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New results for the analysis of linear systems with time-invariant delays

Jianrong Zhang[1], Carl R. Knospe[1,*,†] and Panagiotis Tsiotras[2]

[1] *Department of Mechanical and Aerospace Engineering, University of Virginia, 122 Engineer's Way, P.O. Box 400746, Charlottesville, VA 22904-4746, U.S.A.*
[2] *School of Aerospace Engineering, Georgia Institute of Technology, 270 Ferst Dr., Atlanta, GA 30332, U.S.A.*

SUMMARY

This paper presents a comparison system approach for the analysis of stability and $\mathscr{H}_\infty$ performance of linear time-invariant systems with unknown delays. The comparison system is developed by replacing the delay elements with certain parameter-dependent Padé approximations. It is shown using the special properties of the Padé approximation to e^{-s} that the value sets of these approximations provide outer and inner coverings for that of each delay element and that the robust stability of the outer covering system is a sufficient condition for the stability of the original time delay system. The inner covering system, in turn, is used to provide an upper bound on the degree of conservatism of the delay margin established by the sufficient condition. This upper bound is dependent only upon the Padé approximation order and may be made arbitrarily small. In the single delay case, the delay margin can be calculated explicitly without incurring any additional conservatism. In the general case, this condition can be reduced with some (typically small) conservatism to finite-dimensional LMIs. Finally, this approach is also extended to the analysis of $\mathscr{H}_\infty$ performance for linear time-delay systems with an exogenous disturbance. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: time-delay systems; stability; Padé approximation

1. INTRODUCTION

The analysis of linear time-delay systems (LTDS) has attracted much interest over a half century, especially in the last decade. The recent books [1–3] contain an extensive collection of research results dealing with both delay-dependent and delay-independent stability conditions. Much interest in the literature has focused on searching for sufficient conditions which are numerically tractable but are not too conservative. Many of the stability criteria have been formulated in the time domain, based on Lyapunov's Second Method using Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions [4–9]. Frequency domain

*Correspondence to: Dr. C. R. Knospe, Department of Mechanical and Aerospace Engineering, University of Virginia, 122 Engineer's Way, P.O. Box 400746, Charlottesville, VA 22904-4746, U.S.A.
† E-mail: crk4y@virginia.edu

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techniques for analysis of time-delay systems have also been developed [10], such as polynomial criteria [11–13], matrix pencils [14], integral quadratic constraints [15], the singular value test [16], μ-based criteria [17, 18], etc. Most of these stability tests are for systems with a single delay. In recent years, several stability criteria for the multiple delay case have also been reported. In Reference [19], the traditional μ-framework was extended for time-delay systems to obtain a necessary and sufficient stability condition, which was then relaxed to a convex sufficient condition. For linear systems with commensurate delays, [20] proposed a simple stability test which requires computation of eigenvalues and generalized eigenvalues of constant matrices. For the general multiple delays case, several sufficient stability conditions have also been reported [21, 22].

Many of the existing stability criteria involve, either explicitly or implicitly, covering the uncertain delay elements with some (convex) sets so as to obtain numerically tractable stability conditions [23]. Furthermore, the conservatism of the analysis can be effectively reduced by choosing appropriate covering sets, based on the properties of the delay elements [24].

In this paper, we present a covering set for the non-rational delay element based upon parameter-dependent diagonal Padé approximations of the function e^{-s}. Special properties of these approximations are used to develop both inner and outer coverings that are related via frequency dilation. We demonstrate that a comparison system can be obtained by replacing the delay elements with the outer Padé approximation and that robust stability of the resulting finite-dimensional, parameter-dependent system is sufficient for delay-dependent stability of the original time-delay system. Using the inner approximation, we establish that the degree-of-conservatism of this sufficient condition has an upper bound that is dependent *only* on the order of the outer Padé approximation used and may be made arbitrarily small. Moreover, in the single delay case, the delay margin given by this condition can be calculated explicitly without incurring any additional conservatism. In the general case, this condition can also be reduced with some (typically small) conservatism to finite-dimensional LMIs. Finally, this approach is extended to derive a sufficient condition for the $\mathscr{H}_\infty$ performance for LTDS with exogenous disturbance.

The results of this paper indicate that, by replacing the uncertain delay elements $e^{-\tau_k s}$ with Padé approximations directly, the stability analysis of the resultant finite-dimensional system only gives a *necessary condition*, and hence it does not guarantee, in general, the stability of the original systems. This traditional manner of using Padé approximations for time-delay systems has been used extensively; see [25–28]. However, it can only be used for *small delays* and over a *finite bandwidth* of the system, because the Padé approximations are accurate only when $|\tau_k s|$ is sufficiently small. On the other hand, our covering relation holds for all frequencies. Thus, by using a frequency-dilated version of the Padé approximation, robust stability of the comparison system guarantees the stability of the original time-delay system without imposing any restrictions on the magnitude of the delay and/or the system bandwidth.

Many published Lyapunov-based stability analysis results (see References [6, 7, 9] and the references therein) use a model transformation to transform a time-delay system into a system with distributed delays. The recent results of References [29, 30] demonstrated that this transformation introduces additional dynamics and hence any stability criteria based on this transformation will be inherently conservative if the additional dynamics have unstable poles (in addition to the conservatism induced by the value set covering discussed in Reference [23]). Our result does not involve such model transformation, and therefore does not suffer the inherent conservatism incurring in these Lyapunov-based results.

The notation used in this paper is conventional. Let $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) be the set of all real (complex) $n\times m$ matrices, $\mathbb{C}_+$ ($\bar{\mathbb{C}}_+$) be the open (closed) right-half of the complex plane, $\mathbb{R}_e := \mathbb{R}\cup\{\infty\}$, I_n be $n\times n$ identity matrix, and W^{T} be the transpose of real matrix W. $P>0$ indicates that P is a symmetric and positive definite matrix, and $\|\cdot\|_\infty$ indicates the $\mathscr{H}_\infty$ norm defined by $\|G\|_\infty := \sup_{\omega\in\mathbb{R}}\bar{\sigma}[G(\mathrm{j}\omega)]$ where $\bar{\sigma}(M)$ is the maximum singular value of the complex matrix M. For a transfer function matrix $G(s)$, its minimal realization (A,B,C,D) is denoted by

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

and A is said to be the *kernel matrix* of $G(s)$. If $P=P^{\mathrm{T}}$ and $Q=Q^{\mathrm{T}}$, then

$$\begin{bmatrix} P & M \\ * & Q \end{bmatrix}$$

denotes the symmetric matrix

$$\begin{bmatrix} P & M \\ M^{\mathrm{T}} & Q \end{bmatrix}.$$

For matrices $M=(m_{ij})\in\mathbb{R}^{n_1\times n_1}$ and $N\in\ \mathbb{R}^{n_2\times n_2}$, the Kronecker product is defined by $M\otimes N := (m_{ij}N)$ and the Kronecker sum is defined by $M\oplus N := M\otimes I_{n_2} + I_{n_1}\otimes N$. $\lambda^+_{\max}(M)$ is the maximum positive real eigenvalue of M and $\lambda^+_{\max}(M)=0^+$ when M does not have any positive real eigenvalues. Finally, given a continuous function $w(q):[0,\infty)\to\mathscr{D}$ where $\mathscr{D}=\{z\in\mathbb{C}|\ |z|=1\}$, letting Γ_r be the path created by mapping the interval $q\in[0,r]$ via $w(q)$ to $\mathscr{D}$, we define a *continuous* argument (phase) function for the value $w(r)$ as $\mathrm{Arg}(w(r)) := \arg(w(r)) + 2\pi n(\Gamma_r, 0)$, where $\arg(z)\in(-2\pi,0]$ is the unique argument of $z\in\mathbb{C},\ z\neq 0$ and (Γ,a) is the winding number[‡] of path Γ about a.

2. PRELIMINARIES

Consider the linear, multiple time-delay system given by

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{N} A_k x(t-\tau_k) \tag{1}$$

where the time delays $\tau_k\in[0,\bar{\tau}_k]$, $\bar{\tau}_k>0$, $k=1,\ldots,N$, are constant, unknown and independent of each other. We denote the delay vector by $\boldsymbol{\tau}=[\tau_1\ \cdots\ \tau_N]$, and the delay set $\prod_{k=1}^{N}\times[0,\bar{\tau}_k] := \{[\tau_1\ \cdots\ \tau_N] | \tau_k\in[0,\bar{\tau}_k],\ k=1,\ \ldots,\ N\}$.

The following assumption is necessary when investigating asymptotic stability of the system (1).

[‡] For clockwise paths, winding numbers are negative. See Reference [31] for more details.

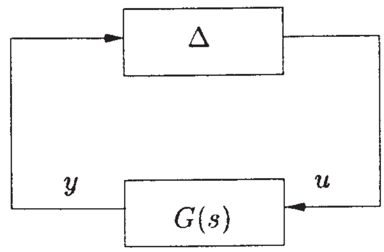

Figure 1. An interconnection system.

Assumption 1
The system (1) free of delays is asymptotically stable, that is, the matrix $\bar{A} := A + \sum_{k=1}^{N} A_k$ is Hurwitz.

Herein, we provide the definitions and preliminary results that will be used later in our analysis. First, we provide the notion of robust stability of a feedback interconnection of a finite-dimensional linear, time-invariant (FDLTI) system and an uncertain system with known uncertainty structure. More on this definition can be found in Reference [32].

Definition 1
Consider a linear, time-invariant (finite-dimensional) system $G(s)$ interconnected with an uncertain block $\Delta \in \underline{\mathbf{\Delta}}$ ($\underline{\mathbf{\Delta}}$ is a set of linear time-invariant stable systems), as shown in Figure 1, denoted as $\sum[G(s), \Delta(s)]$. Then the system is said to be *robustly stable* if $G(s)$ is internally stable, the interconnection is well-posed and it remains internally stable for all $\Delta \in \underline{\mathbf{\Delta}}$.

It is well known that the stability of (1) can be described by its characteristic function (see e.g. References [14, 33–35] and the references therein).

Definition 2
The system (1) is said to be *asymptotically stable* on $\prod_{k=1}^{N}[0, \bar{\tau}_k]$ if and only if

$$\Psi(s, \tau_1, \ldots, \tau_N) \neq 0, \quad \forall s \in \bar{\mathbb{C}}_+, \tau_k \in [0, \bar{\tau}] \tag{2}$$

where $\Psi(s, \tau_1, \ldots, \tau_N) := \det(sI_n - A - \sum_{k=1}^{N} A_k e^{-\tau_k s})$ is the *characteristic function* associated with system (1).

