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New results for boundary value problems of Hadamard-type fractional differential inclusions and integral boundary conditions

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Abstract

In this paper, we study the existence of solutions for a fractional boundary value problem involving Hadamard-type fractional differential inclusions and integral boundary conditions. Our results include the cases for convex as well as non-convex valued maps and are based on standard fixed point theorems for multivalued maps. Some illustrative examples are also presented.

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1 Introduction

The theory of fractional differential equations and inclusions has received much attention over the past years and has become an important field of investigation due to its extensive applications in numerous branches of physics, economics and engineering sciences [1–4]. Fractional differential equations and inclusions provide appropriate models for describing real world problems, which cannot be described using classical integer order differential equations. Some recent contributions to the subject can be seen in [5–21] and references cited therein.

It has been noticed that most of the work on the topic is based on Riemann-Liouville and Caputo-type fractional differential equations. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [22], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains a logarithmic function of arbitrary exponent. Details and properties of the Hadamard fractional derivative and integral can be found in [1, 23–27].

In this paper, we study the following boundary value problem of Hadamard-type fractional differential inclusions:

$$\begin{cases} D^{\alpha}x(t) \in F(t,x(t)), & 1 < t < e, 1 < \alpha \le 2, \\ x(1) = 0, \qquad x(e) = I^{\beta}x(\eta), & 1 < \eta < e, \end{cases} \tag{1.1}$$

where D^α is the Hadamard fractional derivative of order α, I^β is the Hadamard fractional integral of order β and $F : [1,e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

We aim to establish a variety of results for inclusion problem (1.1) by considering the given multivalued map to be convex as well as non-convex valued. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while the third result is obtained by using the fixed point theorem for contractive multivalued maps due to Covitz and Nadler.

We emphasize that the main idea of the present research is to introduce Hadamard-type fractional differential inclusions supplemented with Hadamard-type integral boundary conditions and develop some existence results for the problem at hand. It is imperative to note that our results are absolutely new in the context of Hadamard-type integral boundary value problems and provide a new avenue to the researchers working on fractional boundary value problems.

The paper is organized as follows. In Section 2, we solve a linear Hadamard-type integro-differential boundary value problem and recall some preliminary concepts of multivalued analysis that we need in the sequel. Section 3 contains the main results for problem (1.1). In Section 4, some illustrative examples are discussed.

2 Preliminaries

This section is devoted to the basic concepts of Hadamard-type fractional calculus and multivalued analysis. We also establish an auxiliary lemma to define the solutions for the given problem.

2.1 Fractional calculus

Definition 2.1 [1] The Hadamard derivative of fractional order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s}\,ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 [1] The Hadamard fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{g(s)}{s}\,ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.3 (Auxiliary lemma) *For* $1 < \alpha \leq 2$ *and* $\zeta \in C([1,e],\mathbb{R})$, *the unique solution of the problem*

$$\begin{cases} D^\alpha x(t) = \zeta(t), & 1 < t < e, \\ x(1) = 0, & x(e) = I^\beta x(\eta) \end{cases} \tag{2.1}$$

is given by

$$x(t) = I^{\alpha}\zeta(t) + \frac{(\log t)^{\alpha-1}}{\Omega}\big[I^{\beta+\alpha}\zeta(\eta) - I^{\alpha}\zeta(e)\big], \tag{2.2}$$

where

$$\Omega = \frac{1}{1 - \frac{1}{\Gamma(\beta)}\int_1^{\eta}(\log\frac{\eta}{s})^{\beta-1}\frac{(\log s)^{\alpha-1}}{s}\,ds}. \tag{2.3}$$

Proof As argued in [1], the solution of the Hadamard differential equation in (2.1) can be written as

$$x(t) = I^{\alpha}\zeta(t) + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2}. \tag{2.4}$$

