New results on group classification of nonlinear diffusion-convection equations

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Abstract

Using a new method and transformations of conditional equivalence, we carry out group classification in a class of variable coefficient $(1+1)$-dimensional nonlinear diffusion-convection equations of the general form $f(x)u_t = (D(u)u_x)_x + K(u)u_x$. We obtain new interesting cases of such equations with localized density f, having large invariance algebra. Examples of Lie ansätze and exact solutions of these equations are constructed.

1 Introduction

Solving problems of group classification is interesting not only from mathematical point of view, it also has own applied importance. In physical models there often exist a priori requirements to symmetry groups which follow from physical laws (in particular, from Galilei or relativistic theory). Moreover, modelling differential equations could contain parameters or functions which have been found experimentally and so are not strictly fixed. (It is said that these parameters and functions are arbitrary elements.) At the same time mathematical models have to be enough simple for the aim to analyze and solve them. Solving problems of group classification makes possible to accept for the criterion of applicability the following statement. Modeling differential equations have to admit a group with certain properties or the most abundant symmetry group from possible ones.

In this paper we consider a class of variable coefficient nonlinear diffusion-convection equations of the form

$$f(x)u_t = (g(x)D(u)u_x)_x + K(u)u_x, \tag{1}$$

where $f(x)$, $g(x)$, $D(u)$, $K(u)$ are arbitrary smooth functions of their variables, $f(x)g(x)D(u)\neq 0$. The linear case of (1) ($D, K = \mathrm{const}$) was studied by S. Lie [1] in his classification of linear second-order PDEs with two independent variables. (See also modern treatment on this subject in [2].) That is why we assume below $(D_u, K_u) \neq (0, 0)$, i.e. (1) is a nonlinear equation.

Moreover, using the transformations $\tilde t = t$, $\tilde x = \int g(x)\,dx$, $\tilde u = u$, equation (1) can be reduced to

$$\tilde f(\tilde x)\tilde u_{\tilde t} = (D(u)\tilde u_{\tilde x})_{\tilde x} + K(u)\tilde u_{\tilde x},$$

where $\tilde f(\tilde x) = g(x)f(x)$ and $\tilde g(\tilde x) = 1$. (In an analogous way any equation of form (1) can be reduced to one with $\tilde f(\tilde x) = 1$.) That is why without loss of generality we restrict ourselves to investigation of the equation

$$f(x)u_t = (D(u)u_x)_x + K(u)u_x. \tag{2}$$

Apart from their own theoretical interest equations (2) are used to model a wide variety of phenomena in physics, chemistry, mathematical biology etc. For the case $f(x) = 1$ equation (2) describes the vertical one-dimensional transport of water in homogeneous non-deformable porous media. When $K(u) = 0$ this equation describes stationary motion of a boundary layer of fluid over a flat plate, an eddy of incompressible fluid in porous medium for polytropic relations of gas density and pressure. The outstanding representative of the class of equations (2) is the Burgers equation which is a mathematical model of a great number of physical phenomena. (For more detail refer to [3]–[7].)

Investigation of the nonlinear heat equations using symmetry methods started in 1959 with Ovsiannikov's work [8] where he studied symmetries of equation

$$u_t = (f(u)u_x)_x. \tag{3}$$

In 1987 I.Sh. Akhatov, R.K. Gazizov and N.Kh. Ibragimov [9] classified the equations

$$u_t = G(u_x)u_{xx}. \tag{4}$$

V.A. Dorodnitsyn (1982, [10]) carried out group classification of the equation

$$u_t = (G(u)u_x)_x + g(u). \tag{5}$$

A. Oron, P. Rosenau (1986, [11]) and M.P. Edwards (1994, [12]) presented the most extensive list of symmetries of the equations

$$u_t = (G(u)u_x)_x + f(u)u_x. \tag{6}$$

The results of [8, 10, 11] were generalized by R.M. Cherniha and M.I. Serov (1998, [13]) who classified the nonlinear heat equation with convection term

$$u_t = (G(u)u_x)_x + f(u)u_x + g(u). \tag{7}$$

S.K. El-labany, A.M. Elhanbaly and R. Sabry (2002, [3]) considered some symmetry properties of equation (1).

It should be noted that equations (1)–(7) are particular cases of the more general class of equations

$$u_t = F(t, x, u, u_x)u_{xx} + G(t, x, u, u_x). \tag{8}$$

Group classification of (8) is adduced in [17]. However, since the equivalence group of (8) is essentially wider than ones for (1)–(7) the results of [17] cannot be directly used to symmetry classification of equations (1)–(7). Nevertheless, results of [17] are useful to find additional equivalence transformations in the above classes.

Equations of form (2) have been also investigated with different from classic Lie symmetry points of view. So, potential symmetries of subclasses of (2) where e.g. either $f = 1$ or $K = 0$ were studied by C. Sophocleous [14, 15, 16].

Inspired by the recent work [3] we decided to continue the investigation of Lie symmetries. We carried out the complete group classification, found conditional equivalence transformations and exact solutions of equations (2). We obtained a lot of new interesting cases of extensions of maximal Lie symmetry group for these equations. For example, we determined equations which have the density f localized in the space of x and are invariant with respect to four-dimensional

Lie symmetry algebras. We carried out the reduction on these operators and found some exact solutions.

Our paper is built as follows. First of all (Section 2) we describe the method used here. Then (Sections 3) we significantly enhance the results of [3] and give the complete group classification of class (2). Since the case $f(x) = 1$ has a great variety of applications and was investigated by a number of authors we collect results for this class together in Section 4. Section 5 contains the proof of the main theorem on group classification of class (2). The conditional equivalence transformations are investigated in Section 6. The results of group classification are used to find exact solutions of equations from the class (2) (Section 7).

2 Description of method

Let us describe the classical algorithm of group classification restricting, for simplicity, to the case of one differential equation of the general form

$$L^\theta(x, u_{(n)}) = L(x, u_{(n)}, \theta_{(p)}(x, u_{(n)})) = 0. \tag{9}$$

Here $x = (x_1, \dots, x_l)$ denotes independent variables, u is a dependent variable, $u_{(n)}$ is the set of all the derivatives of the function u with respect to x, which have order no greater than n, including u as the derivative of zero order. L is a fixed function of x, $u_{(n)}$ and $\theta_{(p)}$. $\theta_{(p)}$ denotes the set of all the derivatives of order no greater than p of arbitrary (parametric) functions $\theta(x, u_{(n)}) = (\theta^1(x, u_{(n)}), \dots, \theta^k(x, u_{(n)}))$ satisfying the conditions

$$S(x, u_{(n)}, \theta_{(q)}(x, u_{(n)})) = 0, \quad S = (S_1, \dots, S_r). \tag{10}$$

These conditions are formed by r differential equations on θ where x and $u_{(n)}$ play the role of independent variables. It was follows we call the functions $\theta(x, u_{(n)})$ as arbitrary elements. Denote class of equations of form (9) with the arbitrary elements θ satisfying constraint (10) as $L|_S$.

Let the functions θ be fixed. Each one-parametric group of local point transformations which keeps equation (9) invariant corresponds to an infinitesimal symmetry operator of the form

$$Q = \xi^a(x, u)\partial_{x_a} + \eta(x, u)\partial_u.$$

(here the summation over the repeated indices understood). The complete set of such groups generates the principal group $G^{\max} = G^{\max}(L, \theta)$ of equation (9). The principal group $G^{\max}$ has a corresponding Lie algebra $A^{\max} = A^{\max}(L, \theta)$ of infinitesimal symmetry operators of equation (9). The kernel of principal groups is the group

$$G^{\ker} = G^{\ker}(L, S) = \bigcap_{\theta: S(\theta)=0} G^{\max}(L, \theta)$$

for which the Lie algebra is

$$A^{\ker} = A^{\ker}(L, S) = \bigcap_{\theta: S(\theta)=0} A^{\max}(L, \theta).$$

Let $G^{\text{equiv}} = G^{\text{equiv}}(L, S)$ denote the local transformations group keeping the form of equations from $L|_S$.

The problem of group classification is to find all possible inequivalent cases of extensions of $A^{\max}$, i.e. to list all G^{equiv}-inequivalent values of θ which satisfy equation (10) and the condition $A^{\max}(L,\theta) \neq A^{\ker}$.