Compared with the single-delay case, the analysis of linear systems with multiple delays is much more complicated. As a matter of fact, in the general non-commensurate delays case, this problem is $\mathcal{NP}$-hard [36]. Consequently, it is unlikely to find efficient algorithms to solve this problem exactly in the general case. Our objective is to find sufficient conditions which are numerically tractable but are not too conservative. Moreover, it turns out that the analysis of the stability regions is rather complex even for the case of a scalar differential equation involving only two delays [37]. In particular, this system may have multiple 'maximum' stability margins, i.e., the system may have stability margins which are unbounded in two different directions [38].

This phenomenon complicates our analysis. Herein, we introduce the definition of the *actual delay margin with a proportionality ratio vector*.

Definition 3
Given the proportionality ratio vector $v := [l_1 \ \cdots \ l_N] \in \mathbb{R}^N$, where $\min_{1 \leqslant k \leqslant N} (l_k) = 1$, the *actual delay margin* $\bar{\tau}^*$ for the system (1) is defined by

$$\bar{\tau}^* := \sup\left\{ \bar{\tau} | (1) \text{ is asymptotcially stable on } \prod_{k=1}^{N} [0, l_k \bar{\tau}] \right\}$$

The stability of system (1) is said to be *delay-dependent* if $\bar{\tau}^*$ is finite, and delay-independent otherwise.

Next, we introduce the definition of the *degree of conservatism* of a stability criterion.

Definition 4
Suppose the stability of (1) is delay-dependent with actual delay margin $\bar{\tau}^*$ with respect to the proportionality ratio vector $v := [l_1 \ \cdots \ l_N]$, and $\mathscr{P}$ is a sufficient condition which ensures the asymptotic stability of (1). The *degree of conservatism* (d.o.c.) of $\mathscr{P}$ is defined by

$$\text{d.o.c.}(\mathscr{P}) := \frac{\bar{\tau}^* - \bar{\tau}^*_{\mathscr{P}}}{\bar{\tau}^*}$$

where $\bar{\tau}^*_{\mathscr{P}} := \sup\{\bar{\tau} | \mathscr{P} \text{ is true on } \prod_{k=1}^{N} [0, l_k \bar{\tau}]\}$. Moreover, $\bar{\tau}^*_{\mathscr{P}}$ is said to be the delay *margin guaranteed by* $\mathscr{P}$ with the same proportionality ratio vector v.

The d.o.c.($\mathscr{P}$) gives a *quantitative measure* for the conservatism of the stability criterion $\mathscr{P}$. Notice that $0 \leqslant \text{d.o.c.}(\mathscr{P}) \leqslant 1$. If $\mathscr{P}$ is necessary and sufficient, $\text{d.o.c.}(\mathscr{P}) = 0$.

Our results will make use of the following lemmas.

Lemma 1 (Datko [34])
Given $\tau_1, \ldots, \tau_N \geqslant 0$, the function $\sigma(h) := \sup\{\text{Re}(s) | \Psi(s, h\tau_1, \ldots, h\tau_N) = 0\}$ is continuous for all $h \geqslant 0$.

Lemma 2
The system (1) is asymptotically stable on $\prod_{k=1}^{N} [0, \bar{\tau}_k]$ if and only if

$$\Psi(\mathrm{j}\omega, \tau_1, \ldots, \tau_N) \neq 0, \quad \forall \omega \geqslant 0, \tau_k \in [0, \bar{\tau}_k] \tag{3}$$

Proof
Necessity is obvious. We prove the sufficiency by contradiction. Assume there exist $s_0 \in \mathbb{C}_+$ and $\tau_{k_0} \in [0, \bar{\tau}_k]$, $k = 1, \ldots, N$ such that $\Psi(s_0, \tau_{l_0}, \ldots, \tau_{N_0}) = 0$. Let $\sigma(h) := \sup\{\text{Re}(s) | \Psi(s, h\tau_{l_0}, \ldots, h\tau_{N_0}) = 0\}$. Then we have $\sigma(1) > 0$. Notice that $\bar{A} = A + \sum_{k=1}^{N} A_k$ is Hurwitz, thus $\sigma(0) < 0$. Since from Lemma 1, $\sigma(h)$ is continuous for $h \geqslant 0$, there exists $h_0 \in [0, 1]$ such that $\sigma(h_0) = 0$. Let $\hat{\tau}_k := h_0 \tau_{k_0} \in [0, \bar{\tau}_k]$, $k = 1, \ldots, N$. Then we have $\sup\{\text{Re}(s) | \Psi(s, \hat{\tau}_{k_l}, \ldots, \hat{\tau}_{k_N}) = 0\} = 0$. On the other hand, for any given constant $c > 0$, the number of the roots of $\Psi(s, \hat{\tau}_{k_l}, \ldots, \hat{\tau}_{k_N})$ in the region $-c < \text{Re}(s) < c$ is finite, and these roots, if any, have finite magnitude [35]. Therefore, there exists an $\hat{\omega} \in \mathbb{R}$ such that $\Psi(\mathrm{j}\hat{\omega}, \hat{\tau}_{k_l}, \ldots, \hat{\tau}_{k_N}) = 0$. This contradicts the condition (3) because

the roots of $\Psi(s, \hat{\tau}_{k_1}, \ldots, \hat{\tau}_{k_N})$ are symmetric with respect to the real axis. The proof is hence complete. □

In the sequel, we decompose $A_k = H_k F_k$ where $H_k \in \mathbb{R}^{n\times q_k}$ and $F_k \in \mathbb{R}^{q_k \times n}$ have full rank, and denote $H := [H_1 \ \cdots \ H_N]$, and $F := [F_1^{\mathrm{T}} \ \cdots \ F_N^{\mathrm{T}}]^{\mathrm{T}}$. The following zero exclusion condition will be used for the stability analysis of (1).

Lemma 3 (Zero Exclusion Condition: Multiple Delay Case)
The system (1) is asymptotically stable on $\prod_{k=1}^{N} [0, \bar{\tau}_k]$ if and only if

$$\det[I_q - G(\mathrm{j}\omega)\Phi(\tau, \mathrm{j}\omega)] \neq 0, \quad \forall \omega \geqslant 0, \ \ \tau_k \in [0, \bar{\tau}_k] \tag{4}$$

where $q = q_1 + \cdots + q_N$,

$$G(s) := F(sI_n - \bar{A})^{-1} H = \left[\begin{array}{c|c} \bar{A} & H \\ \hline F & 0 \end{array}\right]$$

and $\Phi(\boldsymbol{\tau}, s) := \mathrm{diag}\{\phi(\tau_1 s) I_{q_1}, \ldots, \phi(\tau_N s) I_{q_N}\}, \ \ \phi(\tau_k s) = \mathrm{e}^{-\tau_k s} - 1.$

Proof
This follows form Lemma 2 immediately. □

Examining the stability of (1) by checking the condition (4) directly is nontrivial, because (4) implies solving a transcendental equation. An indirect but intuitive approach for examining whether (4) holds, is to cover $\Phi(\boldsymbol{\tau}, \mathrm{j}\omega)$ with another set $\underline{\Phi}(\omega)$, that is, to find a value set $\underline{\Phi}(\omega)$ such that for each $\omega \geqslant 0$,

$$\Phi(\boldsymbol{\tau}, \mathrm{j}\omega) \in \underline{\Phi}(\omega), \quad \forall \tau \in \prod_{k=1}^{N} [0, \bar{\tau}_k]$$

Then (4) holds if $\det[I_q - G(\mathrm{j}\omega)\Delta(\mathrm{j}\omega)] \neq 0, \ \forall \Delta(\mathrm{j}\omega) \in \underline{\Phi}(\omega)$, for each $\omega \geqslant 0$. This is satisfied if the interconnection $\sum[G(s), \Delta(s)]$ (referred to as the comparison system in the sequel) is robustly stable. The conservatism of this approach mainly arises from the manner in which the covering set $\underline{\Phi}(\omega)$ is chosen for each frequency ω, based on the properties of the delay element. In Reference [23], it was demonstrated that the unit disk was implicitly used in the Lyapunov-based stability criteria of [4, 7, 6, 9]. In Reference [24], various covering sets, based on a shifted disk and/or a weighting filter were introduced to reduce the conservatism of the analysis.

Herein, we introduce a new less conservative covering set for the delay element $\Phi(\tau, \mathrm{j}\omega)$, which is based on the properties of the diagonal Padé approximation to the delay element.

3. INNER AND OUTER COVERING OF THE DELAY ELEMENT VALUE SET USING PADÉ APPROXIMATION

Since 1970s, Padé approximations have been widely used in various fields, such as physics, chemistry and mathematics. Recently, Padé approximations have also been used for LTDS (see Reference [25] and the references therein). A Padé approximation is a rational approximation to an irrational function and is defined so that its power series expansion matches as many terms of the power series of the approximated function as possible [39].

Next, we develop an inner and outer covering relation for the delay element using a Padé approximation to e^{-s}. This relation is fundamental to the stability analysis developed in the later sections. The primary reason that a Padé approximation is used for our analysis of LTDS is that this choice ensures the approximation of the delay element is stable itself for any order of approximation [40]. Alternative approximations, such as Taylor, do not necessarily enjoy this property.

Consider the mth ($m \geqslant 3$) *diagonal* Padé approximation $R_m(s)$ to e^{-s}. Then we have

$$R_m(s) = \frac{P_m(s)}{Q_m(s)}$$

where

$$P_m(s) = \sum_{k=0}^{m} \frac{(2m-k)!m!(-s)^k}{(2m)!k!(m-k)!}, \quad \text{and} \quad Q_m(s) = P_m(-s).$$

Clearly, for each $\omega \in \mathbb{R}$, $R_m(j\omega)$ is on the unit circle in the complex plane. Now, consider the continuous argument function $\mathrm{Arg}(R_m(j\omega))$ for $R_m(j\omega)$ for all $\omega \geqslant 0$. This satisfies $\mathrm{Arg}(R_m(j\omega))|_{\omega=0} = 0$. The following lemma provides a property of this phase function which plays a key role in proving Lemma 5.

