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New Solitary-Wave Solutions of the Van der Waals Normal Form for Granular Materials via New Auxiliary Equation Method

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Abstract: In this paper, we use general Riccati equation to construct new solitary wave solutions of the Van der Waals normal form, which is one of the most famous models for natural and industrial granular materials. It is very important to understand the static and dynamic characteristics of this models in many application fields. We solve the general Riccati equation through different function transformation, and many new hyperbolic function solutions are obtained. Then, it is substituted into the Van der Waals normal form as an auxiliary equation. Abundant types of solitary-wave solutions are obtained by choosing different coefficient in the general Riccati equation, and some of them have not been found in other documents. The results show that the analysis method we used is very simple and effective for dealing with nonlinear models.

Keywords: Riccati equation; Van der Waals normal form; nonlinear evolution equation; solitary wave solution; auxiliary equation

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1. Introduction

At present, the exploration of solitary waves and solitons is a frontier topic of nonlinear physics. It has penetrated into various fields of natural science, such as hydrodynamics, optics, plasma, condensed matter physics, elementary particle physics, material physics, ocean engineering, astrophysics, and biology, etc. [1–6]. The research shows that most physical laws can establish mathematical models under certain conditions, and many nonlinear identification studies can ultimately be attributed to the nonlinear evolution equations (NLEEs). Therefore, searching for the analytical and numerical solutions of NLEEs, such as exact traveling wave solutions and solitary wave solutions, has very important physical and practical significance for understanding their characteristics, providing wave parameter information and their applications. It is also an important hotspot from mathematical and physical research to constructing exact traveling-wave solutions and solitary-wave solutions of various nonlinear models. The exact solutions of NLEEs are very important for understanding the mechanism of many nonlinear phenomena and processes in different fields of natural science. They help us demonstrate exact solutions through physical images so that we can analyze the structure of various nonlinear complex phenomena, such as existence of peaking regimes, absence or multiplicity of steady states under different conditions, spatial localization of transfer process, and many others.

Since NLEEs have amazing applications in the field of nonlinear science and can understand the behavior of physical phenomena, efficient and powerful methods are needed to analyze and study them in order to construct their exact traveling-wave solutions and solitary-wave solutions. In recent years, great progress has been made in solving NLEEs. In various literature, many powerful and effective methods have been proposed. For example, F-expansion method [7,8], tanh-sech method and the extended tanh-coth method [9,10], Jacobi elliptic function method [11,12], auxiliary equation method [13–15], and so on. These methods obtain many types of traveling-wave solutions and solitary-wave solutions when solving some nonlinear equations. However, some of these methods are effective for some nonlinear models but not for others. The problem is that there is still no one method that can be applied to all models. In these methods, the auxiliary equation method is based on these original methods by introducing auxiliary equations to construct exact solutions of NLEEs. Finding a suitable auxiliary equation can not only greatly simplify the solution process but also obtain various complex forms of exact traveling-wave and solitary-wave interaction solutions. In this research work, we solve the general Riccati equation through several different function transformation and obtain many new types of hyperbolic function solutions, which greatly extend the earlier Riccati equation method [16]. Then, we use this general Riccati equation as an auxiliary equation to solve the Van der Waals normal form [17–20] and obtain many new types of solitary wave interaction solutions. On the one hand, this method greatly simplifies the process of solving the Van der Waals normal form. On the other hand, the solutions of this general Riccati equation can be used as a mapping to solve other equations.

In this paper, we consider the following Van der Waals normal form (see Appendix A for the derivation process):

$$\frac{\partial^2 u}{(\partial t)^2}+\frac{\partial^2}{(\partial x)^2}\left[\frac{\partial^2 u}{(\partial x)^2}-\eta\frac{\partial u}{\partial t}-u^3-\varepsilon u\right]=0, \tag{1}$$

Equation (1) is also Equation (4) in ref. [19] and Equation (6) in ref. [20]. In the refs. [19,20], different methods are used to solve Equation (1), and a large number of exact solutions are obtained. Although these methods are effective in solving Equation (1), there are still some new methods and new solutions to be explored. Next, we will use the new auxiliary equation method to construct abundant exact solitary-wave solutions of the Van der Waals normal form. The frame work of the paper is as follows: Section 2 introduces the method of solving the Van der Waals normal form and establishes how to operate this method for producing new solitary-wave solutions. Section 3 is the conclusion. In the part of the main text, there are some detailed and lengthy derivation processes that are not essential for the general understanding, so the analysis is presented in the Appendixes A and B.

