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NEW RESULTS ON CONTROL OF MULTIBODY SYSTEMS WHICH CONSERVE ANGULAR MOMENTUM

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ABSTRACT. A planar system of rigid bodies interconnected by one degree of freedom rotational joints is considered. This multibody system is referred to as a multilink, and the rigid bodies are referred to as links. The angular momentum of the multilink is conserved but is not necessarily zero. We show that if the number of links is at least four, then periodic joint motions can make the absolute orientation of a specified base link track exactly a specified function of time whose time derivative is periodic. This result on the use of periodic joint motions for orientation *tracking* extends previous work [15], [20], [22] on using periodic joint motions for rest-to-rest reorientation. It has interesting physical consequences. Specifically, in the case of non-zero angular momentum periodic joint motions can maintain the orientation of the base link constant. In the case of zero angular momentum periodic joint motions can change the orientation of the base link at a specified angular rate. We also demonstrate that if the multilink consists of at least three links, then for *any* value of the angular momentum joint motions can reorient the multilink arbitrarily over an *arbitrary* time interval. This result extends similar results in [15] for zero angular momentum and in [20] that apply for nonzero angular momentum but not for an arbitrary time interval. In terms of their control-theoretic aspects, the problems treated in the paper can be viewed as controllability problems for a class of nonlinear control system with time-dependent drift.

1. INTRODUCTION

We address a motion planning problem for a planar system of $N > 1$ rigid bodies interconnected with one degree of freedom frictionless rotational joints in the form of an open kinematic tree (Fig. 1). This system of rigid

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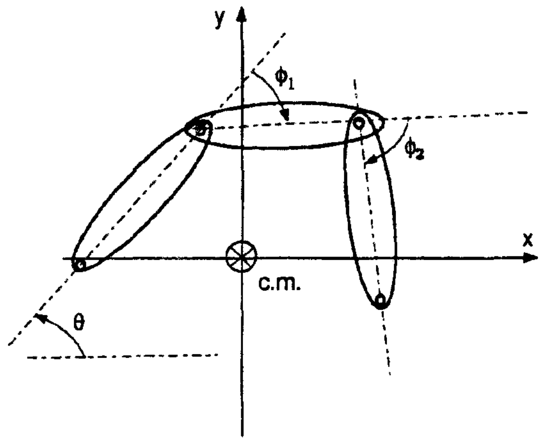

Fig. 1. The multilink

bodies is referred to as a multilink, and the rigid bodies are referred to as links. The origin of a reference frame is fixed at the center of mass of the multilink. We assume that the reference frame is an inertial frame and that the angular momentum of the multilink about the multilink's center of mass is conserved. The configuration of the multilink is specified by its orientation θ which is defined as the absolute orientation of the base link, and by the joint angles ϕ_i, $i = 1, \cdots, (N-1)$, which determine the shape of the multilink (Fig. 1). The joints are controlled by internal (angular momentum preserving) actuators, e.g., by direct drive motors.

A number of control problems for a multilink with zero angular momentum has been considered in the literature; see, e.g., [1]–[3], [11], [15], [17]–[20], [22]. The interest in these problems has been motivated, in part, by space robotics applications [1]–[3], springboard diver dynamics [4], [8], and the falling cat phenomenon [7], [16]. Previous work concentrated mostly on using joint motions for rest-to-rest reconfiguration of the multilink. In particular, for a multilink consisting of at least three nondegenerate links it has been shown that periodic joint motions (i.e., cyclic shape changes) can induce nonzero net orientation changes.

In the present paper we extend this result on the use of periodic joint motions to tracking problems. Specifically, we show that if the multilink consists of at least four nondegenerate links, then periodic joint motions can make the orientation track exactly a specified function of time whose time derivative is periodic. This result has interesting and, at a first glance, counterintuitve consequences. Specifically, in the case of non-zero angular momentum of the multilink periodic joint motions can maintain the orientation of the multilink constant. In the case of zero angular momentum of the multilink periodic joint motions can change the orientation of the multilink at a specified angular rate. We also obtain new results for the problem of reconfiguring the multilink from a given initial shape and orientation to a specified final shape and orientation over a specified time interval. Specifically, we prove that if the multilink consists of at least three nondegenerate links, then for *any* value of the angular momentum joint actuators can reconfigure the multilink arbitrarily over an *arbitrary* time interval. This result extends similar results in [15] for zero angular momentum and in [20] that apply for nonzero angular momentum but not for an arbitrary time interval.

The paper is organized as follows. In Sec. 2 we summarize the equations of motion. In Sec. 3 we present the results for the exact tracking problem. In Sec. 4 we treat the reconfiguration problem. The proofs in Secs. 3–4 are constructive and rely on averaging theory [6], [13, [14]. In Sec. 5 we discuss implications of our results for multibody spacecraft attitude tracking maneuvers, where the spacecraft is modeled as a multilink in orbit. Section 6 contains concluding remarks. Some of these results were reported in an earlier conference paper [10].

Our notation is standard: $\mathbb{R}$ is the set of real numbers, Z^+ is the set of nonnegative integers, a function f is C^n if f is n times continuously differentiable.

2. Equations of motion and preliminaries

The angular momentum of the planar multilink about the center of mass of the multilink is constant but not necessarily zero. Conservation of angular momentum relates the joint angle velocities to the time rate of change of the orientation as [15], [20]

$$\dot{\theta} = \frac{H}{m_{\theta\theta}(\phi)} - \sum_{i=1}^{N-1} \frac{m_{\theta i}(\phi)}{m_{\theta\theta}(\phi)}\dot{\phi}_i, \tag{1}$$

where H denotes the constant value of the angular momentum. The functions $m_{\theta\theta}$ and $m_{\theta i}$, $i = 1, \cdots, N-1$, are real analytic and 2π-periodic in each of their arguments. The function $m_{\theta\theta}$ satisfies $m_{\theta\theta}(\phi) > 0$ for all ϕ.