Lemma 4

For every integer $m \geqslant 3$, the function $(d/d\omega)\,\mathrm{Arg}(R_m(j\omega))$ can be expressed in the following form:

$$\frac{d}{d\omega}\mathrm{Arg}(R_m(j\omega)) = -\frac{T_m(\omega)}{\omega^{2m} + T_m(\omega)}, \quad \forall \omega \in \mathbb{R}$$

where

$$T_m(\omega) = \sum_{k=0}^{m-1} a_k \omega^{2k} \tag{5}$$

and $a_k > 0,\ k = 0, \ldots, m-1$ are independent of ω. (see Appendix for proof)

In the sequel, given constant $\bar{\tau} > 0$ and $\omega \geqslant 0$, we define the following value sets:[§]

$$\mathbf{\Omega}_d(\omega, \bar{\tau}) := \{e^{-j\tau\omega} | \tau \in [0, \bar{\tau}]\}$$

$$\mathbf{\Omega}_o(\omega, \bar{\tau}) := \{R_m(j\theta\alpha_m\omega) | \theta \in [0, \bar{\tau}]\}$$

$$\mathbf{\Omega}_i(\omega, \bar{\tau}) := \{R_m(j\theta\omega) | \theta \in [0, \bar{\tau}]\}$$

where $\alpha_m := \omega_{cm}/2\pi$, and ω_{cm} is the phase crossover frequency of $R_m(j\omega)$ at the -2π line:

$$\omega_{cm} := \min\{\omega > 0 | R_m(j\omega) = 1\}.$$

It can be found that for $m = 3, 4$ and 5, $\alpha_m \approx 1.2329, 1.0315$ and 1.00363, respectively. Since $|R_m(j\omega)| = 1$, for every $\omega > 0$, $\mathbf{\Omega}_d(\omega, \bar{\tau}), \mathbf{\Omega}_o(\omega, \bar{\tau})$ and $\mathbf{\Omega}_i(\omega, \bar{\tau})$ are arcs on the unit circle; see Figure 2.

[§]The subscripts d, o, and i indicate *delay*, *outer*, and *inner*, respectively.

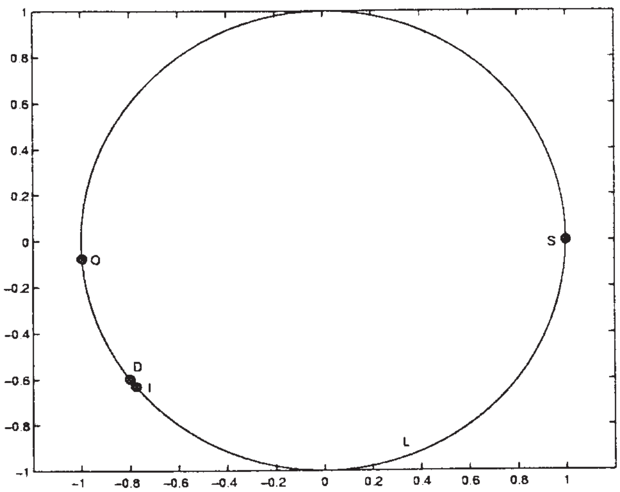

Figure 2. The covering sets generated by a third order Padé approximation ($\omega = 2.5$ rad/s and $\bar{\tau} = 1$ s). Arc $\widehat{SLI}$-set $\mathbf{\Omega}_i(\omega, \bar{\tau})$; Arc $\widehat{SLD}$-set $\mathbf{\Omega}_d(\omega, \bar{\tau})$; Arc $\widehat{SLO}$-set $\mathbf{\Omega}_o(\omega, \bar{\tau})$.

Now, for the continuous argument functions of $e^{-j\omega}$ and $R_m(j\alpha_m\omega)$ for all $\omega \geqslant 0$, we have $\text{Arg}(e^{-j\omega})\,|_{\omega=0} = \text{Arg}(R_m(j\alpha_m\omega))\,|_{\omega=0} = 0$, $\text{Arg}(e^{-j\omega}) = -\omega$, for all $\omega \geqslant 0$. When $m = 3$, the phase functions $\text{Arg}(e^{-j\omega})$, $\text{Arg}(R_m(j\alpha_m\omega))$, and $\text{Arg}(R_m(j\omega))$ are shown in Figure 3. We have the following lemma about the relationship among these functions.

Lemma 5
For every integer $m \geqslant 3$, the phase functions $\text{Arg}(e^{-j\omega})$, $\text{Arg}(R_m(j\alpha_m\omega))$, and $\text{Arg}(R_m(j\omega))$ have the following properties.

(a) $\text{Arg}(e^{-j\omega}) \leqslant \text{Arg}(R_m(j\omega))$, $\quad \forall \omega \in [0, \omega_{cm}]$, and
(b) $\text{Arg}(R_m(j\alpha_m\omega)) \leqslant \text{Arg}(e^{-j\omega})$, $\quad \forall \omega \in [0, 2\pi]$.

Proof
From Lemma 4, we know that there exist constants $a_k > 0$, $k = 0, \ldots, m-1$, such that

$$\frac{d}{d\omega}\text{Arg}(R_m(j\omega)) = -\frac{T_m(\omega)}{\omega^{2m} + T_m(\omega)}, \quad \forall \omega \in \mathbb{R}$$

where

$$T_m(\omega) = \sum_{k=0}^{m-1} a_k \omega^{2k} \tag{6}$$

If follows that

$$\frac{d}{d\omega}\text{Arg}(R_m(j\omega)) \geqslant -1, \quad \forall \omega \in \mathbb{R}$$

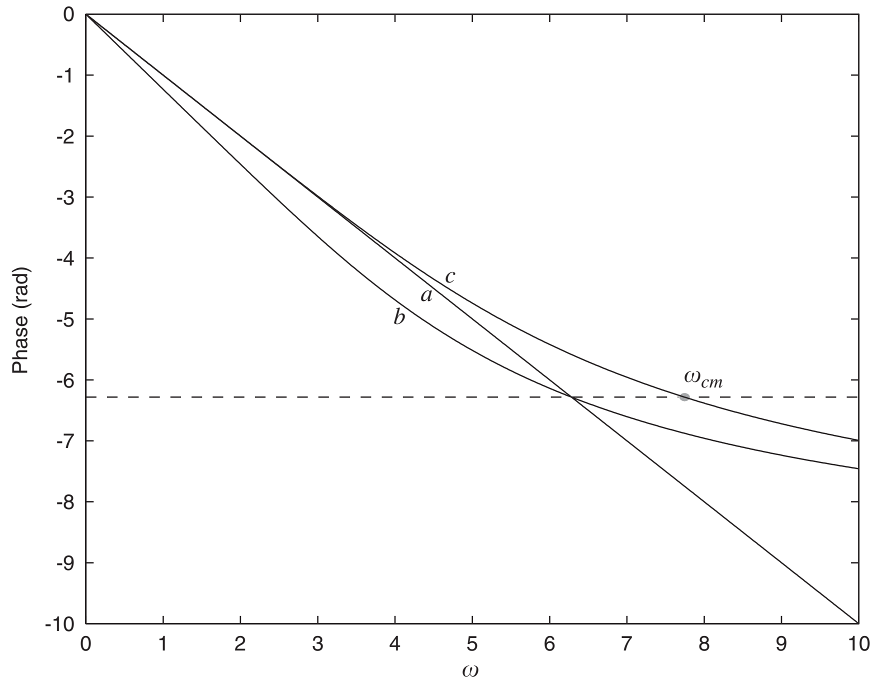

Figure 3. Phase plot for delay and third order Padé approximation. (a) $\mathrm{Arg}(e^{-j\omega})$. (b) $\mathrm{Arg}(R_m(j\alpha_m\omega))$. (c) $\mathrm{Arg}(R_m(j\omega))$.

Then (a) follows from integrating the above inequality from 0 to ω, by using the fact that $\mathrm{Arg}(e^{-j\omega}) = -\omega$ and $\mathrm{Arg}(R_m(j\omega))|_{\omega=0} = 0$.

To show (b), we define

$$g(\omega) := \mathrm{Arg}(R_m(j\omega)) + \frac{1}{\alpha_m}\omega$$

and obtain

$$\begin{aligned}\frac{d^2}{d\omega^2} g(\omega) &= \frac{d^2}{d\omega^2}\mathrm{Arg}(R_m(j\omega))\\ &= -\frac{d}{d\omega}\left[\frac{T_m(\omega)}{\omega^{2m} + T_m(\omega)}\right]\\ &= \frac{\omega^{2m-1}[2mT_m(\omega) - \omega T_m{}'(\omega)]}{[\omega^{2m} + T_m(\omega)]^2}\\ &= \frac{\omega^{2m-1}[a_0 + 2\sum_{k=0}^{m-1} a_k(m-k)\omega^{2k}]}{[\omega^{2m} + T_m(\omega)]^2}\end{aligned}$$

Therefore, $(d^2/d\omega^2)g(\omega) > 0$ for all $\omega > 0$ because $a_0 > 0$ and $a_k(m-k) > 0,\ k = 1, \ldots, m-1$. This indicates that the function $g(\omega)$ is strictly convex in the interval $\omega \in (0, \omega_{cm})$. Clearly,

$g(\alpha_m\omega)$ is strictly convex in the interval $\omega \in (0, 2\pi)$. Because $g(\alpha_m\omega)|_{\omega=0} = g(\alpha_m\omega)|_{\omega=2\pi} = 0$, we have $g(\alpha_m\omega) \leqslant 0$, for all $\omega \in [0, 2\pi]$, which yields (b). $\square$

Next we demonstrate that the function $R_m(s)$ and the above defined sets have several important properties which are summarized in the following lemma.

Lemma 6
For every integer $m \geqslant 3$, the following statements hold:

(a) All poles of $R_m(s)$ are in the open left half complex plane.
(b) Given any $\bar{\tau} \geqslant 0$ and any $\omega \geqslant 0$, $\mathbf{\Omega}_i(\omega, \bar{\tau}) \subseteq \mathbf{\Omega}_d(\omega, \bar{\tau}) \subseteq \mathbf{\Omega}_o(\omega, \bar{\tau})$, and
(c) $\alpha_m \to 1$ as $m \to \infty$.

Proof
(a) The conclusion follows directly from the results of Reference [40].
(b) From Lemma 5, we conclude that

$$\mathbf{\Omega}_i(\omega, \bar{\tau}) \subseteq \mathbf{\Omega}_d(\omega, \bar{\tau}), \quad \forall \omega \in \left[0, \frac{\omega_{cm}}{\bar{\tau}}\right]$$

$$\mathbf{\Omega}_d(\omega, \bar{\tau}) \subseteq \mathbf{\Omega}_o(\omega, \bar{\tau}), \quad \forall \omega \in \left[0, \frac{2\pi}{\bar{\tau}}\right]$$

Also, if $\omega > \omega_{cm}/\bar{\tau}$, both $\mathbf{\Omega}_i(\omega, \bar{\tau})$ and $\mathbf{\Omega}_d(\omega, \bar{\tau})$ cover the entire unit circle. Thus, $\mathbf{\Omega}_i(\omega, \bar{\tau}) = \mathbf{\Omega}_d(\omega, \bar{\tau})$ for all $\omega > \omega_{cm}/\bar{\tau}$. Similarly, for all $\omega > 2\pi/\bar{\tau}$, both $\mathbf{\Omega}_o(\omega, \bar{\tau})$ and $\mathbf{\Omega}_d(\omega, \bar{\tau})$ cover the entire unit circle, hence $\mathbf{\Omega}_o(\omega, \bar{\tau}) = \mathbf{\Omega}_d(\omega, \bar{\tau})$. Thus we have $\mathbf{\Omega}_i(\omega, \bar{\tau}) \subseteq \mathbf{\Omega}_d(\omega, \bar{\tau}) \subseteq \mathbf{\Omega}_o(\omega, \bar{\tau})$, $\forall \omega \geqslant 0$.