Using the given boundary conditions, we find that $c_2 = 0$ and

$$\begin{aligned} I^{\alpha}\zeta(e) + c_1 &= I^{\beta}\big(I^{\alpha}\zeta(s) + c_1(\log s)^{\alpha-1}\big)(\eta) \\ &= I^{\beta+\alpha}\zeta(\eta) + \frac{c_1}{\Gamma(\beta)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\beta-1}\frac{(\log s)^{\alpha-1}}{s}\,ds, \end{aligned}$$

which gives

$$c_1 = \frac{1}{1 - \frac{1}{\Gamma(\beta)}\int_1^{\eta}(\log\frac{\eta}{s})^{\beta-1}\frac{(\log s)^{\alpha-1}}{s}\,ds}\big[I^{\beta+\alpha}\zeta(\eta) - I^{\alpha}\zeta(e)\big]. \tag{2.5}$$

Substituting the values of c_1 and c_2 in (2.4), we obtain (2.2). This completes the proof. □

2.2 Basic concepts of multivalued analysis

Here we outline some basic definitions and results for multivalued maps [28, 29].

Let $C([1,e],\mathbb{R})$ denote a Banach space of continuous functions from $[1,e]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[1,e]}|x(t)|$. Let $L^1([1,e],\mathbb{R})$ be the Banach space of measurable functions $x:[1,e]\to\mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^e |x(t)|\,dt$.

For a normed space $(X,\|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$:

- (i) is *convex* (*closed*) *valued* if $G(x)$ is convex (closed) for all $x \in X$;
- (ii) is *bounded* on bounded sets if $G(\mathbb{B}) = \bigcup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (*i.e.*, $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);
- (iii) is called *upper semicontinuous* (*u.s.c.*) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;
- (iv) G is *lower semicontinuous* (*l.s.c.*) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E;
- (v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$;

(vi) is said to be *measurable* if for every $y \in \mathbb{R}$, the function

$$t \mapsto d\big(y, G(t)\big) = \inf\big\{|y - z| : z \in G(t)\big\}$$

is measurable;

(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by Fix G.

For each $x \in C([1,e], \mathbb{R})$, define the set of selections of F by

$$S_{F,x} := \big\{v \in L^1\big([1,e], \mathbb{R}\big) : v(t) \in F\big(t, x(t)\big) \text{ for a.e. } t \in [1,e]\big\}. \tag{2.6}$$

We define the graph of G to be the set $\mathrm{Gr}(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

Lemma 2.4 [28, Proposition 1.2] *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $\mathrm{Gr}(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semicontinuous.*

Lemma 2.5 [30] *Let X be a separable Banach space. Let $F : [0,1] \times X \to \mathcal{P}_{cp,c}(X)$ be measurable with respect to t for each $x \in X$ and u.s.c. with respect to x for almost all $t \in [1,e]$ and $S_{F,x} \neq \emptyset$ for any $x \in C([1,e], X)$, and let Θ be a linear continuous mapping from $L^1([1,e], X)$ to $C([1,e], X)$. Then the operator*

$$\Theta \circ S_F : C\big([1,e], X\big) \to \mathcal{P}_{cp,c}\big(C\big([1,e], X\big)\big), \qquad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$

is a closed graph operator in $C([1,e], X) \times C([1,e], X)$.

3 Existence results

Definition 3.1 A function $x \in AC^1([1,e], \mathbb{R})$ is called a solution of problem (1.1) if there exists a function $g \in L^1([1,e], \mathbb{R})$ with $g(t) \in F(t, x(t))$, a.e. on $[1,e]$ such that $D^\alpha x(t) = g(t)$, a.e. on $[1,e]$ and $x(1) = 0$, $x(e) = I^\beta x(\eta)$.

3.1 The upper semicontinuous case

Our first main result for Carathéodory case is established via the nonlinear alternative of Leray-Schauder for multivalued maps.

Lemma 3.2 (Nonlinear alternative for Kakutani maps [31]) *Let E be a Banach space, C be a closed convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is an upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there are $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Theorem 3.3 *Assume that*:

(H_1) *$F : [1,e] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is Carathéodory, i.e.,*

(i) *$t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;*

(ii) *$x \mapsto F(t,x)$ is u.s.c. for almost all $t \in [1,e]$;*

(H_2) *there exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([1,e],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \quad \textit{for each } (t,x) \in [1,e] \times \mathbb{R};$$

(H_3) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\psi(M)\|p\|\{\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Omega|}(\frac{(\log\eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + \frac{1}{\Gamma(\alpha+1)})\}} > 1,$$

where Ω *is given by* (2.3).