Assuming that every joint is actuated, equations relating the joint angles and the joint torques can be written in the form

$$F_1(\phi)\ddot{\phi} + F_2(\phi, \dot{\phi})\dot{\phi} + HF_3(\phi)\dot{\phi} + H^2F_4(\phi) = \tau, \tag{2}$$

where $\tau = (\tau_1, \cdots, \tau_{N-1})$ is an $(N-1)$-vector of joint torques. The matrix functions $F_1, F_3, F_4 : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$, $F_2 : \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \to \mathbb{R}^{(N-1)\times(N-1)}$ are 2π-periodic in each of their arguments. Equations (2) can be obtained either by substituting (1) into Lagrange's equations of motion for the multilink or, equivalently, by Routhian reduction; see [15], [20] for details. The triple $(\theta, \phi, \dot{\phi})$ is referred to as the *state* of the multilink.

Let p, r, s be three integers, $1 \leq p, r, s \leq (N-1)$, which identify three specific joints. To state our assumptions and results, it is convenient to introduce two functions of the joint angles

$$\rho_{prs} : \mathbb{R}^{(N-1)} \to \mathbb{R} \quad \text{and} \quad \nu_{pr} : \mathbb{R}^{(N-1)} \to \mathbb{R}.$$

They are defined as

$$\rho_{prs}(\phi) = \frac{\partial \left(\dfrac{m_{\theta s}}{m_{\theta p}}\right)}{\partial \phi_p} \left(\frac{m_{\theta r}}{m_{\theta p}}\right) - \frac{\partial \left(\dfrac{m_{\theta r}}{m_{\theta p}}\right)}{\partial \phi_p} \left(\frac{m_{\theta s}}{m_{\theta p}}\right) + \\ + \frac{\partial \left(\dfrac{m_{\theta s}}{m_{\theta p}}\right)}{\partial \phi_r} - \frac{\partial \left(\dfrac{m_{\theta r}}{m_{\theta p}}\right)}{\partial \phi_s},$$

$$\nu_{pr}(\phi) = \frac{\partial \left(\dfrac{m_{\theta r}}{m_{\theta\theta}}\right)}{\partial \phi_p} - \frac{\partial \left(\dfrac{m_{\theta p}}{m_{\theta\theta}}\right)}{\partial \phi_r}.$$

3. Exact tracking

Throughout this section the following assumptions hold: (A1) $N \geq 4$; (A2) there exist $\phi^0 \in \mathbb{R}^{N-1}$ and three integers p, r, s, $1 \leq p, r, s \leq (N-1), p \neq r \neq s$, such that $m_{\theta p}(\phi^o) \neq 0$ and $\rho_{prs}(\phi^o) \neq 0$; (A3) the scalar tracking objective function $\theta_d(t)$, $t \geq t'$, is C^2 and its time derivative $\dot{\theta}_d$ is T-periodic for some $T > 0$. With the exception of some degenerate situations (e.g., when masses or inertias of some of the links are zero), a multilink with at least four links satisfies (A2). Moreover, in the usual case almost all shapes $\phi^0 \in \mathbb{R}^{N-1}$ satisfy $m_{\theta p}(\phi^0) \neq 0$, $\rho_{prs}(\phi^0) \neq 0$. Assumption (A3) implies that $\theta_d(t)$, $t \geq t'$, can be represented as a sum of a linear function of time and a periodic function of time which is twice continuously differentiable. If (A1)–(A3) hold, the orientation of the multilink can track exactly $\theta_d(t)$, $t \geq t'$, while the joint motions are periodic and joint angle deviations are as small as desired.

Theorem 3.1. *Suppose* (A1)–(A3) *hold and let* $\varepsilon > 0$ *be given. Then there exists a state* $(\theta^*, \phi^*, \dot{\phi}^*) \in \mathbb{R}^{2N-1}$ *and continuous joint torques* $\tau_i(t)$, $i = 1, \cdots, (N-1)$, *defined for* $t \geq t'$, *such that if* $(\theta(t'), \phi(t'), \dot{\phi}(t')) = (\theta_d(t'), \phi^*, \dot{\phi}^*)$, *then the solution of* (1)–(2) *satisfies*: (i) $\theta(t) = \theta_d(t)$ *for* $t \geq t'$; (ii) $\phi_i(t)$, $t \geq t'$, $i = 1, \cdots, (N-1)$, *are* C^2 *and* T*-periodic*; (iii) $|\phi_i(t) - \phi_i^o| < \varepsilon$, *for all* $t \geq t'$, $i = 1, \cdots, (N-1)$.

Proof. The theorem is proved by constructing a twice continuously differentiable T-periodic joint motion $\phi(t),\ t \geq t'$, which satisfies (iii) and which guarantees that if $\theta(t') = \theta_d(t')$, then the solution of Eq. (1) satisfies $\theta(t) = \theta_d(t)$ for all $t \geq t'$. The joint torques can be then obtained by substituting $\phi(t),\ t \geq t'$, into (2).

Without loss of generality, assume that $p = 1, r = 2, s = 3$, and $\rho_{123}(\phi^o) > 0$. If $N > 4$ define

$$\phi_i(t) \equiv \phi_i^o,\ i = 4, \cdots, (N-1),\ t \geq t', \tag{3}$$

and assume, without loss of generality, that $N = 4$.

