New Results on Degree Sequences of Uniform Hypergraphs

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Abstract

A sequence of nonnegative integers is k-graphic if it is the degree sequence of a k-uniform hypergraph. The only known characterization of k-graphic sequences is due to Dewdney in 1975. As this characterization does not yield an efficient algorithm, it is a fundamental open question to determine a more practical characterization. While several necessary conditions appear in the literature, there are few conditions that imply a sequence is k-graphic. In light of this, we present sharp sufficient conditions for k-graphicality based on a sequence's length and degree sum.

Kocay and Li gave a family of edge exchanges (an extension of 2-switches) that could be used to transform one realization of a 3-graphic sequence into any other realization. We extend their result to k-graphic sequences for all $k \geqslant 3$. Finally we give several applications of edge exchanges in hypergraphs, including generalizing a result of Busch et al. on packing graphic sequences.

Keywords: degree sequence, hypergraph, edge exchange, packing

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1 Introduction

A hypergraph H is *k-uniform*, or is a *k-graph*, if every edge contains k vertices. A k-uniform hypergraph is *simple* if there are no repeated edges. Thus, a simple 2-uniform hypergraph is a simple graph. For a vertex v in a k-graph H, the *degree of* v, denoted $d_H(v)$ (or simply $d(v)$ when H is understood) is the number of edges of H that contain v. As with 2-graphs, the list of degrees of vertices in a k-graph H is called the *degree sequence* of H.

Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers. We let $\sigma(\pi)$ denote the sum $\sum_{i=1}^{n} d_i$, and when it is convenient, we write $\pi = (d_1^{m_1}, \ldots, d_n^{m_n})$, where exponents denote multiplicity. If π is the degree sequence of a simple k-graph H, we say π is *k-graphic*, and that H is a *k-realization* of π. When $k = 2$, we will simply say that π is *graphic* and that H is a *realization* of π.

Our work in this area is motivated by the following fundamental problems:

Problem 1.1. Determine an efficient characterization of k-graphic sequences for all $k \geqslant 3$.

Problem 1.2. Investigate the properties of the family of k-realizations of a given sequence.

We will present results relating to each of these problems. Our results are motivated by similar work on graphic sequences. When $k = 2$, there are many characterizations of graphic sequences, including those of Havel [20] and Hakimi [19], and Erdős and Gallai [15]. Sierksma and Hoogeveen [21] list seven criteria and give a unifying proof. For $k \geqslant 3$, Problem 1.1 appears to be much less tractable.

The following theorem from [12] is the only currently known characterization of k-graphic sequences for $k \geqslant 3$.

Theorem 1.3 (Dewdney 1975)**.** *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers. π is k-graphic if and only if there exists a nonincreasing sequence $\pi' = (d'_2, \ldots, d'_n)$ of nonnegative integers such that*

1. *π' is $(k-1)$-graphic,*
2. *$\sum_{i=2}^{n} d'_i = (k-1)d_1$, and*
3. *$\pi'' = (d_2 - d'_2, d_3 - d'_3, \ldots, d_n - d'_n)$ is k-graphic.*

Dewdney's characterization hinges on a relatively simple, yet quite useful, idea that we will utilize in the sequel. Given a vertex v in a hypergraph H, let H_v denote the subgraph of H with vertex set $V(H)$ and edge set consisting of the edges of H that contain v. The *link* of v, $L_H(v)$, is then the hypergraph obtained by deleting v from each edge in H_v. Thus, if H is k-uniform, then $L_H(v)$ is a $(k-1)$-uniform hypergraph, and if H is a graph, then each vertex in $N_H(v)$ gives rise to a 1-edge in $L_H(v)$.

Suppose that $\pi' = (x_1, \ldots, x_n)$ is a $(k-1)$-graphic sequence and $\pi'' = (y_1, \ldots, y_n)$ is a k-graphic sequence. It follows that the sequence

$$\pi = \left(\frac{\sigma(\pi')}{k-1}, x_1 + y_1, \ldots, x_n + y_n \right)$$

is also k-graphic. To see this, let H_1 be a $(k-1)$-realization of π' and H_2 be a k-realization of π'', both with vertex set $\{v_1, \ldots, v_n\}$. Add a new vertex v_0 to each edge in H_1 to obtain the k-graph H_1'. Then $H = H_1' + H_2$ is a realization of π as desired; furthermore, π' is the degree sequence of $L_H(v_0)$.

Considering this process in reverse, a sequence $\pi = (d_1, \ldots, d_n)$ is k-graphic if for some index i there is a $(k-1)$-graphic sequence $\pi' = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ such that

$$\pi'' = (d_1 - x_1, \ldots, d_{i-1} - x_{i-1}, 0, d_{i+1} - x_{i+1}, \ldots, d_n - x_n)$$

is k-graphic. Here, as above, we would be able to construct a realization H of π in which π' is the degree sequence of the link $L_H(v_i)$. Again, note that the i'th term of each sequence is 0, corresponding to the vertex v_i that does not appear in the $(k-1)$-realization of π or the k-realization of π'.

This is the crucial idea of the Havel-Hakimi algorithm, wherein it is proved that it is sufficient to select $\pi' = (0, 1^{d_1}, 0^{n-d_1-1})$ (which is trivially 1-graphic) and therefore one must only determine the graphicality of the residual sequence $(0, d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$. In Theorem 1.3, however, there is no standard form of the "link sequence" that is sufficient to determine if π is k-graphic. Were one able to similarly demonstrate that it suffices to check only one $(k-1)$-graphic π', or even a small number, then this would represent significant progress towards Problem 1.1.

In addition to Theorem 1.3, several other authors have studied Problem 1.1. Bhave, Bam and Deshpande [6] gave an Erdős-Gallai-type characterization of degree sequences of loopless linear hypergraphs. While interesting in its own right, their result does not directly generalize to Problem 1.1. Colbourn, Kocay and Stinson [11] proved that several related problems dealing with 3-graphic sequences are NP-complete. Additionally, several necessary conditions for a sequence to be 3-graphic have been found (see Achuthan, Achuthan, and Simanihuruk [1], Billington [7], and Choudum [10] for some).

Unfortunately, Achuthan et al. [1] showed that the necessary conditions of [1], [7], and [10] are not sufficient. In fact, surprisingly few sufficient conditions for a sequence to be k-graphic exist, despite the apparent interest in the general characterization problem. Inspired by sufficient conditions for 2-graphic sequences given by Yin and Li [29], Aigner and Triesch [2], and Barrus, Hartke, Jao and West [3], we present in Section 2 some general sufficient conditions for k-graphicality when $k \geqslant 3$.

A useful tool for studying graphic sequences is the edge exchange, or 2-switch, where two edges in a graph are replaced with two nonedges while maintaining the degree of each vertex. In particular, this is a key tool in the standard proof of the Havel-Hakimi characterization of graphic sequences [28, p. 45–46]. It is our hope that a better understanding of edge exchanges may lead to an efficient Havel-Hakimi-type characterization of k-graphic sequences. Additionally, edge exchanges have proved vital in approaching a number of problems for graphs in the vein of Problem 1.2 (cf. [4, 8, 9, 16]).

