Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 5 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

New Results and Applications on Robust Stability and Tracking of Pseudo-Quadratic Uncertain MIMO Discrete-Time Systems

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For the above class of systems, some results and systematic methods are provided, which allow to solve, "via majorant system", several analysis problems of robust stability, stabilization and tracking of a generic reference signal with bounded variation in presence of a generic disturbance with bounded variation.

The utility and the efficiency of the main results proposed in this paper are illustrated with three significant examples.

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GJSFR-F Classification : MSC 2010: 11K45, 11R11

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I. Introduction

There exist many discrete-time systems linear but with uncertain parameters, uncertain pseudo-linear and with bounded coefficients, uncertain pseudo-quadratic and with bounded coefficients, having, in several cases, a bounded nonlinear additional term or that are solicited with non standard inputs, for whose not always an equilibrium state or the steady-state response exists.

Regarding this, consider: the demographic, economic, resource and traffic management, environmental, agricultural, biological, medical, sampled systems, etc.

For a given system of the above mentioned significant class, it is important to obtain an estimate of its evolution in a finite or infinite time, for all the initial conditions belonging to a prefixed compact set and for all the values of uncertainties, or to design a controller in order to practically stabilize it or, finally, to design a controller to force this system to track a sufficiently regular prefixed trajectory with a bounded error.

Despite the numerous scientific papers available in literature (e.g. [1]-[16]), some of which also very recent (e.g. [19], [23]-[25]), the following practical limitations remain: *1.* the considered classes of systems are often with little relevant interest to engineers; *2.* the considered signals (references, disturbances, etc.) are almost always standard waveforms (polynomial and/or sinusoidal ones); *3.* the controllers are often not very robust and/or do not allow satisfying more than a single specification.

In this paper a systematic method, in a more general framework with respect to the ones proposed in literature (see e.g. [1]-[16],[19], [23]-[25]), for the analysis, for the practical stabilization and the tracking of a significant class of pseudo-linear and pseudo-quadratic uncertain MIMO systems, with additional bounded nonlinearities and/or bounded disturbances, is considered. Some of these results are an extension of analogous results for the continuous-time systems provided in [18],[21],[22],[26],[27].

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In detail, in Section II the considered class of pseudo-quadratic uncertain MIMO systems is presented, the definition of majorant system is given and, finally, several analysis and synthesis problems are formulated. In Section III some basic theorems are stated, which allow to determine, by calculating the eigenvalues of appropriate matrices only in correspondence of the extreme values of the uncertain parameters and of the vertices of suitable polytopes of the state space, if the system is pseudo-quadratic, a majorant system of a pseudo-quadratic uncertain MIMO system. In Section IV some methods are proposed to analyze the robust practical stability, via majorant system, of a pseudo-quadratic uncertain MIMO system.

In Section V some theorems are provided, which allow to determine a state feedback control law to stabilize a pseudo-quadratic uncertain MIMO system and to design an integral controller with state reaction, that forces a LTI uncertain MIMO system to track a generic reference signal with bounded variation in presence of a generic disturbance with bounded variation too.

In Section VI the main results proposed in this paper are illustrated with three significant examples.

II. Problem formulation

Consider the following class of nonlinear discrete-time dynamic systems

$$\begin{aligned} x_{k+1} &= a(x_k,p,k) + A_0(x_k,w_k,p,k)x_k + \left(\sum_{i=1}^{n} A_i(x_k,w_k,p,k)x_{ik}\right)x_k + B(x_k,p,k)u_k \\ y_k &= C(x_k,p,k)x_k, \end{aligned} \tag{1}$$

where: $k \in Z$ is the time; $x \in R^n$ is the state; $u \in U \subset R^h$, with U compact set, is the control and/or disturbance input; $w \in W \subset R^l$, with W compact set, is a possible parametric input; $y \in R^m$ is the output; $p \in [p^-, p^+] \subset R^\nu$ is the vector of uncertain parameters; $A_i \in R^{n\times n}, i = 0,1,...,n,$ $B \in R^{n\times h}$ and $C \in R^{m\times n}$ are bounded matrices continuous with respect to their arguments, with C of full rank, and multilinear with respect to p and w; $a \in R^n$ is a bounded continuous vector multilinear with respect to p, which models particular nonlinearities of the system.

The following preliminary notations and definitions are introduced:

$$\|x\|_P = \sqrt{x^T P x},\ \ \|x\| = \|x\|_I = \sqrt{x^T x},\ \ S_{P,\rho} = \left\{x : \|x\|_P \le \rho\right\},\ \ C_{P,\rho} = \left\{x : \|x\|_P = \rho\right\},\ \ \hat{C}_{P,\rho} \supseteq C_{P,\rho}, \tag{2}$$

where $P \in R^{n\times n}$ is a symmetric and positive definite (*p.d.*) matrix, x^T is the transpose of $x \in R^n$ and $\hat{C}_{P,\rho}$ is a compact set; $\lambda_{\max}(Q) = \max_{i=1,2,...,n} \|\lambda_i(Q)\|$, where $Q \in R^{n\times n}$;

$$\tau_{\max}(A) = -\frac{1}{\ln\left(\lambda_{\max}(A)\right)}, \quad \text{where } A \in R^{n\times n}. \tag{3}$$

Definition 1. Give the system (1), an hyper-interval $W_I = [w^-, w^+] \subseteq W$, a $\delta \ge 0$ and a *p.d.* symmetric matrix $P \in R^{n\times n}$. A positive first-order system $\rho_{k+1} = f(\rho_k, \delta), \rho_0 = \|x_0\|_P$, $\upsilon_k = \eta(\rho_k)$, where $\rho_k = \|x_k\|_P$, such that $\|y_k\| \le \upsilon_k$, $\forall k \ge 0,\ \forall x_0 \in R^n,\ \forall w_k \in W_I,\ \forall u_k \in U : \|u_k\| \le \delta$ and $\forall p \in [p^-, p^+]$, is said to be majorant system of the system (1).

Since a majorant system of a system belonging to the class of uncertain nonlinear systems (1), that obviously includes also the linear uncertain system, is a first-order time-invariant system, this system can be used to easily solve numerous analysis and synthesis problems. E.g. the analysis of practical stability, the analysis of the performances decreasing of a control system, or its impossibility to guarantee given performances in the hypothesis of deterioration and/or faults of its components (fault tolerance), the robust stabilization, the robust tracking.

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The aim of the present paper is to establish new fundamental results which easily allow to determine a majorant system of the system (1) and to provide systematic methods, via majorant system, to solve, for brevity, only some main problems between the numerous analysis and synthesis above mentioned ones.

III. Basic Theorems

To easily determine and analyze a majorant system of the system (1) the following basic theorems are necessary.

Theorem 1. Let $P \in R^{n\times n}$ be a symmetric *p.d.* matrix, $Q(x,p) \in R^{n\times n}$ be a symmetric *s.p.d.* matrix, $g(x,p) \in R^n$ be a vector, continuous with respect to $x \in R^n$ and $p \in \wp \subset R^{\nu}$, with $\wp$ compact set; then $\forall \rho \ge 0$ it is:

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T Q(x,p)x \le \max_{x\in C_{P,\rho},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^2 \le \max_{x\in \hat{C}_{P,\rho},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^2 \tag{4}$$

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T g(x,p) \le \max_{x\in C_{P,\rho},\, p\in\wp} \sqrt{g(x,p)^T P^{-1} g(x,p)}\,\rho \le \max_{x\in \hat{C}_{P,\rho},\, p\in\wp} \sqrt{g(x,p)^T P^{-1} g(x,p)}\,\rho. \tag{5}$$

Moreover, if $Q(x,p)$ is linear with respect to x it is

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T Q(x,p)x \le \max_{x\in C_{P,1},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^3 \le \max_{x\in \hat{C}_{P,1},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^3, \tag{6}$$

while if $Q(x,p)$ is quadratic with respect to x it turns out to be

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T Q(x,p)x \le \max_{x\in C_{P,1},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^4 \le \max_{x\in \hat{C}_{P,1},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^4. \tag{7}$$