Let

$$U_{\phi^o} = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_i - \phi_i^o| < \delta, i = 1, 2, 3\}$$

be a neighborhood of ϕ^0. Since $\rho_{123}(\phi^o) > 0, m_{\theta 1}(\phi^o) \neq 0$, and the functions ρ_{123} and $m_{\theta 1}$ are continuous, there exist scalars δ, γ, b, and B satisfying $0 < \delta < \varepsilon/4$, $\gamma > 0$, $B > b > 0$,

$$|m_{\theta 1}(\phi)| > \gamma > 0, \quad \text{for all } \phi \in U_{\phi^o},$$

and

$$\frac{8}{\pi}B > \rho_{123}(\phi) > \frac{8}{\pi}b > 0, \quad \text{for all } \phi \in U_{\phi^o}.$$

If $\dot{\theta}(t) = \dot{\theta}_d(t),\ t \geq t'$, Eq. (1) implies

$$\dot{\phi}_1 = \frac{H}{m_{\theta 1}(\phi)} - \frac{m_{\theta\theta}(\phi)}{m_{\theta 1}(\phi)}\dot{\theta}_d - \frac{m_{\theta 2}(\phi)}{m_{\theta 1}(\phi)}\dot{\phi}_2 - \frac{m_{\theta 3}(\phi)}{m_{\theta 1}(\phi)}\dot{\phi}_3. \tag{4}$$

Let l_1 and l_2 be such that

$$l_2 > \frac{H}{m_{\theta 1}(\phi)} - \frac{m_{\theta\theta}(\phi)}{m_{\theta 1}(\phi)}\dot{\theta}_d(t) > l_1 \quad \text{for all } \phi \in U_{\phi^o} \text{ and } t' \leq t \leq t' + T.$$

Select $M \in Z^+$ sufficiently large so that

$$M > \max\{\frac{4l_2T}{\delta}, -\frac{4l_1T}{\delta}, (l_2B - l_1b)\frac{8T}{b\delta}, (l_2b - l_1B)\frac{8T}{b\delta}\}.$$

The outline of the proof is now given. Let $t_i = t' + Ti/M, i = 0, \cdots, M$. We shall show that there exists a C^2 joint motion $(\phi_1(t), \phi_2(t), \phi_3(t)) \in U_{\phi^o}$, $t' \leq t \leq t' + T$, satisfying (4) and such that

$$\phi_1(t_i) = \phi_{1}^o, \quad i = 0, \cdots, M, \tag{5}$$

and

$$\phi_2(t_i) = \phi_2^o, \quad \phi_3(t_i) = \phi_3^o, \quad \dot{\phi}_2(t_i) = \dot{\phi}_3(t_i) = 0, \quad \ddot{\phi}_2(t_i) = \ddot{\phi}_3(t_i) = 0,$$
$$i = 0, \cdots, M. \tag{6}$$

The construction of the joint motion satisfying (5) and (6) is accomplished separately in each interval $t_i \leq t \leq t_{i+1}$. We then extend $(\phi_1(t), \phi_2(t), \phi_3(t))$ constructed for $t' \leq t \leq t' + T$ periodically with period T for $t \geq t' + T$. Since $\dot{\theta}_d(t)$, $t \geq t'$, is T-periodic, (4) implies that $\phi_1(t+T) = \phi_1(t)$ for $t \geq t'$. Therefore, $(\phi_1(t), \phi_2(t), \phi_3(t))$ is C^2, T-periodic, and contained in U_{ϕ^o} for all $t \geq t'$.

We now construct the desired joint motion $(\phi_1(t), \phi_2(t), \phi_3(t))$ for $t_i \leq t \leq t_{i+1}$. The selection of M guarantees the existence of $\alpha > 0$ and $\beta > 0$ such that

$$\frac{\delta M/(4T) - l_2}{B} > \beta^2 > \frac{b\delta M/(8BT) - l_1}{b} \tag{7}$$

and

$$\frac{\delta M/(4T) + l_1}{B} > \alpha^2 > \frac{l_2 + bM\delta/(8BT)}{b}. \tag{8}$$

Consider a family of joint motions $\phi_2(t,k,n)$ and $\phi_3(t,k,n)$ parametrized for $t \in [t_i, t_{i+1}]$ in terms of a real parameter $k \in [-\beta, \alpha]$ and a positive integer n as

$$\phi_2(t,k,n) = \phi_2^o + \left(\frac{t_{i+1} - t_i}{2\pi n}\right)^{-1/2} \int_{t_i}^{t} \big(k_+ v_1(\sigma,n) + (-k)_+ v_2(\sigma,n)\big)\,d\sigma, \tag{9}$$

$$\phi_3(t,k,n) = \phi_3^o + \left(\frac{t_{i+1} - t_i}{2\pi n}\right)^{-1/2} \int_{t_i}^{t} \big(k_+ v_2(\sigma,n) + (-k)_+ v_1(\sigma,n)\big)\,d\sigma, \tag{10}$$

where $(k)_+ = k$ if $k > 0$ and is zero otherwise. Here, in terms of $h = (t_{i+1} - t_i)/(4n)$,

$$v_1(t,n) = \begin{cases} 1 - \cos\left(\dfrac{2\pi(t - t_i)}{h}\right), & t_i \leq t \leq t_i + h, \\ 0, & t_i + h \leq t \leq t_i + 2h, \\ -1 + \cos\left(\dfrac{2\pi(t - t_i - 2h)}{h}\right), & t_i + 2h \leq t \leq t_i + 3h, \\ 0, & t_i + 3h \leq t \leq t_i + 4h, \end{cases}$$

$$v_2(t,n) = \begin{cases} 0, & t_i \leq t \leq t_i + h, \\ 1 - \cos\left(\dfrac{2\pi(t - t_i - h)}{h}\right), & t_i + h \leq t \leq t_i + 2h, \\ 0, & t_i + 2h \leq t \leq t_i + 3h, \\ -1 + \cos\left(\dfrac{2\pi(t - t_i - 3h)}{h}\right), & t_i + 3h \leq t \leq t_i + 4h, \end{cases}$$

and

$$v_1(t,n) = v_1(t-4h,n), \quad v_2(t,n) = v_2(t-4h,n), \quad t_i + 4h \le t \le t_{i+1}.$$

Note that for arbitrary values of the parameters k and n the joint motions $\phi_2(t,k,n)$ and $\phi_3(t,k,n)$ are C^2 for $t_i \le t \le t_{i+1}$ and satisfy

$$\phi_j(t_i,k,n) = \phi_j(t_{i+1},k,n) = \phi_j^o,$$

$$\dot{\phi}_j(t_i,k,n) = \dot{\phi}_j(t_{i+1},k,n) = \ddot{\phi}_j(t_i,k,n) = \ddot{\phi}_j(t_{i+1},k,n) = 0,$$

where $j = 2,3$.