2. Main Steps of the Scheme and Application

It is assumed that Equation (1) has the following traveling-wave solution:

$$u(x,t)=u(\xi),\ \ \xi=kx+\lambda t, \tag{2}$$

where k and λ are wave parameters to be determined. Substituting Equation (2) into Equation (1), integrating twice, and setting the integration constant to zero yields

$$\frac{\lambda^2-\varepsilon k^2}{k^2}u+k^2u''-\eta\lambda u'-u^3=0, \tag{3}$$

where u' means $du/d\xi$. Suppose Equation (3) has the following formal solution:

$$u(\xi)=\sum_{i=0}^{n}a_i z^i(\xi), \tag{4}$$

where a_i are constants determined later, and $z(\xi)$ satisfies the following general Riccati equation:

$$z'(\xi)=p_3z^2(\xi)+q_3z(\xi)+r_3. \tag{5}$$

Here, Equation (5) can have different hyperbolic function solutions for different values of p_3, q_3, and r_3 (see Appendix B for the derivation process).The positive integer n can be obtained by controlling the homogeneous balance between the governing nonlinear term

u^3 and the highest order derivative of u'' in Equation (3). It is easy to know $n = 1$. Thus, the solution of Equation (3) can be expressed as

$$u(\xi) = a_0 + a_1 z(\xi). \tag{6}$$

Substituting (6) into (3) yields a set of algebraic equations for a_0, a_1, k, and λ. Collecting all terms with the same power of $z(\xi)$ together and equating each coefficient to zero, we have

$$\begin{cases} a_1^3 = 2a_1 {p_3}^2 k^2 \\ -3a_0 a_1^2 + 3k^2 a_1 p_3 q_3 - \eta\lambda a_1 p_3 = 0 \\ -3a_0^2 a_1 + k^2 a_1({q_3}^2 + 2p_3 r_3) - \eta\lambda a_1 q_3 + \frac{\lambda^2 - \varepsilon k^2}{k^2} a_1 = 0 \\ -{a_0}^3 + k^2 a_1 q_3 r_3 - \eta\lambda a_1 r_3 + \frac{\lambda^2 - \varepsilon k^2}{k^2} a_0 = 0 \end{cases} \tag{7}$$

These equations are solved by using MATLAB 2014a program (the same as below, also used for drawing in this paper), and a_0, a_1, k, and λ can be obtained as follows:

Case 1

$$a_0 = \pm A\left(\frac{q_3}{\sqrt{2}}\sqrt{\frac{1}{{q_3}^2 - 4p_3 r_3}} + \frac{1}{\sqrt{2}}\right),\ a_1 = \pm\sqrt{2}Ap_3\sqrt{\frac{1}{{q_3}^2 - 4p_3 r_3}},$$
$$k^2 = \frac{A^2}{{q_3}^2 - 4p_3 r_3},\ \lambda = \frac{3A^2}{\eta^2}\sqrt{\frac{1}{{q_3}^2 - 4p_3 r_3}},\ A = \eta\sqrt{\frac{\varepsilon}{9 - 2\eta^2}}.$$

Case 2

$$a_0 = \pm A\left(\frac{q_3}{\sqrt{2}}\sqrt{\frac{1}{{q_3}^2 - 4p_3 r_3}} - \frac{1}{\sqrt{2}}\right),\ a_1 = \pm\sqrt{2}Ap_3\sqrt{\frac{1}{{q_3}^2 - 4p_3 r_3}},$$
$$k^2 = \frac{A^2}{{q_3}^2 - 4p_3 r_3},\ \lambda = -\frac{3A^2}{\eta^2}\sqrt{\frac{1}{{q_3}^2 - 4p_3 r_3}},\ A = \eta\sqrt{\frac{\varepsilon}{9 - 2\eta^2}}$$