More in general, if $Q(x,p)$ is continuous, pseudo-linear with respect to x and with bounded coefficients, i.e. if $Q(x,p) = \sum_{i=1}^{n} Q_i(x,p)x_i$, with bounded $Q_i(x,p)$, then

$$\begin{aligned}\max_{x\in C_{P,\rho},\, p\in\wp} x^T\left(\sum_{i=1}^{n} Q_i(x,p)x_i\right)x &\le \max_{\substack{x\in C_{P,1},\, p\in\wp\\ z\in R^n}} \lambda_{\max}\left(\sum_{i=1}^{n} Q_i(z,p)x_i P^{-1}\right)\rho^3 \le \\ &\le \max_{\substack{x\in \hat{C}_{P,1},\, p\in\wp\\ z\in R^n}} \lambda_{\max}\left(\sum_{i=1}^{n} Q_i(z,p)x_i P^{-1}\right)\rho^3,\end{aligned} \tag{8}$$

whereas if $Q(x,p)$ is continuous, pseudo-quadratic with respect to x and with bounded coefficients, i.e. if $Q(x,p) = \sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}(x,p)x_i x_j$, with bounded $Q_{ij}(x,p)$, then

$$\begin{aligned}\max_{x\in C_{P,\rho},\, p\in\wp} x^T\left(\sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}(x,p)x_i x_j\right)x &\le \max_{\substack{x\in C_{P,1},\, p\in\wp\\ z\in R^n}} \lambda_{\max}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}(z,p)x_i x_j P^{-1}\right)\rho^4 \le \\ &\le \max_{\substack{x\in \hat{C}_{P,1},\, p\in\wp\\ z\in R^n}} \lambda_{\max}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}(z,p)x_i x_j P^{-1}\right)\rho^4.\end{aligned} \tag{9}$$

Proof. Note that, if $f(x) \in R$ is a continuous function with respect to $x \in R^n$ and X_1, X_2, $X_1 \subset X_2$, are compact subsets of R^n, it is $\max_{x \in X_1} f(x) \le \max_{x \in X_2} f(x)$. Moreover, since P is *p.d.*, there exists a symmetric nonsingular matrix S such that $P = S^2$. Hence, by posing $z = Sy$, it is

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T Q(x,p)x \le \max_{y\in C_{P,\rho}, x\in C_{P,\rho},\, p\in\wp} y^T Q(x,p)y = \max_{z\in C_{I,\rho}, x\in C_{P,\rho},\, p\in\wp} z^T S^{-1}Q(x,p)S^{-1}z = \max_{z\in C_{I,\rho}, x\in C_{P,\rho},\, p\in\wp} \lambda_{\max}(S^{-1}Q(x,p)S^{-1})z^T z =$$
$$= \max_{x\in C_{P,\rho},\, p\in\wp} \lambda_{\max}(SS^{-1}Q(x,p)S^{-1}S^{-1})\rho^2 = \max_{x\in C_{P,\rho},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^2 \le \max_{x\in \hat{C}_{P,\rho},\, p\in\wp} \lambda_{\max}(Q(x,p)P^{-1})\rho^2, \qquad (10)$$

and so (4). Similarly

$$\max_{x\in C_{P,\rho},\, p\in\wp} x^T g(x,p) \le \max_{y\in C_{P,\rho}, x\in C_{P,\rho},\, p\in\wp} y^T g(x,p) = \max_{z\in C_{I,\rho}, x\in C_{P,\rho},\, p\in\wp} z^T S^{-1} g(x,p) \le$$
$$\le \max_{z\in C_{I,\rho}, x\in C_{P,\rho},\, p\in\wp} \|z\|\|S^{-1}g(x,p)\| = \max_{x\in C_{P,\rho},\, p\in\wp} \sqrt{g(x,p)^T P^{-1} g(x,p)}\,\rho \le \max_{x\in \hat{C}_{P,\rho},\, p\in\wp} \sqrt{g(x,p)^T P^{-1} g(x,p)}\,\rho, \qquad (11)$$

and hence (5).
Inequalities (6) easily follow from the fact that, if $Q(x,p)$ is linear with respect to x, $Q(x,p)\big|_{x\in C_{P,\rho}} = Q(x,p)\big|_{x\in C_{P,1}}\rho$. Inequalities (7), (8), (9) analogously follow.

Remark 1. Theorem 1 can be easily generalized to the case in which $Q(x,p)$ is a homogeneous function of degree ν with respect to x.

Remark 2. Clearly, if $Q(x,p)$ and $g(x,p)$ are independent of x, inequalities (4) and (5) hold with the equal sign. Moreover, if $Q(x,p)$ depends on x, it is quite difficult to compute $\max_{x\in C_{P,\rho}} x^T Q(x,p)x$ because $x^T Q(x,p)x\big|_{x\in C_{P,\rho}}$ has, in general, different points of relative maximum, of relative minimum and of "inflection"; the second and third members of (4), ((6),(8)), ((7),(9)) allow an easier computation of an upper bound on $x^T Q(x,p)x\big|_{x\in C_{P,\rho}}$ proportional to ρ^2, ρ^3, ρ^4 , respectively, as it will be shown later on. A similar talking is valid if $g(x,p)$ depends on x.

Theorem 2. Let $P \in R^{n\times n}$ be a *p.d.* symmetrix matrix, $C \in R^{m\times n}$ be a matrix with rank m and $B \in R^{n\times h}$ be a matrix with rank h. Then

$$\upsilon = \|Cx\| \le \sqrt{\lambda_{\max}(CP^{-1}C^T)}\,\rho, \quad \forall x \in C_{P,\rho} \qquad (12)$$

$$\|Bu\|_P \le \sqrt{\lambda_{\max}(B^T PB)}\,\|u\|, \quad \forall u \in R^h. \qquad (13)$$

Proof. By taking into account that, if F is a real matrix $m\times n$ with rank m, the matrix $F^T F$ has $n-m$ null eigenvalues and m positive eigenvalues equal to the ones of FF^T and, by posing $z = Sx$, where S is a symmetric nonsingular matrix such that $P = S^2$, it is

$$\upsilon = \|Cx\| = \sqrt{x^T C^T Cx} = \sqrt{z^T S^{-1} C^T C S^{-1} z} \le \sqrt{\lambda_{\max}\left(S^{-1}C^T C S^{-1}\right)}\,\|z\| =$$
$$= \sqrt{\lambda_{\max}\left(CS^{-1}S^{-1}C^T\right)}\,\|Sx\| = \sqrt{\lambda_{\max}\left(CP^{-1}C^T\right)}\,\rho \qquad (14)$$

and so (12). The proof of (13) easily follows.

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Theorem 3. Let $A = \sum_{i_1, i_2, ..., i_\mu \in \{0,1\}} A_{i_1 i_2 ... i_l} \pi_1^{i_1} \pi_2^{i_2} ... \pi_\mu^{i_\mu} \in R^{nxn}$ be a matrix multilinearly depending on the parameters $[\pi_1 \ \ \pi_2 \ \ ... \ \ \pi_\mu]^T = \pi \in \prod = \{\pi \in R^\mu : \pi^- \le \pi \le \pi^+\}$ and let $P \in R^{n \times n}$ be a symmetric *p.d.* matrix. Then the maximum of $\lambda_{\max\limits_{\pi \in \prod}}(QP^{-1})$, where $Q = A^T PA$, is attained in one of the 2^μ vertices of $\prod$.

Proof. Note that for a constant $\pi_j, j \ne i$, it is $A = A_0 + \pi_i A_1, \ \pi_i \in [\pi_i^-, \ \pi_i^+]$. Moreover, since for Theorem 1 and Remark 2 it is $\lambda_{\max}(QP^{-1}) = \max\limits_{x \in C_{P,1}} x^T Qx$, it turns out to be

$$\begin{aligned} &\max_{\pi_i \in [\pi_i^-, \pi_i^+]} \lambda_{\max}\left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)P^{-1}\right) = \max_{\pi_i \in [\pi_i^-, \pi_i^+], \, x \in C_{P,1}} x^T \left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)\right)x = \\ &\max_{\pi_i \in [\pi_i^-, \pi_i^+], \, x \in C_{P,1}} x^T \left(A_1^T PA_1 \pi_i^2 + \left(A_0^T PA_1 + A_1^T PA_0\right)\pi_i + A_0^T PA_0\right)x. \end{aligned} \tag{15}$$

Therefore, said $\hat{\pi}_i, \ \hat{x}$ the point of maximum of $f(\pi_i, x) = x^T \left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)\right)x\Big|_{x \in C_{P,1}, \pi_i \in [\pi_i^-, \pi_i^+]}$, it is

$$\begin{aligned} &\max_{\pi_i \in [\pi_i^-, \pi_i^+]} \lambda_{\max}\left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)P^{-1}\right) = \max_{\pi_i \in [\pi_i^-, \pi_i^+], \, x \in C_{P,1}} x^T \left((A_0 + \pi_i A_1)^T P(A_0 + \pi_i A_1)\right)x = \\ &\max_{\pi_i \in [\pi_i^-, \pi_i^+]} \left(\hat{x}^T A_1^T PA_1 \hat{x} \pi_i^2 + 2\hat{x}^T A_0^T PA_1 \hat{x} \pi_i + \hat{x}^T A_0^T PA_0 \hat{x}\right) = \max_{\pi_i \in [\pi_i^-, \pi_i^+]} \left(\hat{c}_2 \pi_i^2 + \hat{c}_1 \pi_i + \hat{c}_0\right). \end{aligned} \tag{16}$$

Since $\hat{c}_2 = \hat{x}^T A_1^T PA_1 \hat{x} \ge 0$, it is

$$\max_{\pi_i \in [\pi_i^-, \pi_i^+]} \left(\hat{c}_2 \pi_i^2 + \hat{c}_1 \pi_i + \hat{c}_0\right) = \max\left\{\hat{c}_2(\pi_i^-)^2 + \hat{c}_1 \pi_i^- + \hat{c}_0, \quad \hat{c}_2(\pi_i^+)^2 + \hat{c}_1 \pi_i^+ + \hat{c}_0\right\}. \tag{17}$$

The proof easily follows from (16) and (17).