Then problem (1.1) *has at least one solution on* $[1,e]$.

Proof In view of Lemma 2.3, we define an operator $\mathcal{F} : C([1,e],\mathbb{R}) \to \mathcal{P}(C([1,e],\mathbb{R}))$ by

$$\begin{aligned}\mathcal{F}(x) = \bigg\{ h \in C([1,e],\mathbb{R}) : h(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}\,ds \\ &+ \frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{g(s)}{s}\,ds \right. \\ &\left. - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s)}{s}\,ds\right]\bigg\}\end{aligned} \tag{3.1}$$

for $g \in S_{F,x}$ (defined by (2.6)). Observe that the fixed points of the operator $\mathcal{F}$ correspond to the solutions of problem (1.1). We will show that $\mathcal{F}$ satisfies the assumptions of the Leray-Schauder nonlinear alternative (Lemma 3.2). The proof consists of several steps.

Step 1. $\mathcal{F}(x)$ *is convex for each* $x \in C([1,e],\mathbb{R})$.

This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

Step 2. $\mathcal{F}$ *maps bounded sets* (*balls*) *into bounded sets in* $C([1,e],\mathbb{R})$.

For a positive number ρ, let $B_\rho = \{x \in C([1,e],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $C([1,e],\mathbb{R})$. Then, for each $h \in \mathcal{F}(x)$, $x \in B_\rho$, there exists $g \in S_{F,x}$ such that

$$\begin{aligned}h(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}\,ds + \frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{g(s)}{s}\,ds\right. \\ &\left.\quad - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s)}{s}\,ds\right].\end{aligned}$$

Then we have

$$\begin{aligned}|h(t)| &\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|g(s)|}{s}\,ds \\ &\quad + \frac{1}{|\Omega|}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{|g(s)|}{s}\,ds + \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|g(s)|}{s}\,ds\right] \\ &\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{p(s)\psi(\|x\|)}{s}\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{|\Omega|}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{p(s)\psi(\|x\|)}{s}\,ds\right.\\
&\left.+ \frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{p(s)\psi(\|x\|)}{s}\,ds\right]\\
&\le \frac{\psi(\|x\|)\|p\|}{\Gamma(\alpha+1)} + \frac{\psi(\|x\|)\|p\|}{|\Omega|}\left(\frac{(\log\eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right).
\end{aligned}
$$

Thus

$$
\|h\| \le \psi(\rho)\|p\|\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\left(\frac{(\log\eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right)\right\}.
$$

Step 3. *$\mathcal{F}$ maps bounded sets into equicontinuous sets of* $C([1,e],\mathbb{R})$.

Let $\tau_1, \tau_2 \in [1,e]$ with $\tau_1 < \tau_2$ and $u \in B_\rho$, where B_ρ is a bounded set of $C([1,e],\mathbb{R})$ as in Step 2. For each $h \in \mathcal{F}(u)$, we obtain

$$
\begin{aligned}
&\left|h(\tau_2) - h(\tau_1)\right|\\
&\le \frac{\psi(\rho)\|p\|}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}\,ds - \int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right|\\
&+ \psi(\rho)\|p\|\left|\frac{(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{1}{s}\,ds\right.\right.\\
&\left.\left.- \frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right]\right|\\
&\le \frac{\psi(\rho)\|p\|}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_1}{s}\right)^{\alpha-1} - \left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\right]\frac{1}{s}\,ds\right|\\
&+ \frac{\psi(\rho)\|p\|}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right|\\
&+ \psi(\rho)\|p\|\left|\frac{(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{1}{s}\,ds\right.\right.\\
&\left.\left.- \frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}\,ds\right]\right|.
\end{aligned}
$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $\tau_2 - \tau_1 \to 0$. In view of Steps 1-3, the Arzelá-Ascoli theorem applies and hence $\mathcal{F}: C([1,e],\mathbb{R}) \to \mathcal{P}(C([1,e],\mathbb{R}))$ is completely continuous.

