. L. Glaisher, *Congruences relating to the sums of products of the first n numbers and to other sums of products*, Quart. J. Math. **31** (1900), p. 231.

[9] A. B. Goncharov, *Galois symmetries of fundamental groupoids and noncommutative geometry*, Duke Math. J. **128** (2005), no. 2, 209–284, DOI 10.1215/S0012-7094-04-12822-2. MR2140264 (2007b:11094)

[10] Henry W. Gould, *Combinatorial identities*, Henry W. Gould, Morgantown, W.Va., 1972. A standardized set of tables listing 500 binomial coefficient summations. MR0354401 (50 #6879)

[11] Kh. Hessami Pilehrood and T. Hessami Pilehrood, *Simultaneous generation for zeta values by the Markov-WZ method*, Discrete Math. Theor. Comput. Sci. **10** (2008), no. 3, 115–123. MR2457055 (2009j:11133)

[12] Kh. Hessami Pilehrood and T. Hessami Pilehrood, *Series acceleration formulas for beta values*, Discrete Math. Theor. Comput. Sci. **12** (2010), no. 2, 223–236. MR2676672 (2011j:05020)

[13] Kh. Hessami Pilehrood and T. Hessami Pilehrood, *Bivariate identities for values of the Hurwitz zeta function and supercongruences*, Electron. J. Combin. **18** (2011), no. 2, Paper 35, 30. MR2900448

[14] Kh. Hessami Pilehrood, T. Hessami Pilehrood, *Congruences arising from Apéry-type series for zeta values*, Adv. in Appl. Math. **49** (2012), no. 3-5, 218–238. MR3017957

[15] Kh. Hessami Pilehrood, T. Hessami Pilehrood, and R. Tauraso, *Congruences concerning Jacobi polynomials and Apéry-like formulae*, Int. J. Number Theory **8** (2012), no. 7, 1789–1811, DOI 10.1142/S1793042112501035. MR2968951

[16] Michael E. Hoffman, *Multiple harmonic series*, Pacific J. Math. **152** (1992), no. 2, 275–290. MR1141796 (92i:11089)

[17] Michael E. Hoffman, *The algebra of multiple harmonic series*, J. Algebra **194** (1997), no. 2, 477–495, DOI 10.1006/jabr.1997.7127. MR1467164 (99e:11119)

[18] Michael E. Hoffman, *Algebraic aspects of multiple zeta values*, Zeta functions, topology and quantum physics, Dev. Math., vol. 14, Springer, New York, 2005, pp. 51–73, DOI 10.1007/0-387-24981-8_4. MR2179272 (2006g:11185)

[19] M. E. Hoffman, *Quasi-symmetric functions and* mod *p multiple harmonic sums,* arXiv:math.NT/0401319.

[20] Kenneth Ireland and Michael Rosen, *A classical introduction to modern number theory*, 2nd ed., Graduate Texts in Mathematics, vol. 84, Springer-Verlag, New York, 1990. MR1070716 (92e:11001)

[21] Emma Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. of Math. (2) **39** (1938), no. 2, 350–360, DOI 10.2307/1968791. MR1503412

[22] D. Leshchiner, *Some new identities for $\zeta(k)$*, J. Number Theory **13** (1981), no. 3, 355–362, DOI 10.1016/0022-314X(81)90020-2. MR634205 (83k:10072)

[23] Yasuo Ohno and Noriko Wakabayashi, *Cyclic sum of multiple zeta values*, Acta Arith. **123** (2006), no. 3, 289–295, DOI 10.4064/aa123-3-5. MR2263259 (2007g:11113)

[24] Zhi-Hong Sun, *Congruences concerning Bernoulli numbers and Bernoulli polynomials*, Discrete Appl. Math. **105** (2000), no. 1-3, 193–223, DOI 10.1016/S0166-218X(00)00184-0. MR1780472 (2001m:11022)

[25] K. Tasaka, S. Yamamoto, *On some multiple zeta-star values of one-two-three indices,* arXiv:1203.1115v1 [math.NT]

[26] Roberto Tauraso, *More congruences for central binomial coefficients*, J. Number Theory **130** (2010), no. 12, 2639–2649, DOI 10.1016/j.jnt.2010.06.004. MR2684486 (2011h:11018)

[27] Roberto Tauraso and Jianqiang Zhao, *Congruences of alternating multiple harmonic sums*, J. Comb. Number Theory **2** (2010), no. 2, 129–159. MR2907787 (2012m:11005)

[28] Tomohide Terasoma, *Mixed Tate motives and multiple zeta values*, Invent. Math. **149** (2002), no. 2, 339–369, DOI 10.1007/s002220200218. MR1918675 (2003h:11073)

[29] D. V. Vasil′ev, *Some formulas for the Riemann zeta function at integer points*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1** (1996), 81–84 (Russian); English transl., Moscow Univ. Math. Bull. **51** (1996), no. 1, 41–43. MR1489497

[30] J. A. M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, Internat. J. Modern Phys. A **14** (1999), no. 13, 2037–2076, DOI 10.1142/S0217751X99001032. MR1693541 (2000g:65001)

[31] M. Waldschmidt, *A course on multizeta values,* The Institute of Mathematical Sciences, Special year in number theory at IMS, Chennai (India), April 2011; available at http://www.math.jussieu.fr/~miw/articles/pdf/MZV2011IMSc.pdf

[32] J. Wolstenholme, *On certain properties of prime numbers,* Quart. J. Pure Appl. Math. **5** (1862), 35–39.

[33] Don Zagier, *Values of zeta functions and their applications*, First European Congress of Mathematics, Vol. II (Paris, 1992), Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 497–512. MR1341859 (96k:11110)

[34] D. Zagier, *Evaluation of the multiple zeta values $\zeta(2,\ldots,2,3,2,\ldots,2)$*, Ann. of Math. (2) **175** (2012), no. 2, 977–1000, DOI 10.4007/annals.2012.175.2.11. MR2993756

[35] Jianqiang Zhao, *Wolstenholme type theorem for multiple harmonic sums*, Int. J. Number Theory **4** (2008), no. 1, 73–106, DOI 10.1142/S1793042108001146. MR2387917 (2008m:11006)

[36] Jianqiang Zhao, *Mod p structure of alternating and non-alternating multiple harmonic sums*, J. Théor. Nombres Bordeaux **23** (2011), no. 1, 299–308 (English, with English and French summaries). MR2780631 (2012g:11160)

[37] J. Zhao, *Finiteness of p-divisible sets of multiple harmonic sums,* Annales des Sciences Mathematiques du Quebec, published online Dec. 2, 2011.

[38] J. Zhao, *private communication.* June 2012.

[39] S. A. Zlobin, *Generating functions for the values of a multiple zeta function*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **2** (2005), 55–59, 73 (Russian, with Russian summary); English transl., Moscow Univ. Math. Bull. **60** (2005), no. 2, 44–48. MR2152592 (2006d:11104)

Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 4R2

E-mail address: `hessamik@gmail.com`

Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 4R2

E-mail address: `hessamit@gmail.com`

Current address: Department of Mathematics and Computing Science, Saint Mary's University, Halifax, Nova Scotia, Canada B3H 3C3

Dipartimento di Matematica, Università di Roma "Tor Vergata", 00133 Roma, Italy

E-mail address: `tauraso@mat.uniroma2.it`