Let $\phi_1(t,k,n)$ denote the solution of Eq. (4), where $\phi_2(t,k,n)$ and $\phi_3(t,k,n)$, $t_i \le t \le t_{i+1}$, are defined by Eqs. (9) and (10), and $\phi_1(t_i,k,n) = \phi_1^o$. We want to show that there exists a choice of parameters $k = \hat{k}$ and $n = \hat{n}$ such that $\phi_1(t_{i+1},\hat{k},\hat{n}) = \phi_1^o$.

By applying the averaging technique developed in [6], [14] to Eq. (4), where $\phi_2(t,k,n)$ and $\phi_3(t,k,n)$ are defined by Eqs. (9) and (10), it can be shown that there exists $M_1 \in Z^+$ such that for all $n > M_1$ and for all $k \in [-\beta,\alpha]$,

$$|\bar{\phi}_j(t,k) - \phi_j(t,k,n)| < b\delta/(16B), \quad t_i \le t \le t_{i+1}, \quad j = 1,2,3, \quad (11)$$

where $\bar{\phi}_j(t,k)$, $t_i \le t \le t_{i+1}$, $j = 1,2,3$, solve the averaged equations

$$\begin{cases} \dot{\bar{\phi}}_1 = \dfrac{H}{m_{\theta 1}(\bar{\phi})} + \dfrac{m_{\theta\theta}(\bar{\phi})}{m_{\theta 1}(\bar{\phi})}\dot{\theta}_d - \operatorname{sign}(k)k^2\dfrac{\pi}{8}\rho_{123}(\bar{\phi}), \quad \bar{\phi}_1(t_i,k) = \phi_1^o, \\ \bar{\phi}_2 = \phi_2^o, \\ \bar{\phi}_3 = \phi_3^o. \end{cases}$$

Inequalities (7) and (8) guarantee that the solution of the averaged equations exists for $t_i \le t \le t_{i+1}$ and

$$|\bar{\phi}_1(t,k) - \phi_1^o| < \delta/4, \quad \text{for all } t_i \le t \le t_{i+1} \text{ and all } k \in [-\beta,\alpha],$$

$$\bar{\phi}_1(t_{i+1},-\beta) - \phi_1^o > b\delta/(8B) > 0,$$

$$\bar{\phi}_1(t_{i+1},\alpha) - \phi_1^o < -b\delta/(8B) < 0.$$

Let $\hat{n} > M_1$ be selected. The estimate (11) implies that

$$\phi(t,k,\hat{n}) \in U_{\phi^o}, \text{ for all } t_i \le t \le t_{i+1} \text{ and for all } k \in [-\beta,\alpha],$$

$$\phi_1(t_{i+1},-\beta,\hat{n}) - \phi_1^o > b\delta/(16B) > 0,$$

and

$$\phi_1(t_{i+1},\alpha,\hat{n}) - \phi_1^o < -b\delta/(16B) < 0.$$

Since the solution $\phi_1(t,k,\hat{n})$ of Eq. (4), where $\phi_2(t,k,\hat{n})$ and $\phi_3(t,k,\hat{n})$, $t_i \le t \le t_{i+1}$, are defined by Eqs. (9) and (10) and $\phi_1(t_i,k,\hat{n}) = \phi_1^o$, depends continuously on the parameter k it follows that there exists a value of the parameter $k = \hat{k} \in [-\beta, \alpha]$ such that

$$\phi_1(t_{i+1}, \hat{k}, \hat{n}) = \phi_1^o.$$

This completes the proof of Theorem 3.1. □

To summarize, the proof shows that appropriately constructed small amplitude high frequency periodic joint motions result in $\theta(t) = \theta_d(t),\ t \geq t'$. The two main ingredients of the proof are an averaging estimate and a fixed point argument.

We now comment on some interesting implications of Theorem 3.1. First, for any value of the angular momentum, the orientation of the multilink can track an arbitrary tracking objective satisfying (A3), while the joint motions are periodic and joint angle deviations are as small as desired. In fact, for sufficiently small ε in (iii), it would appear to an external observer that the multilink behaves as a rigid body with fixed shape ϕ^o, acted on by an "external" torque that causes the orientation of the multilink to track exactly the tracking objective. Furthermore, Theorem 3.1 implies that in the case of non-zero angular momentum the orientation of the base link can be maintained constant while at the same time the links do not rotate on "average" relative to one another. In the case of zero angular momentum it is possible that the base link rotates at a non-zero angular rate relative to an inertial frame while the links do not rotate on "average" relative to one another. These conclusions should be contrasted with possible solutions of the problem using a momentum wheel to control the absolute orientation $\theta(t)$ of the base link. Specifically, to guarantee $\theta(t) = 0,\ t \geq t'$, for nonzero angular momentum or to guarantee $\theta(t) = \Omega(t - t'),\ t \geq t',\ \Omega \neq 0$, for zero angular momentum a momentum wheel must necessarily have nonzero "average" rotation.

Our next remark concerns the conclusion in (iii). Assume the shape ϕ^o is collision-free, i.e., the links do not intersect and do not touch each other. Then, exact tracking can be accomplished without link collisions. For that, it is necessary to select joint motions according to the construction procedure in the proof of Theorem 3.1 for ε sufficiently small.