(c) Let $m \geqslant 5$. First, it can be easily seen from part (a) of Lemma 5 that $\omega_{cm} \geqslant 2\pi$. On the other hand, for every $\omega > 0$, it can be verified that $T_{m+1}(\omega) > \omega^2 T_m(\omega)$ where $T_m(\omega)$ is given by (6). Hence, using Lemma 4, we have

$$\begin{aligned}
|\mathrm{Arg}(R_m(\mathrm{j}\omega_{cm}))| &= \left| \int_0^{\omega_{cm}} \left[\frac{\mathrm{d}}{\mathrm{d}\omega} \mathrm{Arg}(R_m(\mathrm{j}\omega)) \right] \mathrm{d}\omega \right| \\
&= \int_0^{\omega_{cm}} \frac{T_m(\omega)}{\omega^{2m} + T_m(\omega)} \mathrm{d}\omega \\
&\geqslant \int_0^{\omega_{cm}} \frac{T_5(\omega)}{\omega^{10} + T_5(\omega)} \mathrm{d}\omega
\end{aligned}$$

Then by the definition of ω_{cm}, we conclude that $2\pi \leqslant \omega_{cm} \leqslant \omega_{cm}|_{m=5} < 2.0073\pi$. In view of the fact $(4/e)m > \omega_{cm}$, it follows from the results of Reference [41] that

$$\begin{aligned}
|\mathrm{e}^{-\mathrm{j}\omega_{cm}} - 1| &= |\mathrm{e}^{-\mathrm{j}\omega_{cm}} - R_m(\mathrm{j}\omega_{cm})| \\
&\leqslant \left(\frac{\omega_{cm} e}{4m}\right)^{2m+1} < \left(\frac{4.286}{m}\right)^{2m+1}
\end{aligned}$$

In addition, there exists $\xi_m \in [2\pi, \omega_{cm}]$ such that

$$|e^{-j\omega_{cm}} - 1| = \left|2\sin\frac{\omega_{cm}}{2}\right| = |\omega_{cm} - 2\pi|\left|\cos\frac{\xi_m}{2}\right|$$
$$\geqslant |\omega_{cm} - 2\pi|\cos\frac{2.0073\pi}{2}$$

Therefore,

$$|\alpha_m - 1| = \frac{1}{2\pi}|\omega_{cm} - 2\pi| < 0.16\left(\frac{4.286}{m}\right)^{2m+1} \tag{7}$$

which implies that $\alpha_m \to 1$ as $m \to \infty$.

It should be noted that the above proof has an important implication. When $\omega > \omega_{cm}/\bar{\tau}$, the two sets $\mathbf{\Omega}_i(\omega, \bar{\tau})$ and $\mathbf{\Omega}_d(\omega, \bar{\tau})$ both cover the unit circle and consequently $\mathbf{\Omega}_i(\omega, \bar{\tau}) = \mathbf{\Omega}_d(\omega, \bar{\tau})$. Similarly, for every $\omega > 2\pi/\bar{\tau}$, both $\mathbf{\Omega}_o(\omega, \bar{\tau})$ and $\mathbf{\Omega}_d(\omega, \bar{\tau})$ cover the unit circle, and hence $\mathbf{\Omega}_d(\omega, \bar{\tau}) = \mathbf{\Omega}_o(\omega, \bar{\tau})$. The fact that the inner and outer covering relation in (b) of Lemma 6 holds *for every frequency* $\omega \geqslant 0$ is of fundamental importance for our analysis and is in contrast with the traditional manner of using rational approximations, in which the accuracy and validity can only be ensured *for a certain finite frequency range*.

4. STABILITY ANALYSIS

We now will take advantage of the outer covering developed in the preceding section to create a finite-dimensional, parameter-dependent comparison system, the robust stability of which will guarantee the stability of the original time-delay system. The inner approximation will be employed to establish a necessary condition for stability, and this will allow the establishment of an upper bound on the degree-of-conservatism of the sufficient condition. Toward this end, we replace the delay elements $e^{-\tau_k s}$ with the real rational functions $R_m(\theta_k \alpha_m s)$ and $R_m(\theta_k s)$ and denote the resulting finite-dimensional interconnection systems as $\sum_o(\theta) := \sum(G(s), P_\theta(\alpha_m s))$ and $\sum_i(\theta) := \sum(G(s), P_\theta(s))$, respectively, where $P_\theta(s) := \text{diag}\{[R_m(\theta_1 s) - 1]I_{q1}, \ldots, R_m(\theta_N s) - 1]I_{qN}\}$. This changes the analysis problem from one of examining the stability of a family of infinite-dimensional systems (parameterized by τ_k) to one of examining the stability of a family of finite-dimensional systems (parameterized by θ_k). Note, however, that the variables τ_k and θ_k are not in any sense equivalent.

The following theorem gives a *sufficient* condition for the stability of (1).

Theorem 1

The system (1) is asymptotically stable on $\prod_{k=1}^N [0, \bar{\tau}_k]$, if the comparison system $\sum_o(\boldsymbol{\theta})$ is robustly stable for $\boldsymbol{\theta} \in \prod_{k=1}^N [0, \bar{\tau}_k]$.

Proof

If $\sum_o(\theta)$ is robustly stable for $\boldsymbol{\theta} \in \prod_{k=1}^N [0, \bar{\tau}_k]$, then

$$\det[I_q - G(j\omega)P_\theta(j\alpha_m\omega)] \neq 0, \quad \forall \omega \geqslant 0, \ \boldsymbol{\theta} \in \prod_{k=1}^N [0, \bar{\tau}_k]$$

which can be rewritten as

$$\det[I_q - G(j\omega)\,\mathrm{diag}\{[\delta_1(\omega)-1]I_{q_1},\ldots,[\delta_N(\omega)-1]I_{q_N}\}] \neq 0$$
$$\forall \omega \geqslant 0, \delta_k(\omega) \in \mathbf{\Omega}_o(\omega, \bar{\tau}_k)$$

From Lemma 6, $\mathbf{\Omega}_d(\omega, \bar{\tau}_k) \subseteq \mathbf{\Omega}_o(\omega, \bar{\tau}_k)$. Hence

$$\det[I_q - G(j\omega)\,\mathrm{diag}\{[\delta_1(\omega)-1]I_{q_1},\ldots,[\delta_N(\omega)-1]I_{q_N}\}] \neq 0$$
$$\forall \omega \geqslant 0,\ \ \delta_k(\omega) \in \mathbf{\Omega}_d(\omega, \bar{\tau}_k) \tag{8}$$

which implies that

$$\det[I_q - G(j\omega)\Phi(\boldsymbol{\tau}, j\omega)] \neq 0, \quad \forall \omega \geqslant 0,\ \boldsymbol{\tau} \in \prod_{k=1}^{N} [0, \bar{\tau}_k]$$

Thus from Lemma 3, (1) is asymptotically stable on $\prod_{k=1}^{N} [0, \bar{\tau}_k]$. □

The following theorem provides a *necessary* condition for the stability of (1) which will be used to check the d.o.c. of our analysis result.

Theorem 2
If (1) is asymptotically stable on $\prod_{k=1}^{N} [0, \bar{\tau}_k]$, then $\sum_i(\theta)$ is robustly stable for $\theta \in \prod_{k=1}^{N} [0, \bar{\tau}_k]$.

The proof of this theorem is rather technical and is given in Appendix B.

Next, we show that the d.o.c. of Theorem 1 is bounded by a function of α_m.

Theorem 3
The d.o.c. of Theorem 1 with any proportionality ratio vector v satisfies

$$\text{d.o.c.(Theorem 1)} \leqslant \frac{\alpha_m - 1}{\alpha_m} \tag{9}$$

Moreover, d.o.c.(Theorem 1) $\to 0$ as $m \to \infty$.

Proof
Let $\bar{\tau}^*$ be the actual delay margin of (1), and $\bar{\tau}^*_{\text{Theorem 1}}$ be the delay margin guaranteed by Theorem 1, both with the same proportionality ratio vector v. Let

$$T_i := \sup\left\{ \bar{\tau}_i \,\middle|\, \sum_i(\theta) \text{ is robustly stable for } \theta \in \prod_{k=1}^{N} [0, l_k \bar{\tau}_i] \right\}$$

Then, clearly, we have $T_i = \alpha_m \bar{\tau}^*_{\text{Theorem 1}}$. In addition, from Theorem 2, $\sum_i(\theta)$ is robustly stable for $\theta \in \prod_{k=1}^{N} [0, \bar{\tau}_k]$ whenever (1) is asymptotically stable on $\prod_{k=1}^{N} [0, \bar{\tau}_k]$. Therefore,

$$T_i \geqslant \bar{\tau}^*$$

which immediately yields (9). □

Remark 1
For $m = 3$, 4 and 5, we have $(\alpha_m - 1)/\alpha_m \approx 18.9$, 3.05 and 0.361%, respectively. This upper bound depends only on the order of Padé approximation used and it can be reduced to any

desired degree by choosing higher order m. This d.o.c. upper bound is independent of $\bar{\tau}^*$, A and A_k.

Remark 2
Theorem 2 indicates that, by replacing the delay elements $e^{-\tau_k s}$ with Padé approximations directly, the stability analysis of the resultant finite-dimensional system only gives a necessary condition, and hence it does not guarantee, in general, the stability of the original time-delay systems. This manner of using Padé approximations for time-delay systems has been used extensively; see [25–28]. However, it can only be used for small delays and over a finite bandwidth of the system, because the Padé approximations are accurate only when $|\tau_k s|$ is sufficiently small. On the other hand, Theorem 1 may be used for analysis without any restriction in its use on the delays or on the bandwidth of the system. This is due to the fact that the covering relations hold for all frequencies.

The following corollary follows immediately from Theorem 3 and inequality (7). This result gives an explicit bound on the d.o.c. as a function of m.

Corollary 1
For $m \geqslant 5$, the d.o.c. of Theorem 1 with any proportionality ratio vector v satisfies

$$\text{d.o.c.(Theorem 1)} < 0.16\left(\frac{4.286}{m}\right)^{2m+1} \tag{10}$$

4.1. The singularity problem

The comparison system $\sum_o(\theta)$ is free of delays, but it has parametric (real) uncertainties θ_k. Some care must be taken when examining its robust stability. When $\theta_k = 0$ for some $k = 1, \dots, N$, the system dynamics suffer a fundamental and abrupt change. The singularity of the system at $\theta_k = 0$ obviously complicates the employment of Theorem 1 for analysis. Next, we discuss several approaches for dealing with this issue. We first consider the single delay case, and then we address the multiple delays case. For convenience, in the sequel, let the minimal realization of $P_k(s) := [R_m(\alpha_m s) - 1]I_{q_k}$ be

$$P_k(s) = \left[\begin{array}{c|c} A_{P_k} & B_{P_k} \\ \hline C_{P_k} & D_{P_k} \end{array}\right]$$

$A_s := \bar{A} + \sum_{k=1}^{N} H_k D_{P_k} F_k, B_{sk} := B_{P_k} F_k$, $C_{sk} := H_k C_{P_k}$ and denote n_k as the order of A_{P_k}.