From Theorem 3 the next Corollary easily follows.

Corollary 1. Let $A = \sum_{i_1, i_2, ..., i_\mu \in \{0,1,2\}} A_{i_1 i_2 ... i_l} \pi_1^{i_1} \pi_2^{i_2} ... \pi_\mu^{i_\mu} \in R^{nxn}$ be a matrix multiquadratic depending on the parameters $[\pi_1 \ \ \pi_2 \ \ ... \ \ \pi_\mu]^T = \pi \in \prod = \{\pi \in R^\mu : \pi^- \le \pi \le \pi^+\}$ and $P \in R^{n \times n}$ a symmetric *p.d.* matrix. Then an upper bound of the maximum of $\lambda_{\max\limits_{\pi \in \prod}}(QP^{-1})$, where $Q = A^T PA$, is equal to the maximum of $\lambda_{\max}(A_e^T PA_e P^{-1})$, attained in one of the $2^{2\mu}$ vertices of $\prod \times \prod$, where A_e is obtained from matrix A by substituting the product $\pi_i \pi_{\mu+i}$, $i = 1, 2, ..., \mu$, to π_i^2.

Theorem 3 can be generalized as follows.

Theorem 4. Let A be the matrix

$$A = \sum_{i_1, i_2, ..., i_\mu \in \{0,1\}} A_{i_1 i_2 ... i_\mu} g_1(\pi_1)^{i_1} g_2(\pi_2)^{i_2} ... g_\mu(\pi_l)^{i_\mu} \in R^{nxn}, \tag{18}$$

in which $[\pi_1 \ \ \pi_2 \ \ ... \ \ \pi_\mu]^T = \pi \in \prod = \{\pi \in R^\mu : \pi^- \le \pi \le \pi^+\}$ and each function $g_i, i = 1, 2, .., \mu,$ is continuous with respect to π_i, and let $P \in R^{n \times n}$ be a *p.d.* symmetric matrix. Then the maximum of $\lambda_{\max}(QP^{-1})$, where $Q = A^T PA$, is attained in one of 2^μ vertices of Γ, where

$$\Gamma = \left\{ \gamma \in R^{\mu} : \min[g_1 ... g_{\mu}] \le \gamma \le \max[g_1 ... g_{\mu}] \right\}. \tag{19}$$

Proof. The proof follows from Theorem 3, by making the change of variable $\gamma = g(\pi) = \left[g_1(\pi_1)\ g_2(\pi_2) ... g_{\mu}(\pi_{\mu}) \right]$ and by noting that $\max_{\pi \in \Pi} \lambda_{\max}\left(Q(g(\pi))P^{-1} \right) = \max_{\gamma \in \Gamma} \lambda_{\max}\left(Q(\gamma)P^{-1} \right)$.

Remark 3. For Theorem 3 and Corollary 1 the computation of $\max_{x \in \hat{C}_{P,\rho}} \lambda_{\max}(Q(x,p)P^{-1})$, if $Q(x,p)$ is linear or quadratic with respect to x, is very easy if $\hat{C}_{P,1} \supseteq C_{P,1}$ is an hyper-rectangle (or a polytope decomposable into hyper-rectangles). To this aim, to compute the vertices of $\hat{C}_{P,1}$, it is easy to prove that the point of contact of the hyper-line orthogonal to the versor $e_i, i = 1, 2, ..., n$, of R^n and tangent to the hyper-ellipse $C_{P,1}$ (see Fig. 1) is

$$p_i = \frac{P^{-1} e_i}{\sqrt{e_i^T P^{-1} e_i}}. \tag{20}$$

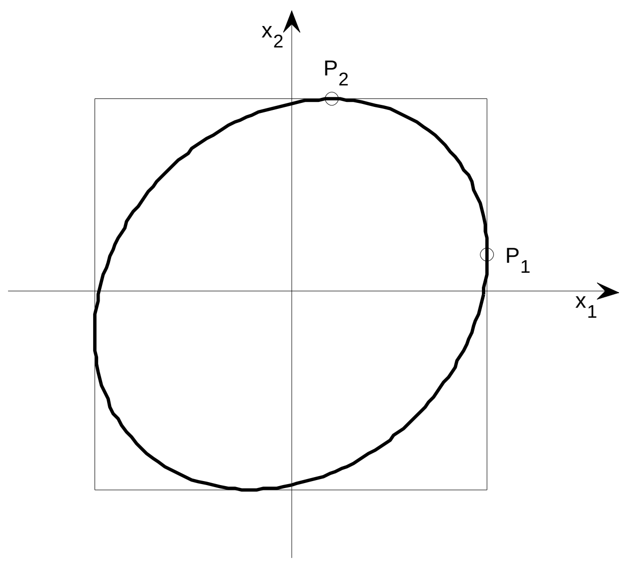

Figure 1 : Points of contact of the hyper-rectangle circumscribed to $C_{P,1}$.

Theorem 5. Consider the quadratic discrete-time system

$$\rho_{k+1} = a_2 \rho_k^2 + a_1 \rho_k + a_0, \quad a_0 \ge 0,\ a_1 \ge 0,\ a_2 \ge 0, \quad \rho_0 \ge 0. \tag{21}$$

If $a_1 > 1$ the system (21) is unstable. If $0 < a_1 < 1$ and $a_2 = 0, \forall \rho_0 \ge 0$ the system evolves toward the equilibrium point $\rho_e = \frac{a_0}{1 - a_1}$ with time constant $\tau = -\frac{1}{\ln a_1}$. Finally, if $a_2 > 0,\ 0 < a_1 < 1$ and $(1 - a_1)^2 - 4a_2 a_0 > 0$, said ρ_1, ρ_2, $\rho_1 < \rho_2$, the roots of the algebraic equation $a_2 \rho^2 + (a_1 - 1)\rho + a_0 = 0$, $\forall \rho_0 \in [0, \rho_2)$, the system evolves toward the equilibrium point $\rho_e = \rho_1$ with time constant of the linearized system equal to $\tau_l = -\frac{1}{\ln \alpha_1}$, where $\alpha_1 = a_1 + 2a_2 \rho_e$.

Proof. If $a_1 > 1$ it is always $\rho_{k+1} > \rho_k$ and, hence, the system is unstable. The remaining part of the proof easily follows by making standard manipulations and from Fig. 2.

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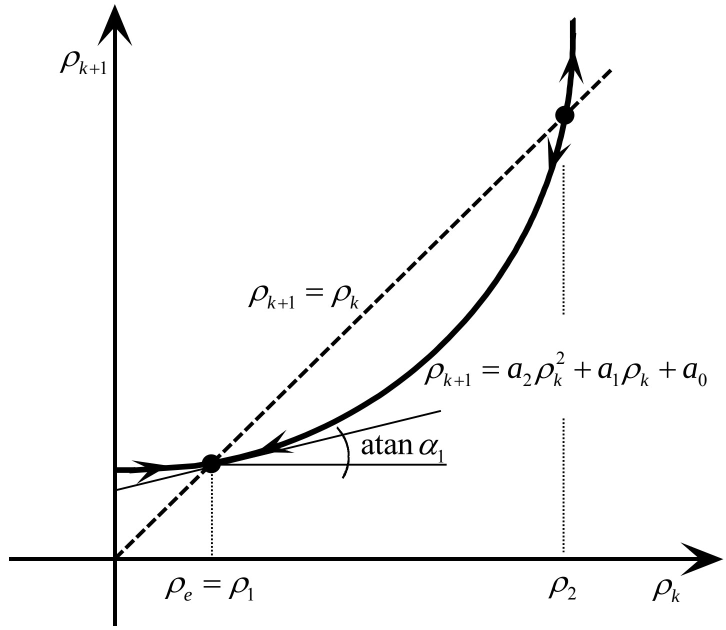

Figure 2 : Graphical illustration of Theorem 5.