While the proof of Theorem 3.1 is constructive, the construction procedure is rather complex. In terms of computations, simpler approaches may be desirable. Such approaches are available. One approach is sketched below. Suppose that $\theta(t) = \theta_d(t)$ for all $t \geq t'$. Assuming $m_{\theta 1}(\phi) \neq 0$ we can solve (1) for $\dot{\phi}_1$, in terms of $\dot{\theta}_d$ and $\dot{\phi}_2, \cdots, \dot{\phi}_{(N-1)}$, as

$$\dot{\phi}_1 = \frac{H}{m_{\theta 1}(\phi)} - \frac{m_{\theta\theta}(\phi)}{m_{\theta 1}(\phi)}\dot{\theta}_d - \sum_{i=2}^{N-1} \frac{m_{\theta i}(\phi)}{m_{\theta 1}(\phi)}\dot{\phi}_i. \tag{12}$$

The approach relies on the assumption that the joint motions $\phi_2(t), \cdots, \phi_{N-1}(t),\ t \geq t'$, can be parametrized as

$$\phi_i(t) = \sum_{j=0}^{n} \gamma_j^i \psi_j^i(t), \quad i = 2, \cdots, (N-1), \tag{13}$$

where ψ_j^i, $j = 0, \cdots, n$, are specified twice continuously differentiable functions that are T-periodic for $t \geq t'$. The constants $\gamma_j^i \in \mathbb{R}$, $j = 1, \cdots, n$, $i = 2, \cdots, (N-1)$, are determined from the condition

$$\phi_1(t' + T) - \phi_1(t') = 0, \tag{14}$$

The difference $\phi_1(t'+T) - \phi_1(t')$ is a function of γ_j^i and $\phi_1(t')$. Thus Eq. (14) is a finite-dimensional root-finding problem. If the set of functions ψ_i^j is sufficiently rich, a solution to (14) exists and can be determined numerically. If multiple solutions are available, a specific solution can be determined by solving a constrained optimization problem. It is also possible to incorporate link collision constraints and $m_{\theta 1}(\phi) \neq 0$ into the above formulation using the interior penalty function method [5]. Since $\dot{\phi}_2, \cdots, \dot{\phi}_{(N-1)}$ and $\dot{\theta}_d$ are T-periodic, (12) and (14) imply that $\phi_1(t+T) = \phi_1(t)$ for all $t \geq t'$. After $\phi(t),\ t \geq t'$, has been obtained, the joint torques can be computed from (2).

4. Reconfiguration

Suppose an initial state $(\theta_0, \phi_0, \dot{\phi}_0) \in \mathbb{R}^{2N-1}$ at a time instant t_0, and a final state $(\theta^*, \phi^*, \dot{\phi}^*) \in \mathbb{R}^{2N-1}$ at a time instant t' are given. The reconfiguration problem is to determine joint torques $\tau_i(t)$, $i = 1, \cdots, (N-1)$, $t_0 \leq t \leq t'$, such that the solution of Eqs. (1) and (2) with $\theta(t_0) = \theta_0,\ \phi(t_0) = \phi_0,\ \dot{\phi}(t_0) = \dot{\phi}_0$ satisfies

$$\theta(t') = \theta^*, \quad \phi_i(t') = \phi_i^*, \quad \text{and} \quad \dot{\phi}_i(t') = \dot{\phi}_i^*, \quad i = 1, \cdots, (N-1). \tag{15}$$

Our use of $(\theta^*, \phi^*, \dot{\phi}^*)$ and t' in formulations of both the exact tracking problem and the reconfiguration problem is not accidental. It is motivated by the fact that a reconfiguration of the multilink is typically required prior to exact tracking to match the orientation and the shape with the required at $t = t'$ for the exact tracking. This combination of reconfiguration and exact tracking is illustrated in Sec. 5 by a spacecraft example.

Throughout this section we assume: (S1) $N \geq 3$; (S2) there exist two fixed integers p and r, $1 \leq p, r \leq (N-1)$, $p \neq r$, and $\phi^c \in \mathbb{R}^{(N-1)}$ such that $\nu_{pr}(\phi^c) \neq 0$. The assumption (S2) can be interpreted as the angular momentum nonintegrability condition [15], [20]. It is satisfied for a multilink with at least three links, unless the multilink is degenerate (e.g., masses or inertias of some of the links are zero) [15]. In the usual case almost all shapes ϕ^c satisfy $\nu_{pr}(\phi^c) \neq 0$.

The reconfiguration problem was solved in [11], [15] for the case of zero angular momentum and $\dot{\phi}_0 = \dot{\phi}^* = 0$. In the present paper we provide a reconfiguration procedure for the general case when the angular momentum is not necessarily zero. While the proposed procedure can be viewed as an extension to the case $H \neq 0$ of a similar procedure in [15], the arguments which we justify it with are different. They are based on the averaging theory used to obtain an estimate of the orientation drift induced by small amplitude high frequency periodic joint motions. A related work on using averaging for attitude control includes reference [12]. However, the results in [12] do not apply to the problem at hand because of the different form of the equations.

A specific reconfiguration procedure is now described. Let $t_0 < t_1 < t_2 < t'$ be a partition of the interval $[t_0, t']$. The desired joint motion is defined in terms of $n \in Z^+$, $k \in \mathbb{R}$, and $\mu \in \{-1, +1\}$ as

$$\phi(n,k,\mu,t) = \begin{cases} \phi^-(t), & \text{if } t_0 \le t \le t_1, \\ \tilde{\phi}(n,k,\mu,t), & \text{if } t_1 \le t \le t_2, \\ \phi^+(t), & \text{if } t_2 \le t \le t', \end{cases} \tag{16}$$

The function $\phi^-(t),\ t_0 \le t \le t'$, is C^2 and satisfies

$$\phi^-(t_0) = \phi_0, \quad \dot{\phi}^-(t_0) = \dot{\phi}_0, \quad \phi^-(t_1) = \phi^c, \quad \dot{\phi}^-(t_1) = 0. \tag{17}$$

The function $\phi^+(t),\ t_0 \le t \le t'$, is C^2 and satisfies

$$\phi^+(t_2) = \phi^c, \quad \dot{\phi}^+(t_2) = 0, \quad \phi^+(t') = \phi^*, \quad \dot{\phi}^+(t') = \dot{\phi}^*. \tag{18}$$

There exist multiple choices of ϕ^- and ϕ^+. One obvious choice is to use cubic polynomials. Other choices can be made based on optimality or link collision-avoidance considerations.