In the approach used here group classification is implementation of the following algorithm [2, 18]:

1. From the infinitesimal Lie invariance criterion we find the system of determining equations on the coefficients of Q. It is possible some of the determining equations does not contain arbitrary elements and therefore can be integrated at ones. Others (i.e. equations containing arbitrary elements explicitly) are called classifying equations. The main difficulty of group classification is the need to solve classifying equations with respect to coefficients of the operator Q and arbitrary elements simultaneously.

2. The next step involves finding the kernel algebra $A^{\ker}$ of principal groups of equations from $L|_S$. After decomposing the determining equations with respect to all the unconstrained derivatives of arbitrary elements one obtains a system of partial differential equations only for coefficients of the infinitesimal operator Q. Solving this system yields the algebra $A^{\ker}$.

3. In order to construct the equivalence group G^{equiv} of the class $L|_S$ we have to investigate the local symmetry transformations of system (9), (10), considering it as a system of partial differential equations with respect to θ with the independent variables x, $u_{(n)}$. Usually one considers only trasformations being projectible on the space of the variables x and u. Although in the case θ depending, at most, on these variables it can be assumed the transformations of them depend on θ too. After restricting ourselves to studying the connected component of unity in G^{equiv}, we can use the Lie infinitesimal method. To find the complete equivalence group (including discrete trasformations) we are supposed to use the more complicated direct method.

4. If $A^{\max}$ is an extension of $A^{\ker}$ (i.e. when $A^{\max}(L,\theta) \neq A^{\ker}$) then the classifying equations define a system of nontrivial equations on θ. Depending on their form and number we obtain different cases of extensions of $A^{\ker}$. To integrate completely the determining equations we have to investigate a large number of such cases. In order to avoid cumbersome enumeration of possibilities in solving the determining equations we use a method which involves compatibility investigation of the classifying equations [19]–[22].

The results of applying of above algorithm is a list of equations with their Lie invariance algebras. The problem of group classification is assumed to be completely solved if

i) the list contains all the possible inequivalent cases of extensions;

ii) all the equations from the list are mutually inequivalent with respect to the transformations from G^{equiv};

iii) the obtained algebras are the maximal invariance algebras of their equations.

In the list there can exist equations being mutually equivalent with respect to local transformations which do not belong to G^{equiv}. Knowledge of such additional equivalences allows to simplify further investigation of $L|_S$ essentially. Constructing of them can be considered as the fifth step of the algorithm of group classification. Then, the above enumeration of requirements to the resulting list of classification can be completed the following one:

iv) all the possible additional equivalences between the listed equations are constructed in explicit form.

One from the ways to find additional equivalences is based on the fact that equivalent equations have equivalent maximal invariance algebras. The second way is to study conditional equivalence transformations in the class $L|_S$. Let us give their definition. Consider a system

$$S'(x, u_{(n)}, \theta_{(q')}(x, u_{(n)})) = 0, \quad S' = (S'_1, \ldots, S'_{r'}), \tag{11}$$

formed by r' differential equations on θ with x and $u_{(n)}$ as independent variables. Let $G^{\rm equiv}(L, (S, S'))$ denote the equivalence group of the subclass $L|_{S,S'}$ of $L|_S$, where the functions θ satisfy systems (10) and (11) simultaneously.

Notion 1. *We call the transformations from $G^{\rm equiv}(L, (S, S'))$ as (strong) conditional equivalence transformations of class $L|_S$. The local transformations which transform equations from $L|_{S,S'}$ to $L|_S$ are called weak conditional equivalence transformations of class $L|_S$.*

3 Results of classification

Consider a one-parameter Lie group of local transformations in (t, x, u) with an infinitesimal operator of the form

$$Q = \xi^t(t, x, u)\partial_t + \xi^x(t, x, u)\partial_x + \eta(t, x, u)\partial_u,$$

which keeps equation (2) invariant. The Lie criteria of infinitesimal invariance yields the following determining equations for ξ^t, ξ^x and η

$$\begin{aligned}
&\xi^t_x = \xi^t_u = \xi^x_u = 0,\\
&D\eta_{uu} + D_u\eta_u - D_u(2\xi^x_x - \xi^t_t) + D_{uu}\eta - \frac{f_x}{f}D_u\xi^x = 0,\\
&2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = \frac{D_u}{D}\eta,\\
&f\eta_t - K\eta_x - D\eta_{xx} = 0,\\
&K\left(\frac{f_x}{f}\xi^x + \xi^x_x - \xi^t_t\right) + D(\xi^x_{xx} - 2\eta_{xu}) - 2D_u\eta_x - K_u\eta - f\xi^x_t = 0.
\end{aligned} \tag{12}$$

Investigating the compatibility of system (12) we obtain an additional equation $\eta_{uu} = 0$ without arbitrary elements. Taking into account this equation, system (12) can be rewritten in the form

$$\xi^t_x = \xi^t_u = \xi^x_u = 0, \quad \eta_{uu} = 0, \tag{13}$$

$$2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = \frac{D_u}{D}\eta, \tag{14}$$

$$D\eta_{xx} + K\eta_x - f\eta_t = 0, \tag{15}$$

$$(D_uK - K_uD)\frac{\eta}{D} - K\xi^x_x - 2D_u\eta_x + D\xi^x_{xx} - f\xi^x_t - 2D\eta_{xu} = 0. \tag{16}$$

Integration of equations (13) not containing arbitrary elements results in

$$\xi^t = \xi^t(t), \quad \xi^x = \xi^x(t, x), \quad \eta = \eta^1(t, x)u + \eta^0(t, x). \tag{17}$$

Thus, group classification of (2) reduces to solving classifying conditions (14)–(16).

Splitting system (14)–(16) with respect to the arbitrary elements and their non-vanishing derivatives gives the equations $\xi^t_t = 0$, $\xi^x = 0$, $\eta = 0$ on the coefficients of operators from $A^{\rm ker}$ of (2). As a result, the following theorem is true.

Theorem 1. *The Lie algebra of the kernel of principal groups of (2) is* $A^{\ker} = \langle\partial_t\rangle$.

The next step of algorithm of group classification is the finding of equivalence transformations of class (2). To find these transformations, we have to investigate Lie symmetries of system which consists from equation (2) and additional conditions

$$f_t = f_u = 0, \quad D_t = D_x = 0, \quad K_t = K_x = 0.$$

Using the classical Lie approach we find the invariance algebra of the system above, which forms the Lie algebra of $G^{\rm equiv}$ for class (2). Thus we obtain the following statement.

Theorem 2. *The Lie algebra of* $G^{\rm equiv}$ *for class (2) is*

$$A^{\rm equiv} = \langle \partial_t,\ \partial_x,\ \partial_u,\ t\partial_t + f\partial_f,\ x\partial_x - 2f\partial_f - K\partial_K,\ u\partial_u,\ f\partial_f + K\partial_K + D\partial_D\rangle. \tag{18}$$

Therefore, $G^{\rm equiv}$ contains the continuous transformations:

$$\tilde t = te^{\varepsilon_4} + \varepsilon_1,\ \tilde x = xe^{\varepsilon_5} + \varepsilon_2,\ \tilde u = ue^{\varepsilon_6} + \varepsilon_3,$$
$$\tilde f = fe^{\varepsilon_4 - 2\varepsilon_6 + \varepsilon_7},\ \tilde D = De^{\varepsilon_7},\ \tilde K = Ke^{-\varepsilon_5 + \varepsilon_7},$$

where $\varepsilon_1, \dots, \varepsilon_7$ are arbitrary constants. For class (2) there also exists a nontrivial group of discrete equivalence transformations generated by four involutive transformations of changing sign in the sets $\{t, D, K\}$, $\{x, K\}$, $\{u\}$ and $\{f, D, K\}$. It can be proved by the direct method that $G^{\rm equiv}$ coincides with the group generated by the both continuous and discrete above transformations.

Theorem 3. *The complete set of inequivalent with respect to the transformations from* $G^{\rm equiv}$ *equations (2) with* $A^{\max} \neq A^{\ker}$ *is exhausted by cases given in Tables 1-3.*

In Tables 1–3 we list all possible $G^{\rm equiv}$-inequivalent sets of functions $f(x)$, $D(u)$, $K(u)$ and corresponding invariance algebras. Numbers with the same arabic figure correspond to cases which are equivalent with respect to a local equivalence transformation. Explicit formulas for these transformations are adduced after the Tables. Moreover, the cases numbered with different arabic figures are inequivalent with respect to local equivalence transformations. In order to simplify presented results, in the case $f(x) = 1$ we just use the conditional equivalence transformation $\tilde x = x - \varepsilon t$, $\tilde K = K + \varepsilon$ (the other variables are not transformed) from $G_1^{\rm equiv}$ (see Section 4). Other conditional equivalence transformations are considered in Section 6.