Theorem 6. Let $A \in R^{n\times n}$ be a matrix with ν real distinct eigenvalues λ_i, $i=1, \ldots, \nu$, and with $\mu = \frac{n-\nu}{2}$ distinct pair of complex conjugate eigenvalues $\lambda_{h\pm} = \alpha_h \pm j\omega_h$, $h=1, \ldots, \mu$; moreover, let u_i, $i=1, \ldots, \nu$, and $u_{h\pm} = u_{ah} \pm ju_{bh}$, $h=1, \ldots, \mu$, be the associated eigenvectors. Then, by denoting with Z^* the conjugate transpose of the matrix of the eigenvectors $Z = \left[u_1 \ldots u_\nu \; u_{a1} + ju_{b1} \; u_{a1} - ju_{b1} \ldots u_{a\mu} + ju_{b\mu} \; u_{a\mu} - ju_{b\mu}\right]$ and with $\Lambda = diag\left(\lambda_1, \ldots, \lambda_\nu, \alpha_1 + j\omega_1, \alpha_1 - j\omega_1, \ldots, \alpha_\mu + j\omega_\mu, \alpha_\mu - j\omega_\mu\right)$ the diagonal matrix of the eigenvalues, the matrix

$$P = \left(ZZ^*\right)^{-1} = \left[\sum_{i=1}^{\nu} u_i u_i^T + 2\sum_{h=1}^{\mu}\left(u_{ah}u_{ah}^T + u_{bh}u_{bh}^T\right)\right]^{-1} \tag{22}$$

turns out to be always *p.d.*; moreover it is

$$\lambda_{\max}\left(QP^{-1}\right) = \lambda_{\max}^2\left(A\right) \Rightarrow \tau_{\max}\left(A\right) = -\frac{2}{\ln\left(\lambda_{\max}\left(QP^{-1}\right)\right)}, \tag{23}$$

where $Q = A^T PA$.

Proof. As, for hypothesis, the eigenvalues of A are distinct, the matrix of the eigenvectors Z is nonsingular. Hence the matrix ZZ^* is *p.d.* and, therefore, also its inverse P is *p.d.* . Moreover, since $A = Z\Lambda Z^{-1}$, it is

$$QP^{-1} = A^T PAP^{-1} = A^* PAP^{-1} = \left(Z^*\right)^{-1}\Lambda^* Z^*\left(Z^*\right)^{-1} Z^{-1}Z\Lambda Z^{-1}ZZ^* = \left(Z^*\right)^{-1} Z^{-1}Z\Lambda Z^{-1} = \left(Z^*\right)^{-1}\left(\Lambda^*\Lambda\right)Z^*. \tag{24}$$

Hence the eigenvalues of QP^{-1} are $\lambda_j^*\lambda_j = \left\|\lambda_j\right\|^2$, $j=1,\ldots,n$, from which the proof follows.

IV. Stability Analysis

First, consider the linear and time-invariant uncertain MIMO system

$$
\begin{aligned}
&x_{k+1} = A(p)x_k + B(p)u_k,\quad y_k = C(p)x_k,\quad \text{with}\\
&A(p) = \sum_{i_1,i_2,\dots,i_\nu\in\{0,1\}} A_{i_1 i_2\dots i_l}\, p_1^{i_1} p_2^{i_2}\dots p_\nu^{i_\nu} \in R^{nxn},\ B(p) = \sum_{i_1,i_2,\dots,i_\nu\in\{0,1\}} B_{i_1 i_2\dots i_l}\, p_1^{i_1} p_2^{i_2}\dots p_\nu^{i_\nu} \in R^{nxn},\\
&C(p) = \sum_{i_1,i_2,\dots,i_\nu\in\{0,1\}} C_{i_1 i_2\dots i_l}\, p_1^{i_1} p_2^{i_2}\dots p_\nu^{i_\nu} \in R^{nxn},\ p\in[p^-,p^+]\subset R^\nu.
\end{aligned}
\tag{25}
$$

It is well-known that this system is asymptotically stable if $a = \max_{p\in[p^-,p^+]} \lambda_{\max}(A) < 1$. If the goal is only to study the asymptotic stability without calculating a, then the Jury criterion to the characteristic polynomial $d(\lambda,p) = \det(\lambda I - A(p))$ can be applied or it is possible to use one of the several methods to establish the definite positivity of the matrix $P(p)$, which is solution of the Lyapunov equation $A^T(p)P(p)A(p) - P(p) = -Q$, with Q $p.d.$. As it can be easily realized, both the methods are very onerous because of the strong nonlinearity with respect to the parameters p both of $d(\lambda,p)$ and $P(p)$. Clearly the computation of a is even more onerous.

Let P be a $p.d.$ symmetric matrix and fixed a $\rho_k \ge 0,\ \forall x_k \in C_{P,\rho_k}$ from the first of (25) for $u_k = 0$ and from Theorem 1 easily follows that

$$
\rho_{k+1} = \|x_{k+1}\|_P = \sqrt{x_k^T A^T(p)PA(p)x_k}\Big|_{x_k\in C_{p,\rho_k}} \le a\rho_k \Rightarrow \rho_k \le a^k \rho_0,
\tag{26}
$$

where

$$
a = \sqrt{\max_{p\in[p^-,p^+]} \lambda_{\max}(Q(p)P^{-1})} = \sqrt{\max_{p\in V_p} \lambda_{\max}(Q(p)P^{-1})},\ Q(p) = A^T(p)PA(p),
\tag{27}
$$

in which V_p is the set of the 2^ν vertices of the hyper-rectangle $[p^-, p^+]$.

It is interesting to note that the last of (26) provides an upper bound of the free evolution of the system (25) $\forall p\in[p^-,p^+]$. Clearly the goodness of this bound depends on P; a not appropriate matrix P could provide a value of a greater than 1 also when $p^- = p^+$ and the system is asymptotic stable.

If for a given $\hat p\in[p^-,p^+]$ the matrix $A(\hat p)$ has distinct eigenvalues, condition almost always verified, the relative matrix P given by (22) is always $p.d$ and for (23) it is always that $a = \sqrt{\lambda_{\max}(Q(\hat p)P^{-1})} = \lambda_{\max}(A(\hat p))$, also when $A(\hat p)$ has not all the eigenvalues with magnitude less than 1.

From this reasoning, from the theorems stated in Section III and from the fact that $\|x_1 + x_2\|_P \le \|x_1\|_P + \|x_2\|_P,\ \forall x_1, x_2 \in R^n$, the following theorems easily derive.

Theorem 7. Give a matrix $A(p) = \sum_{i_1,i_2,\dots,i_\nu\in\{0,1\}} A_{i_1 i_2\dots i_l}\, p_1^{i_1} p_2^{i_2}\dots p_\nu^{i_\nu} \in R^{nxn}$, $p\in\wp\subset R^\nu$, with $\wp$ a compact set. An estimate of the $\max_{p\in\wp} \lambda_{\max}(A(p))$ can be obtained by covering the set $\wp$ with N hyper-rectangles $[p_i^-, p_i^+]$ and by computing the maximum of $\left\{a_i : a_i = \sqrt{\max_{p\in V_{ip}} \lambda_{\max}\left(Q(p_i)P_i^{-1}\right)},\ Q(p_i) = A^T(p_i)P_iA(p_i),\ i = 1,2,..,N\right\}$, where $p_i = \dfrac{p_i^- + p_i^+}{2}$ or it is a near value, $P_i = \left(Z_i Z_i^*\right)^{-1}$, where Z_i is the eigenvectors matrix of $A(p_i)$ and V_{ip} is the set of vertices of $[p_i^-, p_i^+]$.

Theorem 8. Suppose there exist a $\hat{p} \in [p^-, p^+]$ such that the dynamic matrix $A(\hat{p})$ of the multivariable uncertain system (25) has distinct eigenvalues; said $P = \left(ZZ^*\right)^{-1}$, where Z is the eigenvectors matrix of $A(\hat{p})$, a majorant system of the system (25) is

$$\rho_{k+1} = a\rho_k + b\delta, \quad \upsilon_k = c\rho_k , \tag{28}$$

in which: $\rho_k = \|x_k\|_P$, $\delta \geq \|u_k\|$, $\upsilon_k = \|y_k\|$, $a = \sqrt{\max_{p\in V_p} \lambda_{\max}\left(Q(p)P^{-1}\right)}$, $Q(p) = A^T(p)PA(p)$, $b = \sqrt{\max_{p\in V_p} \lambda_{\max}\left(B^T(p)PB(p)\right)}$, $c = \sqrt{\max_{p\in V_p} \lambda_{\max}\left(C(p)P^{-1}C^T(p)\right)}$, where V_p is the set of the 2^ν vertices of the hyper-rectangle $[p^-, p^+]$.

Remark 4. By using the theorems stated in Section III, Theorem 8 can be easily extended also to the case in which the system (25) is pseudo-linear, i.e. to the case in which the matrices A, B, C depend also on x_k and k and they are bounded.

Now consider the quadratic uncertain MIMO system

$$\begin{aligned} x_{k+1} &= A_0(p)x_k + \left(\sum_{i=1}^{n} A_i(p)x_{ik}\right)x_k + B(p)u_k \\ y_k &= C(p)x_k, \end{aligned} \tag{29}$$

where $A_i(p), i = 0,1,\ldots,n$, $B(p)$, $C(p)$ are multilinear functions of $p \in [p^-, p^+]$. The following theorem holds.