By Lemma 2.4, $\mathcal{F}$ will be upper semicontinuous (u.s.c.) if we prove that it has a closed graph since $\mathcal{F}$ is already shown to be completely continuous.

Step 4. *$\mathcal{F}$ has a closed graph.*

Let $x_n \to x_*$, $h_n \in \mathcal{F}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{F}(x_*)$. Associated with $h_n \in \mathcal{F}(x_n)$, there exists $g_n \in S_{F,x_n}$ such that for each $t \in [1,e]$,

$$
\begin{aligned}
h_n(t) = {}& \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g_n(s)}{s}\,ds + \frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{g_n(s)}{s}\,ds\right.\\
&\left.- \frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g_n(s)}{s}\,ds\right].
\end{aligned}
$$

Thus we have to show that there exists $g_* \in S_{F,x_*}$ such that for each $t \in [1,e]$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g_*(s)}{s}\,ds + \frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\beta+\alpha-1} \frac{g_*(s)}{s}\,ds - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g_*(s)}{s}\,ds\right].$$

Let us consider the linear operator $\Theta : L^1([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ given by

$$\begin{aligned} g \mapsto \Theta(g)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s}\,ds + \frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\beta+\alpha-1} \frac{g(s)}{s}\,ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s)}{s}\,ds\right]. \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{(g_n(s) - g_*(s))}{s}\,ds \right. \\ &\quad + \frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\beta+\alpha-1} \frac{(g_n(s) - g_*(s))}{s}\,ds \right. \\ &\quad \left.\left. - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{(g_n(s) - g_*(s))}{s}\,ds\right]\right\| \to 0 \end{aligned}$$

as $n \to \infty$.

Thus, it follows by Lemma 2.5 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g_*(s)}{s}\,ds + \frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\beta+\alpha-1} \frac{g_*(s)}{s}\,ds - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g_*(s)}{s}\,ds\right]$$

for some $g_* \in S_{F,x_*}$.

Step 5. *We show that there exists an open set* $U \subseteq C([1,e],\mathbb{R})$ *with* $x \notin \lambda\mathcal{F}(x)$ *for any* $\lambda \in (0,1)$ *and all* $x \in \partial U$.

Let $x \in \lambda\mathcal{F}(x)$ for some $\lambda \in (0,1)$. Then there exists $g \in L^1([1,e],\mathbb{R})$ with $g \in S_{F,x}$ such that, for $t \in [1,e]$, we have

$$x(t) = \lambda\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s}\,ds + \lambda\frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\beta+\alpha-1} \frac{g(s)}{s}\,ds - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s)}{s}\,ds\right].$$

Using the computations of the second step above, we have

$$\|x\| \le \psi\left(\|x\|\right)\|p\|\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\left(\frac{(\log \eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right)\right\}.$$

Consequently, we have

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\{\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Omega|}(\frac{(\log\eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)}+\frac{1}{\Gamma(\alpha+1)})\}}\leq 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U=\{x\in C([1,e],\mathbb{R}):\|x\|<M\}.$$

Note that the operator $\mathcal{F}:\overline{U}\to\mathcal{P}(C([1,e],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x\in\lambda\mathcal{F}(x)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.2), we deduce that $\mathcal{F}$ has a fixed point $x\in\overline{U}$ which is a solution of problem (1.1). This completes the proof. □

3.2 The lower semicontinuous case

In what follows, we consider the case when F is not necessarily convex valued and obtain the existence result by combining the nonlinear alternative of Leray-Schauder type with the selection theorem due to Bressan and Colombo [32] for lower semicontinuous maps with decomposable values.