Thus, by selecting different solutions of Equation (5), the new solitary-wave solutions of Equation (1) can be written as

$$u_{11}(\xi) = \sqrt{2}k - \sqrt{2}k\frac{sinh(\xi) + cosh(\xi)}{sinh(\xi) + cosh(\xi) + r} \tag{8}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq 0$.

$$u_{12}(\xi) = -\sqrt{2}k\frac{sinh(\xi) + cosh(\xi)}{sinh(\xi) + cosh(\xi) + r} \tag{9}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq 0$. Equations (8) and (9) are the same type of solitary-wave solutions. Figure 1a shows the three-dimensional diagrams of Equation (8), which represent the kink solitary-wave solution. Figure 1b shows that the amplitude and velocity of this solitary wave remain unchanged during propagation.

$$u_{21}(\xi) = \sqrt{2}k\frac{sinh(\xi) - cosh(\xi)}{sinh(\xi) - cosh(\xi) + r} \tag{10}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq 0$.

$$u_{22}(\xi) = -\sqrt{2}k + \sqrt{2}k\frac{sinh(\xi) - cosh(\xi)}{sinh(\xi) - cosh(\xi) + r} \tag{11}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq 0$. Equations (10) and (11) are the same type of solitary-wave solutions. Figure 2 shows the three-dimensional and two-dimensional diagrams of Equation (10). Since the denominator of the solution has –$cosh(\xi)$, the solution will have singularity at a certain space-time position. Moreover, taking different values of r can control the appearance of singularity in different space-time positions.

$$u_{31}(\xi) = \sqrt{2}k - \sqrt{2}k\frac{1}{r[sinh(\xi) - cosh(\xi)] + 1} \tag{12}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq 0$.

$$u_{32}(\xi) = -\sqrt{2}k\frac{1}{r[sinh(\xi) - cosh(\xi)] + 1} \tag{13}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq 0$. Although the solutions of Equations (12) and (13) are different from those of Equations (10) and (11), they each represent the same type of solitary-wave solutions because r can take any constant that is not zero.

$$u_{41}(\xi) = \sqrt{2}k\frac{1}{r[sinh(\xi) + cosh(\xi)] + 1} \tag{14}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq 0$.

$$u_{42}(\xi) = -\sqrt{2}k + \sqrt{2}k\frac{1}{r[sinh(\xi) + cosh(\xi)] + 1} \tag{15}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq 0$. The solutions of Equations (14) and (15) also represent the same type of solitary-wave solutions as Equations (8) and (9).

$$u_{51}(\xi) = \sqrt{2}k - \frac{\sqrt{2}}{2}k\frac{sinh(\xi) + cosh(\xi) + r}{sinh(\xi) + r} \tag{16}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, and $r^2 = -1$.

$$u_{52}(\xi) = -\frac{\sqrt{2}}{2}k\frac{sinh(\xi) + cosh(\xi) + r}{sinh(\xi) + r} \tag{17}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, and $r^2 = -1$. The solutions of Equations (16) and (17) represent the traveling-wave solutions of Equation (3) in complex space.

$$u_{61}(\xi) = \sqrt{2}k - \frac{\sqrt{2}}{2}k\frac{sinh(\xi) + cosh(\xi) + r}{cosh(\xi) + r} \tag{18}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, and $r^2 = 1$.

$$u_{62}(\xi) = -\frac{\sqrt{2}}{2}k\frac{sinh(\xi) + cosh(\xi) + r}{cosh(\xi) + r} \tag{19}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, and $r^2 = 1$. Equations (18) and (19) are the same type of solitary-wave solutions. Owing to $\frac{1}{2}\frac{sinh(\xi)+cosh(\xi)\pm 1}{cosh(\xi)\pm 1} = \frac{sinh(\xi)+cosh(\xi)}{sinh(\xi)+cosh(\xi)\pm 1}$, in fact, these two sets of solutions are two sets of special solutions of Equations (8) and (9) under the conditions of $r = 1$ and $r = -1$.