Theorem 9. Suppose that there exists a $\hat{p} \in [p^-, p^+]$ such that the matrix $A_0(\hat{p})$ in (29) has distinct eigenvalues; by choosing $P = \left(ZZ^*\right)^{-1}$, where Z is the eigenvectors matrix of $A_0(\hat{p})$, a majorant system of the system (29) turns out to be

$$\rho_{k+1} = a_1\rho_k + a_2\rho_k^2 + b\delta, \quad \upsilon_k = c\rho_k , \tag{30}$$

where: $\rho_k = \|x_k\|_P$, $\delta \geq \|u_k\|$, $\upsilon_k = \|y_k\|$, $a_1 = \sqrt{\max_{p\in V_p} \lambda_{\max}\left(Q_1(p)P^{-1}\right)}$, $Q_1(p) = A_0^T(p)PA_0(p)$,

$a_2 = \sqrt{\max_{p\in V_p, x\in V_x} \lambda_{\max}\left(Q_2(p,x)P^{-1}\right)}$, $Q_2(p,x) = \left(\sum_{i=1}^{n} A_i^T(p)x_i\right)P\left(\sum_{i=1}^{n} A_i(p)x_i\right)$, $b = \sqrt{\max_{p\in V_p} \lambda_{\max}\left(B^T(p)PB(p)\right)}$,

$c = \sqrt{\max_{p\in V_p} \lambda_{\max}\left(C(p)P^{-1}C^T(p)\right)}$, V_p is the set of the 2^ν vertices of the hyper-rectangle $[p^-, p^+]$ and V_x is the set of the 2^n vertices of the hyper-rectangle $[x^-, x^+]$ circumscribed to the hyper-circle $C_{P,1}$.

Proof. The proof easily follows from Theorems 1, 2, 3.

Remark 5. By using the results stated in Section III, Theorems 9 can be easily extended also to the case in which the system (29) is pseudo-quadratic, i.e. to the case in which the matrices A_i, B, C depends also on x_k and k and they are bounded.

Remark 6. Give the system $x_{k+1} = f(x_k, p)$, with $f(0,p) = 0$, being

$$\begin{aligned} x_{k+1} &= \partial f/\partial x_k\big|_{x_k=0} x_k + 1/2\left[x_k^T\left\{\partial^2 f_1/\partial x_{ik}\partial x_{jk}\right\}\Big|_{x_k=\xi} x_k \ \ldots\ x_k^T\left\{\partial^2 f_n/\partial x_{ik}\partial x_{jk}\right\}\Big|_{x_k=w_k} x_k\right]^T = \\ &= A_0(p)x_k + \left(\sum_{i=1}^{n} A_i(w_k,p)x_{ik}\right)x_k,\ 0 \leq w_k \leq x_k, \end{aligned} \tag{31}$$

if the second order partial derivatives of f are bounded then, by using the stated theorems, it is easy and systematic to estimate a region of asymptotic stability (RAS) of the origin and, moreover, to estimate the degree of linearity of the system by comparing a_1 to a_2 of a related majorant system.

V. Robust Stabilization and Tracking of a Pseudo-Quadratic Uncertain Mimo System

The problem of stabilization and tracking of an uncertain discrete-time system is very complex and, in some cases, it is impossible to solve, above all when the interval of uncertainties is very wide, unless an identification method of the parameters is used. It is sufficient to consider, e.g., the system $x_{k+1} = ax_k + u_k$, with a uncertain parameter belonging to an interval of amplitude greater than 2.
By considering bounded enough uncertainties, in many cases, the following results are very useful.

Theorem 10. Give the system (29), suppose that the state is measurable and that there exists a $\hat{p} \in [p^-, p^+]$ such that the couple $(A_0(\hat{p}),\ B(\hat{p}))$ is reachable. Said $\bar{\Lambda} = \{\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_n\}$ a symmetric set of n distinct complex numbers with $\lambda_{\max}(\lambda_i) \le 1$, a possible control law to stabilize or to increase the RAS of the system (29) is the following

$$u_k = -K_0 x_k - \left(\sum_{i=1}^{n} K_i x_{ik} \right) x_k, \tag{32}$$

where K_0 is such that $eig\left(A_0(\hat{p}) - B(\hat{p})K_0\right) = r\bar{\Lambda},\ r \in [0, 1)$ and K_i, with $i = 1, 2, ..., n,$ is such that to minimize $\|A_i(\hat{p}) - B(\hat{p})K_i\|_P$, where $P = \left(ZZ^*\right)^{-1}$, in which Z is the eigenvectors matrix of $A_0(\hat{p}) - B(\hat{p})K_0$.

Remark 7. If $rank\left(B(\hat{p})\right) > 1$, by posing $K0 = K_0$, $Ac = A_0(\hat{p})$, $Bc = B(\hat{p})$, $L = r\bar{\Lambda}$, the matrix K_0 can be computed by using the Matlab command *K0=place(Ac,Bc,L)*, based on the algorithm in [3], that uses the extra degrees of freedom to find a solution that minimizes the sensitivity of the closed-loop poles to perturbation in $A_0(\hat{p})$ or $B(\hat{p})$.
Instead, by posing $Ai = A_i(\hat{p})$, $Ki = K_i$, the matrices $K_i, i = 1, 2, ..., n,$ can be computed with the Matlab commands: *S=P^.5; Ki=pinv(S*Bc,S*Ai)* .
Finally the value of r can be used to reduce both a_1 and a_2, or to maximize $(1 - a_1)/a_2$.

Consider now the LTI uncertain MIMO plant described by

$$x_{k+1} = A(p)x_k + B(p)u_k + E(p)d_k, \quad y_k = C(p)x_k + D(p)d_k, \tag{33}$$

where: $x_k \in R^n$ is the state, $u_k \in R^r$ is the control signal, $d_k \in R^l$ is the disturbance, $y_k \in R^m$ is the output, $A(p), B(p), E(p), C(p), D(p)$ are matrices of suitable dimensions, which are multilinear functions of the parameter vector $p \in \wp$. Suppose that $\wp = \left\{p : p \in \left[p^-, p^+\right]\right\} \subset R^\nu$ is an hyper-rectangle and that the following reachable condition

$$rank\left[B(\hat{p}) \quad A(\hat{p})B(\hat{p}) \quad ... \quad A^{n-1}(\hat{p})B(\hat{p})\right] = n \tag{34}$$

is satisfied for at least a $\hat{p} \in \wp$.
In the following, for simplicity of notations, the dependency of $A(p), B(p), E(p), C(p), D(p)$ on p will be omitted.

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A main goal is :

1) to state new results to estimate, $\forall p \in \wp$ and for an assigned controller with integral (I) action of the system (33), the maximum time constant and the maximum tracking error of a generic reference signal r_k, with bounded variation $\delta r_k = r_k - r_{k-1}$, in presence of a generic disturbance d_k with bounded variation $\delta d_k = d_k - d_{k-1}$ too;

2) to design, if possible, a LTI controller such that to force the uncertain MIMO system (33) to track, with prefixed maximum time constant and maximum error, a generic reference signal r_k, with bounded variation, in presence of a generic disturbance d_k with bounded variation, $\forall p \in \wp$.

Regarding this, it is important to note that a relevant class of reference signals r_k with generic behavior but with bounded variation is very recurring in the practice and easily feasible by using digital technologies (see Fig. 3).

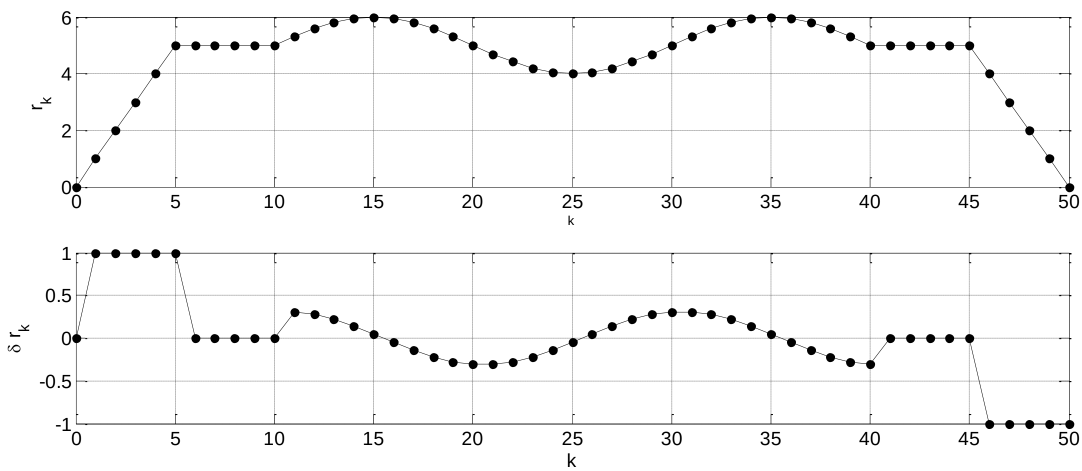

Figure 3 : A possible reference signal r_k with its bounded variation δr_k.