4.2. Single delay case: An explicit delay margin formula

First, we consider the special case when $N = 1$. In this single delay case, the singularity issue can be dealt with in a straightforward manner. We demonstrate that the delay margin $\bar{\tau}^*_{\text{Theorem 1}}$ guaranteed by Theorem 1 can be *explicitly calculated without incurring any additional conservatism in the single delay case*. Then, we conclude that if $\bar{\tau}_1 < \bar{\tau}^*_{\text{Theorem 1}}$, the system (1) is asymptotically stable for all $\tau_1 \in [0, \bar{\tau}_1]$.

Theorem 4
Suppose that $N = 1$ and the system (1) is asymptotically stable for all $\tau_1 \in [0, \bar{\tau}_a]$, where $\bar{\tau}_a > 0$. Then the delay margin guaranteed by Theorem 1 with proportionality ratio vector $v = 1$, is given by

$$\bar{\tau}^*_{\text{Theorem 1}} = \frac{\bar{\tau}_a}{\alpha_m} + \frac{1}{\lambda^+_{\max}(-(M_0 \oplus M_0)^{-1}(M_1 \oplus M_1))} \tag{11}$$

where $M_0 := \begin{bmatrix} \frac{\bar{\tau}_a}{\alpha_m} A_s & C_{s1} \\ \frac{\bar{\tau}_a}{\alpha_m} B_{s1} & A_{P_1} \end{bmatrix}$ and $M_1 := \begin{bmatrix} A_s & 0 \\ B_{s1} & 0 \end{bmatrix}$.

To prove Theorem 4, we need the following lemma.

Lemma 7 (Barmish [42])
Let $\hat{M}(q) := M_0 + qM_1$, where M_0 and M_1 are constant square matrices. Suppose M_0 is Hurwitz and let

$$\bar{q}^* := \sup\{\bar{q} \mid \hat{M}(q) \text{ is Hurwitz for all } q \in [0, \bar{q}]\}$$

Then

$$\bar{q}^* = \frac{1}{\lambda^+_{\max}(-(M_0 \oplus M_0)^{-1}(M_1 \oplus M_1))}$$

Proof of Theorem 4
When $N = 1$, it can be easily verified that the comparison system $\sum_o(\theta_1)$ can be realized by

$$\begin{aligned} \dot{x} &= \bar{A}x + H_1 u \\ y &= F_1 x \\ \dot{x}_{P_1} &= \theta_1^{-1} A_{P_1} x_{P_1} + \theta_1^{-1/2} B_{P_1} y \\ u &= \theta_1^{-1/2} C_{P_1} x_{P_1} + D_{P_1} y \end{aligned}$$

The closed-loop system is given as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{P_1} \end{bmatrix} = A_L(\theta_1) \begin{bmatrix} x \\ x_{P_1} \end{bmatrix}$$

where

$$A_L(\theta_1) := \begin{bmatrix} A_s & \theta_1^{-1/2} C_{s1} \\ \theta_1^{-1/2} B_{s1} & \theta_1^{-1} A_{P_1} \end{bmatrix}$$

is the kernel matrix of $\sum_o(\boldsymbol{\theta}_1)$. Notice that for $\theta_1 > 0$, $A_L(\theta_1)$ is Hurwitz if and only if

$$\theta_1 A_L(\theta_1) = \begin{bmatrix} \theta_1 A_s & \theta_1^{1/2} C_{s1} \\ \theta_1^{1/2} B_{s1} & A_{P_1} \end{bmatrix}$$

is Hurwitz. Let $E_{\theta_1} := \mathrm{diag}\{\theta_1^{-1/2} I_n, I_{n_1}\}$. Then $A_L(\theta_1)$ is Hurwitz if and only if the matrix $\hat{A}(\theta_1) := E_{\theta_1}\theta_1 A_L(\theta_1) E_{\theta_1}^{-1}$ is Hurwitz. Then, $\hat{A}(\theta_1)$ can be rewritten as

$$\hat{A}(\theta_1) = M_0 + \left(\theta_1 - \frac{\bar{\tau}_a}{\alpha_m}\right) M_1$$

Since (1) is asymptotically stable for all $\tau_1 \in [0, \bar{\tau}_a]$, from Theorem 2, $\sum_i(\theta_1)$ is asymptotically stable for $\theta_1 \in [0, \bar{\tau}_a]$, which implies that $\sum_o(\theta_1)$ is asymptotically stable for $\theta_1 \in [0, \bar{\tau}_a/\alpha_m]$. Hence $M_0 = A_L(\bar{\tau}_a/\alpha_m)$ is Hurwitz. The conclusion then follows immediately from Lemma 7 and Theorem 2. □

4.3. General case: A new LMI delay-dependent stability criterion

In this section, we present a delay-dependent stability criterion based on a parameter-dependent Lyapunov matrix for system (1). Some additional conservatism is introduced, but the resultant stability condition is formulated as a finite set of LMIs, which can be solved efficiently [18].

In the sequel, let $F_c := \prod_{k=1}^{N} [0, \bar{\tau}_k]$, and $F_o := \prod_{k=1}^{N} (0, \bar{\tau}_k]$. The following theorem indicates that under an additional condition, the robust stability of $\sum_o(\theta)$ on F_o implies its robust stability on the closed set F_c.

Lemma 8

$\sum_o(\theta)$ is asymptotically stable on F_o if and only if for every $\theta \in F_o$ there exists a positive definite matrix $X(\theta)$ satisfying the Lyapunov inequality

$$A_L(\theta)^{\mathrm{T}} X(\theta) + X(\theta) A_L(\theta) < -I$$

where

$$A_L(\theta) := \begin{bmatrix} A_s & \theta_1^{-1/2} C_{s1} & \cdots & \theta_N^{-1/2} C_{sN} \\ \theta_1^{-1/2} B_{s1} & \theta_1^{-1} A_{P_1} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \theta_N^{-1/2} B_{sN} & 0 & 0 & \theta_N^{-1} A_{P_N} \end{bmatrix}$$

is the kernel of the closed loop system $\sum_o(\theta)$. If, in addition, the matrix $X(\theta)$ is bounded for all $\theta \in F_o$, then $\sum_o(\theta)$ is asymptotically stable on F_c.

Proof

See Appendix C. □

Next, we present a delay-dependent stability condition for system (1). This condition is formulated in terms of a (finite) set of LMIs.

Theorem 5

The system (1) is asymptotically stable on $\prod_{k=1}^{N} [0, \bar{\tau}_k]$, if there exist a constant $\varepsilon > 0$, positive definite matrices $Y_i \in \mathbb{R}^{n \times n}$ and $X_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots, N$, and matrices $W_k \in \mathbb{R}^{n \times n_k}$, $k = 1, \ldots, N$, such that for each vertex $\theta_l^{\#}$, $l = 1, \ldots, 2^N$ of the polytope $\prod_{k=1}^{N} [0, \bar{\tau}_k]$,

$$\Pi(\theta_l^{\#}) < 0 \tag{12}$$

and

$$X(\theta_l^{\#}) > 0 \tag{13}$$

where

$$\Pi(\theta) := \begin{bmatrix} \Pi_{1,1}(\theta) & \Pi_{1,2}(\theta) & \cdots & \Pi_{1,N+1}(\theta) \\ * & \Pi_{2,2}(\theta) & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & \Pi_{N+1,N+1}(\theta) \end{bmatrix}$$

$$X(\theta) := \begin{bmatrix} Y(\theta) & \theta_1^{1/2} W_1 & \cdots & \theta_N^{1/2} W_N \\ * & X_1 & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & X_N \end{bmatrix}$$

with $\Pi_{1,1}(\theta) := Y(\theta)A_s + A_s^{\mathrm{T}} Y(\theta) + \sum_{k=1}^{N} (W_k B_{sk} + B_{sk}^{\mathrm{T}} W_k^{\mathrm{T}}) + \varepsilon I_n$, $\Pi_{1,k+1}(\theta) := Y(\theta)C_{sk} + W_k A_{P_k} + \theta_k A_s^{\mathrm{T}} W_k + B_{sk}^{\mathrm{T}} X_k$, $\Pi_{k+1,k+1}(\theta) := \theta_k W_k^{\mathrm{T}} C_{sk} + \theta_k C_{sk}^{\mathrm{T}} W_k + X_k A_{P_k} + A_{P_k}^{\mathrm{T}} X_k + \varepsilon \theta_k I_{n_k}$, $k = 1, \ldots, N$, and $Y(\theta) := Y_0 + \sum_{k=1}^{N} \theta_k Y_k$.

Proof
First, notice that $\Pi(\theta)$ is convex in θ because it is affine in θ. Hence (12) implies that

$$\Pi(\theta) < 0, \quad \forall \theta \in F_c \tag{14}$$

Using the properties of Shur Complement, Equation (13) is equivalent to $\Theta((\theta_l^{\#}) > 0$, where $\Theta(\theta) := Y(\theta) - \sum_{k=1}^{N} \theta_k W_k X_k^{-1} W_k^{\mathrm{T}}$. Since $\Theta(\theta)$ is convex in θ, we obtain $\Theta(\theta) > 0$, $\forall \theta \in F_c$, which is equivalent to $X(\theta) > 0$, $\forall \theta \in F_c$. For any $\theta \in F_o$, multiplying (14) on both sides by $E_\theta := \mathrm{diag}\{I_n, \theta_1^{-1/2} I_{n_1}, \ldots, \theta_N^{-1/2} I_{n_N}\}$ yields $E_\theta \Pi(\theta) E_\theta < 0$ which immediately gives

$$A_L(\theta)^{\mathrm{T}} X(\theta) + X(\theta) A_L(\theta) < -\varepsilon I$$

where $A_L(\theta)$ is the kernel of the closed loop system $\sum_o(\theta)$. Since $X(\theta)$ is bounded for all $\theta \in F_o$, by Theorem 8, $\sum_o(\theta)$ is also asymptotically stable on F_c. Hence, by Theorem 1, the system (1) is asymptotically stable on F_c. □

5. $\mathscr{H}_\infty$ PERFORMANCE OF LTDS

Our comparison system approach via value set covering can be easily extended to the analysis of other properties of time-delay systems. As an example, in this section we address the problem of $\mathscr{H}_\infty$ performance of LTDS. For simplicity, we only consider the single delay case, but our analysis results can be generalized easily to systems with multiple delays.