Below for convenience we use double numeration T.N of classification cases and local equivalence transformations, where T denotes the number of table and N does the number of case (or transformation) in Table T. The notion "equation T.N" is used for the equation of form (2) where the parameter-functions take values from the corresponding case.

The operators from Tables 1–3 form bases of the maximal invariance algebras iff the corresponding sets of the functions f, D, K are $G^{\rm equiv}$-inequivalent to ones with more abundant invariance algebras. For example, in Case 3.1 $(\mu,\nu) \neq (0,0)$ and $\lambda \neq -1$ if $\nu = 0$. And in Case 3.2 $(\mu,\nu) \notin \{(-2,-2),(0,1)\}$ and $\nu \neq 0$. Similarly, in Case 2.1 the constraint set on the parameters μ, ν and λ coincides with the one for Case 3.1, and we can assume that $\mu = 1$ or $\nu = 1$. In Case 2.2 we consider $\nu = 1$ immediately.

After analyzing the obtained results, we can formulate the following theorem.

Theorem 4. *If an equation of form (2) is invariant with respect to a Lie algebra of dimension no less than 4 then it can be reduced by means of local transformations to one with* $f(x) = 1$.

Table 1: Case $D(u)-\forall$

N	$K(u)$	$f(x)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	∂_t
2a	$\forall$	$e^{\varepsilon x}$	$\partial_t,\ \varepsilon t\partial_t+\partial_x$
2b	D	$e^{-2x+\gamma e^{-x}}$	$\partial_t,\ \gamma t\partial_t-e^x\partial_x$
2c	D	$e^{-2x}(e^{-x}+\gamma)^\nu$	$\partial_t,\ (\nu+2)t\partial_t-(e^{-x}+\gamma)e^x\partial_x$
2d	0	$\|x\|^\nu$	$\partial_t,\ (\nu+2)t\partial_t+x\partial_x$
2e	1	x^{-1}	$\partial_t,\ e^{-t}(\partial_t-x\partial_x)$
3a	0	1	$\partial_t,\ \partial_x,\ 2t\partial_t+x\partial_x$
3b	D	e^{-2x}	$\partial_t,\ 2t\partial_t-\partial_x,\ e^x\partial_x$

Here $\gamma,\nu\neq 0,\ \ \varepsilon=0,1 \bmod G^{\rm equiv},\ \ \gamma=\pm 1\ \bmod G^{\rm equiv}$. Additional equivalence transformations:

1. $2b\to 2a$ ($K=0,\ \varepsilon=1$): $\tilde t=t,\ \tilde x=\gamma e^{-x},\ \tilde u=u$;
2. $2c\to 2a$ ($K=-D/(\nu+2),\ \varepsilon=1$): $\tilde t=t,\ \tilde x=(\nu+2)\ln|e^{-x}+\gamma|,\ \tilde u=u$;
3. $2d\to 2a$ ($K=-D/(\nu+2),\ \varepsilon=1$): $\tilde t=t,\ \tilde x=(\nu+2)\ln|x|,\ \tilde u=u$;
4. $2e\to 2a$ ($K=-D,\ \varepsilon=1$): $\tilde t=e^t,\ \tilde x=\ln|x|+t,\ \tilde u=u$;
5. $3b\to 3a$: $\tilde t=t,\ \tilde x=e^{-x},\ \tilde u=u$.

Table 2: Case $D(u)=e^{\mu u}$

N	μ	$K(u)$	$f(x)$	Basis of $A^{\max}$
1	$\forall$	$e^{\nu u}$	$\|x\|^\lambda$	$\partial_t,\ (\lambda\mu-\lambda\nu+\mu-2\nu)t\partial_t+(\mu-\nu)x\partial_x+\partial_u$
2	$\forall$	e^u	1	$\partial_t,\ \partial_x,\ (\mu-2)t\partial_t+(\mu-1)x\partial_x+\partial_u$
3	1	u	1	$\partial_t,\ \partial_x,\ t\partial_t+(x-t)\partial_x+\partial_u$
4	1	εe^u	$\forall$	$\partial_t,\ t\partial_t-\partial_u$
5a	1	0	$f^1(x)$	$\partial_t,\ t\partial_t-\partial_u,\ \alpha t\partial_t+(\beta x^2+\gamma_1 x+\gamma_0)\partial_x+\beta x\partial_u$
5b	1	e^u	$f^2(x)$	$\partial_t,\ t\partial_t-\partial_u,\ \alpha t\partial_t-(\beta e^{-x}+\gamma_1+\gamma_0 e^x)\partial_x+\beta e^{-x}\partial_u$
5c	1	1	x^{-1}	$\partial_t,\ x\partial_x+\partial_u,\ e^{-t}(\partial_t-x\partial_x)$
6a	1	0	1	$\partial_t,\ t\partial_t-\partial_u,\ 2t\partial_t+x\partial_x,\ \partial_x$
6b	1	e^u	e^{-2x}	$\partial_t,\ t\partial_t-\partial_u,\ 2t\partial_t-\partial_x,\ e^x\partial_x$
6c	1	e^u	$e^{-2x}(e^{-x}+\gamma)^{-3}$	$\partial_t,\ t\partial_t-\partial_u,\ (e^{-x}+\gamma)e^x\partial_x+\partial_u,\ -(e^{-x}+\gamma)^2e^x\partial_x+(e^{-x}+\gamma)\partial_u$
6d	1	0	x^{-3}	$\partial_t,\ t\partial_t-\partial_u,\ x\partial_x-\partial_u,\ x^2\partial_x+x\partial_u$

Here $\lambda\neq 0,\ \ \varepsilon\in\{0,1\} \bmod G^{\rm equiv},\ \ \alpha,\beta,\gamma_1,\gamma_0=\mathrm{const}$ and

$$f^1(x)=\exp\left\{\int\frac{-3\beta x-2\gamma_1+\alpha}{\beta x^2+\gamma_1 x+\gamma_0}\,dx\right\},\qquad f^2(x)=\exp\left\{\int\frac{\beta e^{-x}-2\gamma_0 e^x-\alpha}{\beta e^{-x}+\gamma_1+\gamma_0 e^x}\,dx\right\}.$$

Additional equivalence transformations:

1. $5b\to 5a$: $\tilde t=t,\ \tilde x=e^{-x},\ \tilde u=u$;
2. $5c\to 5a$ ($\alpha=\gamma_0=1,\ \beta=\gamma_1=0,\ f^1=x^{-1}$): $\tilde t=e^t,\ \tilde x=e^t x,\ \tilde u=u$;
3. $6b\to 6a$: $\tilde t=t,\ \tilde x=e^{-x},\ \tilde u=u$;
4. $6c\to 6a$: $\tilde t=t,\ \tilde x=-1/(e^{-x}+\gamma),\ \tilde u=u-\ln|e^{-x}+\gamma|$;
5. $6d\to 6a$: $\tilde t=t,\ \tilde x=-1/x,\ \tilde u=u-\ln|x|$.