Let $h = (t_2 - t_1)/(4n)$. The function $\tilde{\phi}(n,k,\mu,t)$ is defined for $t_1 \le t \le t_2$ in terms of parameters $n \in Z^+$, $k \in \mathbb{R}$, and $\mu \in \{-1, +1\}$ as

$$\begin{aligned} \tilde{\phi}_r(n,k,\mu,t) &= \phi_r^c + \frac{k}{h}\int_{t_1}^{t} \big(\mu_+ v_1(\sigma,n) + (-\mu)_+ v_2(\sigma,n)\big)d\sigma, \\ \tilde{\phi}_p(n,k,\mu,t) &= \phi_p^c + \frac{k}{h}\int_{t_1}^{t} \big(\mu_+ v_2(\sigma,n) + (-\mu)_+ v_1(\sigma,n)\big)d\sigma, \\ \tilde{\phi}_j(n,k,t) &= \phi_j^c, \quad \text{if } j = 1, \cdots, (N-1),\ j \neq r, p\ , \end{aligned} \tag{19}$$

where $(\mu)_+ = \mu$ if $\mu > 0$ and is zero otherwise. Here,

$$v_1(t,n) = \begin{cases} 1 - \cos\left(\dfrac{2\pi(t-t_1)}{h}\right), & t_1 \le t \le t_1 + h, \\ 0, & t_1 + h \le t \le t_1 + 2h, \\ -1 + \cos\left(\dfrac{2\pi(t-t_1-2h)}{h}\right), & t_1 + 2h \le t \le t_1 + 3h, \\ 0, & t_1 + 3h \le t \le t_1 + 4h, \end{cases}$$

$$v_2(t,n) = \begin{cases} 0, & t_1 \le t \le t_1 + h, \\ 1 - \cos\left(\dfrac{2\pi(t-t_1-h)}{h}\right), & t_1 + h \le t \le t_1 + 2h, \\ 0, & t_1 + 2h \le t \le t_1 + 3h, \\ -1 + \cos\left(\dfrac{2\pi(t-t_1-3h)}{h}\right), & t_1 + 3h \le t \le t_1 + 4h, \end{cases}$$

and

$$v_1(t,n) = v_1(t-4h,n), \quad v_2(t,n) = v_2(t-4h,n), \quad t_1 + 4h \le t \le t_2.$$

By construction,

$$\phi(n,k,\mu,t_1) = \phi(n,k,\mu,t_2) = \phi^c, \quad \dot\phi(n,k,\mu,t_1) = \dot\phi_j(n,k,\mu,t_2) = 0.$$

Let $\theta(n,k,\mu,t)$, $t_0 \le t \le t'$, denote the solution to equation (1), where $\phi(t) = \phi(n,k,\mu,t)$ is defined by equation (16) and $\theta(n,k,\mu,t_0) = \theta_0$.

Lemma 4.1. *For all $\hat n \in Z^+$ sufficiently large there exists a choice of parameters $k = \hat k \in \mathbb{R}$ and $\mu = \hat\mu \in \{-1,+1\}$ such that $\theta(\hat n,\hat k,\hat\mu,t') = \theta^*$.*

Proof. Since $\phi^-(t)$, $t_0 \le t \le t_1$ and $\phi^+(t)$, $t_2 \le t \le t'$ are bounded and do not depend on n, k, and μ and since $m_{\theta\theta}(\phi) > 0$ for all ϕ, the differences $\Delta_1 = \theta(n,k,\mu,t_1) - \theta(n,k,\mu,t_0)$ and $\Delta_2 = \theta(n,k,\mu,t') - \theta(n,k,\mu,t_2)$ are bounded and do not depend on the values of n, k, and μ. Let $\bar\theta(\lambda,\mu,t)$, $t_1 \le t \le t_2$, $k \in \mathbb{R}$, $\mu \in \{-1,+1\}$, denote the solution to

$$\dot{\bar\theta} = \frac{H}{m_{\theta\theta}(\phi^c)} - \lambda^2\mu\nu_{pr}(\phi^c)\frac{\pi}{8}, \quad \bar\theta(\lambda,\mu,t_1) = \theta(n,k,\mu,t_1). \tag{20}$$

Let $\varepsilon > 0$ be given and, without loss of generality, assume that $\nu_{pr}(\phi^c) > 0$. Equation (20) implies that we can select $\lambda = \lambda_1 < 0$ so that

$$\Delta_1 + \bar\theta(\lambda_1,-1,t_2) - \bar\theta(\lambda_1,-1,t_1) + \Delta_2 > \theta^* + \varepsilon$$

and $\lambda = \lambda_2 > 0$ so that

$$\Delta_1 + \bar\theta(\lambda_2,1,t_2) - \bar\theta(k_2,1,t_1) + \Delta_2 < \theta^* - \varepsilon.$$

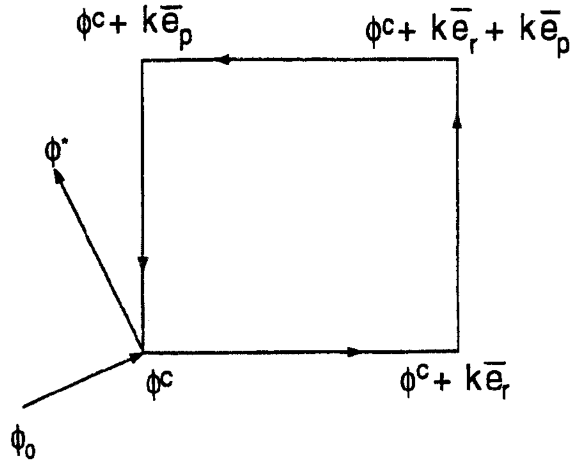

Fig. 2. Reconfiguration maneuver.