Remark 8. Note that, nowadays, the reference signal of a control system (e.g. manufacturing systems, …) is in general a non standard (not polynomial and/or cisoidal) signal whose variation is the desired "working velocity" (clearly finite, even if the planners and the builders make a great effort to make it as higher as it is possible).

The problems 1) and 2), not suitably solved in literature ([1]-[16],[19], [23]-[25]), are very important from a theoretical and practical point of view.

Among the several controllers available in literature, for brevity, in the following it will be considered only the well-known state feedback control scheme with an I action of Fig. 4.

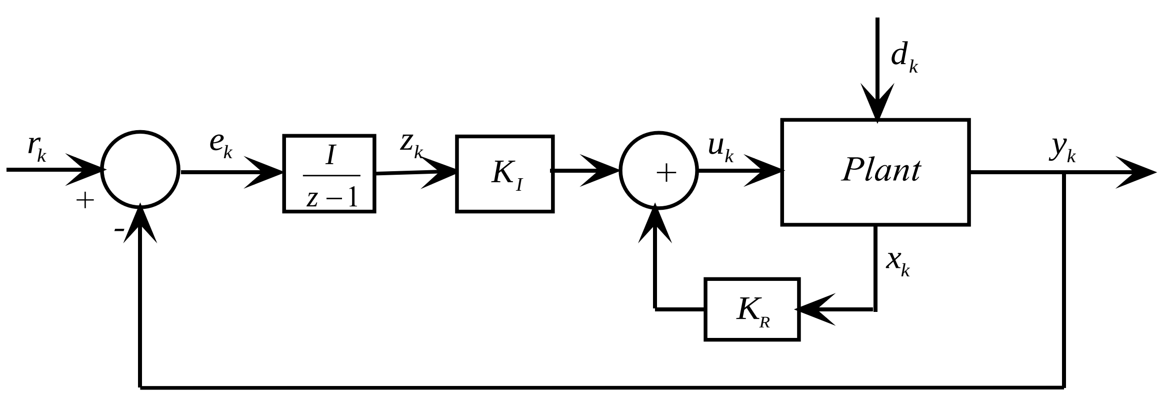

Figure 4 : State feedback control scheme with an I control action.

As regards suppose that there exists at least a $\hat{p} \in \wp$ such that, in addition to (34), also the following condition is satisfied

$$rank\begin{bmatrix} I-A & B \\ C & 0 \end{bmatrix} = n+m. \tag{35}$$

From the control scheme of Fig. 4 it easily derives that:

$$\begin{aligned} &x_{k+1} = Ax_k + Bu_k + Ed_k, \quad u = K_I z_k + K_R x_k, \quad e_k = r_k - Cx_k - Dd_k \\ &z_{k+1} = z_k + r_k - y_k = z_k + r_k - Cx_k - Dd_k\ , \end{aligned} \tag{36}$$

from which:

$$\xi_{k+1} = A_c\xi_k + B_c r_k + E_c d_k, \quad e_k = C_c\xi_k + r_k - Dd_k\ , \tag{37}$$

where:

$$\begin{aligned} &A_c = \begin{bmatrix} A+BK_R & BK_I \\ -C & I \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad E_c = \begin{bmatrix} E \\ -D \end{bmatrix}, \\ &C_c = \begin{bmatrix} -C & 0 \end{bmatrix}, \quad \xi = \begin{bmatrix} x \\ z \end{bmatrix}. \end{aligned} \tag{38}$$

In order to solve the problems 1) and 2) the following preliminary important result is necessary.

Theorem 11. The control system (37) can be described also by:

$$\zeta_{k+1} = A_c\zeta_k + B_c\delta r_{k+1} + E_c\delta d_{k+1}, \quad e_k = H_c\zeta_k, \quad H_c = \begin{bmatrix} 0 & I \end{bmatrix}, \tag{39}$$

or equivalently by

$$C_c\left(zI - A_c\right)^{-1}\left[B_c\ E_c\right] + \left[I\ -D\right] = H_c\left(zI - A_c\right)^{-1}\left[B_c\ E_c\right](z-1), \tag{40}$$

where $\delta r_{k+1} = r_{k+1} - r_k$, $\delta d_{k+1} = d_{k+1} - d_k$, $e_k = r_k - y_k$ is the tracking error and I is the identity matrix of appropriate order.

Proof. Posed

$$\left(zI - A_c\right)^{-1} = \begin{bmatrix} F_1 & F_2 \\ C & (z-1)I \end{bmatrix}^{-1} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}, \tag{41}$$

by using a known formula of the inverse of a partitioned matrix (see e.g. [17]), it is easy to prove that

$$\begin{aligned} &G_1 = \left(F_1 - \frac{F_2 C}{z-1}\right)^{-1}, \quad G_2 = \left(F_1 - \frac{F_2 C}{z-1}\right)^{-1}\frac{F_2}{z-1}, \\ &G_3 = -\frac{C}{z-1}\left(F_1 - \frac{F_2 C}{z-1}\right)^{-1}, \quad G_4 = \frac{1}{z-1}\left(I - C\left(F_1 - \frac{F_2 C}{z-1}\right)^{-1}\frac{F_2}{z-1}\right). \end{aligned} \tag{42}$$

From (38), from the last of (39) and by using (42), after tedious steps, (40) follows and hence the proof.

Remark 9. Note that

$$A_c = \begin{bmatrix} A+BK_R & BK_I \\ -C & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} -K_R & -K_I \end{bmatrix} = A_0 - B_0 K. \tag{43}$$

Therefore, if there exists a $\hat{p} \in \wp$ such that (34) and (35) are satisfied, the eigenvalues of A_c can be arbitrarily assigned.

Remark 10. Theorem 11 is very significant because it allows to evaluate or estimate, via majorant system, the tracking error e_k.

Now the following main result, concerning the robust tracking, can be stated.

Theorem 12. Let K_I, K_R be two matrices such that the matrix A_c, for $p = \hat{p}$, has distinct eigenvalues with magnitudes less than one. Then a majorant system of the system (39) with respect to the norm $\|\zeta\|_P$ with $P = \left(ZZ^*\right)^{-1}$, where Z is the eigenvectors matrix of A_c for $p = \hat{p}$, is

$$\rho_{k+1} = a_c \rho_k + b_c \max\|\delta r_{k+1}\| + e_c \max\|\delta d_{k+1}\|, \quad \|e_k\| = h_c \rho_k, \tag{44}$$

in which: $a_c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(Q(p)P^{-1}\right)}$, $Q(p) = A_c^T(p) P A^T(p)$, $b_c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(B_c^T(p) P B(p)\right)}$, $e_c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(E_c^T(p) P E_c(p)\right)}$, $h_c = \sqrt{\max_{p \in V_p} \lambda_{\max}\left(H_c(p) P^{-1} H_c^T(p)\right)}$, being V_p the set of the 2^ν vertices of the hyper-rectangle $[p_-, p_+]$.

Proof. The proof is standard.

Remark 11. Note that, if $\wp = \hat{p}$, then the time constant of the majorant system $\tau = -1/\ln a_c$ is positive and coincides, for Theorem 6, with the maximum time constant of the control system. Moreover, "at steady-state", the tracking error satisfies relation

$$\|e_k\| \le \frac{h_c b_c}{1-a_c} \max\|\delta r_{k+1}\| + \frac{h_c e_c}{1-a_c} \max\|\delta d_{k+1}\|. \tag{45}$$

Remark 12. Clearly, if the initial state of the control system is not null and/or r_k and/or d_k are "discontinuous" in zero, the tracking error e_k has an additional term, whose practical duration depends on the time constant τ of the majorant system.

VI. Examples

The following examples show the great utility and efficiency of the results stated in the previous sections.