Definition 3.4 Let A be a subset of $I\times\mathbb{R}$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J}\times\mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in I and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

Definition 3.5 A subset $\mathcal{A}$ of $L^1(I,\mathbb{R})$ is decomposable if for all $u,v\in\mathcal{A}$ and measurable $\mathcal{J}\subset I$, the function $u\chi_{\mathcal{J}}+v\chi_{I-\mathcal{J}}\in\mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Lemma 3.6 [32] *Let Y be a separable metric space, and let $N:Y\to\mathcal{P}(L^1(I,\mathbb{R}))$ be a lower semicontinuous* (*l.s.c.*) *multivalued operator with nonempty closed and decomposable values. Then N has a continuous selection, that is, there exists a continuous function* (*single-valued*) $h:Y\to L^1(I,\mathbb{R})$ *such that* $h(x)\in N(x)$ *for every* $x\in Y$.

Theorem 3.7 *Assume that* (H_2), (H_3) *and the following condition holds*:

(H_4) $F:[1,e]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
  (a) $(t,x)\mapsto F(t,x)$ *is* $\mathcal{L}\otimes\mathcal{B}$ *measurable*,
  (b) $x\mapsto F(t,x)$ *is lower semicontinuous for each* $t\in[1,e]$.

Then problem (1.1) *has at least one solution on* $[1,e]$.

Proof It follows from (H_2) and (H_4) that F is of l.s.c. type [33]. Then, by Lemma 3.6, there exists a continuous function $f:AC([1,e],\mathbb{R})\to L^1([1,e],\mathbb{R})$ such that $f(x)\in\mathcal{S}(x)$ for all $x\in C([1,e],\mathbb{R})$, where $\mathcal{S}:C([1,e]\times\mathbb{R})\to\mathcal{P}(L^1([1,e],\mathbb{R}))$ is the Nemytskii operator associated with F, defined as

$$\mathcal{S}(x)=\{w\in L^1([1,e],\mathbb{R}):w(t)\in F(t,x(t))\text{ for a.e. }t\in[1,e]\}.$$

Consider the problem

$$\begin{cases} D^{\alpha}x(t) = f(x(t)), & t \in [1,e], \\ x(1) = 0, & x(e) = I^{\beta}x(\eta). \end{cases} \tag{3.2}$$

Observe that if $x \in AC([1,e],\mathbb{R})$ is a solution of problem (3.2), then x is a solution to problem (1.1). In order to transform problem (3.2) into a fixed point problem, we define an operator $\overline{\mathcal{F}}$ as

$$\begin{aligned} \overline{\mathcal{F}}x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(x(s))}{s}\,ds \\ & + \frac{(\log t)^{\alpha-1}}{\Omega} \left[\frac{1}{\Gamma(\beta+\alpha)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\beta+\alpha-1} \frac{f(x(s))}{s}\,ds \right. \\ & \left. - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{f(x(s))}{s}\,ds \right]. \end{aligned}$$

It can easily be shown that $\overline{\mathcal{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.3. So we omit it. This completes the proof. □

3.3 The Lipschitz case

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{ \sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b) \right\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space (see [34]).

Definition 3.8 A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d\big(N(x),N(y)\big) \le \gamma d(x,y) \quad \text{for each } x,y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

To show the existence of solutions for problem (1.1) with a nonconvex valued right-hand side, we need a fixed point theorem for multivalued maps due to Covitz and Nadler [35].

Lemma 3.9 [35] *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Theorem 3.10 *Assume that the following conditions hold*:

(H_5) *$F : [1,e] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,x) : [1,e] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.*

(H_6) $H_d(F(t,x),F(t,\bar{x})) \leq m(t)|x-\bar{x}|$ *for almost all* $t \in [1,e]$ *and* $x,\bar{x} \in \mathbb{R}$ *with* $m \in C([1,e],\mathbb{R}^+)$ *and* $d(0,F(t,0)) \leq m(t)$ *for almost all* $t \in [1,e]$.

Then problem (1.1) *has at least one solution on* $[1,e]$ *if*

$$\|m\|\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\left(\frac{(\log\eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right)\right\}<1.$$

Proof We transform problem (1.1) into a fixed point problem by means of the operator $\mathcal{F}: C([1,e],\mathbb{R}) \to \mathcal{P}(C([1,e],\mathbb{R}))$ defined by (3.1) and show that the operator $\mathcal{F}$ satisfies the assumptions of Lemma 3.9. The proof will be given in two steps.