Table 3: Case $D(u) = u^\mu$

N	μ	$K(u)$	$f(x)$	Basis of $A^{\max}$
1	$\forall$	u^ν	$\|x\|^\lambda$	$\partial_t,\ (\mu+\lambda\mu-2\nu-\lambda\nu)t\partial_t+(\mu-\nu)x\partial_x+u\partial_u$
2	$\forall$	u^ν	1	$\partial_t,\ \partial_x,\ (\mu-2\nu)t\partial_t+(\mu-\nu)x\partial_x+u\partial_u$
3	$\forall$	$\ln u$	1	$\partial_t,\ \partial_x,\ \mu t\partial_t+(\mu x-t)\partial_x+u\partial_u$
4	$\forall$	εu^μ	$\forall$	$\partial_t,\ \mu t\partial_t-u\partial_u$
5a	$\forall$	0	$f^3(x)$	$\partial_t,\ \mu t\partial_t-u\partial_u,$
				$\alpha t\partial_t+((1+\mu)\beta x^2+\gamma_1 x+\gamma_0)\partial_x+\beta xu\partial_u$
5b	$\forall$	u^μ	$f^4(x)$	$\partial_t,\ \mu t\partial_t-u\partial_u,$
				$\alpha t\partial_t-((1+\mu)\beta e^{-x}+\gamma_1+\gamma_0 e^x)\partial_x+\beta e^{-x}u\partial_u$
5c	$\mu\neq-3/2$	1	x^{-1}	$\partial_t,\ e^{-t}(\partial_t-x\partial_x),\ \mu x\partial_x+u\partial_u$
6a	$\mu\neq-4/3$	0	1	$\partial_t,\ \mu t\partial_t-u\partial_u,\ \partial_x,\ 2t\partial_t+x\partial_x$
6b	$\mu\neq-4/3$	u^μ	e^{-2x}	$\partial_t,\ \mu t\partial_t-u\partial_u,\ 2t\partial_t-\partial_x,\ e^x\partial_x$
6c	-1	0	$e^{\gamma x}$	$\partial_t,\ t\partial_t+u\partial_u,\ \partial_x-\gamma u\partial_u,\ 2t\partial_t+x\partial_x-\gamma xu\partial_u$
6d	-1	u^{-1}	$e^{-2x+\gamma e^{-x}}$	$\partial_t,\ t\partial_t+u\partial_u,\ e^x\partial_x+\gamma u\partial_u,\ 2t\partial_t-\partial_x-\gamma e^{-x}\partial_u$
6e	$\mu\neq-4/3,-1$	0	$\|x\|^{-\frac{4+3\mu}{1+\mu}}$	$\partial_t,\ \mu t\partial_t-u\partial_u,\ (2+\mu)t\partial_t-(1+\mu)x\partial_x,$
				$(1+\mu)x^2\partial_x+xu\partial_u$
6f	$\mu\neq-4/3,-1$	u^μ	$\dfrac{e^{-2x}}{(e^{-x}+\gamma)^{\frac{4+3\mu}{1+\mu}}}$	$\partial_t,\ \mu t\partial_t-u\partial_u,\ (2+\mu)t\partial_t+(1+\mu)(e^{-x}+\gamma)e^x\partial_x,$
				$-(1+\mu)(e^{-x}+\gamma)^2e^x\partial_x+(e^{-x}+\gamma)u\partial_u$
6g	$-3/2$	1	x^{-1}	$\partial_t,\ e^{-t}(\partial_t-x\partial_x),\ 3x\partial_x-2u\partial_u,\ e^t(x^2\partial_x-2xu\partial_u)$
7a	$-4/3$	0	1	$\partial_t,\ 4t\partial_t+3u\partial_u,\ \partial_x,\ 2t\partial_t+x\partial_x,\ x^2\partial_x-3xu\partial_u$
7b	$-4/3$	$u^{-4/3}$	e^{-2x}	$\partial_t,\ 4t\partial_t+3u\partial_u,\ 2t\partial_t-\partial_x,\ e^{-x}(\partial_x+3u\partial_u),\ e^x\partial_x$
8	0	u	1	$\partial_t,\ \partial_x,\ 2t\partial_t+x\partial_x-u\partial_u,\ t\partial_x-\partial_u,$
				$t^2\partial_t+tx\partial_x-(tu+x)\partial_u$

Here $\varepsilon = 0, 1 \bmod G^{\text{equiv}}$, $\lambda\neq 0$, $\alpha,\beta,\gamma_1,\gamma_0=\text{const}$ and

$$f^3(x)=\exp\left\{\int\frac{-(4+3\mu)\beta x-2\gamma_1+\alpha}{(1+\mu)\beta x^2+\gamma_1 x+\gamma_0}\,dx\right\},\qquad f^4(x)=\exp\left\{\int\frac{(2+\mu)\beta e^{-x}-2\gamma_0 e^x-\alpha}{(1+\mu)\beta e^{-x}+\gamma_1+\gamma_0 e^x}\,dx\right\}.$$

$\mu\neq 0$ for Cases 4–6. Additional equivalence transformations:

1. 5b $\to$ 5a: $\tilde t=t,\ \tilde x=e^{-x},\ \tilde u=u$;
2. 5c $\to$ 5a ($\alpha=\gamma_0=1,\ \beta=\gamma_1=0,\ f^1=x^{-1}$): $\tilde t=e^t,\ \tilde x=e^t x,\ \tilde u=u$;
3. 6b $\to$ 6a: $\tilde t=t,\ \tilde x=e^{-x},\ \tilde u=u$;
4. 6c $\to$ 6a: $\tilde t=t,\ \tilde x=x,\ \tilde u=e^{\gamma x}u$;
5. 6d $\to$ 6a: $\tilde t=t,\ \tilde x=e^{-x},\ \tilde u=e^{\gamma e^{-x}}u$;
6. 6e $\to$ 6a: $\tilde t=t,\ \tilde x=-1/x,\ \tilde u=|x|^{-\frac{1}{1+\mu}}u$;
7. 6f $\to$ 6a: $\tilde t=t,\ \tilde x=-1/(e^{-x}+\gamma),\ \tilde u=|e^{-x}+\gamma|^{-\frac{1}{1+\mu}}u$;
8. 6g $\to$ 6a: $\tilde t=e^t,\ \tilde x=-e^{-t}/x,\ \tilde u=|e^t x|^{-\frac{1}{1+\mu}}u$;
9. 7b $\to$ 7a: $\tilde t=t,\ \tilde x=e^{-x},\ \tilde u=u$.

Table 4: Case $f(x) = 1$.

N	$D(u)$	$K(u)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	$\partial_t,\ \partial_x$
2	$\forall$	0	$\partial_t,\ \partial_x,\ 2t\partial_t + x\partial_x$
3	$e^{\mu u}$	e^u	$\partial_t,\ \partial_x,\ (\mu-2)t\partial_t + (\mu-1)x\partial_x + \partial_u$
4	e^u	u	$\partial_t,\ \partial_x,\ t\partial_t + (x-t)\partial_x + \partial_u$
5	e^u	0	$\partial_t,\ \partial_x,\ t\partial_t - \partial_u, 2t\partial_t + x\partial_x$
6	u^μ	u^ν	$\partial_t,\ \partial_x,\ (\mu-2\nu)t\partial_t + (\mu-\nu)x\partial_x + u\partial_u$
7a	u^μ	0	$\partial_t,\ \partial_x,\ \mu t\partial_t - u\partial_u, 2t\partial_t + x\partial_x$
7b	u^{-2}	u^{-2}	$\partial_t,\ \partial_x,\ 2t\partial_t + u\partial_u, e^{-x}(\partial_x + u\partial_u)$
8	$u^{-4/3}$	0	$\partial_t,\ \partial_x,\ 4t\partial_t + 3u\partial_u, 2t\partial_t + x\partial_x,\ x^2\partial_x - 3xu\partial_u$
9	u^μ	$\ln u$	$\partial_t,\ \partial_x,\ \mu t\partial_t + (\mu x - t)\partial_x + u\partial_u$
10	1	u	$\partial_t,\ \partial_x,\ t^2\partial_t + tx\partial_x - (tu+x)\partial_u,\ 2t\partial_t + x\partial_x - u\partial_u,\ t\partial_x - \partial_u$

Here $\mu, \nu = \text{const.}$ $(\mu, \nu) \neq (-2, -2)$, $(0, 1)$ and $\nu \neq 0$ for N = 6. $\mu \neq -4/3$ for N = 7a. Case 7b can be reduced to 7a by means of the conditional equivalence transformation $\tilde{t} = t$, $\tilde{x} = e^x$, $\tilde{u} = e^{-x}u$.

4 Group classification for subclass with $f(x) = 1$

Class (2) includes a subclass of equations of the general form

$$u_t = (D(u)u_x)_x + K(u)u_x. \tag{19}$$

(i.e. the function f is assumed to be equal to 1 identically). Symmetry properties of equations (19) were studied in [11, 12]. But we do not know any work containing correct and exhaustive investigation on the subject. Now let us single out the results of group classification of equations (19) from the above section.

Theorem 5. *The Lie algebra of the kernel of principal groups of (19) is* $A_1^{\ker} = \langle \partial_t, \partial_x \rangle$.

Theorem 6. *The Lie algebra of* $G_1^{\rm equiv}$ *for the class (19) is*

$$A_1^{\rm equiv} = \langle \partial_t,\ \partial_x,\ \partial_u,\ u\partial_u,\ t\partial_x - \partial_K,\ 2t\partial_t + x\partial_x - K\partial_K,\ t\partial_t - D\partial_D - K\partial_K \rangle. \tag{20}$$

$G_1^{\rm equiv}$ is generated by the transformations:

$$\begin{aligned} &\tilde{t} = t\varepsilon_4^2\varepsilon_5 + \varepsilon_1, \quad \tilde{x} = x\varepsilon_4 + \varepsilon_7 t + \varepsilon_2, \quad \tilde{u} = u\varepsilon_6 + \varepsilon_3, \\ &\tilde{D} = D\varepsilon_5^{-1}, \quad \tilde{K} = K\varepsilon_4^{-1}\varepsilon_5^{-1} - \varepsilon_7, \end{aligned} \tag{21}$$

where $\varepsilon_1, \ldots, \varepsilon_7$ are arbitrary constants, $\varepsilon_4\varepsilon_5\varepsilon_6 \neq 0$.