$$u_{71}(\xi) = 2\sqrt{2}rk\frac{1}{[sinh(\xi) + cosh(\xi)]^2 + r} \tag{20}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq 0$.

$$u_{72}(\xi) = -2\sqrt{2}k + 2\sqrt{2}rk\frac{1}{[sinh(\xi) + cosh(\xi)]^2 + r} \tag{21}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq 0$. The solutions of Equations (20) and (21) represent the anti-kink solitary-wave solution (see Figure 3).

$$u_{81}(\xi) = 2\sqrt{2}k - 2\sqrt{2}rk\frac{1}{[sinh(\xi) - cosh(\xi)]^2 + r} \tag{22}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq 0$.

$$u_{82}(\xi) = -2\sqrt{2}rk\frac{1}{[sinh(\xi) - cosh(\xi)]^2 + r} \tag{23}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq 0$. Since r can take any number that is not zero, essentially, these two sets of solutions and Equations (20) and (21) represent the same type of solitary-wave solutions.

$$u_{91}(\xi) = 2\sqrt{2}rk\frac{[sinh(\xi) - cosh(\xi)]^2}{r[sinh(\xi) - cosh(\xi)]^2 + 1} \tag{24}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq 0$.

$$u_{92}(\xi) = -2\sqrt{2}k + 2\sqrt{2}rk\frac{[sinh(\xi) - cosh(\xi)]^2}{r[sinh(\xi) - cosh(\xi)]^2 + 1} \tag{25}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq 0$. These two sets of solutions represent the same type of solitary-wave solutions as Equations (22) and (23).

$$u_{101}(\xi) = 2\sqrt{2}k - 2\sqrt{2}rk\frac{[sinh(\xi) + cosh(\xi)]^2}{r[sinh(\xi) + cosh(\xi)]^2 + 1} \tag{26}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq 0$.

$$u_{102}(\xi) = -2\sqrt{2}rk\frac{[sinh(\xi) + cosh(\xi)]^2}{r[sinh(\xi) + cosh(\xi)]^2 + 1} \tag{27}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq 0$. These two sets of solutions represent the same type of solitary-wave solutions as Equations (20) and (21).

$$u_{111}(\xi) = (-\sqrt{2}r + \sqrt{2})k + \sqrt{2}(-1 + r^2)k\frac{cosh(\xi)}{sinh(\xi) + rcosh(\xi)} \tag{28}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \in R$.

$$u_{112}(\xi) = (-\sqrt{2}r - \sqrt{2})k + \sqrt{2}(-1 + r^2)k\frac{cosh(\xi)}{sinh(\xi) + rcosh(\xi)} \tag{29}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq \pm 1$. These two sets of solutions contain many kinds of solitary-wave solutions due to the different values of r; for example, if

$r = 0$, $u_{111}(\xi) = \pm\sqrt{2}k \mp \sqrt{2}k\coth(\xi)$, they represent the singular solitary-wave solutions (see Figure 4c), and these solutions are the same as Equations (14)–(21) in ref. [20]. In this reference, the Riccati equation is used as an auxiliary equation to solve the Van der Waals normal form. When $r = 2$ and $k = -0.189$, Equation (28) shows kink solitary-wave solutions (see Figure 4a). If $r = 2$ and $k = 0.189$, Equation (28) shows the anti-kink solitary wave solutions (see Figure 4b).

$$u_{121}(\xi) = (-\sqrt{2}r + \sqrt{2})k + \sqrt{2}(-1 + r^2)k\frac{sinh(\xi)}{cosh(\xi) + rsinh(\xi)} \tag{30}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq \pm 1$.

$$u_{122}(\xi) = (-\sqrt{2}r - \sqrt{2})k + \sqrt{2}(-1 + r^2)k\frac{sinh(\xi)}{cosh(\xi) + rsinh(\xi)} \tag{31}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $r \neq \pm 1$. These two sets of solutions also contain many kinds of solitary-wave solutions due to the different values of r with the changes of r, and Equations (30) and (31) also represent different solitary-wave solutions. When $r = 0$, $u_{121}(\xi) = \pm\sqrt{2}k \mp \sqrt{2}k\tanh(\xi)$, these solutions are also the same as Equations (14)–(21) in ref. [20].