Example 1. Consider the system

$$x_{k+1} = \left(A_0 + A_1 p_1 + A_2 p_2 + A_{12} p_1 p_2 + A_{11} p_1^2\right) x_k + \left(B_0 + p_1 \sin x_{1k} B_1\right) u_k, \quad y_k = C x_k, \text{ where}$$

$$A_0 = \begin{bmatrix} 0.6677 & 0.3864 \\ -0.2338 & 0.6265 \end{bmatrix}, A_1 = \begin{bmatrix} 0.1259 & -0.1492 \\ 0.1623 & 0.1654 \end{bmatrix}, A_2 = \begin{bmatrix} 0.0529 & -0.0886 \\ -0.1610 & 0.0188 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.0915 & -0.0685 \\ 0.0930 & 0.0941 \end{bmatrix}, \tag{46}$$

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$$A_{11}=\begin{bmatrix}0.0433 & 0.1000\\ -0.0406 & -0.1035\end{bmatrix},\ B_0=\begin{bmatrix}0.0187\\ 0.1801\end{bmatrix},\ B_1=\begin{bmatrix}-0.0112\\ 0.1080\end{bmatrix},\ C=\begin{bmatrix}1 & 0\end{bmatrix},\ p_1\in[0.9 \quad 1.1],\ p_2\in[0.9 \quad 1.1].$$

By posing $\hat{p}_1=1,\ \hat{p}_2=1$ and by using Theorem 6 with $A=A(\hat{p})=A_0+A_1+A_2+A_{12}+A_{11}$, it is

$$P=\begin{bmatrix}1.3373 & 0.7606\\ 0.7606 & 1.4895\end{bmatrix}. \tag{47}$$

By using this P, Theorem 8 and Remark 4, a majorant system of the system (46) turns out to be:

$$\rho_{k+1}=a\rho_k+b\delta,\quad \upsilon_k=c\rho_k\ , \tag{48}$$

where, for Theorem 3, Corollary 1 and Theorem 4, it is:

$$\begin{aligned}
a&=\sqrt{\max_{\substack{\pi_1=0.9,\,1.1\\ \pi_2=0.9,\,1.1\\ \pi_3=0.9,\,1.1}}\lambda_{\max}\left(A^T(\pi)PA(\pi)P^{-1}\right)},\quad A(\pi)=A_0+A_1\pi_1+A_2\pi_2+A_{12}\pi_1\pi_2+A_{11}\pi_1\pi_3\\
b&=\sqrt{\max_{\substack{\gamma_1=0.9,1.1\\ \gamma_2=-1,\,1}}B^T(\gamma)B(\gamma)},\quad B=B_0+\gamma_1\gamma_2B_1\\
c&=\sqrt{CP^{-1}C};
\end{aligned} \tag{49}$$

hence $\rho_{k+1}=0.9574\rho_k+0.3688\delta,\quad \upsilon_k=1.0266\rho_k$.

From (48) it follows that $|y_k|\le\dfrac{cb}{1-a}\delta+ca^k\left(\rho_o-\dfrac{b}{1-a}\delta\right),\quad \forall x_0:\|x_0\|_P\le\rho_0$ and $\forall u_k:|u_k|\le\delta$; hence the system (46) is externally asymptotically stable and for $u_k=0$ also internally asymptotically stable.

Note that if $u_k=0$ the system (46) is linear and time invariant. The characteristic polynomial of $A(p)=A_0+A_1p_1+A_2p_2+A_{12}p_1p_2+A_{11}p_1^2$ is of the type

$$\begin{aligned}
d(\lambda,p)&=\det\left(\lambda I-A(p)\right)=\lambda^2+\left(\alpha_0+\alpha_1p_1+\alpha_2p_2+\alpha_3p_1p_2+\alpha_4p_1^2\right)\lambda+\\
&+\left(\beta_0+\beta_1p_1+\beta_2p_2+\beta_3p_1p_2+\beta_4p_1^2+\beta_5p_2^2+\beta_6p_1^2p_2+\beta_7p_1p_2^2+\beta_8p_1^3+\beta_9p_1^2p_2^2+\beta_{10}p_1^3p_2+\beta_{11}p_1^4\right);
\end{aligned}$$

therefore, it is very onerous both to establish the asymptotic stability of $x_{k+1}=A(p)x_k$ and to determine $a=\max_{p\in[p^-,\,p^+]}\lambda_{\max}(A(p))$. Numerically it is $a=0.9315$, while by using the proposed method, very efficient (see Theorem 3 and Corollary 1), an upper bound of a turns out to be $a=0.9574$. By using Corollary 1 and Theorem 7 with four rectangles, an upper bound of a turns out to be $a=0.9458$.

Example 2. Consider the system

$$\begin{aligned}
y_{k+1}&=\left(p_1a_{11}+p_2a_{12}\right)y_k+\left(p_1a_{21}+p_2a_{22}\right)y_{k-1}+g_{11}(y_k,y_{k-1},p_1,p_2,k)y_k^2+g_{12}(y_k,y_{k-1},p_1,p_2,k)y_ky_{k-1}+\\
&+g_{22}(y_k,y_{k-1},p_1,p_2,k)y_{k-1}^2+\left(b_0+p_1b_1\right)u_k,
\end{aligned} \tag{50}$$

where: $a_{11}=-0.730,\ a_{12}=-0.490,\ a_{21}=1.310,\ a_{22}=0.870,\ b_0=0.200,\ b_1=0.050;$
$p_1\in[0.9,\quad 1.1],\ p_2\in[0.8,\quad 1.2],\ g_{11}\in[0.08,\quad 0.12],\ g_{12}\in[0.07,\quad 0.13],\ g_{22}\in[-0.12,\quad -0.08].$

By posing $x_{1k}=y_k,\ x_{2k}=y_{k-1},\ \gamma_1=g_{11},\ \gamma_2=g_{12},\ \gamma_3=g_{22}$, system (50) can be put in the form

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ p_1 a_{21} + p_2 a_{22} & p_1 a_{11} + p_2 a_{12} \end{bmatrix} x_k + \left(\begin{bmatrix} 0 & 1 \\ \gamma_1 & \gamma_2/2 \end{bmatrix} x_{1k} + \begin{bmatrix} 0 & 1 \\ \gamma_2/2 & \gamma_3 \end{bmatrix} x_{2k} \right) x_k + \begin{bmatrix} 0 \\ b_0 + b_1 p_1 \end{bmatrix} u_k \qquad (51)$$
$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k .$$

By applying Theorem 9 with $\hat{p}_1 = 1, \hat{p}_2 = 1,$ a majorant system of the system (50) turns out to be

$$\rho_{k+1} = 1.749\rho_k + 1.413\rho_k^2 + 1.504\delta, \quad \upsilon_k = 1.0484\rho_k . \qquad (52)$$

This system is clearly unstable $\forall \delta \ge 0$, since $a_1 = 1.749 > 1$ (see Theorem 5).

By considering the closed-loop system of (51) with the control law

$$u_k = -\begin{bmatrix} -3.9075 & 5.7213 \end{bmatrix} x_k - \left(\begin{bmatrix} 0.400 & 0.200 \end{bmatrix} x_{1k} + \begin{bmatrix} 0.200 & -0.400 \end{bmatrix} x_{2k} \right) x_k + v_k , \qquad (53)$$

obtained by applying Theorem 10 with $\hat{p}_1 = 1, \hat{p}_2 = 1,$ $\hat{\gamma}_1 = 0.1, \hat{\gamma}_2 = 0.1, \hat{\gamma}_3 = -0.1,$ $r\bar{\Lambda} = \{0.3749 + j0.3203, \ 0.3749 - j0.3203\}$, a majorant system of the closed-loop system is

$$\rho_{k+1} = 0.8046\rho_k + 0.1488\rho_k^2 + 0.6276\delta, \quad \upsilon_k = 0.6254\rho_k , \qquad (54)$$

where

$$\rho = \|x\|_P , \ P = \begin{bmatrix} 1.4728 & -2.2708 \\ -2.2708 & 6.0579 \end{bmatrix} . \qquad (55)$$

From the first of (54) and from Theorem 5 it can be deducted that, whatever the values of uncertainties are, if $v_k = 0 \Rightarrow \delta = 0, \ \forall x_0 : \|x_0\|_P < 1.3135$, it is $\|x_k\|_P < \|x_0\|_P , \forall k > 0$, and $\lim_{k \to \infty} x_k = 0$ (see Fig 5a). Moreover the time constant of the linearized of the majorant system (54) turns out to be $\tau = 4.5995$.
Always from the first of (54) and from Theorem 5 it can be deducted that, for any values of uncertainties, $\forall v_k : |v_k| \le 0.09$ and $\forall x_0 : \|x_0\|_P \le 0.4296$, x_k remains always in the ellipse $C_{P,\,0.4296}$, while $\forall v_k : |v_k| \le 0.09$ and $\forall x_0 : \|x_0\|_P \in (0.4296 , \ 0.8839)$, it is $\|x_k\|_P < \|x_0\|_P , \forall k > 0$ and for k big enough x_k goes in the ellipse $C_{P,\,0.4296}$ (see Fig 5b).

Moreover the time constant of the linearized of the majorant system (54) turns out to be $\tau = 14.2978$.