In Section 3, we give a small collection of elementary edge exchanges that can be applied to transform any realization of a k-graphic sequence into any other realization. This extends a result of Kocay and Li on 3-graphic sequences. As an application of these edge exchanges, we prove a result about packing of k-graphic sequences. Busch et al. [8] proved that graphic sequences pack under certain degree and length conditions. We extend their result to k-graphic sequences.

2 Sufficient conditions on length

2.1 Results

In this section, we give a new sufficient condition for a sequence to be k-graphic, and we give several corollaries that are inspired by previous results on graphic sequences. We say that a nonincreasing sequence is *near-regular* if the difference between the first and last terms of the sequence is at most 1. The main result of this section shows that if the beginning of a sequence is near-regular, which will be made more precise later, then the sequence is k-graphic. For graphs, this is a simple consequence of the Erdős-Gallai inequalities (see for example Lemma 2.1 of [29]), but for $k \geqslant 3$ the situation is more complex.

We will state our theorems here, discuss their sharpness in Section 2.2, and present the proofs in Section 2.3.

Theorem 2.1. *Let π be a nonincreasing sequence with maximum entry Δ and t entries that are at least $\Delta - 1$. If k divides $\sigma(\pi)$ and*

$$\binom{t-1}{k-1} \geqslant \Delta, \tag{1}$$

then π is k-graphic.

This result, combined with a variety of classical and new ideas, yields three immediate corollaries. While poset methods had earlier been used for degree sequences, Aigner and Triesch [2] systematized this approach. Bauer et al. [5] also used posets to study degree sequences, but with a different order relation. Using the same poset as Aigner and Triesch, we show that with a large enough degree sum, the near-regular condition is unnecessary.

Corollary 2.2. *Let π be a nonincreasing sequence with maximum term Δ, and let p be the minimum integer such that $\Delta \leqslant \binom{p-1}{k-1}$. If k divides $\sigma(\pi)$ and $\sigma(\pi) \geqslant (\Delta-1)p+1$, then π is k-graphic.*

This lower bound on the sum of the sequence immediately gives the following sufficient condition on the length of a sequence, analogous to the result of Zverovich and Zverovich [30] for graphs.

Corollary 2.3. *Let π be a nonincreasing sequence with maximum term Δ and minimum term δ, and let p be the minimum integer such that $\Delta \leqslant \binom{p-1}{k-1}$. If k divides $\sigma(\pi)$ and π has length at least $\frac{(\Delta-1)p-\Delta+\delta+1}{\delta}$, then π is k-graphic.*

Finally, if we know a little bit more about the sequence, we can refine the length condition. A sequence is *gap-free* if it has entries with all values between the largest entry Δ and the smallest entry δ. The graphicality of gap-free sequences was studied by Barrus, Hartke, Jao, and West [3].

Corollary 2.4. *Let π be a gap-free sequence with maximum term Δ and minimum term $\delta = 1$, and let p be the minimum integer such that $\Delta \leqslant \binom{p-1}{k-1}$. If $\sigma(\pi)$ is divisible by k and π has length at least $(\Delta - 1)(p - \Delta/2) + 1$, then π is k-graphic.*

The following lemma gives a simple necessary condition for a sequence to be k-graphic, and additionally gives information about the k-realizations of a sequence. This is used to show the sharpness of Corollaries 2.2 and 2.3, and also in Section 3.

Lemma 2.5. *If $\pi = (d_1, \ldots, d_n)$ is a k-graphic sequence, then*

$$\sum_{i=1}^{t} d_i \leqslant k\binom{t}{k} + (k-1)\sum_{j=t+1}^{n} d_j$$

for $k \leqslant t \leqslant n$. If equality holds, then the t vertices of highest degree in any k-realization of π form a clique, and any edge not contained in the clique contains exactly one vertex outside the clique.

2.2 Sharpness

Theorem 2.1 is sharp, as can by seen by examination of the sequence (Δ^M), for any Δ.

For Corollary 2.2, consider the sequence

$$\pi_j = \left(\left(\binom{j-1}{k-1} - (k-1)\right)^{j-1}, \binom{j-1}{k-1} - k - j + 1\right),$$

where $j \geqslant k+2$. The maximum term Δ_j of π_j satisfies $\binom{j-2}{k-1} < \Delta_j \leqslant \binom{j-1}{k-1}$, so in the terminology of Corollary 2.2, $p = j$. We also have $\sigma(\pi_j) = (\Delta_j - 1)j$, and k divides $\sigma(\pi_j)$. However, π_j is not k-graphic, for the inequality in Lemma 2.5 does not hold for π_j when $t = j - 1$.

To see that Corollary 2.2 is also sharp when $p = k + 1$, consider the sequence $\pi = (k, (k-1)^k)$. This is realized by a complete k-graph on $k+1$ vertices with one edge removed, and has degree sum $\sigma(\pi) = k^2$. Subtracting one from each of the last k terms yields the sequence $\pi' = (k, (k-2)^k)$, with degree sum $k^2 - k$. This is not k-graphic: suppose H is a k-realization of π'. Let S be the set of vertices of degree $k-2$ in H, and let v be the vertex of degree k. Every edge containing v must also contain a $(k-1)$-subset of

S. There are exactly k of these. However, this means each vertex in S must have degree $k-1$.

Corollary 2.3 is best possible up to a factor depending only on k. To see this, first note that $\binom{p-2}{k-1} < \Delta$, so $p < (k-1)e\Delta^{1/(k-1)} + 2$. Then the minimum length required by the corollary is bounded above by

$$\frac{1}{\delta}\left((\Delta-1)((k-1)e\Delta^{1/(k-1)}+2) - \Delta + \delta + 1\right)$$

which, when $\Delta > \delta$, is at most

$$\frac{\Delta^{1+1/(k-1)}}{\delta}\left(e(k-1)(1-\frac{1}{\Delta}) + \frac{2}{\Delta^{1/(k-1)}}\right).$$

Now, as Δ becomes large, this quantity is bounded above by

$$\frac{C\Delta^{1+1/(k-1)}}{\delta},$$

where C depends only on k. Thus, a weaker but simpler form of Corollary 2.3 is: If π is a nonincreasing sequence with maximum term Δ and minimum term δ such that $\delta \neq \Delta$, k divides $\sigma(\pi)$, and the length of π is at least $\frac{C\Delta^{1+1/(k-1)}}{\delta}$, then π is k-graphic.

Now, consider the sequence $\pi = (\Delta^M, \delta^m)$, where $\delta < \Delta$ and $M = c_1\Delta^{1/(k-1)}$ for some $c_1 < k/e^2$. By Lemma 2.5, if π has length less than $\left\lceil \frac{M\Delta - k\binom{M}{k}}{(k-1)\delta} \right\rceil + M$, then it is not k-graphic. A lower bound on this expression is:

$$\begin{aligned}
\frac{M\Delta - k\binom{M}{k}}{(k-1)\delta} + M &\geqslant \frac{M\Delta - k\left(\frac{Me}{k}\right)^k}{(k-1)\delta} + M \\
&= \frac{c_1\Delta^{1+1/(k-1)} - k\left(\frac{c_1 e\Delta^{1/(k-1)}}{k}\right)^k}{(k-1)\delta} + c_1\Delta^{1/(k-1)} \\
&= \frac{\Delta^{1+1/(k-1)}}{\delta}\left(\frac{c_1 - k\left(\frac{c_1 e}{k}\right)^k}{(k-1)}\right) + c_1\Delta^{1/(k-1)} \\
&= c_2\frac{\Delta^{1+1/(k-1)}}{\delta} + c_1\Delta^{1/(k-1)},
\end{aligned}$$

where $c_2 = \frac{c_1 - k\left(\frac{c_1 e}{k}\right)^k}{(k-1)}$. Thus, if the length of π is less than $c_2\frac{\Delta^{1+1/(k-1)}}{\delta}$, π is not k-graphic. Comparing this to the result in the previous paragraph establishes our claim.