Let $\varkappa(\lambda, n) = \lambda\sqrt{\dfrac{(t_2 - t_1)\pi}{8n}}$. By applying the method of averaging [6], [14] to Eq. (1), where $\phi(t) = \phi(n, \varkappa(\lambda, n), \mu, t)$, it can be shown that for any $\delta > 0$ there exists $M \in Z^+$ such that for any $n > M$,

$$\begin{gathered} \left|\theta(n, \varkappa(\lambda_1, n), -1, t) - \bar{\theta}(\lambda_1, -1, t)\right| < \delta, \\ \left|\theta(n, \varkappa(\lambda_2, n), 1, t) - \bar{\theta}(\lambda_2, 1, t)\right| < \delta, \\ t_1 \leq t \leq t_2. \end{gathered} \tag{21}$$

Select $\delta < \varepsilon/4$ and $\hat{n} > M$. Then,

$$\theta(\hat{n}, \varkappa(\lambda_1, \hat{n}), \mathrm{sign}(\varkappa(\lambda_1, \hat{n})), t') > \theta^*$$

and

$$\theta(\hat{n}, \varkappa(\lambda_2, \hat{n}), \mathrm{sign}(\varkappa(\lambda_2, \hat{n})), t') < \theta^*.$$

By continuous dependence, there exists a value of the parameter $k = \hat{k}$ such that $\theta(\hat{n}, \hat{k}, \mathrm{sign}(\hat{k}), t') = \theta^*$. This completes the proof of the lemma. $\square$

The lemma leads to a two-step procedure for the reconfiguration maneuver. First, we select $n = \hat{n} \in Z^+$ sufficiently large so that together the orientation differences $\theta(\hat{n}, k, 1, t_2) - \theta(\hat{n}, k, 1, t_1)$ and $\theta(\hat{n}, k, -1, t_2) - \theta(\hat{n}, k, -1, t_1)$ range from 0 to 2π (mod 2π) as the parameter k varies. Then, we solve the nonlinear root finding problem $\theta(\hat{n}, k, \mu, t') = \theta^*$ (mod 2π) for the parameters $k = \hat{k} \in \mathbb{R}$ and $\mu = \hat{\mu} \in \{-1, 1\}$. The joint torques which accomplish the reconfiguration are computed by substituting $\phi(\hat{n}, \hat{k}, \hat{\mu}, t)$, $t_0 \leq t \leq t'$, into Eq. (2).

The reconfiguration maneuver is accomplished in three steps (Fig. 2). In the first step (time duration $t_0 \le t \le t_1$), the multilink is transferred from the initial shape and initial joint angle velocities to the shape ϕ^c and zero angle velocities. In the second step (time duration $t_1 \le t \le t_2$), joint angles ϕ_p and ϕ_r are forced to trace a square path with side size $|\hat{k}|$, $\hat{n}$ times. The direction in which the square path is traced is determined by $\hat{\mu} = 1$ or $\hat{\mu} = -1$. In the third step (time duration $t_2 \le t \le t'$), the multilink is transferred from the shape ϕ^c and zero joint angle velocities to the final shape and final joint angle velocities.

The orientation difference $\theta(n,k,\mu,t_2) - \theta(n,k,\mu,t_1)$ has a number of properties which can be exploited. For example, Eq. (1) implies that

$$\theta(n,k,\mu,t_2) - \theta(n,k,\mu,t_1) = n\Big(\theta(1,k,\mu,t_2) - \theta(1,k,\mu,t_1)\Big).$$

If the angular momentum is zero, then by writing $\theta(n,k,\mu,t_2)-\theta(n,k,\mu,t_1)$ from (1) as a line integral along the path of joint angles, it can be shown that $\theta(n,k,\mu,t_2) - \theta(n,k,\mu,t_1)$ does not depend on t_1 and t_2 and

$$\theta(n,k,\mu,t_2) - \theta(n,k,\mu,t_1) = n\mu\Big(\theta(1,k,1,t_2) - \theta(1,k,1,t_1)\Big).$$

In the literature the orientation differences resulting from periodic joint motions (or cyclic shape changes) are referred to as phases [11], [16], [19], [20], and are, typically, represented as a sum of two terms. The first term is called the geometric phase [20] and it can be written as a line integral along the path of joint angles. The geometric phase depends only on the path of the joint angles and is independent of how fast or slowly the joint angles are changed along this path. The second term is called the dynamic phase [20] and it can be nonzero only if the angular momentum is nonzero. For the multilink with at least three links the geometric phase term can be made arbitrary large by periodic joint motions of sufficiently high frequency. The dynamic phase term remains bounded on bounded time intervals, independently of the joint motions. Thus periodic joint motions can reorient the multilink arbitrarily. This observation is made precise in the proof of Lemma 4.1 using an averaging estimate, and a fixed point argument.

We formally summarize the main result of this section.

Theorem 4.2. *Suppose* (S1), (S2) *hold. Then, for an arbitrary initial state* $(\theta_0, \phi_0, \dot{\phi}_0)$ *and an arbitrary final state* $(\theta^*, \phi^*, \dot{\phi}^*)$, *for arbitrary time instants* t_0 *and* t' *satisfying* $t' > t_0$, *there exist joint torques* $\tau_i(t)$, $i = 1, \cdots, N-1$, *such that if* $\theta(t_0) = \theta_0$, $\phi(t_0) = \phi_0$, *and* $\dot{\phi}(t_0) = \dot{\phi}_0$, *then the solution of Eqs.* (1) *and* (2) *satisfies* $\theta(t') = \theta^*$, $\phi(t') = \phi^*$, $\dot{\phi}(t') = \dot{\phi}^*$.

Actually, by modifying slightly the construction of ϕ^- and ϕ^+, the conclusion of Theorem 4.2 can be strengthened. Specifically, the joint torques which accomplish the reconfiguration can be made continuous. Suppose the shapes ϕ_0, ϕ^c, ϕ^* are collision-free, i.e., the links do not intersect or touch each other. Then, the reconfiguration maneuver can be accomplished without link collisions.