5.1. Problem description

Consider a LTDS (denoted as $\sum_d$ in the sequel) subject to exogenous disturbance given by

$$\dot{x}(t) = Ax(t) + A_1 x(t-\tau) + Bw(t)$$

$$z(t) = Cx(t)$$

where $A, A_1 \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times n_w}$, and $C_1 \in \mathbb{R}^{n_w \times n}$ are constant matrices, $\tau \in [0, \bar{\tau}]$ is constant and unknown time-delay, and $w(t) \in \mathcal{L}_2[0,\infty)$ is an exogenous disturbance. The $\mathcal{H}_\infty$ performance problem is to examine if the system is asymptotically stable for all $\tau \in [0,\bar{\tau}]$ and satisfies

$$\|T(s,\tau)\|_\infty \leqslant \gamma \quad \forall \tau \in [0,\bar{\tau}] \tag{15}$$

where $T(s,\tau)$ is the transfer function from the disturbance vector w to performance vector z, and $\gamma > 0$ is the performance measure.

To proceed with our analysis, we first provide the following definition.

Definition 5

Suppose that a system $\sum$ has an uncertain constant parameter $\xi \in \Xi$, Ξ a compact set, and that $T(s,\xi)$ is the transfer function from the disturbance $w \in \mathcal{L}_2[0,\infty)$ to the performance vector z. If $\sum$ is asymptotically stable for all $\xi \in \Xi$, then the worst case $\mathcal{H}_\infty$ performance γ^* of $\sum$ is defined as

$$\gamma^* := \max_{\xi\in\Xi} \|T(s,\xi)\|_\infty \tag{16}$$

5.2. Analysis of $\mathcal{H}_\infty$ performance

To simplify the presentation, we will employ frequency domain descriptions. Decompose $A_1 = HF$ where $H \in \mathbb{R}^{n\times q}$ and $F \in \mathbb{R}^{q\times n}$ have full rank. First notice that the LTDS $\sum_d$ can be rewritten as

$$sX(s) = \bar{A}X(s) + HV(s) + BW(s)$$

$$V(s) = [(\mathrm{e}^{-s\tau} - 1)F]X(s)$$

$$Z(s) = CX(s)$$

Replacing $\mathrm{e}^{-s\tau}$ with $R_m(\theta\alpha_m s)$ and $R_m(\theta s)$, we obtain the following two systems $\sum_o$ and $\sum_i$.

System $\sum_o$:

$$sX(s) = \bar{A}X(s) + HV(s) + BW(s)$$

$$V(s) = [R_m(\theta\alpha_m s)I_q - I_q]FX(s)$$

$$Z(s) = CX(s)$$

System $\sum_i$:

$$sX(s) = \bar{A}X(s) + HV(s) + BW(s)$$

$$V(s) = [R_m(\theta s)I_q - I_q]FX(s)$$

$$Z(s) = CX(s)$$

Then we have the following theorem regarding the relation among the worst case $\mathcal{H}_\infty$ performances of the systems $\sum_d$, $\sum_o$ and $\sum_i$.

Theorem 6
Suppose $\sum_o$ is asymptotically stable for all $\theta \in [0, \bar{\tau}]$. Then the worst case $\mathscr{H}_\infty$ performances of the systems $\sum_d$, $\sum_o$ and $\sum_i$ satisfy $\gamma_i^* \leqslant \gamma_d^* \leqslant \gamma_o^*$.

Proof
Since $\sum_o$ is asymptotically stable for all $\theta \in [0, \bar{\tau}]$, from Theorems 1 and 2, we know that $\sum_d$ is asymptotically stable for all $\tau \in [0, \bar{\tau}]$, and $\sum_i$ is asymptotically stable for all $\theta \in [0, \bar{\tau}]$. Therefore, the transfer functions $T_d(s, \tau) = C[sI - (A + A_1 e^{-\tau s})^{-1}]B$, $T_o(s, \theta) = C[sI - (A + A_1 R_m(\theta\alpha_m s))^{-1}]B$, and $T_i(s, \theta) = C[sI - (A + A_1 R_m(\theta s))^{-1}]B$ are analytic in s and bounded in the closed right half complex plane. By the maximum modulus theorem [43],

$$\|T_d(s, \tau)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}[T_d(j\omega, \tau)], \quad \forall \tau \in [0, \bar{\tau}]$$

$$\|T_o(s, \theta)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}[T_o(j\omega, \theta)], \quad \forall \theta \in [0, \bar{\tau}]$$

$$\|T_i(s, \theta)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}[T_i(j\omega, \theta)], \quad \forall \theta \in [0, \bar{\tau}]$$

Therefore, $\gamma_d^* := \max_{\tau \in [0,\bar{\tau}]} \sup_{\omega \in \mathbb{R}} \bar{\sigma}[T_d(j\omega, \tau)]$, $\gamma_o^* := \max_{\theta \in [0,\bar{\tau}]} \sup_{\omega \in \mathbb{R}} \bar{\sigma}[T_o(j\omega, \theta)]$, and $\gamma_i^* := \max_{\theta \in [0,\bar{\tau}]} \sup_{\omega \in \mathbb{R}} \bar{\sigma}[T_i(j\omega, \theta)]$. Since $\bar{\sigma}[T_d(j\omega, \tau)]$ is a continuous function in ω and τ, and $\bar{\sigma}[T_d(j\omega, \tau)] \to 0$ as $|\omega| \to \infty$, we know that there exist a *finite* number $\tilde{\omega} \in \mathbb{R}$ and $\tilde{\tau} \in [0, \bar{\tau}]$ such that $\gamma_d^* = \bar{\sigma}[T_d(j\tilde{\omega}, \tilde{\tau})]$. Since $\mathbf{\Omega}_d(\tilde{\omega}, \bar{\tau}) \subseteq \mathbf{\Omega}_o(\tilde{\omega}, \bar{\tau})$, there exists $\tilde{\theta} \in [0, \bar{\tau}]$ such that $e^{-j\tilde{\omega}\tilde{\tau}} = R_m \times (j\alpha_m \tilde{\theta}\tilde{\omega})$ which implies that $T_o(j\tilde{\omega}, \tilde{\theta}) = T_d(j\tilde{\omega}, \tilde{\tau})$. Hence, $\gamma_d^* = \bar{\sigma}[T_d(j\tilde{\omega}, \tilde{\tau})] \leqslant \max_{\theta \in [0,\bar{\tau}]} \sup_{\omega \in \mathbb{R}} \bar{\sigma}[T_o(j\omega, \theta)] = \gamma_o^*$.

Similarly, we can show that $\gamma_i^* \leqslant \gamma_d^*$ and the proof is complete. □

Theorem 7
Let the minimal realization of $P(s) := [R_m(\alpha_m s) - 1]I_q$ be

$$P(s) = \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & D_P \end{array}\right]$$

$A_s := A + A_1 + HD_P F$, $B_s := B_P F$, $C_s := HC_P$ and denote n_P as the order of A_P. Then the system $\sum_d$ is asymptotically stable for any constant time-delay $\tau \in [0, \bar{\tau}]$, and satisfies the $\mathscr{H}_\infty$ performance bound $\gamma_d^* \leqslant \gamma$, if there exist matrices $X_0 = X_0^{\mathrm{T}} > 0$, $X_0 \in \mathbb{R}^{n \times n}$, $X_1 = X_1^{\mathrm{T}} \in \mathbb{R}^{n \times n}$, $X_{22} = X_{22}^{\mathrm{T}} > 0$, $X_{22} \in \mathbb{R}^{n_P \times n_P}$ and $X_{12} \in \mathbb{R}^{n \times n_P}$ such that

$$\Pi(0) < 0, \quad \Pi(\bar{\tau}) < 0 \tag{17}$$

and

$$\begin{bmatrix} X_0 + \bar{\tau} X_1 & \bar{\tau} X_{12} \\ \bar{\tau} X_{12}^{\mathrm{T}} & \bar{\tau} X_{22} \end{bmatrix} > 0 \tag{18}$$

where

$$\Pi(\theta) := \begin{bmatrix} \Pi_{11}(\theta) & \Pi_{12}(\theta) & (X_0 + \theta X_1)B & C^{\mathrm{T}} \\ * & \Pi_{22}(\theta) & \theta X_{12}^{\mathrm{T}} B & 0 \\ * & * & -\gamma I_{n_w} & 0 \\ * & 0 & 0 & -\gamma I_{n_w} \end{bmatrix}$$

$\Pi_{11}(\theta) := (X_0 + \theta X_1)A_s + X_{12}B_s + A^{\mathrm{T}}(X_0 + \theta X_1) + B_s^{\mathrm{T}} X_{12}^{\mathrm{T}}$, $\Pi_{12}(\theta) := (X_0 + \theta X_1)C_s + X_{12}A_P + \theta A_s^{\mathrm{T}} X_{12} + B_s^{\mathrm{T}} X_{22}$, and $\Pi_{22}(\theta) := \theta X_{12}^{\mathrm{T}} C_s + \theta C_s^{\mathrm{T}} X_{12} + X_{22}A_P + A_P^{\mathrm{T}} X_{22}$.

Proof

First of all, notice that $\Pi(\theta)$ is convex in θ because it is affine in θ. Hence (17) implies that

$$\Pi(\theta) < 0, \quad \forall \theta \in (0, \bar{\tau}] \tag{19}$$

Similarly, Equation (16), $X_0 > 0$, as well as $X_2 > 0$, imply that

$$\begin{bmatrix} X_0 + \theta X_1 & \theta X_{12} \\ \theta X_{12}^{\mathrm{T}} & \theta X_{22} \end{bmatrix} > 0, \quad \forall \theta \in (0, \bar{\tau}]$$

Pre- and post-multiplying by $E_\theta := \mathrm{diag}\{I_n, \theta^{-1/2} I_{n_P}, I_{n_w}, I_{n_w}\}$, the above inequality is equivalent to

$$X(\theta) := \begin{bmatrix} X_0 + \theta X_1 & \theta^{1/2} X_{12} \\ \theta^{1/2} X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix} > 0, \quad \forall \theta \in (0, \bar{\tau}]$$

Multiplying (19) on both sides by E_θ yields

$$E_\theta \Pi(\theta) E_\theta < 0 \tag{20}$$

which immediately gives

$$A_L(\theta)^{\mathrm{T}} X(\theta) + X(\theta) A_L(\theta) < 0$$

where

$$A_L(\theta) := \begin{bmatrix} A_s & \theta^{-1/2} C_s \\ \theta^{-1/2} B_s & \theta^{-1} A_P \end{bmatrix}$$