2.3 Proofs

Proof of Theorem 2.1. We will show that π is k-graphic by constructing an appropriate $(k-1)$-graphic link sequence and k-graphic residual sequence, as described in the introduction following Theorem 1.3.

First, note that if $\Delta = 1$, then $\pi = (1^{mk}, 0^{n-mk})$ for some integer m. This sequence is realized by a set of m disjoint edges and $n - mk$ isolated vertices. Thus, we can assume that $\Delta > 1$, and in particular, $t > k$. When $k = 2$, the result follows from the Erdős-Gallai inequalities, so we assume $k \geqslant 3$.

Consider the least k for which the theorem does not hold. Among all nonincreasing sequences that do not satisfy the theorem for this k, consider those that have the smallest maximum term, and let $\pi = (d_0, \ldots, d_{n-1})$ be one such sequence that minimizes the multiplicity of the largest term, Δ.

Let

$$c = \max \left\{ i \in \mathbb{Z} : \sum_{j=1}^{n-1} \max\{0, d_j - i\} \geqslant (k-1)\Delta \right\}.$$

Note that $c \leqslant \Delta - 1$, and we further claim that $c \geqslant 0$. Indeed, if $\Delta \geqslant k$, then $\sum_{j=1}^{n-1} d_j \geqslant (t-1)(\Delta - 1) \geqslant k(\Delta - 1) \geqslant (k-1)\Delta$. If $\Delta < k$, then since $t > k$ there are k terms in the set $\{d_2, \ldots, d_t\}$ that are at least $\Delta - 1$. Their sum is at least $k(\Delta - 1)$. Let $A = \sigma(\pi) - \Delta - k(\Delta - 1)$. Since k divides $\sigma(\pi)$, k must also divide $A + \Delta$, so $A \geqslant k - \Delta$. Then $\sum_{i=1}^{n-1} d_j = A + k(\Delta - 1) \geqslant k - \Delta + k(\Delta - 1) = (k-1)\Delta$. Thus, $c \geqslant 0$.

Define the sequence $L' = (l'_1, \ldots, l'_{n-1})$ by $l'_j = \max\{0, d_j - c\}$, and let $s = \sigma(L') - (k-1)\Delta$. Create the link sequence L by subtracting 1 from each of the first s terms of L'. That is, $L = (l_1, \ldots, l_{n-1})$, where

$$l_i = \begin{cases} l'_i - 1 & \text{if } 1 \leqslant i \leqslant s, \\ l'_i & \text{if } i > s. \end{cases}$$

Finally, let $R = (r_1, \ldots, r_{n-1})$ be the residual sequence, given by $r_j = d_j - l_j$ for $j = 1, \ldots, n-1$. It suffices to show that L is $(k-1)$-graphic and R is k-graphic, as adding a new vertex v_0 to each edge of a $(k-1)$-realization of L and then combining the resulting k-graph with a k-realization of R gives a k-realization of π.

Let m be the number of nonzero entries of L'. First suppose $m \geqslant t - 1$. Since $(d_0, \ldots, d_{t-1})$ is near-regular, the construction of L' implies that $(l'_1, \ldots, l'_{t-1})$ is near-regular, and so $(l_1, \ldots, l_{t-1})$ is near-regular. Let Δ_L be the largest term in $(l_1, \ldots, l_{t-1})$. We now bound Δ_L in order to show that L meets the conditions of the theorem and thus is $(k-1)$-graphic by the minimality of π. Since $\sigma(L) = (k-1)\Delta$ and $m \geqslant t-1$, we have

$$\Delta_L = \left\lceil \frac{\sum_{i=1}^{t-1} l_i}{t-1} \right\rceil \leqslant \left\lceil \frac{(k-1)\Delta}{t-1} \right\rceil \leqslant \left\lceil \frac{k-1}{t-1}\binom{t-1}{k-1} \right\rceil = \left\lceil \binom{t-2}{k-2} \right\rceil = \binom{t-2}{k-2}.$$

Therefore, L satisfies the conditions of the theorem, and by the minimality of π, it is $(k-1)$-graphic. If $m < t-1$, then we must have $c = \Delta - 1$, which means $L' = (1^m, 0^{n-1-m})$ and $L = (1^{(k-1)\Delta}, 0^{n-1-(k-1)\Delta})$. This sequence has a $(k-1)$-realization consisting of Δ disjoint edges and $n - 1 - (k-1)\Delta$ isolated vertices.

Now we turn our attention to R. Since $r_i = d_i - l_i$ and $l_i = d_i - c$ or $l_i = d_i - c - 1$ for $i \leqslant m$, we see that $r_i = c$ or $r_i = c+1$ for $i \leqslant m$. Thus, $R = ((c+1)^s, c^{m-s}, d_{m+1}, \ldots, d_{n-1})$. Note that $\sigma(R) = s + mc + \sum_{i=m+1}^{n-1} d_i = \sum_{i=1}^{n-1} d_i - (k-1)\Delta = \sigma(\pi) - k\Delta$, so k divides

$\sigma(R)$. If $\Delta_R \leqslant \binom{m-1}{k-1}$, then the minimality of π implies that R is k-graphic, so showing this inequality is our goal.

Note that $c+1 \leqslant \Delta$. Suppose first that $c+1 < \Delta$. If $m > t-1$, then

$$c+1 < \Delta \leqslant \binom{t-1}{k-1} \leqslant \binom{m-1}{k-1}$$

and we have our result. Since $c < \Delta - 1$, we have $m \geqslant t-1$, and so we may assume that $m = t-1$. In this case,

$$L = \left(\left\lceil \frac{k-1}{t-1}\Delta \right\rceil^{s}, \left\lfloor \frac{k-1}{t-1}\Delta \right\rfloor^{m-s}, 0^{n-1-m} \right).$$

Hence

$$\begin{aligned} \Delta_R \leqslant \Delta - \left\lfloor \frac{k-1}{t-1}\Delta \right\rfloor &< \Delta - \frac{k-1}{t-1}\Delta + 1 \\ &= \left(1 - \frac{k-1}{t-1}\right)\Delta + 1 \\ &\leqslant \frac{t-k}{t-1}\binom{t-1}{k-1} + 1 = \binom{t-2}{k-1} + 1, \end{aligned}$$

so $\Delta_R \leqslant \binom{t-2}{k-1} = \binom{m-1}{k-1}$. Thus, R is k-graphic.