$$u_{131}(\xi) = (-\frac{\sqrt{2}}{2}r + \frac{\sqrt{2}}{2})k + \frac{\sqrt{2}}{2}(-1 + r^2)k\frac{cosh(\xi) \pm 1}{sinh(\xi) + r[cosh(\xi) \pm 1]} \tag{32}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq \pm 1$

$$u_{132}(\xi) = (-\frac{\sqrt{2}}{2}r - \frac{\sqrt{2}}{2})k + \frac{\sqrt{2}}{2}(-1 + r^2)k \tag{33}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq \pm 1$. These two sets of solutions also represent different kink/anti-kink solitary-wave solutions and singular solitary-wave solutions under different r values and different $\pm$ sign choices.

$$u_{141}(\xi) = \left(-\frac{\sqrt{2}}{2}r + \frac{\sqrt{2}}{2}\right)k + \frac{\sqrt{2}}{2}(-1 + r^2)k\frac{sinh(\xi) \pm i}{cosh(\xi) + r[sinh(\xi) \pm i]} \tag{34}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, $i^2 = -1$, and $r \neq \pm 1$.

$$u_{142}(\xi) = \left(-\frac{\sqrt{2}}{2}r - \frac{\sqrt{2}}{2}\right)k + \frac{\sqrt{2}}{2}(-1 + r^2)k\frac{sinh(\xi) \pm i}{cosh(\xi) + r[sinh(\xi) \pm i]} \tag{35}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, $i^2 = -1$, and $r \neq \pm 1$. These two sets of solutions represent the traveling wave solutions of Equation (3) in complex space.

$$u_{151}(\xi) = \left(-\frac{\sqrt{2}}{2}r + \frac{\sqrt{2}}{2}\right)k + \frac{\sqrt{2}}{2}(-1 + r^2)k\frac{sinh(\xi)}{rsinh(\xi) + cosh(\xi) \pm 1} \tag{36}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq \pm 1$.

$$u_{152}(\xi) = \left(-\frac{\sqrt{2}}{2}r - \frac{\sqrt{2}}{2}\right)k + \frac{\sqrt{2}}{2}(-1 + r^2)k\frac{sinh(\xi)}{rsinh(\xi) + cosh(\xi) \pm 1} \tag{37}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, and $r \neq \pm 1$. These two sets of solutions and Equations (32) and (33) represent the same type of solitary-wave solutions because r can take any number.

$$u_{161}(\xi) = \left(-\frac{\sqrt{2}}{2}r + \frac{\sqrt{2}}{2}\right)k + \frac{\sqrt{2}}{2}(-1 + r^2)k\frac{cosh(\xi)}{rcosh(\xi) + sinh(\xi) \pm i} \tag{38}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{3k^2}{\eta^2}$, $k = \pm A$, $i^2 = -1$, and $r \neq \pm 1$.

$$u_{162}(\xi) = \left(-\frac{\sqrt{2}}{2}r - \frac{\sqrt{2}}{2}\right)k + \frac{\sqrt{2}}{2}(-1 + r^2)k\frac{cosh(\xi)}{rcosh(\xi) + sinh(\xi) \pm i} \tag{39}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{3k^2}{\eta^2}$, $k = \pm A$, $i^2 = -1$, and $r \neq \pm 1$. These two sets of solutions represent the same types of traveling-wave solutions of Equation (3) as Equations (34) and (35) in complex space.

$$u_{171}(\xi) = (-2\sqrt{2}r + 2\sqrt{2})k + 2\sqrt{2}(-1 + r^2)k\frac{2sinh^2(\xi) + 1}{2sinh(\xi)cosh(\xi) + r[2sinh^2(\xi) + 1]} \tag{40}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{12k^2}{\eta^2}$, $k = \pm\frac{A}{4}$, and $r \neq \pm 1$.