The complete set of $G_1^{\rm equiv}$-inequivalent cases of extensions $A^{\max}$ of equations (19) is given in Table 4.

5 Proof of Theorem 3

Our method is based on the fact that substitution of the coefficients of any operator from $A^{\max} \backslash A^{\ker}$ into the classifying equations results in nonidentity equations for arbitrary elements. In the problem under consideration, the procedure of looking over the possible cases mostly depends on equation (14). For any operator $Q \in A^{\max}$ equation (14) gives some equations on D of the general form

$$(au + b)D_u = cD, \tag{22}$$

where $a, b, c = \text{const}$. In the whole for all operators from $A^{\max}$ the number k of such independent equations is no greater then 2 otherwise they form an incompatible system on D. k is an invariant value for the transformations from G^{equiv}. Therefore, there exist three inequivalent cases for the value of k: $k = 0$, $k = 1$, $k = 2$. Let us consider these possibilities in more details, ommiting cumbersome calculations.

$\boldsymbol{k = 0}$ (Table 1). Then the coefficients of any operator from $A^{\max}$ are to satisfy the system

$$\eta = 0, \quad 2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = 0, \quad -K\xi^x_x + D\xi^x_{xx} - f\xi^x_t = 0. \tag{23}$$

Let us suppose that $K \notin \langle 1,\ D\rangle$. It follows from the last equation of the system (23) that $\xi^x_x = \xi^x_t = 0$. Therefore, the second equation is a nonidentity equation for f of the form $f_x = \mu f$ without fail. Solving this equation yields Case 2a.

Now let $K \in \langle 1,\ D\rangle$, i.e. $K = \varepsilon D + \beta$ where $\varepsilon \in \{0, 1\}$, $\beta = \text{const}$. Then the last equation of (23) can be decomposed to the following ones

$$\xi^x_{xx} = \varepsilon\xi^x_x, \quad \beta\xi^x_x + f\xi^x_t = 0.$$

The equation $(\xi^x(f_x/f + 2\varepsilon))_x = 0$ is a differential consequence of the reduced determining equations. Therefore, the condition $f_x/f + 2\varepsilon = 0$ is a classifying one.

Suppose this condition is true, i.e. $f = e^{-2\varepsilon x} \mod G^{\text{equiv}}$. There exist three different possibilities for values of the parameters ε and β:

$$\varepsilon = 1,\ \beta \neq 0; \quad \varepsilon = 1,\ \beta = 0; \quad \varepsilon = 0 \text{ (then } \beta = 0 \mod G^{\text{equiv}}_1),$$

which yield Cases 2a, 3b and 3a respectively.

Let $\varepsilon = 0$ and $f_x/f \neq 0$. Then either our consideration is reduced to Case 2a or $f = x^\mu \mod G^{\text{equiv}}$ where $\mu \neq 0$. Depending on value of the parameter β ($\beta = 0$ or $\beta \neq 0$ and then $\mu = -1$) we obtain Case 2d or Case 2e.

Let $\varepsilon = 1$ and $f_x/f \neq -2$. Then $b = 0$ and $f_x/f = (C_1e^x + C_0)^{-1} - 2$ where we assume $C_1 \neq 0$ to exclude Case 2a. Integrating the latter equation depend on vanishing C_0 and results in Cases 2b and 2d.

$\boldsymbol{k = 1}$. Then $D \in \{e^u, u^\mu, \mu \neq 0\} \mod G^{\text{equiv}}$ and there exists $Q \in A^{\max}$ with $\eta \neq 0$.

Let us investigate the first possibility $D = e^u$ (Table 2). Equation (14) implies $\eta_u = 0$, i.e. $\eta = \eta(t, x)$. Therefore, equations (14)–(16) can be wrote as

$$\begin{aligned} &2\xi^x_x - \xi^t_t + \frac{f_x}{f}\xi^x = \eta, \qquad e^u\eta_{xx} + K\eta_x - f\eta_t = 0, \\ &(K - K_u)\eta - K\xi^x_x - f\xi^x_t + e^u(\xi^x_{xx} - 2\eta_x) = 0. \end{aligned} \tag{24}$$

The latter equation looks with respect to K like $K_u = \nu K + be^u + c$, where $\nu, b, c = \text{const}$. Therefore, K is to take one from five values.

1. $K = e^{\nu u} + \varkappa_1 e^u + \varkappa_0 \mod G^{\text{equiv}}$, where $\nu \neq 0, 1$. (Here and below $\varkappa_i = \text{const}$, $i = 0, 1$.) Then $\eta = \text{const}$, $\varkappa_1 = 0$, and either $\varkappa_0 = 0$ if $f \neq \text{const}$ or $\varkappa_0 = 0 \mod G^{\text{equiv}}_1$ if $f = \text{const}$ that imply $\xi^x_t = 0$, $\xi^t_{tt} = 0$, therefore $f = |x|^\lambda \mod G^{\text{equiv}}$ (Cases 1 and 2).

2. $K = u + \varkappa_1 e^u + \varkappa_0$. In the way analogous to the previous case we obtain $\varkappa_1 = 0$, $f = 1 \mod G^{\text{equiv}}$, $\varkappa_0 = 0 \mod G^{\text{equiv}}_1$ (Case 3).

3. $K = ue^u + \varkappa_1 e^u + \varkappa_0 \mod G^{\text{equiv}}$. It follows from system (24) that $\eta = 0$ for any operator from $A^{\max}$, i.e. we have the contradictory with assumption $\eta \neq 0$ for some operator from $A^{\max}$.

4. $K=e^u+\varkappa_0$. Then $\eta^1=\zeta^1(t)e^{-x}+\zeta^0(t)$, $\xi^x=\sigma^1(t)e^x+\sigma^0(t)-\zeta^1(t)e^{-x}$. It can be proved that $\zeta^1_t=\zeta^0_t=\sigma^1_t=\xi^t_{tt}=0$, either $\varkappa_0=0$ if $f\neq\mathrm{const}$ or $\varkappa_0=0 \bmod G^{\rm equiv}_1$ if $f=\mathrm{const}$, and therefore $\sigma^0_t=0$. The first equation of (24) implies that the function f is to satisfy l ($l=0,1,2$) equations of the form

$$\frac{f_x}{f}=\frac{\beta e^{-x}-\alpha-2\gamma_0e^x}{\beta e^{-x}+\gamma_1+\gamma_0e^x}$$

with non-proportional sets of constant parameters $(\alpha,\beta,\gamma_0,\gamma_1)$. The values $l=0$ and $l=1$ correspond to Cases 4 ($\varepsilon=1$) and 5b. $l=2$ and there is an additional extention of $A^{\max}$ in comparison with $l=1$ iff f is a solution of the equation

$$\frac{f_x}{f}=\frac{\lambda_2e^{-x}}{\lambda_1e^{-x}+\lambda_0}-2,$$

where either $\lambda_2=0$ or $\lambda_2=3\lambda_1\neq0$. Integrating the latter equation gives Cases 6b and 6c.

5. $K=\varkappa_0$. Then $\eta^1=\zeta^1(t)x+\zeta^0(t)$, $\xi^x=\sigma^1(t)x+\sigma^0(t)+\zeta^1(t)x^2$. It follows from compatibility of system (24) that $\eta_t=\xi^x_t=0$ if $f\notin\{x^{-1},1\} \bmod G^{\rm equiv}$ or $\varkappa_0=0$. The values $f=x^{-1}$, $\varkappa_0\neq0$ result in Case 5c. If $f\notin\{x^{-1},1\} \bmod G^{\rm equiv}$ and $\varkappa_0=0$, we obtain Case 1 with $\nu=0$. If $f=\mathrm{const}$ then $\varkappa_0=0 \bmod G^{\rm equiv}_1$. Below $\varkappa_0=0$. The first equation of (24) holds that the function f is a solution of a system of l ($l=0,1,2$) equations of the form

$$\frac{f_x}{f}=\frac{-3\beta x+\alpha-2\gamma_1}{\beta x^2+\gamma_1x+\gamma_0}$$

with non-proportional sets of constant parameters $(\alpha,\beta,\gamma_0,\gamma_1)$. The values $l=0$ and $l=1$ correspond to Cases 4 ($\varepsilon=0$) and 5a. Additional extensions for $l=2$ exist iff f is a solution of the equation

$$\frac{f_x}{f}=\frac{\lambda_2}{\lambda_1x+\lambda_0}.$$

where either $\lambda_2=0$ or $\lambda_2=3\lambda_1\neq0$. These possibilities result in Cases 6a and 6d.