Hence, from Theorem 1, $\sum_d$ is asymptotically stable for any constant time-delay $\tau \in [0, \bar{\tau}]$. Furthermore, (20) gives

$$\begin{bmatrix} A_L(\theta)^{\mathrm{T}} X(\theta) + X(\theta) A_L(\theta) & X(\theta) \begin{bmatrix} B \\ 0 \end{bmatrix} & \begin{bmatrix} C^{\mathrm{T}} \\ 0 \end{bmatrix} \\ * & -\gamma I_{n_w} & 0 \\ * & 0 & -\gamma I_{n_w} \end{bmatrix} < 0$$

which, by the Bounded Real Lemma [18], is equivalent to

$$\left\| [C \quad 0][sI_{n+n_P} - A_L(\theta)]^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \right\|_\infty < \gamma \tag{21}$$

It can be easily verified that

$$\left[\begin{array}{c|c} A_L(\theta) & \begin{bmatrix} B \\ 0 \end{bmatrix} \\ \hline [C \quad 0] & 0 \end{array}\right]$$

is a minimal realization of $\sum_o$, thus we have $\|T_o(s,\theta)\|_\infty < \gamma$, $\forall \theta \in (0, \bar{\tau}]$. In addition, we have

$$\begin{aligned} T(s,\theta) &= [C \quad 0][sI_{n+n_P} - A_L(\theta)]^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= C[sI_n - A_s - C_s(\theta s - A_P)^{-1} B_s]^{-1} B \end{aligned}$$

thus

$$\begin{aligned} \lim_{\theta \to 0^+} T(s,\theta) &= C(sI_n - A_s + C_s A_P^{-1} B_s)^{-1} B \\ &= C(sI_n - \bar{A})^{-1} \ B = T(s, 0) \end{aligned}$$

and we can conclude that $\gamma_o^* \leqslant \gamma$. Using Theorem 6, we have $\gamma_d^* \leqslant \gamma_o^* \leqslant \gamma$. $\square$

6. NUMERICAL EXAMPLES

To examine the effectiveness of our approach, we compare our stability analysis results with those similar criteria published elsewhere [6, 7, 9, 16] including a previous result by the authors [24]. The calculations were performed on a Pentium 200 MHz PC by using the MATLAB LMI Control Toolbox [44].

Example 1 (Park [9])
Consider the system (1) with a single delay and

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

Table I summarizes the maximal delay margin calculated by several methods along with the CPU time for each method. We can see that using higher order Padé approximation reduces the conservatism of our analysis but requires more computation effort. For this example the delay margin from Theorem 5 coincides with the value calculated by Theorem 4, and hence in this case the basis function used in deriving Theorem 5 does not introduce additional conservatism. Note that we used the bisection method with Theorem 5 to calculate the maximal delay margin. The relatively larger computational time that Theorem 5 requires is largely due to the 10 iterations used in the bisection search. This is not necessary if we use Theorem 5 to examine the stability for a given $\bar{\tau}_1$ rather than determine the delay margin.

Table I. Comparison of several methods.

Methods		Maximal delay margin	d.o.c.	CPU Time (s)
Nyquist Criterion*		6.172	0	N/A
Niculescu *et al.* [7][†]		0.956	84.5	3.74
Li *et al.* [6][†]		0.9984	83.8	1.91
Park [9][†]		4.358	29.4	4.80
Zhang *et al.* [24][†,‡]		5.542	10.2	4.80
	$m = 3$	5.021	18.7%	0.45
Theorem 4[§]	$m = 4$	5.985	3.0%	1.00
	$m = 5$	6.150	0.36%	2.60
	$m = 3$	5.020	18.7%	8.67
Theorem 5[§]	$m = 4$	5.985	3.0%	16.7
	$m = 5$	6.150	0.36%	26.7

**ad hoc* approach was used.
[†] The generalized eigenvalue minimization algorithm was used.
[‡] Covering disk $D(-0.251; 0.749)$ was used.
[§] Bisection search with 10 iterations was performed.

Example 2 (Statistical Performance)
Using a 5th order Padé approximation, we compared the statistical performance of our LMI condition Theorem 5 with that of References [6, 9] and [24] by examining 1000 randomly-generated second order single delay systems.[¶] The computed delay margins are compared with the actual values from the MIMO Nyquist Criterion and their distribution is shown in Figure 4. We find that for 97.3% of these systems, our new result gives the d.o.c. less than 10%. We note that with the next best performing criterion of Reference [24], less than 50% of the cases have d.o.c. below 10%. The average d.o.c. for Theorem 5 is 1.52%.

Example 3 (Multiple-delay case)
Consider the following system with two independent delays

$$\dot{x}(t) = -x(t - \tau_1) - x(t - \tau_2) \tag{22}$$

For the stability analysis problem, the actual stability boundary may be computed numerically using the analytical result [45]. In Figure 5, this boundary is plotted along with the stability regions (boxes) that are determined from Theorem 5 (with a 5th order Padé Approximation) with varying proportionality ratio vector. Note that the stability region found for each proportionality ratio is essentially as large as it could be, given the stability boundary (i.e., the corner of the stability region box nearly touches the stability boundary in each case). For instance, with the proportionality ratio vector $v = [1\ 1]$, the actual delay margin was found in [45] to be $\pi/4$. The recent result [22] provides a delay margin of 0.7071 which has d.o.c. of 10%. Using Theorem 5, we found a delay margin of 0.7825, and for this example the d.o.c. of our LMI condition is only 0.37%.

[¶] For each test case, $A + A_1$ is Hurwitz and the stability of the system is delay-dependent.

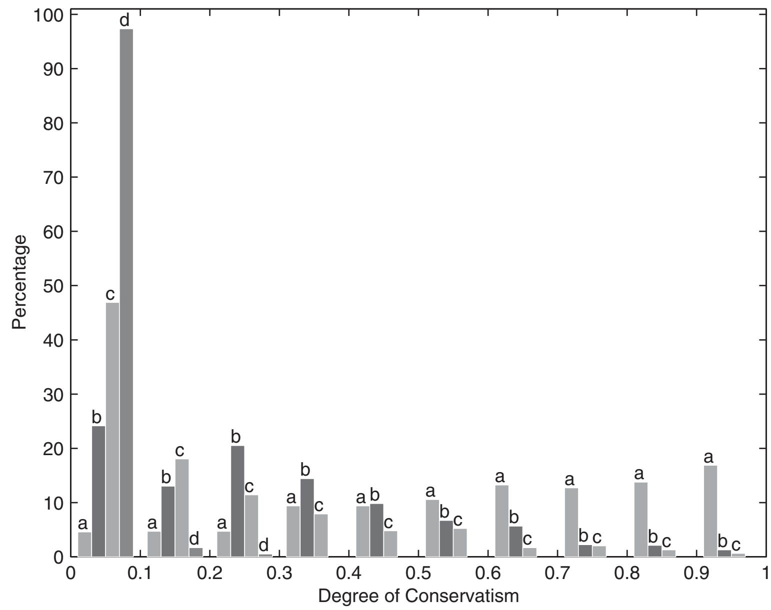

Figure 4. Performance of several criteria. (a) Result of Reference [6]. (b) Result of Reference [9]. (c) Result of Reference [24]. (d) Theorem 5.

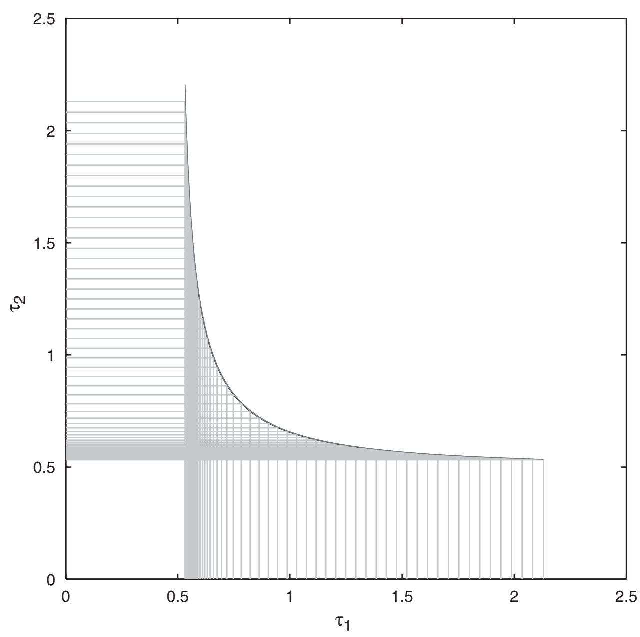

Figure 5. Stability boundary of system (22) and stability regions guaranteed by LMI with varying proportionality ratio.

7. CONCLUSIONS

In this paper, we presented a comparison system approach for analysis of linear systems with time-invariant delays. In the comparison system, the uncertain delay elements are replaced by a parameter dependent Padé approximation to the exponential function. By using inner and outer covering sets for the non-rational delay elements, we demonstrated that replacing the delay elements with the traditional form of a diagonal Padé approximation leads to a necessary condition for stability of the time-delay system, while replacing with a frequency-dilated Padé approximation results in a sufficient condition. The degree of conservatism of this sufficient condition has an *a priori* known upper bound which depends only on the order of the Padé approximation used and can be made arbitrarily small. Moreover, in the single delay case, the delay margin can be calculated explicitly without incurring any additional conservatism. In the general case, this condition can also be reduced with some (typically small) conservatism to finite-dimensional LMIs. Finally, this approach is also extended for analysis of the $\mathscr{H}_\infty$ performance of linear time-delay systems with an exogenous disturbance.