If $c+1 = \Delta$, then $m \leqslant t-1$. In this case, any terms of π that are equal to $\Delta - 1$ become 0 in L'. So $\pi = (\Delta^{m+1}, (\Delta-1)^{t-1-m}, d_t, \ldots, d_{n-1})$, and if $m = t-1$, there are no terms in π equal to $\Delta - 1$. Then, $R = (\Delta^s, (\Delta-1)^{t-1-s}, d_t, \ldots, d_n)$, and we need to show that $\Delta \leqslant \binom{t-2}{k-1}$. Since $L' = (1^m, 0^{n-1-m})$ and $L = (1^{(k-1)\Delta}, 0^{n-1-(k-1)\Delta})$, we have $(k-1)\Delta \leqslant m \leqslant t-1$. Thus, $(k-1)\Delta \leqslant t-1$, so $\Delta \leqslant \frac{t-1}{k-1}$. When $t > k+1$, we have $\frac{t-1}{k-1} \leqslant \binom{t-2}{k-1}$. If $t = k+1$, then $\Delta \leqslant \frac{k}{k-1}$, and since Δ is an integer, $\Delta \leqslant 1 = \binom{t-2}{k-1}$. Thus, $\Delta \leqslant \binom{t-2}{k-1}$, and by the minimality of π, R is k-graphic. □

To prove the first corollary, we require some additional terminology. The *dominance order*, $\preceq$, is defined on the set $D(n,\sigma)$ of nonnegative nonincreasing sequences with length n and sum σ. For two elements $\pi = (d_1, \ldots, d_n)$ and $\pi' = (d'_1, \ldots, d'_n)$ of $D(n,\sigma)$, we say $\pi \preceq \pi'$ if $\sum_{i=1}^{m} d_i \leqslant \sum_{i=1}^{m} d'_i$ for all $1 \leqslant m \leqslant n$. In this poset the set of k-graphic sequences forms an ideal (a downward-closed set) (see [2] for $k=2$ and [14] or [22] for $k \geqslant 3$).

We are now prepared to give our proofs.

Proof of Corollary 2.2. Suppose $\sigma(\pi) = m \geqslant (\Delta-1)p+1$. Then there is a sequence π' that has the same sum and maximum degree such that $\pi \preceq \pi'$ in the dominance order, but the first p terms of π' form a near-regular sequence. By Theorem 2.1, π' is k-graphic, and since k-graphic sequences form an ideal, π is also k-graphic. □

Proof of Corollary 2.4. A gap-free sequence of length n with minimum term 1 has degree sum at least $\sum_{i=1}^{\Delta} i + (n-\Delta) = \binom{\Delta+1}{2} + (n-\Delta)$. Using this sum in Corollary 2.2 and solving for n yields the result. □

Proof of Lemma 2.5. Choose a set S of t vertices in a k-realization H of π. The subgraph induced by S has degree sum at most $k\binom{t}{k}$. A vertex w in $V(H) \setminus S$ contributes at most $(k-1)d_H(w)$ to the degree sum of vertices in S. Thus, $\displaystyle\sum_{i=1}^{t} d_i \leqslant k\binom{t}{k} + (k-1)\sum_{j=t+1}^{n} d_j$. If equality holds, each vertex w outside S contributes exactly $(k-1)d_H(w)$ to $\sum_{i=1}^{t} d_i$. Thus, every edge containing w consists of w as the only vertex outside S and $k-1$ vertices from S. Any edge whose vertex set is not contained in S thus consists of only one vertex outside S, as claimed. □

3 Edge exchanges

3.1 Edge exchanges in graphs and hypergraphs

An edge exchange is any operation that deletes a set of edges in a k-realization of π and replaces them with another set of edges, while preserving the original vertex degrees. When i edges are exchanged, we call this an *i-switch*. The 2-switch operation has been used to prove many results about graphic sequences; for examples see [4, 8, 9, 16].

For completeness, we now formally define edge exchanges in hypergraphs. Let F_1 and F_2 be k-graphs on the same vertex set S such that for every $x \in S$, $d_{F_1}(x) = d_{F_2}(x)$. Let H be a k-multihypergraph containing a subgraph F_1' on vertex set T such that $F_1' \cong F_1$ via the isomorphism $\phi : T \to S$. The *edge exchange* $\epsilon(F_1, F_2)$ applied to H replaces the edges of F_1' with the edges of a subgraph F_2' that is isomorphic to F_2 by the same map ϕ.

Define $\mathcal{M}_k(\pi)$ to be the set of k-uniform multihypergraphs that realize a sequence π, and $\mathcal{S}_k(\pi) \subseteq \mathcal{M}_k(\pi)$ be the set of simple k-realizations of π. Let $\mathcal{F} \subseteq \mathcal{M}_k(\pi)$ and $\mathcal{Q}$ be a collection of edge exchanges such that $\epsilon(F_1, F_2) \in \mathcal{Q}$ if and only if $\epsilon(F_2, F_1) \in \mathcal{Q}$. Then $G(\mathcal{F}, \mathcal{Q})$ is the graph whose vertices are the elements of $\mathcal{F}$, with an edge between vertices H_1 and H_2 if and only if H_1 can be obtained from H_2 by an edge exchange in $\mathcal{Q}$. Note that the symmetry condition imposed on $\mathcal{Q}$ permits us to define $G(\mathcal{F}, \mathcal{Q})$ as an undirected graph.

3.2 Navigating the space of k-realizations

Let e and e' be distinct edges in a k-graph G, and choose vertices $u \in e \setminus e'$ and $v \in e' \setminus e$. The operation $e \overset{u}{\underset{v}{\rightleftharpoons}} e'$ deletes the edges e and e' and adds the edges $e - u + v$ and $e' - v + u$ (where $e - u + v$ denotes the set $e - \{u\} \cup \{v\}$); see Figure 1. Denote this family of edge exchanges by $\mathcal{Q}_k^*$.

Petersen [25] showed that given any pair of realizations of a graphic sequence, one can be obtained from the other by a sequence of 2-switches. This result simply says that $G(\mathcal{S}_2(\pi), \mathcal{Q}_2^*)$ is connected. Kocay and Li [23] proved a similar result for 3-graphs, namely that any pair of 3-graphs with the same degree sequence can be transformed into each other using edge exchanges from $\mathcal{Q}_3^*$. However, unlike in the graph case, intermediate hypergraphs obtained while applying edge exchanges from $\mathcal{Q}_3^*$ may have multiple edges. In other words, $G(\mathcal{M}_3(\pi), \mathcal{Q}_3^*)$ is connected.

We extend Kocay and Li's result to arbitrary $k \geqslant 3$.