Consider the case $D=u^\mu$ (Tables 3). Equation (14) implies $\eta^0=0$, i.e. $\eta=\eta^1(t,x)u$. Therefore, system (14)–(16) can be wrote as

$$\begin{aligned}&2\xi^x_x-\xi^t_t+\frac{f_x}{f}\xi^x=\mu\eta^1, \qquad u^\mu\eta^1_{xx}+K\eta^1_x-f\eta^1_t=0,\\ &(\mu K-uK_u)\eta^1-K\xi^x_x+(\xi^x_{xx}-2(\mu+1)\eta^1_x)u^\mu-f\xi^x_t=0.\end{aligned}\tag{25}$$

The latter equation looks with respect to K like $uK_u=\nu K+bu^\mu+c$, where $\nu,b,c=\mathrm{const}$. Therefore, K is to take one from five values.

1. $K=u^\nu+\varkappa_1u^\mu+\varkappa_0 \bmod G^{\rm equiv}$, where $\nu\neq0,\mu$. Equations (25) imply $\eta^1=\mathrm{const}$, $\xi^x=(\mu-\nu)\eta^1x+\sigma(t)$, $\varkappa_1\xi^x_x=0$ (therefore, $\varkappa_1=0$ since $\eta^1=0$), $f=|x|^\lambda \bmod G^{\rm equiv}$, $\xi^t_t=(\mu+\lambda\mu-2\nu-\lambda\nu)\eta^1$, $\lambda\sigma=0$, and either $\varkappa_0=0$ if $\lambda\neq0$ (Case 1) or $\varkappa_0=0 \bmod G^{\rm equiv}_1$ if $\lambda=0$ (Case 2).

2. $K=\ln u+\varkappa_1u^\mu+\varkappa_0 \bmod G^{\rm equiv}$. In the way analogous to the previous case we obtain $\varkappa_1=0$, $f=1 \bmod G^{\rm equiv}$, $\varkappa_0=0 \bmod G^{\rm equiv}_1$ (Case 3).

3. $K=u^\mu\ln u+\varkappa_1u^\mu+\varkappa_0 \bmod G^{\rm equiv}$. It follows from system (25) that $\eta=0$ for any operator from $A^{\max}$, i.e. we have the contradictory with assumption $\eta\neq0$ for some operator from $A^{\max}$.

4. $K=u^\mu+\varkappa_0 \mod G^{\rm equiv}$. Then $\eta^1=\zeta^1(t)e^{-x}+\zeta^0(t)$, $\xi^x=\sigma^1(t)e^x+\sigma^0(t)-(\mu+1)\zeta^1(t)e^{-x}$. It can be proved that $\zeta^1_t=\zeta^0_t=\sigma^1_t=\xi^t_{tt}=0$, either $\varkappa_0=0$ if $f\neq{\rm const}$ or $\varkappa_0=0 \mod G^{\rm equiv}_1$ if $f={\rm const}$, and therefore $\sigma^0_t=0$. The first equation of (25) implies that the function f is to satisfy l ($l=0,1,2$) equations of the form

$$\frac{f_x}{f}=\frac{(\mu+2)\beta e^{-x}-\alpha-2\gamma_0e^x}{(\mu+1)\beta e^{-x}+\gamma_1+\gamma_0e^x}$$

with non-proportional sets of constant parameters $(\alpha,\beta,\gamma_0,\gamma_1)$. The values $l=0$ and $l=1$ correspond to Cases 4 ($\varepsilon=1$) and 5b. $l=2$ iff f is a solution of the equation

$$\frac{f_x}{f}=\frac{\lambda_2e^{-x}}{\lambda_1e^{-x}+\lambda_0}-2.$$

Looking over the inequivalent possibilities of integrating the latter equation results in Cases 6b, 6d, 6f, 7b.

5. $K=\varkappa_0$. Then $\eta^1=\zeta^1(t)x+\zeta^0(t)$, $\xi^x=\sigma^1(t)x+\sigma^0(t)+(\mu+1)\zeta^1(t)x^2$. It follows from compatibility of system (25) that $\eta_t=\xi^x_t=0$ if $f\not\in\{x^{-1},1\} \mod G^{\rm equiv}$ or $\varkappa_0=0$. The values $f=x^{-1}$, $\varkappa_0\neq0$ result in Cases 5c and 6g. If $f\not\in\{x^{-1},1\} \mod G^{\rm equiv}$ and $\varkappa_0=0$, we obtain Case 1 with $\nu=0$. If $f={\rm const}$ then $\varkappa_0=0 \mod G^{\rm equiv}_1$. Below $\varkappa_0=0$. The first equation of (25) holds that the function f is a solution of a system of l ($l=0,1,2$) equations of the form

$$\frac{f_x}{f}=\frac{-(3\mu+4)\beta x+\alpha-2\gamma_1}{(\mu+1)\beta x^2+\gamma_1x+\gamma_0}$$

with non-proportional sets of constant parameters $(\alpha,\beta,\gamma_0,\gamma_1)$. The values $l=0$ and $l=1$ correspond to Cases 4 ($\varepsilon=0$) and 5a. $l=2$ iff f is a solution of the equation

$$\frac{f_x}{f}=\frac{\lambda_2}{\lambda_1x+\lambda_0}.$$

Looking over the inequivalent possibilities of integrating the latter equation results in Cases 6a, 6c, 6e, 7a.

***k* = 2.** Assumption on two independent equations of form (22) on D yields $D={\rm const}$, i.e. $D=1 \mod G^{\rm equiv}$. $K_u\neq0$ (otherwise, equation (2) is linear). Equations (14)–(16) can be wrote as

$$\begin{gathered}2\xi^x_x-\xi^t_t+\frac{f_x}{f}\xi^x=0,\qquad \eta_{xx}+K\eta_x-f\eta_t=0,\\ -K_u\eta-K\xi^x_x+\xi^x_{xx}-f\xi^x_t-2\eta^1_x=0.\end{gathered}\tag{26}$$

The latter equation looks with respect to K like $(au+b)K_u=cK+d$, where $a,b,c,d={\rm const}$. Therefore, to within transformations from $G^{\rm equiv}$ K is to take one from four values:

$$K=u^\nu+\varkappa_0,\ \nu\neq0,1;\quad K=\ln u+\varkappa_0;\quad K=e^u+\varkappa_0;\quad K=u.$$

Classification for these values is carried out in the way like the above. The obtained extensions can be entered in either Table 2 or Table 3.

The problem of the group classification of equation (2) is completely solved.

Table 5: Conditional equivalence algebras

Conditions	Basis of A^{equiv}
$K=D$	$\partial_t,\ \partial_x,\ \partial_u,\ u\partial_u,\ t\partial_t+f\partial_f,\ e^x(\partial_x-2f\partial_f),\ f\partial_f+D\partial_D$
$K=D=e^u$	$\partial_t,\ t\partial_t+f\partial_f,\ \partial_x,\ \partial_u+f\partial_f,\ x\partial_x-2f\partial_f,\ x^2\partial_x+x\partial_u-3x\partial_f$
$D=e^u$, $K=0$	$\partial_t,\ t\partial_t+f\partial_f,\ \partial_x,\ \partial_u+f\partial_f,\ x\partial_x-2f\partial_f,\ x^2\partial_x+x\partial_u-3x\partial_f$
$D=K=u^\mu$	$\partial_t,\ t\partial_t+f\partial_f,\ \partial_x,\ \partial_u+\mu f\partial_f,\ e^x(\partial_x-2f\partial_f),\ e^{-x}((1+\mu)\partial_x-u\partial_u+(2+\mu)f\partial_f)$
$D=u^\mu$, $K=0$	$\partial_t,\ t\partial_t+f\partial_f,\ \partial_x,\ \partial_u+\mu f\partial_f,\ x\partial_x-2f\partial_f,\ (1+\mu)x^2\partial_x+xu\partial_u-(4+3\mu)xf\partial_f$

6 Conditional equivalence transformations

When we imposed some restrictions on arbitrary elements we can find additional equivalence transformations named conditional equivalence transformations (see Notion 1). A mentioned above, the most simple way to find such equivalences between previously classified equations is based on the fact that equivalent equations have equivalent maximal invariance algebras. A more systematic way is to classify these transformations using the infinitesimal or the direct methods. Examples of conditional equivalence algebras calculated by the infinitesimal method are listed in Table 5.