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Appendix A: Proof of Lemma 4

Proof
From the results of Reference [40], we have

$$|Q_m(\mathrm{j}\omega)|^2 = \frac{(m!)^2}{[(2m)!]^2}\sum_{k=0}^{m} a_k\omega^{2k}, \quad \omega\in\mathbb{R} \tag{A1}$$

where $a_k = \frac{[2(m-k)]!(2m-k)!}{k![(m-k)!]^2} > 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}s}R_m(s) = -R_m(s) + (-1)^m\frac{(m!)^2}{[(2m)!]^2}\frac{s^{2m}}{Q_m^2(s)} \tag{A2}$$

Since $Q_m(\mathrm{j}\omega) = P_m(-\mathrm{j}\omega)$, from (A1) and (A2) we get

$$\begin{aligned}\frac{\frac{\mathrm{d}}{\mathrm{d}\omega}R_m(\mathrm{j}\omega)}{R_m(\mathrm{j}\omega)} &= -1 + \frac{(m!)^2}{[(2m)!]^2}\frac{\omega^{2m}}{|Q_m(\mathrm{j}\omega)|^2}\\ &= -1 + \frac{\omega^{2m}}{\omega^{2m} + \sum_{k=0}^{m-1} a_k\omega^{2k}}\end{aligned}$$

Note that $|R_m(\mathrm{j}\omega)| \equiv 1$, $\log R_m(\mathrm{j}\omega) = \mathrm{j}\mathrm{Arg}\, R_m(\mathrm{j}\omega)$, $\omega\in\mathbb{R}$. Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Arg}\, R_m(\mathrm{j}\omega) &= \frac{\mathrm{d}}{\mathrm{d}s}(\log R_m(s))|_{s=\mathrm{j}\omega}\\ &= \frac{\frac{\mathrm{d}}{\mathrm{d}\omega}R_m(\mathrm{j}\omega)}{R_m(\mathrm{j}\omega)} = -\frac{T_m(\omega)}{\omega^{2m}+T_m(\omega)}\end{aligned}$$ □

Appendix B: Proof of Theorem 2

Proof

If (1) is asymptotically stable on $\prod_{k=1}^{N}[0, \bar{\tau}_k]$, then

$$\det[I_q - G(\mathrm{j}\omega)\Delta(\omega)] \neq 0, \quad \forall \omega \geqslant 0, \delta_k(\omega) \in \mathbf{\Omega}_d(\omega, \bar{\tau}_k)$$

where $\Delta(\omega) := \mathrm{diag}\,\{(\delta_1(\omega)-1)I_{q_1}, \ldots, (\delta_N(\omega)-1)I_{q_N}\}$. In view of the fact that $\mathbf{\Omega}_i(\omega, \bar{\tau}_k) \subseteq \mathbf{\Omega}_d(\omega, \bar{\tau}_k)$, for every $\delta_k(\omega) \in \mathbf{\Omega}_i(\omega, \bar{\tau}_k)$, we have $\delta_k(\omega) \in \mathbf{\Omega}_d(\omega, \bar{\tau}_k)$. Therefore

$$\det[I_q - G(\mathrm{j}\omega)\Delta(\omega)] \neq 0, \quad \forall \omega \geqslant 0, \ \delta_k(\omega) \in \mathbf{\Omega}_i(\omega, \bar{\tau}_k)$$

which can be rewritten as

$$\det[I_q - G(\mathrm{j}\omega)\Delta_\theta(\omega)] \neq 0, \quad \forall \omega \geqslant 0, \ \theta_k \in [0, \bar{\tau}_k]$$

where $\Delta_\theta(\omega) := \mathrm{diag}\{(R_m(\mathrm{j}\theta_1\omega)-1)I_{q_1}, \ldots, (R_m(\mathrm{j}\theta_k\omega)-1)I_{q_N}\}$. Define $p(s) := R_m(s) - 1 = N(s)/D(s)$, then the above condition is equivalent to

$$\rho_\theta(\mathrm{j}\omega, h) \neq 0, \quad \forall \omega \geqslant 0, \ \theta_k \in [0, \bar{\tau}_k], h \in [0, 1] \tag{B1}$$

where $\rho_\theta(s, h)$ is the characteristic function associated with the closed loop system $\sum_i(\theta)$, given by

$$\rho_\theta(s, h) := \det\left[(sI_n - \bar{A}) - \sum_{k=1}^{N} A_k \frac{N(h\theta_k s)}{D(h\theta_k s)}\right]$$

Since $\bar{A}$ is Hurwitz, all of its eigenvalues are located in the open left half complex plane $\mathbb{C}_-$. The eigenvalues of a matrix continuously depend on its elements, hence there exists a sufficient small $\varepsilon > 0$ such that the eigenvalues of the matrix $\bar{A} + \sum_{k=1}^{N} z_k A_k$ remain in $\mathbb{C}_-$, for all $|z_k| \leqslant \varepsilon$.

Next, we fix θ. The condition (B1) indicates that for any $h \in [0, 1]$, there are no roots of $\rho_\theta(s, h)$ located on the jω axis. For $h = 0, N(h\theta_k s) = 0$, $D(h\theta_k s) = 1$, and the roots of $\rho_\theta(s, h)$ are all located in $\mathbb{C}_-$. Next, we will prove by contradiction that for $h \in (0, 1]$, there exist no roots of $\rho_\theta(s, h)$ located in the open right half plane $\mathbb{C}_+$. Assume that for $h_1 \in (0, 1]$, there is a zero $\lambda_1(h_1) \in \mathbb{C}_+$ of $\rho_\theta(s, h_1)$. Since the degree of $\rho_\theta(s, h)$ is invariant for $h > 0$, the zeros of $\rho_\theta(s, h)$ are continuous in h. Hence $\lambda_1(h)$ remains in $\mathbb{C}_+$ for all $h \in (0, 1]$ since it cannot cross the jω axis. On the other hand, because $\lim_{hs \to 0} p(h\theta_k s) = 0$, there exists a constant $\beta > 0$ such that $|p \times (h\theta_k s)| \leqslant \varepsilon$, for all $|hs| < \beta$ and all $k \in \{1, \ldots, N\}$. In addition, $p(s)$ is stable and proper, hence for every $\mathrm{Re}(s) \geqslant 0$, $|p(s)| \leqslant \sup_{\mathrm{Re}(z) \geqslant 0} |p(z)| = \sup_{\omega \in R} |p(\mathrm{j}\omega)| = 2$. Also, since $\lambda_1(h)$ is a zero of $\rho_\theta \times (s, h)$, from Matrix Theory [46], $|\lambda_1(h)| \leqslant \|\bar{A} + \sum_{k=1}^{N} A_k p(h\theta_k \lambda_1(h))\|_\infty \leqslant \|\bar{A}\|_\infty + \sum_{k=1}^{N} |p(h\theta_k \lambda_1 \times (h))| \, \|A_k\|_\infty \leqslant c$, where $c := \|\bar{A}\|_\infty + 2\sum_{k=1}^{N} \|A_k\|_\infty > 0$ is a constant. Then, we can choose a sufficiently small $h_2 \in (0, 1]$ such that $|h_2| < \frac{\beta}{c}$. Then $|h_2 \lambda_1(h_2)| < \beta$ and hence $|z_k| \leqslant \varepsilon$, where $z_k := p(h_2 \theta_k \lambda_1(h_2))$. Therefore, $\lambda_1(h_2) \in \mathbb{C}_+$ is an eigenvalue of $\bar{A} + \sum_{k=1}^{N} z_k A_k$. This contradicts the fact that all eigenvalues of $\bar{A} + \sum_{k=1}^{N} z_k A_k$ are in $\mathbb{C}_-$, for all $|z_k| \leqslant \varepsilon$. The proof is thus complete. □

Appendix C: Proof of Lemma 8

To prove Lemma 8, we need the following results.

Lemma C.1
Suppose $A \in \mathbb{R}^{n\times n}$ is Hurwitz and satisfies the Lyapunov inequality

$$A^{\mathrm{T}}P + PA < -Q$$

where P and Q are symmetric and positive definite matrices. Then every eigenvalue λ_i of A satisfies

$$\mathrm{Re}(\lambda_i(A)) < -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \quad i = 1, \ldots, n$$

Proof
Take $\mu := \lambda_{\min}(Q)/2\lambda_{\max}(P)$. Then we have $2\mu P \leqslant Q$, or,

$$-Q + 2\mu P \leqslant 0$$

Since $A^{\mathrm{T}}P + PA < -Q$, we have

$$(A + \mu I)^{\mathrm{T}}P + P(A + \mu I) < -Q + 2\mu P$$

Hence $(A + \mu I)^{\mathrm{T}}P + P(A + \mu I) < 0$. Therefore, all eigenvalues of $A + \mu I$ are in the open left half complex plane, or

$$\mathrm{Re}(\lambda_i(A)) < -\mu = -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \qquad \square$$

Proof of Lemma 8
The proof for the first part is trivial. To prove the second part, we notice that since $X(\theta)$ is bounded for all $\theta \in F_o$, there exists a constant $c_0 > 0$ such that $\lambda_{\max}(X(\theta)) \leqslant c_0$ for all $\theta \in F_o$. Hence from Lemma C.1, we conclude that all the closed loop poles of $\sum_o(\theta)$ remain to the left of the vertical line $\mathrm{Re}(s) = -1/2c_0 < 0$ for all $\theta \in F_o$.

Define $p(s) := R_m(\alpha_m s) - 1 = N(s)/D(s)$, and suppose $\rho_\theta(s)$ is the characteristic function associated with the closed loop system $\sum_o(\theta)$, given by

$$\rho(s, \theta_1, \theta_2, \ldots, \theta_N) := \det\left[sI_n - \bar{A} - \sum_{k=1}^{N} A_k \frac{N(\theta_k s)}{D(\theta_k s)}\right]$$

Next, let $c_1 := \max\{\mathrm{Re}(s) | D(s) = 0\}$. Then $c_1 < 0$ since $D(s)$ is Hurwitz. Let $c_2 = \max\{-\frac{1}{2c_0}, \max_{1\leqslant k\leqslant N}\{\frac{c_1}{\bar{\tau}_k}\}\}$, then it is clear that $\rho_\theta(s)$ is analytic in the region $\mathrm{Re}(s) > c_2$.

Clearly, we only need to show that $\sum_o(\boldsymbol{\theta})$ is asymptotically stable on the boundary of F_c. Without loss of generality, it suffices to show that $\rho_\theta(s)$ does not have roots in the closed right half plane $\bar{\mathbb{C}}_+$ for any $L > 1$ with $\theta_k = 0$ for all $k \in \{1, \ldots, L-1\}$, and $0 < \theta_k \leqslant \bar{\tau}_k$ for all $k \in \{L, \ldots, N\}$. To this end, we fix θ_k, $k \in \{L, \ldots, N\}$, and assume there exists s_0, such that $\rho(s_0, 0, \ldots, 0, \theta_L, \ldots, \theta_N) = 0$ and $\mathrm{Re}(s_0) \geqslant 0$. Notice that $\rho(s, \theta_1, \ldots, \theta_N)$ can be represented in the form

$$\rho(s, \theta_1, \theta_2, \ldots, \theta_N) = \phi(s) + \eta(s, \theta_1, \ldots, \theta_N)$$

where $\phi(s) := \rho(s, 0, \ldots, 0, \theta_L, \ldots, \theta_N)$. Then, there exists a sufficiently small neighborhood F of s_0, such that F is a closed set and located entirely in the region $\mathrm{Re}(s) > c_2$, and $\phi(s)$ does not vanish on the boundary of F. Since $\eta(s, \theta_1, \ldots, \theta_N)$ is continuous in θ and $\eta(s, 0, \ldots, 0, \theta_L, \ldots, \theta_N) = 0$, we know that for sufficiently small $\theta_k = \tilde{\theta} > 0$, $k = 1, \ldots, L-1$, we have that $|\eta(s, \theta_1,$

$\ldots, \theta_N)| < |\phi(s)|$ on the boundary of F. By Rouché's Theorem [31], $\rho(s, \tilde{\theta}, \ldots, \tilde{\theta}, \theta_L, \ldots, \theta_N)$ has at least one root s in F. This is the contradiction we seek. □

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