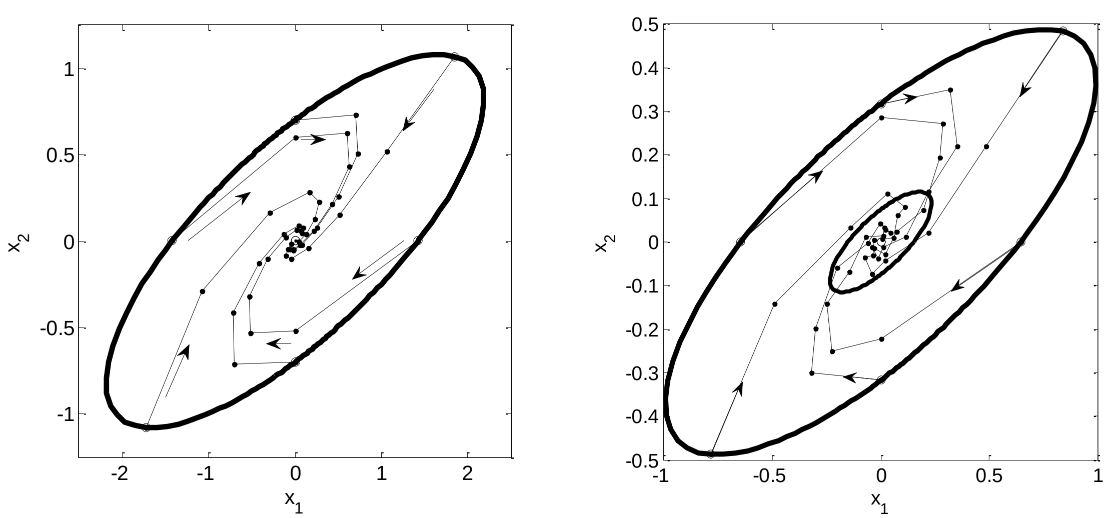

Figure 5a : RAS for $v_k = 0$.

Figure 5b : Practical RAS for $v_k = 0$.

Example 3. Consider the control system of the traffic of the road network in Fig. 6. By denoting with x_{ik} the distance of the vehicle i $(i=1,2)$ from the next one $i+1$ at the instant kT, where T is the sampling time,

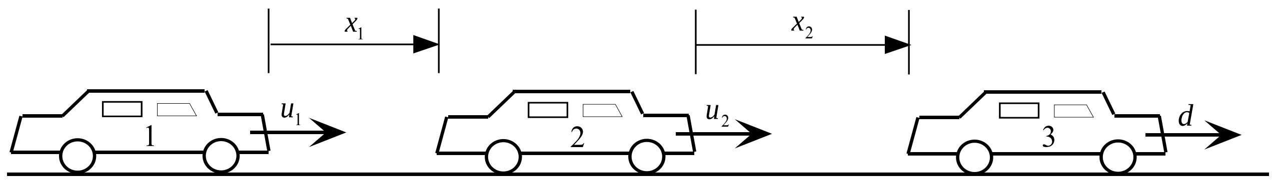

Figure 6 : Road network.

with u_{ik} the stretch of road that the vehicle i $(i=1,2)$ must cover in the interval $[kT,(k+1)T]$ and with d_k the stretch of road that the head vehicle (number 3 in Fig. 6) will run in the interval $[kT,(k+1)T]$, it is:

$$x_{1k+1}=x_{1k}-(1+p_1)u_{1k}+(1+p_2)u_{2k},\quad x_{2k+1}=x_{2k}-(1+p_2)u_{2k}+d_k, \tag{56}$$

where p_1, p_2 are the relative errors of actuation. From (56) it is:

$$x_{k+1}=Ax_k+B(p)u_k+Ed_k,\quad y_k=C_kx_k, \tag{57}$$

where:

$$A=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad B(p)=\begin{bmatrix}-1&1\\0&-1\end{bmatrix}+\begin{bmatrix}-1&0\\0&0\end{bmatrix}p_1+\begin{bmatrix}0&1\\0&-1\end{bmatrix}p_2,\quad E=\begin{bmatrix}0\\1\end{bmatrix},\quad C=\begin{bmatrix}1&0\\0&1\end{bmatrix}. \tag{58}$$

An easier model is the one in which for the vehicle number 1 the signal u_{2k} is a disturbance, i.e. in the hypothesis of a decentralized control. In this case it is:

$$x_{ik+1}=x_{ik}-(1+p_i)u_{ik}+d_{ik},\quad i=1,2,\quad d_{1k}=u_{2k},\quad d_{2k}=d_k, \tag{59}$$

By using the decentralized control law

$$z_{ik+1}=z_{ik}+(r_{ik}-y_{ik}),\quad u_{ik}=-0.5429z_{ik}+1.2929y_{ik},\quad i=1,2, \tag{60}$$

obtained such that the eigenvalues of A_{ic} for $\hat{p}_1=\hat{p}_2=0$ are $0.5e^{\pm j\frac{\pi}{4}}$, in the hypothesis that $p_1\in[-0.05,0.05]$, $p_2\in[-0.05,0.05]$, upper bounds of the tracking errors can be determined by using the majorant systems

$$\rho_{ik+1}=0.5760\rho_{ik}+2.6085\max|\delta r_{ik+1}|+2.4843|\delta d_{ik+1}|,\quad |e_{ik}|=1.1385\rho_{ik},\quad i=1,2, \tag{61}$$

computed with Theorem 12.

In Fig. 7 the simulation results of the controlled road network, in the hypothesis of $p_1=0.05,\ p_2=-0.05,\ r_{1k}=10\cdot 1_k,\ r_{2k}=20\cdot 1_k,\ d_k=\left(20+2\sin\frac{k}{10}-2\cos\frac{k}{20}\right)\cdot 1_k,\ \zeta_0=0$, are shown. After the transient phase, due to the initial "discontinuous" of r_{1k},r_{2k} and d_k, it is $|e_{2k}|\le 0.5827$, while from the majorant system it turns out to be that $|e_{2k}|\le 1.9993$.

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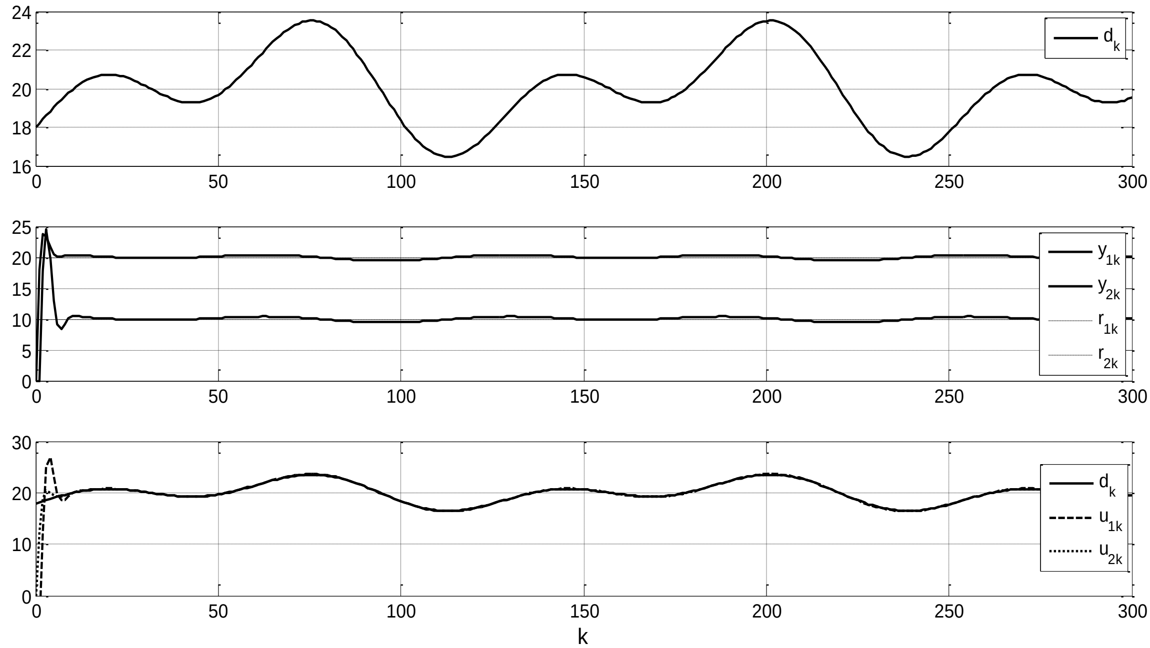

Figure 7 : Errors and control signals of the controlled road network.

VII. Conclusions and Future Developments

In this paper several basic theorems have been stated and proved. They allow to determine, by calculating the eigenvalues of suitable matrices only in correspondence of the vertices of appropriate polytopes, a majorant system of a pseudo-quadratic uncertain MIMO system.

By using the provided results, systematic methods have been derived, which allow to solve, via majorant system, several analysis and synthesis problems about the robust stability, robust stabilization and tracking of a generic reference signal with bounded variation, in presence of a generic disturbance with bounded too. The presented examples have shown the utility and the efficiency of the main proposed results.

Future developments are going on in the direction of the fault tolerance and of the robust tracking in the hypothesis of a non measurable state.

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