Theorem 3.1. *If π is any sequence of nonnegative integers with a k-multihypergraph realization, then $G(\mathcal{M}_k(\pi), \mathcal{Q}_k^*)$ is connected.*

Proof. Suppose there exists a sequence π with a k-multihypergraph realization for which $G(\mathcal{M}_k(\pi), \mathcal{Q}_k^*)$ is not connected. For two k-multihypergraphs H and F in $G(\mathcal{M}_k(\pi), \mathcal{Q}_k^*)$, let $R(H, F)$ be the subgraph of H with $E(R(H, F)) = E(H) \setminus E(F)$ and $B(H, F)$ be the subgraph of F with $E(B(H, F)) = E(F) \setminus E(H)$. Since $G(\mathcal{M}_k(\pi), \mathcal{Q}_k^*)$ is not connected, there are two k-multihypergraphs realizing π that are in different components of this graph. Now, among all such pairs of k-multihypergraphs, choose the pair H_1 and H_2 that minimize $|E(H_1) \triangle E(H_2)|$, and subject to this, that maximize $|e_r \cap e_b|$ for edges $e_r \in E(R(H_1, H_2))$ and $e_b \in E(B(H_1, H_2))$. Let $i = |e_r \cap e_b|$, and let $R = R(H_1, H_2)$ and $B = B(H_1, H_2)$. We refer to the edges of R as *red* and the edges of B as *blue*.

Since $e_r \neq e_b$, there are vertices $u \in e_r \setminus e_b$ and $v \in e_b \setminus e_r$. As H_1 and H_2 are realizations of π, $d_{H_1}(x) = d_{H_2}(x)$ for any vertex x. Note, if we let $d_P(x)$ equal the number of edges incident to x that appear in both H_1 and H_2, then $d_R(x) = d_{H_1}(x) - d_P(x)$ and $d_B(x) = d_{H_2}(x) - d_P(x)$. Thus $d_R(x) = d_B(x)$, and we may assume without loss of generality that $d_B(u) \geqslant d_B(v)$; otherwise $d_R(v) \geqslant d_R(u)$, and the roles of u and v, and red and blue, may be switched in the remainder of the proof.

We claim that u must be in some blue edge e' such that $v \notin e'$ and such that $e' + v - u$ is not a blue edge. Note that $e_b' = e_b + u - v$ is not a blue edge, for otherwise e_r and e_b' are red and blue edges, respectively, and intersect in $i + 1$ vertices. Since $d_B(u) \geqslant d_B(v)$ and $v \in e_b$ while $u \notin e_b$, we know u is in at least one blue edge that does not contain v. If $e + v - u$ is a blue edge for every blue edge e containing u but not v, then $d_B(u)$ is less than $d_B(v)$, a contradiction. Thus the edge e' exists.

Now we apply $e' \underset{v}{\overset{u}{\rightleftharpoons}} e_b$ to H_2 to obtain the k-multihypergraph H_2'. Let B' be the subgraph of H_2' such that $E(B') = E(H_2') \setminus E(H_1)$. Note that H_2 and H_2' are adjacent in $G(\mathcal{M}_k(\pi), \mathcal{Q}_k^*)$, so H_1 and H_2' must be in different components. However, there is a red edge $e_r \in E(R)$ and a blue edge $e_b' \in E(B')$ that intersect in $i + 1$ places. If $i + 1 < k$,

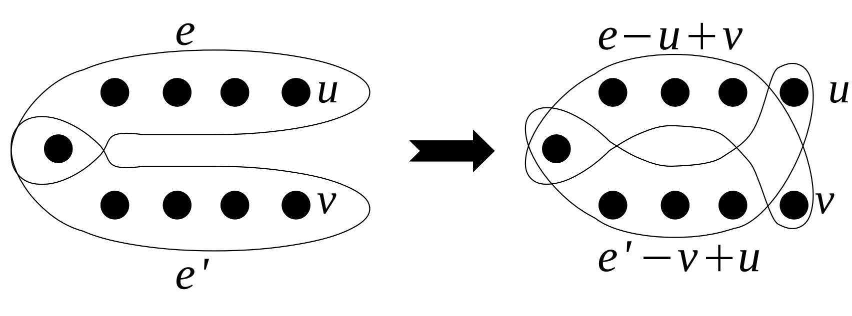

Figure 1: The operation $e \underset{v}{\overset{u}{\rightleftharpoons}} e'$.

this contradicts our choice of H_1 and H_2 maximizing edge intersections. If $i+1=k$, then $e'_b = e_r$ and $|E(H_1) \triangle E(H'_2)| < |E(H_1) \triangle E(H_2)|$, again contradicting our choice of H_1 and H_2. $\square$

We have shown that for a k-graphic sequence π, there is a path between simple k-realizations of π in $G(\mathcal{M}_k(\pi), \mathcal{Q}^*_k)$. This path may go through multihypergraph realizations, unlike in the result of Petersen. In the proof of Theorem 3.1, we argue that e_b+u-v and $e'+v-u$ are not blue edges, but either edge may still be an edge of H_2. Hence performing the edge exchange may in fact result in duplicating an edge of H_2. It is unknown whether $G(\mathcal{S}_k(\pi), \mathcal{Q})$ is connected for some small collection $\mathcal{Q}$ of edge exchanges.

For a positive integer i, let $\mathcal{E}_i$ be the collection containing all j-switches for $j \leqslant i$. Gabelman [18] gave an example of a 3-graphic sequence π whose simple realizations cannot be transformed into each other using only 2-switches, without creating multiple edges. That is, $G(\mathcal{S}_3(\pi), \mathcal{E}_2)$ is not connected. We extend his example to $k \geqslant 3$, which shows we cannot replace $\mathcal{M}_k$ with $\mathcal{S}_k$ in Theorem 3.1.

Proposition 3.2. *For each $k \geqslant 3$ there is a k-graphic sequence π such that $G(\mathcal{S}_k(\pi), \mathcal{E}_{k-1})$ is not connected.*

Proof. Consider the following matrix A of real numbers:

$$A = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k-1} & -y_1 \\ x_{2,1} & x_{2,2} & \cdots & x_{2,k-1} & -y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{k-1,1} & x_{k-1,2} & \cdots & x_{k-1,k-1} & -y_{k-1} \\ -z_1 & -z_2 & \cdots & -z_{k-1} & w \end{bmatrix}$$

where

$$y_j = \sum_{i=1}^{k-1} x_{j,i}, \qquad z_j = \sum_{i=1}^{k-1} x_{i,j}, \qquad \text{and} \qquad w = \sum_{i,j} x_{i,j}.$$

We choose the terms $x_{i,j}$ so that if a set of k entries of the matrix sums to zero, then those entries must be from a single row or column. This can be done by choosing the $x_{i,j}$'s to be linearly independent over $\mathbb{Q}$, or by choosing them to be powers of some sufficiently small ϵ.

We form a hypergraph H on a set V of k^2 vertices as follows: weight each vertex with a different entry of the matrix. The edges of H are the k-sets whose total weight is positive, and the k-sets corresponding to the rows of A. By construction of the matrix A, the only k-sets that have zero weight correspond to rows and columns. Thus the only k-sets that are non-edges either have negative weight or correspond to columns.

The degree sequence of H is not uniquely realizable, as the k-switch that adds the k-sets corresponding to columns of A to the edge set while removing the edges corresponding to rows gives another realization. However, we show that we cannot apply an i-switch to H for any $i < k$.