To find the complete collection of additional local equivalence transformations including both continious and discrete ones, we are to use the direct method. Moreover, application of this method allows us to describe all the local transformations that are possible for pairs of equations from the class under consideration. Now we formulate a number of simple but very useful lemmas. (We mean the condition of nonsingularity is satisfied.)

Lemma 1. *With respect to t any local transformation between two evolutionary second-order equations (i.e. equations of the form $u_t=H(t,x,u,u_x,u_{xx})$ where $H_{u_{xx}}\neq 0$) depends only on t.*

Lemma 2. *Any local transformation between two evolutionary second-order quasi-linear equations having the form $u_t=F(t,x,u)u_{xx}+G(t,x,u,u_x)$ where $G\neq 0$ is projectable, i.e. $\tilde t=T(t)$, $\tilde x=X(t,x)$, $\tilde u=U(t,x,u)$.*

Lemma 3. *Any local transformation between two equations from class (2) is linear with respect to u: $\tilde t=T(t)$, $\tilde x=X(t,x)$, $\tilde u=U^1(t,x)u+U^0(t,x)$, and with respect to transformations from G^{equiv} we can assume the coefficient D is not changed.*

Lemma 4. *$(U_t,U_x)\neq(0,0)$ for a local transformation between two equations from class (2) only if $D\in\{u^\mu,\ e^u\}$* mod G^{equiv}.

As an example of discrete equivalence transformations we can give the following one:

$$\tilde u=u-x,\ \tilde x=-x$$

in the couple of equations

$$u_t=e^u(u_{xx}+u_x^2+u_x)\quad\text{and}\quad e^{-x}u_t=e^u(u_{xx}+u_x^2+u_x).$$

Moreover, this transformation is a discrete invariance transformation for the equation

$$e^{-x/2}u_t=e^u(u_{xx}+u_x^2+u_x).$$

Table 6: Reduced ODEs for (27). $\alpha \neq 0,\ \varepsilon = \pm 1$.

N	Subalgebra	Ansätze $u =$	ω	Reduced ODE
1	$\langle Q_1 \rangle$	$\varphi(\omega)$	x	$(e^{\varphi}\varphi')' = 0$
2	$\langle Q_2 \rangle$	$\varphi(\omega) - \ln\|t\|$	x	$(e^{\varphi}\varphi')' = -1$
3	$\langle Q_3 \rangle$	$\varphi(\omega)$	t	$\varphi' = 0$
4	$\langle Q_4 \rangle$	$\varphi(\omega) + 2\ln\|x\|$	t	$\varphi' = 2e^{\varphi}$
5	$\langle Q_2 + \alpha Q_4 \rangle$	$\varphi(\omega) + \frac{2\alpha+1}{\alpha}\ln\|x\|$	$\ln x - \alpha \ln\|t\|$	$e^{\varphi + \frac{\omega}{\alpha}}\left(\varphi'' + (\varphi')^2 + \frac{3\alpha-2}{\alpha}\varphi' + \frac{2\alpha^2-3\alpha+1}{\alpha^2}\right) = -\alpha\varphi'$
6	$\langle Q_4 + \varepsilon Q_1 \rangle$	$\varphi(\omega) + 2\ln\|x\|$	$xe^{-\varepsilon t}$	$e^{\varphi}((\varphi'' + (\varphi')^2)\omega^2 + 4\omega\varphi' + 2) = -\varepsilon\varphi'\omega$
7	$\langle Q_2 + \varepsilon Q_3 \rangle$	$\varphi(\omega) - \varepsilon x$	$x - \varepsilon \ln\|t\|$	$e^{\varphi - \omega\varepsilon}(\varphi'' + (\varphi')^2 - 2\varepsilon\varphi' + 1) = -\varepsilon\varphi'$
8	$\langle Q_1 + \varepsilon Q_3 \rangle$	$\varphi(\omega)$	$x - \varepsilon t$	$e^{\varphi}(\varphi'' + (\varphi')^2) = -\varepsilon\varphi'$

We also investigated some transformations into other classes of reaction-diffusion equations. So, using the discrete transformation $\tilde{u} = u + x/2,\ \tilde{x} = -x$ we can reduce the equation

$$e^{-x/2}u_t = e^u(u_{xx} + u_x^2 + u_x).$$

to the reaction-diffusion equation

$$\tilde{u}_t = (e^{\tilde{u}}\tilde{u}_{\tilde{x}})_{\tilde{x}} - \frac{1}{4}e^{\tilde{u}},$$

from the classification of Dorodnitsyn [10].

7 Exact solutions

We now turn to present some exact solutions of (2). Using our classification with respect to all the possible local transformations (i.e. not only with respect to ones from G^{equiv}), at first we can obtain solutions of simpler equations (e.g. 6a from Tables 2 or 3). Then we transform them to solutions of more complicated equations (such as 6b, 6c, ...). To construct exact solutions of the equations under consideration, we use both the classical Lie–Ovsiannikov algorithm and non-classical methods.

So, consider equation 2.6a

$$u_t = (e^u u_x)_x. \tag{27}$$

Let us remind that for (27) the basis of $A^{\max}$ is formed by the operators

$$Q_1 = \partial_t,\ Q_2 = t\partial_t - \partial_u,\ Q_3 = \partial_x,\ Q_4 = x\partial_x + 2\partial_u.$$

The non-zero commutators of pairs of these operators are only $[Q_1, Q_2] = Q_1$ and $[Q_3, Q_4] = Q_3$. Therefore $A^{\max}$ is a realization of the algebra $2A_{2.1}$ [23]. All the possible inequivalent (with respect to inner automorphisms) one-dimensional subalgebras of $2A_{2.1}$ [24] are exhausted by the ones listed in Table 6 along with the corresponding ansätze and the reduced ODEs.

We succeeded to solve the equations 6.1–6.4 and 6.8. Thus we have the following solutions of (27):

$$u = \ln(c_1 x + c_0), \qquad u = \ln(-x^2/2 + c_1 x + c_0) - \ln|t|, \qquad u = \ln\frac{x^2}{c_0 - 2t},$$

Table 7: Reduced ODEs for (29). $\alpha \neq 0,\ \varepsilon = \pm 1$.

N	Subalgebra	Ansätze $u =$	ω	Reduced ODE
1	$\langle Q_1 \rangle$	$\varphi(\omega)$	x	$(\varphi^{-1}\varphi')' = 0$
2	$\langle Q_2 \rangle$	$\varphi(\omega)t$	x	$(\varphi^{-1}\varphi')' = \varphi$
3	$\langle Q_3 \rangle$	$\varphi(\omega)$	t	$\varphi' = 0$
4	$\langle Q_4 \rangle$	$\varphi(\omega)/x^2$	t	$\varphi' = 2$
5	$\langle Q_2 + \alpha Q_4 \rangle$	$\varphi(\omega)t^{1-2\alpha}$	x/t^{α}	$(\varphi^{-1}\varphi')' = -\alpha\omega\varphi' + (1-2\alpha)\varphi$
6	$\langle Q_4 + \varepsilon Q_1 \rangle$	$\varphi(\omega)/x^2$	$xe^{-\varepsilon t}$	$(\varphi^{-1}\varphi')'\omega^2 - 2 = -\varepsilon\omega\varphi'$
7	$\langle Q_2 + \varepsilon Q_3 \rangle$	$\varphi(\omega)t$	$x - \varepsilon \ln\|t\|$	$(\varphi^{-1}\varphi')' = \varphi - \varepsilon\varphi'$
8	$\langle Q_1 + \varepsilon Q_3 \rangle$	$\varphi(\omega)$	$x - \varepsilon t$	$(\varphi^{-1}\varphi')' = -\varepsilon\varphi'$

$$u = \varphi(x - \varepsilon t) \quad \text{where} \int \frac{e^{\varphi}}{c_1 - \varepsilon\varphi} d\varphi = \omega + c_0.$$