Step 1. $\mathcal{F}(x)$ is nonempty and closed for every $v \in S_{F,x}$.

Since the set-valued map $F(\cdot,x(\cdot))$ is measurable with the measurable selection theorem (*e.g.*, [36, Theorem III.6]), it admits a measurable selection $v:[1,e]\to\mathbb{R}$. Moreover, by assumption (H_6), we have

$$\bigl|v(t)\bigr| \leq m(t)+m(t)\bigl|x(t)\bigr|,$$

that is, $v \in L^1([1,e],\mathbb{R})$ and hence F is integrably bounded. Therefore, $S_{F,x} \neq \emptyset$. Moreover, $\mathcal{F}(x) \in \mathcal{P}_{cl}(C([1,e],\mathbb{R}))$ for each $x \in C([1,e],\mathbb{R})$. Indeed, let $\{u_n\}_{n\geq 0} \in \mathcal{F}(x)$ be such that $u_n \to u$ ($n\to\infty$) in $C([1,e],\mathbb{R})$. Then $u \in C([1,e],\mathbb{R})$ and there exists $g_n \in S_{F,u_n}$ such that, for each $t\in[1,e]$,

$$\begin{aligned}u_n(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g_n(s)}{s}\,ds\\&\quad+\frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{g_n(s)}{s}\,ds\right.\\&\quad\left.-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g_n(s)}{s}\,ds\right].\end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that g_n converges to g in $L^1([1,e],\mathbb{R})$. Thus, $g \in S_{F,u}$ and for each $t \in [1,e]$, we have

$$\begin{aligned}u_n(t) &\to u(t)\\&= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}\,ds\\&\quad+\frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{g(s)}{s}\,ds\right.\\&\quad\left.-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s)}{s}\,ds\right].\end{aligned}$$

Hence, $u \in \mathcal{F}(x)$.

Step 2. Next we show that there exists $\delta<1$ such that

$$H_d\bigl(\mathcal{F}(x),\mathcal{F}(\bar{x})\bigr) \leq \delta\|x-\bar{x}\| \quad \text{for each } x,\bar{x} \in AC\bigl([1,e],\mathbb{R}\bigr).$$

Let $x,\bar{x}\in AC([1,e],\mathbb{R})$ and $h_1\in\mathcal{F}(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[1,e]$,

$$\begin{aligned} h_1(t) = {}&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_1(s)}{s}\,ds \\ &+\frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{v_1(s)}{s}\,ds\right. \\ &\left.-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_1(s)}{s}\,ds\right]. \end{aligned}$$

By (H_6), we have

$$H_d\bigl(F(t,x),F(t,\bar{x})\bigr)\le m(t)\bigl|x(t)-\bar{x}(t)\bigr|.$$

So, there exists $w(t)\in F(t,\bar{x}(t))$ such that

$$\bigl|v_1(t)-w(t)\bigr|\le m(t)\bigl|x(t)-\bar{x}(t)\bigr|,\quad t\in[1,e].$$

Define $U:[1,e]\to\mathcal{P}(\mathbb{R})$ by

$$U(t)=\bigl\{w\in\mathbb{R}:\bigl|v_1(t)-w(t)\bigr|\le m(t)\bigl|x(t)-\bar{x}(t)\bigr|\bigr\}.$$

Since the multivalued operator $U(t)\cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [36]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t)\in F(t,\bar{x}(t))$ and for each $t\in[1,e]$, we have $|v_1(t)-v_2(t)|\le m(t)|x(t)-\bar{x}(t)|$.