5. Application to multibody spacecraft attitude control problems

In this section we briefly comment on applicability of our results to multibody spacecraft exact orientation tracking maneuvers. A frequent attitude control objective for spacecraft in orbit is to ensure that it remains pointed in the direction of a specified target, e.g., a fixed star, the center of the Earth, or another satellite, as may be required for space observations and communications. A traditional approach to such problems is to use external actuation, gas jets or momentum wheels, to produce external moments on the spacecraft. Even without such external actuation it is still possible to accomplish a variety of pointing maneuvers by cyclicly changing the shape of the spacecraft, e.g., by moving an appendage or a robotic arm in a periodic fashion relative to the spacecraft body.

For the case of zero angular momentum it has been known for some time that cyclic shape changes are capable of inducing net orientation drift. In the present paper we showed that cyclic shape changes can be used for *exact* orientation tracking. A typical spacecraft exact orientation tracking maneuver would consist of two phases. During the first phase the spacecraft is reconfigured so as to match its orientation and shape with those required for exact tracking. During the second phase cyclic shape changes make the orientation follow exactly a specified tracking objective whose time derivative is periodic. Details of several such spacecraft maneuvers are reported in [10].

6. Concluding remarks

In this paper we developed new results on control of multibody systems which conserve angular momentum. Prior results in the literature on the use of periodic joint motions are extended to exact tracking problems and to reconfiguration problems for the case of nonzero angular momentum. Further extensions are available. For example, the assumption that the angular momentum of the multilink is constant can be relaxed. It is sufficient to assume that the angular momentum is *prescribed* as a function of joint angles and time that is periodic in time.

In terms of their control-theoretic aspects, the problems addressed in this paper can be viewed as controllability problems for a class of nonlinear control systems with time-varying drift. The control systems may have no equilibria, and state constraints may be imposed. It appears that in these controllability problems standard Lie-algebraic tools [9], [21] do not apply. However, the direct approach of this paper, based on explicit control input parametrization, averaging, and a fixed point argument, has been shown to be effective in addressing these controllability problems.

REFERENCES

1. S. Dubowsky and E. Papadopoulos, The kinematics, dynamics and control of free-flying and free-floating space robotic systems. *IEEE Trans. Robot. Automat.* **9** (1993), No. 5, 531-543.

2. C. Fernandes, L. Gurvits, and Z. Li, Near-optimal nonholonomic motion planning for a system of coupled rigid bodies. *IEEE Trans. Automat. Control* **39** (1994), No. 3, 450-464.

3. ______, Attitude control of space platform/manipulator system using internal motion. *Space Robot. Dynam. and Control.* Y. Xu and T. Kanade, eds., *Kluwer*, 1993, 131-159.

4. C. Frohlich, Do springboard divers violate angular momentum conservation? *Am. J. Phys.* **47** (1979), 583–592.

5. E. Gilbert and D. Johnson, Distance functions and their application to robot path planning in the presence of obstacles. *IEEE J. Robot. Automat.* **RA-1** (1985), No. 1.

6. L. Gurvits and Z. X. Li, Smooth time-periodic feedback solutions for nonholonomic motion planning. In: Nonholonomic Motion Planning. Z. Li and J. F. Canny, eds., *Kluwer*, 1993, 53–108.

7. T. R. Kane and M. P. Scher, A dynamical explanation of the falling cat phenomenon. *Int. J. Solid Struct.* **5** (1969), 663–670.

8. ______, Human self-rotation by means of limb movements. *J. Biomech.* **3** (1970), 39–49.

9. M. Kawski, High order local controllability. In: Nonlinear Controllability and Optimal Control. H. J. Sussmann, ed., *Pure Appl. Math.* **133**, 431–467.

10. I. Kolmanovsky, N. H. McClamroch, and V. T. Coppola, Exact tracking for a planar multilink in space using internal actuation. *Proc. Am. Control Conf.* **1** (1994), 143-147.

11. P. S. Krishnaprasad, Geometric phases and optimal reconfiguration for multibody systems. *Proc. Am. Control Conf.* **3** (1990), 2440-2444.

12. N. E. Leonard and P. S. Krishnaprasad, Averaging for attitude control and motion planning. *Proc. 32nd Conf. Decision and Control* **4** (1993), 3098–3104.

13. W. Liu, Averaging theorems for highly oscillatory differential equations and iterated Lie brackets. *Rept. SYCON*-93-14, *Rutgers Center for Syst. and Control* (1993).

14. P. Lochak and C. Meunier, Multiphase averaging for classical systems. *Springer*, 1988.

15. M. Reyhanoglu and N. H. McClamroch, Planar reorientation maneuvers of space multibody systems using internal controls. *J. Guidance, Control Dynam.* **15** (1992), No. 6, 1475-1480.

16. R. Montgomery, The isoholonomic problem and some applications. *Commun. Math. Phys.* **128** (1990), 565–592.

17. M. Reyhanoglu and N. H. McClamroch, Nonlinear attitude control of planar structures in space using only internal controls. *Fields Inst. Commun.* **2** (1993), 91-100.

18. A. Shapere and F. Wilczek, Gauge kinematics of deformable bodies. *Geometr. Phases in Phys., World Scientific, Singapore*, 1989, 449-460.

19. N. Sreenath, Y. G. Oh, P. S. Krishnaprasad, and J. E. Marsden, The dynamics of coupled planar rigid bodies. *Dynam. Stability Syst.* **3** (1988), 25-49.

20. N. Sreenath, Nonlinear control of planar multibody systems in shape space. *Math. Control Signals Syst.* **5** (1992), No. 4, 343-363.

21. H. J. Sussmann, A general theorem on local controllability. *SIAM J. Control Optimiz.* **25** (1987), 1588–1594.

22. G. C. Walsh and S. S. Sastry, On reorienting linked rigid bodies using internal motions. *Proc. IEEE Trans. Robot. Automat.* **11** (1995), No. 1.

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