Note that in any edge exchange that replaces a set F_1 of edges with a set F_2 of nonedges,

$$\sum_{e\in F_1}\sum_{v\in e} wt(v) = \sum_{v\in V}(\deg_{F_1}(v))wt(v) = \sum_{v\in V}(\deg_{F_2}(v))wt(v) = \sum_{e\in F_2}\sum_{v\in e} wt(v).$$

Since edges of F_1 have nonnegative weight and nonedges of F_2 have nonpositive weight, we conclude that the edges of F_1 must have zero weight and thus correspond to rows of A, and the nonedges of F_2 have zero weight and correspond to columns of A. But no proper subset of edges corresponding to rows can be swapped for a proper subset of nonedges corresponding to columns, because this does not maintain the degree of every vertex. $\square$

This result immediately suggests the following problem:

Problem 3.3. Determine the smallest cardinality of a collection $\mathcal{Q}$ such that $G(\mathcal{S}_k(\pi), \mathcal{Q})$ is connected for every k-graphic sequence π.

Results for graphs suggest several different possible approaches. Is there a finite collection that works? Would it be sufficient to add all possible k-switches? Or would it suffice to add just the k-switch suggested by Proposition 3.2 to $\mathcal{E}_{k-1}$?

3.3 Applications

3.3.1 Obtaining a "good" realization

One consequence of the Havel-Hakimi characterization of 2-graphic sequences is that any graphic sequence has a realization in which a specified vertex v is adjacent to vertices whose degrees are the highest-degree vertices in the graph. This elementary fact has been proved in several places, for instance [17]. Motivated by this, we prove the following.

Theorem 3.4. *Let $\pi = (d_1, \dots, d_n)$ be a nonincreasing k-graphic sequence, and let H be a k-realization of π on vertices $\{v_1, \dots, v_n\}$ such that $d(v_i) = d_i$ for each $i, 1 \leqslant i \leqslant n$. Let $i < j$ and suppose there is an edge e in H such that v_j is in e but v_i is not in e. Then there is a realization H' of π such that $e - v_j + v_i$ is an edge in H'.*

Proof. If $e - v_j + v_i$ is already an edge in H, we are done. So we can assume this edge does not exist. Since $d_i \geqslant d_j$, there is an edge f such that v_i is in f but v_j is not. Additionally, some such f has the property that $f - v_i + v_j$ is not an edge in H. Perform the exchange $e \underset{v_i}{\overset{v_j}{\rightleftharpoons}} f$. This does not create any multi-edges, so we have the desired realization. $\square$

An immediate corollary of this result is that for any vertex v of positive degree, there is a k-realization of π such that v is in an edge with the $k-1$ remaining vertices of highest degree. Thus, there is always a realization of π in which the k vertices of highest degree are in a single edge. If we could prove the existence of a k-realization in which the link of a vertex contains only the highest degree vertices, then we would be able to obtain a Havel-Hakimi-type characterization of k-graphic sequences.

3.3.2 Packing k-graphic sequences

Two n-vertex graphs G_1 and G_2 *pack* if they can be expressed as edge-disjoint subgraphs of the complete graph K_n. Kostochka, Stocker, and Hamburger [24], and Pilśniak and Woźniak [26, 27] recently studied packing of hypergraphs. Busch et al. [8] extended the idea of graph packing to graphic sequences. We utilize edge exchanges to examine related questions for hypergraphic sequences.

Let π_1 and π_2 be k-graphic sequences with $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$. We say that π_1 and π_2 *pack* if there exist edge-disjoint k-graphs G_1 and G_2 on vertex set $\{v_1, \ldots, v_n\}$ such that $d_{G_1}(v_i) = d_i^{(1)}$ and $d_{G_2}(v_i) = d_i^{(2)}$ for all i. When we discuss packing of graphic sequences, the sequences need not be nonincreasing; however, no reordering of the indices is allowed.

Dürr, Guiñez, and Matamala [13] showed that the problem of packing two graphic sequences is NP-complete, and we show that the same conclusion holds when considering k-graphic sequences for $k \geqslant 3$.

Theorem 3.5. *The degree-sequence packing problem for k-graphs is NP-complete for all $k \geqslant 2$.*

Proof. The degree-sequence packing problem for $k \geqslant 2$ is in NP since the certificate giving realizations that pack can easily be checked in polynomial time. NP-hardness for $k = 2$ is shown in [13]. For $k \geqslant 3$ we show that any instance of the degree-sequence packing problem for 2-graphs can be reduced to an instance of the degree-sequence packing problem for k-graphs. Given 2-graphic sequences π_1 and π_2, add $k-2$ new entries to each sequence to create sequences π_1^k and π_2^k, with each new entry of π_i^k equal to $\frac{1}{2}\sigma(\pi_i)$. Then, any k-realization of π_i^k has the same number of edges as a 2-realization of π_i, and each of the $k-2$ vertices associated with the new entries must appear in every edge. Hence there is a one-to-one correspondence between 2-realizations of the original sequences and k-realizations of the new sequences. $\square$

Given the computational complexity of the overarching problem, it is natural to seek sufficient conditions that ensure a pair of k-graphic sequences pack. Busch et al. showed that if π_1 and π_2 are graphic sequences and $\Delta \leqslant \sqrt{2\delta n} - (\delta - 1)$, where Δ and δ are the largest and smallest entries in $\pi_1 + \pi_2$, then π_1 and π_2 pack. We prove a similar result for k-graphic sequences when $k \geqslant 3$.

For a vertex v in a k-graph H, we define the neighborhood of v, $N_H(v)$, to be the set of vertices that are in at least one edge with v. Similarly, for a set $S = \{v_1, \ldots, v_m\}$ of vertices in H, the neighborhood of S is $N_H(S) = \cup_{i=1}^m N_H(v_i)$. When H is understood, we write $N(v)$. Also, let $H[S]$ denote the subgraph of H induced by the vertices in S.

Theorem 3.6. *Fix an integer $k \geqslant 2$. There exist constants c_1, c_2 (depending only on k) such that if π_1 and π_2 are k-graphic sequences each with length n that satisfy*

$$n > c_1 \frac{\Delta^{k/(k-1)}}{\delta} + c_2 \Delta, \tag{2}$$

where Δ and δ are the maximum and minimum entries of $\pi_1 + \pi_2$, then π_1 and π_2 pack.

Proof. Among all k-realizations of π_1 and π_2, let H_1 and H_2 be k-realizations such that the number of double edges in $H_1 \cup H_2$ is minimized. We may assume that $H_1 \cup H_2$ has at least one multiple edge, lest H_1 and H_2 give rise to a packing. Let $H = H_1 \cup H_2$, $e = \{v_1, \ldots, v_k\}$ be a double edge in H, and $I = V(H) \setminus \bigcup_{i=1}^k N_H(v_i)$. Taking $c_2 > k^2 - k$, inequality (2) implies that $I \neq \emptyset$. Let $Q = N_H(I)$.

If there is some edge f that contains more than one vertex of I, say i_1 and i_2, then the 2-switch $e \underset{i_1}{\overset{v_1}{\rightleftharpoons}} f$ reduces the number of double edges, contradicting the choice of H_1 and H_2. Hence, each edge including a vertex of I consists of that vertex and $k-1$ vertices of Q.