Using them we can construct solutions for Cases 2.6b–2.6d easily. For example, with the transformation 2.4 we obtain the corresponding solutions for the more complicated and interesting equation

$$\frac{e^x}{(\gamma e^x + 1)^3} u_t = e^u (u_x)_x + e^u u_x \tag{28}$$

having localized density (Case 2.6c):

$$u = \ln\left|\frac{c_1}{e^{-x} + \gamma} + c_0\right| - \ln|e^{-x} + \gamma|, \quad u = \ln\left(-\frac{1}{2(e^{-x}+\gamma)^2} - \frac{c_1}{e^{-x}+\gamma} + c_0\right) - \ln|e^{-x}+\gamma| - \ln|t|,$$

$$u = -\ln|e^{-x} + \gamma|^3 (c_0 - 2t).$$

A singular value of the parameter μ for Case 3.6a is $\mu = -1$. So, the equation

$$u_t = \left(\frac{u_x}{u}\right)_x \tag{29}$$

is distinguished by the reduction procedure. Moreover, it is the equation from subclass 3.6a that the Cases 3.6c and 3.6d are reduced to it. The invariance algebra of (29) is generated by the operators

$$Q_1 = \partial_t,\ Q_2 = t\partial_t + u\partial_u,\ Q_3 = \partial_x,\ Q_4 = x\partial_x - 2u\partial_u$$

and is a realization of the algebra $2A_{2.1}$ too. The reduced ODEs for (29) are listed in Table 7. After integrating Cases 7.1–7.4 we have the following solutions of (29):

$$u = c_0 e^{c_1 x}, \qquad u = \frac{t}{(c \pm x/\sqrt{2})^2}, \qquad u = \frac{2t(c_0 c_1)^2 e^{\pm 4 c_1 x}}{(1 - c_0 e^{\pm 2 c_1 x})^2},$$

$$u = t(c_1 \tan(c_0 \pm c_1 x) - c_1^2)/2, \qquad u = \frac{2t + c}{x^2}.$$

Analogously to previous case by means of transformations 3.5 we obtain exact solutions of equation 3.6d which look like:

$$u = c_0 e^{c_1 x - \gamma e^{-x}}, \qquad u = \frac{t e^{-\gamma e^{-x}}}{(c \pm x/\sqrt{2})^2}, \qquad u = \frac{2t(c_0 c_1)^2 e^{\pm 4 c_1 x - \gamma e^{-x}}}{(1 - c_0 e^{\pm 2 c_1 x})^2},$$

Table 8: Reduced ODEs for (30). $\mu \neq -1, -4/3,\ \alpha \neq 0,\ \varepsilon = \pm 1$.

N	Subalgebra	Ansätze $u =$	ω	Reduced ODE
1	$\langle Q_1 \rangle$	$\varphi(\omega)$	x	$(\varphi^\mu \varphi')' = 0$
2	$\langle Q_2 \rangle$	$\varphi(\omega) t^{-1/\mu}$	x	$(\varphi^\mu \varphi')' = -\varphi/\mu$
3	$\langle Q_3 \rangle$	$\varphi(\omega)$	t	$\varphi' = 0$
4	$\langle Q_4 \rangle$	$\varphi(\omega)/x^{2/\mu}$	t	$\varphi' = \frac{2(2+\mu)}{\mu^2} \varphi^{\mu+1}$
5	$\langle Q_2 + \alpha Q_4 \rangle$	$\varphi(\omega) t^{(2\alpha-1)/\mu}$	x/t^α	$(\varphi^\mu \varphi')' = \frac{2\alpha-1}{\mu}\varphi - \alpha\omega\varphi'$
6	$\langle Q_4 + \varepsilon Q_1 \rangle$	$\varphi(\omega) e^{2\varepsilon t}$	$x e^{-\varepsilon t}$	$(\varphi^\mu \varphi')' = -\mu\varepsilon\omega\varphi' + 2\varepsilon\varphi$
7	$\langle Q_2 + \varepsilon Q_3 \rangle$	$\varphi(\omega) t^{-1/\mu}$	$\mu\varepsilon x - \ln t$	$\mu^2(\varphi^\mu \varphi')' + \varphi' + \varphi/\mu = 0$
8	$\langle Q_1 + \varepsilon Q_3 \rangle$	$\varphi(\omega)$	$x - \varepsilon t$	$(\varphi^\mu \varphi')' = -\varepsilon\varphi'$

$$u = t(c_1 \tan(c_0 \pm c_1 x) - c_1^2) e^{-\gamma e^{-x}}/2, \qquad u = \frac{2t + c}{x^2} e^{-\gamma e^{-x}}.$$

Another example of equation with localized density is given by Case 3.6f . To look for exact solutions for it, at first we reduce the equation 3.6a

$$u_t = (u^\mu u_x)_x\,. \tag{30}$$

As in the previous cases the invariance algebra of (30)

$$A^{\max} = \langle Q_1 = \partial_t,\ Q_2 = \mu t \partial_t - u\partial_u,\ Q_3 = \partial_x,\ Q_4 = \mu x \partial_x + 2u\partial_u \rangle$$

is a realization of the algebra $2A_{2.1}$. The result of reduction (30) under inequivalent subalgebras of $A^{\max}$ is written down in Table 8.

For some of the reduced equations we can construct the general solutions. For other ones we succeeded to find the particular solutions only. These solutions are following:

$$u = (c_1 x + c_0)^{1/(\mu+1)}, \qquad u = \left(\frac{c_0 \pm \sqrt{-2\mu/(\mu+2)}\, x}{\sqrt{t}} \right)^{2/\mu},$$

$$u = \left(\frac{x^2}{c - 2(2+\mu)t/\mu} \right)^{1/\mu}, \qquad u = \left(c \pm \frac{\mu}{2} \sqrt{\frac{-2\varepsilon}{\mu+2}} (x - \varepsilon t) \right)^{2/\mu}.$$

All the results of Tables 7, 8 as well as constructed solutions can be extended to equations 3.6b–3.6g by means of local equivalence transformations. So for the equation

$$\frac{e^{-2x}}{(e^{-x} + \gamma)^{\frac{4+3\mu}{1+\mu}}} u_t = (u^\mu u_x)_x + u^\mu u_x \tag{31}$$

(Case 3.6f) by means of transformations 3.7 we obtain exact solutions in the form

$$u = (c_1 x + c_0)^{1/(\mu+1)}, \qquad u = \left(\frac{c_0(e^{-x} + \gamma) \pm \sqrt{-2\mu/(\mu+2)}}{\sqrt{t}(e^{-x} + \gamma)} \right)^{2/\mu} (e^{-x} + \gamma)^{1/(1+\mu)},$$

$$u = (e^{-x} + \gamma)^{-\frac{\mu+2}{\mu(\mu+1)}} (c - 2(2+\mu)t/\mu)^{-1/\mu},$$

$$u = \left(c \pm \frac{\mu}{2} \sqrt{\frac{-2\varepsilon}{\mu+2}} \left(\frac{1}{e^{-x} + \gamma} + \varepsilon t \right) \right)^{2/\mu} (e^{-x} + \gamma)^{1/(1+\mu)}.$$

A number of exact solutions were constructed for equations from class (19) ($f = 1$) by meas of nonclassical methods. Starting from them and using local transformations of conditional equivalence we can obtain non-Lie exact solutions for more complicate equations (Cases 6b, 6c, ...).

So, King [26] suggested to look for solutions of the equation $u_t = (u^{-1/2}u_x)_x$ (3.6a, $\mu = -1/2$) in the form $u = (\varphi^1(x)t + \varphi^0(x))^2$ where the functions $\varphi^1(x)$ and $\varphi^0(x)$ satisfy the system of ODEs $\varphi^1_{xx} = (\varphi^1)^2$, $\varphi^0_{xx} = \varphi^0\varphi^1$. A particular solution of this system is

$$\varphi^1 = \frac{6}{x^2}, \qquad \varphi^0 = \frac{c_1}{x^2} + \frac{c_2}{x^3}.$$

Therefore, equation 3.6f with $\mu = -1/2$ has the particular solution

$$u = (6t + c_1 + c_2e^{-x})^2(e^{-x} + \gamma)^6.$$

Galaktionov [27] used the transformation $u \to 1/u$ to find exact solutions of (29). Using this transformation one can reduce (29) to the equation

$$u_t = u^2(u^{-1}u_x)_x.$$

King [28] and Pukhnachev [29] obtained some interesting solutions of the latter equation. One of these solutions is a travelling wave of the form $u = 1 + ce^{mx-m^2t}$ which generates the exact solution of (29):

$$u = \frac{1}{1 + ce^{mx-m^2t}}.$$

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