For each $t\in[1,e]$, let us define

$$\begin{aligned} h_2(t) = {}&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_2(s)}{s}\,ds \\ &+\frac{(\log t)^{\alpha-1}}{\Omega}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{v_2(s)}{s}\,ds\right. \\ &\left.-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_2(s)}{s}\,ds\right]. \end{aligned}$$

Thus,

$$\begin{aligned} \bigl|h_1(t)-h_2(t)\bigr| \le {}&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|v_1(s)-v_2(s)|}{s}\,ds \\ &+\frac{(\log t)^{\alpha-1}}{|\Omega|}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{|v_1(s)-v_2(s)|}{s}\,ds\right. \\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|v_1(s)-v_2(s)|}{s}\,ds\right] \\ \le {}&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{m(s)\|x-\bar{x}\|}{s}\,ds \\ &+\frac{1}{|\Omega|}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\beta+\alpha-1}\frac{m(s)\|x-\bar{x}\|}{s}\,ds\right. \end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{m(s)\|x-\bar{x}\|}{s}\,ds\Bigg]$$
$$\le\|m\|\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\left(\frac{(\log\eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right)\right\}\|x-\bar{x}\|.$$

Hence,

$$\|h_1-h_2\|\le\|m\|\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\left(\frac{(\log\eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right)\right\}\|x-\bar{x}\|.$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$H_d(\mathcal{F}(x),\mathcal{F}(\bar{x}))\le\delta\|x-\bar{x}\|$$
$$\le\|m\|\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\left(\frac{(\log\eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right)\right\}\|x-\bar{x}\|.$$

Since $\mathcal{F}$ is a contraction, it follows by Lemma 3.9 that $\mathcal{F}$ has a fixed point x which is a solution of (1.1). This completes the proof. □

4 Examples

In this section we present some concrete examples to illustrate our results.

Let us consider the boundary value problem

$$\begin{cases} D^{3/2}x(t)\in F(t,x(t)), & 1<t<e,\\ x(1)=0, & x(e)=I^{3/2}x(2).\end{cases}\tag{4.1}$$

Here $\alpha=3/2$, $\beta=3/2$, $\eta=2$,

$$\Omega=\frac{1}{1-\frac{1}{\Gamma(\beta)}\int_1^\eta(\log\frac{\eta}{s})^{\beta-1}\frac{(\log s)^{\alpha-1}}{s}\,ds}=\frac{4}{4-\sqrt{\pi}(\log 2)^2}\approx 1.27$$

and

$$\omega=\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\left(\frac{(\log\eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right)\approx 1.39.$$

Example 4.1 Let $F:[1,e]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$x\to F(t,x)=\left[\frac{x^4}{x^4+2}+e^{-x^2}+t+2,\frac{|x|}{|x|+1}+t^2+\frac{1}{2}\right].\tag{4.2}$$

For $f\in F$, we have

$$|f|\le\max\left(\frac{x^4}{x^4+2}+e^{-x^2}+t+2,\frac{|x|}{|x|+1}+t^2+\frac{1}{2}\right)\le 5,\quad x\in\mathbb{R}.$$

Here $\|F(t,x)\|_{\mathcal{P}}:=\sup\{|y|:y\in F(t,x)\}\le 5=p(t)\psi(\|x\|)$, $x\in\mathbb{R}$, with $p(t)=1$, $\psi(\|x\|)=5$. It is easy to verify that $M>6.95$. Then, by Theorem 3.3, problem (4.1) with $F(t,x)$ given by (4.2) has at least one solution on $[1,e]$.

Example 4.2 Consider the multivalued map $F : [1,e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$x \to F(t,x) = \left[0, \frac{1}{7}(t+1)\sin x + \frac{2}{7}\right]. \tag{4.3}$$

Then we have

$$\sup\{|u| : u \in F(t,x)\} \le \frac{1}{7}(t+1) + \frac{2}{7}$$

and

$$H_d\big(F(t,x), F(t,\bar{x})\big) \le \frac{1}{7}(t+1)|x - \bar{x}|.$$

Let $m(t) = \frac{1}{7}(t+1)$. Then $H_d(F(t,x), F(t,\bar{x})) \le m(t)|x - \bar{x}|$, and $\|m\|\omega \approx 0.74 < 1$. By Theorem 3.10, problem (4.1) with $F(t,x)$ given by (4.3) has at least one solution on $[1,e]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, SKN and AA contributed to each part of this work equally and read and approved the final version of the manuscript.

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