Let $Q_i = N_{H_i}(I)$ for $i \in \{1, 2\}$. Suppose Q_1 is not a clique in H. That is, let $A = \{y_1, \ldots, y_k\}$ be a set of vertices in Q_1 that is not an edge in H. Since each y_j is in Q_1, for each j with $1 \leqslant j \leqslant k$ there is an edge $f_j \in H_1$ that contains both y_j and some vertex of I. Let $\mathcal{E} = \{f_1, \ldots, f_k\}$ be a set of such edges in H_1, where it is possible that some f_j's are equal. Now we can repeatedly perform 2-switches of the form $e \underset{y_j}{\overset{v_j}{\rightleftharpoons}} f_j$ until one copy of e is replaced by the new edge $\{y_1, \ldots, y_k\}$, in the following way. First, do the exchange $S_1 = e \underset{y_1}{\overset{v_1}{\rightleftharpoons}} f_1$ to obtain edges $e_1 = e - v_1 + y_1$ and $f_1' = f_1 - y_1 + v_1$. The edge e_1 may already exist in H, but it will be removed in the next step. The edge f_1' cannot exist in H, as it contains both a vertex of e and a vertex of I. Having performed edge exchanges S_1 through S_j, the next exchange is $S_{j+1} = e_{j+1} \underset{y_{j+1}}{\overset{v_{j+1}}{\rightleftharpoons}} f_{j+1}$, unless $f_{j+1} = f_p$ for some $p \leqslant j$. In that case, $f_{j+1} = f_p$ is no longer an edge, but has been transformed into the edge $f_p' = f_p - y_p + v_p$. Then $S_{j+1} = e_{j+1} \underset{y_{j+1}}{\overset{v_{j+1}}{\rightleftharpoons}} f_p'$, and the new edges created in this exchange are $e_{j+1} = e_j - v_j + y_j$ and $f_j' = f_p' - y_j + v_j$. After the k^{th} iteration of this process, we have created the edge consisting of the vertices in A, and removed one of the copies of e, while no new double edges have been created. Since this contradicts our choice of H_1 and H_2, the vertices of A must already form an edge, so Q_1 is a clique. The same argument shows that Q_2 is a clique.

Let $v_i \in e$ and $x \in Q$, and suppose that $e - v_i + x$ is not an edge in H. Let f be an edge containing x and a vertex of I. Then the switch $e \underset{x}{\overset{v_i}{\rightleftharpoons}} f$ reduces the number of double edges in H. Hence every vertex of Q is in an edge with each of the $(k-1)$-subsets of e.

Let $q = |Q|$ and $r = |E(H[Q])|$. Since Q_1 and Q_2 are cliques, $r \geqslant 2\binom{q/2}{k}$. Counting the degrees of vertices in Q, we have

$$\begin{aligned} \Delta q &\geqslant kq + (k-1)\delta|I| + kr \\ &\geqslant kq + (k-1)\delta|I| + 2k\binom{q/2}{k}. \end{aligned}$$

Rearranging gives

$$|I| \leqslant \frac{(\Delta - k)q - 2k\binom{q/2}{k}}{(k-1)\delta}. \tag{3}$$

By the principle of inclusion-exclusion, we also know that

$$\begin{aligned} |I| &= n - \left|\bigcup_{i=1}^{k} N_H(v_i)\right| \qquad (4) \\ &= n + \sum_{s=1}^{k}(-1)^s \sum_{\substack{B\subseteq e\\ |B|=s}} \left|\bigcap_{v\in B} N_H(v)\right|. \end{aligned}$$

For any subset B of e, we have that all of Q and $e \setminus B$ are in the common neighborhood of B in H; thus

$$q + k - |B| \leqslant \left|\bigcap_{v\in B} N_H(v)\right|.$$

On the other hand, the size of this common neighborhood is maximized when all vertices in B have the same neighborhood; hence

$$\left|\bigcap_{v\in B} N_H(v)\right| \leqslant (k-1)(\Delta-2) + k - |B|.$$

Using these inequalities in (4), we have

$$\begin{aligned} |I| &\geqslant n - \sum_{s \text{ odd}} \binom{k}{s}\left((k-1)(\Delta-2)+k-s\right) + \sum_{s \text{ even}} \binom{k}{s}(q+k-s) \\ &= n + \sum_{s=1}^{k}(-1)^s(k-s)\binom{k}{s} - (k-1)(\Delta-2)\sum_{s \text{ odd}}\binom{k}{s} + q\sum_{s \text{ even}}\binom{k}{s}. \end{aligned}$$

Applying the binomial theorem, this becomes

$$\begin{aligned} |I| &\geqslant n - k - (\Delta-2)(k-1)\left(2^{k-1}\right) + q\left(2^{k-1}-1\right) \qquad (5) \\ &= n - \Delta(k-1)\left(2^{k-1}\right) + q\left(2^{k-1}-1\right) + (k-1)\left(2^{k}-1\right) - 1. \end{aligned}$$

Combining equations (3) and (5) yields

$$\begin{aligned} &(k-1)\delta\Big(n - \Delta(k-1)\left(2^{k-1}\right) + (k-1)\left(2^{k}-1\right) - 1\Big) \\ &\qquad \leqslant (\Delta-k)q - (k-1)\delta\left(2^{k-1}-1\right)q - 2k\binom{q/2}{k} \qquad (6) \\ &\qquad \leqslant \Delta q - 2k\frac{q^k}{(2k)^k}. \end{aligned}$$

Without loss of generality, suppose $|Q_1| \geqslant |Q_2|$, and let $q_1 = |Q_1|$. Since Q_1 is a clique, $\binom{q_1-1}{k-1} \leqslant \Delta$, so $q_1 \leqslant c'\Delta^{1/(k-1)}$ for some constant c' depending only on k. Then, since $Q = Q_1 \cup Q_2$, we have $q \leqslant 2q_1 \leqslant 2c'\Delta^{1/(k-1)} = c\Delta^{1/(1-k)}$. Inequality (6) now becomes

$$n \leqslant \left(c - \frac{c^k}{(2k)^{k-1}}\right)\frac{\Delta^{k/(k-1)}}{\delta} + \left((k-1)2^{k-1}\right)\Delta.$$

This establishes the theorem, with $c_1 = \left(c - \frac{c^k}{(2k)^{k-1}}\right)$ and $c_2 = \left((k-1)2^{k-1}\right)$. $\square$

When $\delta = o\left(\Delta^{1/(k-1)}\right)$, the bound in Theorem 3.6 reduces to

$$n > c\frac{\Delta^{k/(k-1)}}{\delta}$$

for $c = c_1 + c_2$. We show that for δ in this range, Theorem 3.6 is best possible up to the choice of c.

Fix k and δ and choose an integer $x \gg \delta$ such that $\frac{x-k}{\delta(k-1)}$ is an integer. Form a complete k-graph on x vertices; set aside k of these vertices to form the set B, and let T be the set of remaining vertices. Add an independent set I of order

$$\frac{(x-k)}{\rho(k-1)\delta}\binom{x-1}{k-1},$$

where $\rho > 1$ is chosen such that $\frac{1}{\rho}\binom{x-1}{k-1}$ is an integer. Partition T into sets $T_1, \ldots, T_r$, each of size $\delta(k-1)$, where $r = \frac{x-k}{\delta(k-1)}$, and partition I into sets $I_1, \ldots, I_r$ of size $\frac{1}{\rho}\binom{x-1}{k-1}$. For each vertex $v \in I_j$, create edges $e_1, \ldots, e_\delta$, where each edge consists of v and $k-1$ distinct vertices of T_j. Thus, $N(v) = T_j$ and each vertex in T_j is in exactly one edge with each vertex of I_j. Finally, add an independent set of size $x - k + |I|$.

We now have a k-graph H where each vertex in T has degree

$$\binom{x-1}{k-1} + \frac{1}{\rho}\binom{x-1}{k-1} = \left(1 + \frac{1}{\rho}\right)\binom{x-1}{k-1},$$

each vertex in B has degree $\binom{x-1}{k-1}$, and each vertex in I has degree δ.

Consider two orderings of the degree sequence of H:

$$\pi_1 = \left(\binom{x-1}{k-1}^k, \left(\left(1+\frac{1}{\rho}\right)\binom{x-1}{k-1}\right)^{x-k}, 0^{x-k}, \delta^{|I|}, 0^{|I|}\right)$$
$$\pi_2 = \left(\binom{x-1}{k-1}^k, 0^{x-k}, \left(\left(1+\frac{1}{\rho}\right)\binom{x-1}{k-1}\right)^{x-k}, 0^{|I|}, \delta^{|I|}\right).$$

Note that n, the length of sequences π_1 and π_2, is

$$\begin{aligned} n &= 2x - k + 2|I| \\ &= 2x - k + \frac{2(x-k)}{\rho(k-1)\delta}\binom{x-1}{k-1} \\ &= \Theta(x^k/\delta). \end{aligned}$$

In $\pi_1 + \pi_2$ the minimum degree is δ and the maximum degree is $\Delta = 2\binom{x-1}{k-1} = \Theta(x^{k-1})$. Hence $\Delta = \Theta((\delta n)^{(k-1)/k})$.

In any realization of π_1, Lemma 2.5 implies that the vertices of degree greater than δ must form a clique. Since the k vertices of B must be in this clique, those vertices must form an edge in any realization of π_1. The same argument applies to π_2, hence the sequences do not pack.

References

[1] N. Achuthan, N. Achuthan, and M. Simanihuruk. On 3-uniform hypergraphic sequences. *J. Combin. Math. Combin. Comput.*, 14:3–13, 1993.

[2] M. Aigner and E. Triesch. Realizability and uniqueness in graphs. *Discrete Math.*, 136(1-3):3–20, 1994. Trends in discrete mathematics.

[3] M. Barrus, S. Hartke, K. Jao, and D. West. Length thresholds for graphic lists given fixed largest and smallest entries and bounded gaps. *Discrete Math.*, 312(9):1494–1501, 2012.

[4] M. Barrus, S. Hartke, and M. Kumbhat. Graph classes characterized both by forbidden subgraphs and degree sequences. *J. Graph Theory*, 57(2):131–148, 2008.

[5] D. Bauer, H. Broersma, J. van den Heuvel, N. Kahl, and E. Schmeichel. Toughness and vertex degrees. *J. Graph Theory*, 2012. doi: 10.1002/jgt.21639.

[6] N. Bhave, B. Bam, and C. Deshpande. A characterization of degree sequences of linear hypergraphs. *J. Indian Math. Soc. (N.S.)*, 75(1-4):151–160 (2009), 2008.

[7] D. Billington. Conditions for degree sequences to be realizable by 3-uniform hypergraphs. *J. Combin. Math. Combin. Comput.*, 3:71–91, 1988.

[8] A. Busch, M. Ferrara, S. Hartke, M. Jacobson, H. Kaul, and D. West. Packing of graphic n-tuples. *J. Graph Theory*, 70(1):29–39, 2012.

[9] Y. C. Chen. A short proof of Kundu's k-factor theorem. *Discrete Math.*, 71(2):177–179, 1988.

[10] S. Choudum. On graphic and 3-hypergraphic sequences. *Discrete Math.*, 87(1):91–95, 1991.

[11] C. Colbourn, W. Kocay, and D. Stinson. Some NP-complete problems for hypergraph degree sequences. *Discrete Appl. Math.*, 14(3):239–254, 1986.

[12] A. Dewdney. Degree sequences in complexes and hypergraphs. *Proc. Amer. Math. Soc.*, 53(2):535–540, 1975.

[13] C. Dürr, F. Guiñez, and M. Matamala. Reconstructing 3-colored grids from horizontal and vertical projections is NP-hard. In *Algorithms—ESA 2009*, volume 5757 of *Lecture Notes in Comput. Sci.*, pages 776–787. Springer, Berlin, 2009.

[14] A. Duval and V. Reiner. Shifted simplicial complexes are Laplacian integral. *Trans. Amer. Math. Soc.*, 354(11):4313–4344 (electronic), 2002.

[15] P. Erdős and T. Gallai. Graphs with prescribed degrees of vertices (hungarian). *Mat. Lapok*, 11:264–274, 1960.

[16] M. Ferrara, T. LeSaulnier, C. Moffatt, and P. Wenger. On the sum necessary to ensure a degree sequence is potentially H-graphic. *Arxiv preprint arXiv:1203.4611*, 2012.

[17] D. Fulkerson, A. Hoffman, and M. McAndrew. Some properties of graphs with multiple edges. *Canad. J. Math.*, 17:166–177, 1965.

[18] I. Gabelman. *The functional behavior of majority (threshold) elements.* PhD thesis, Syracuse U, 1961.

[19] S. Hakimi. On realizability of a set of integers as degrees of the vertices of a linear graph. I. *J. Soc. Indust. Appl. Math.*, 10:496–506, 1962.

[20] V. Havel. A remark on the existence of finite graphs (czech). *Časopis Pěst. Mat.*, 80:477–480, 1955.

[21] H. Hoogeveen and G. Sierksma. Seven criteria for integer sequences being graphic. *J. Graph Theory*, 15:223–231, 1991.

[22] C. Klivans and V. Reiner. Shifted set families, degree sequences, and plethysm. *Electron. J. Combin.*, 15(1):Research Paper 14, 35, 2008.

[23] W. Kocay and P. C. Li. On 3-hypergraphs with equal degree sequences. *Ars Combin.*, 82:145–157, 2007.

[24] A. Kostochka, C. Stocker, and P. Hamburger. A hypergraph version of a graph packing theorem by bollobás and eldridge. *J. Graph Theory*, 2012. doi: 10.1002/jgt.21706.

[25] J. Petersen. Die Theorie der regularen Graphen, *Acta Math.* **15** (1891),193-220.

[26] M. Pilśniak and M. Woźniak. A note on packing of two copies of a hypergraph. *Discuss. Math. Graph Theory*, 27(1):45–49, 2007.

[27] M. Pilśniak and M. Woźniak. On packing of two copies of a hypergraph. *Discrete Math. Theor. Comput. Sci.*, 13(3):67–74, 2011.

[28] D. West. *Introduction to Graph Theory.* Prentice Hall, Inc., Upper Saddle River, NJ, second edition, 2001.

[29] J.-H. Yin and J.-S. Li. Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size. *Disc. Math.*, 301:218–227, 2005.

[30] I. Zverovich and V. Zverovich. Contributions to the theory of graphic sequences. *Discrete Math.*, 105(1-3):293–303, 1992.