$$u_{172}(\xi) = (-2\sqrt{2}r - 2\sqrt{2})k + 2\sqrt{2}(-1 + r^2)k\frac{2sinh^2(\xi) + 1}{2sinh(\xi)cosh(\xi) + r[2sinh^2(\xi) + 1]} \tag{41}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{12k^2}{\eta^2}$, $k = \pm\frac{A}{4}$, and $r \neq \pm 1$. Although these two sets of solutions are the new set ones of Equation (3) that we have obtained, due to $\frac{2sinh^2(\xi)+1}{2sinh(\xi)cosh(\xi)+r[2sinh^2(\xi)+1]} = \frac{cosh(\xi)}{sinh(\xi)+rcosh(\xi)}$, however, they are the same as Equations (28) and (29).

$$u_{181}(\xi) = (-2\sqrt{2}r + 2\sqrt{2})k + 2\sqrt{2}(-1 + r^2)k\frac{2sinh(\xi)cosh(\xi)}{2rsinh(\xi)cosh(\xi) + 2sinh^2(\xi) + 1} \tag{42}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{12k^2}{\eta^2}$, $k = \pm\frac{A}{4}$, and $r \neq \pm 1$.

$$u_{182}(\xi) = \left(-2\sqrt{2}r - 2\sqrt{2}\right)k + 2\sqrt{2}(-1 + r^2)k\frac{2sinh(\xi)cosh(\xi)}{2rsinh(\xi)cosh(\xi) + 2sinh^2(\xi) + 1} \tag{43}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{12k^2}{\eta^2}$, $k = \pm\frac{A}{4}$, and $r \neq \pm 1$. These two sets of solutions and Equations (40) and (41) represent the same type of solutions because of the arbitrariness of the value of r.

$$u_{191}(\xi) = \left(-\frac{\sqrt{2}}{2}\delta + \sqrt{2}\right)k + \sqrt{2}(\frac{\delta^2}{2} - 2)k\frac{sinh(\xi)[cosh(\xi) \pm 1]}{2sinh^2(\xi) + \delta sinh(\xi)cosh(\xi) \pm \delta sinh(\xi) \pm 2cosh(\xi) + 2} \tag{44}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $\delta \in R$.

$$u_{192}(\xi) = \left(-\frac{\sqrt{2}}{2}\delta - \sqrt{2}\right)k + \sqrt{2}(\frac{\delta^2}{2} - 2)k\frac{sinh(\xi)[cosh(\xi) \pm 1]}{2sinh^2(\xi) + \delta sinh(\xi)cosh(\xi) \pm \delta sinh(\xi) \pm 2cosh(\xi) + 2} \tag{45}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $\delta \in R$. These two sets of solutions are the new ones of Equation (3). When $\delta = 0$, $u_{191}(\xi) = \pm\sqrt{2}k \mp \sqrt{2}k\tanh(\xi)$, which is the same as u_{121} when $r = 0$. These two sets of solutions also represent different kink/anti-kink solitary-wave solutions and singular solitary-wave solutions under different δ values and different $\pm$ sign choices (see Figure 5).

$$u_{201}(\xi) = \left(-\frac{\sqrt{2}}{2}\delta + \sqrt{2}\right)k + \sqrt{2}(\frac{\delta^2}{2} - 2)k\frac{cosh(\xi)[sinh(\xi) \pm i]}{2sinh^2(\xi) + \delta cosh(\xi)(sinh(\xi) \pm i) \pm \delta icosh(\xi) \pm 2isinh(\xi)} \tag{46}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $\delta \in R$.

$$u_{202}(\xi) = \left(-\frac{\sqrt{2}}{2}\delta - \sqrt{2}\right)k + \sqrt{2}(\frac{\delta^2}{2} - 2)k\frac{cosh(\xi)[sinh(\xi) \pm i]}{2sinh^2(\xi) + \delta cosh(\xi)(sinh(\xi) \pm i) \pm \delta icosh(\xi) \pm 2isinh(\xi)} \tag{47}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{6k^2}{\eta^2}$, $k = \pm\frac{A}{2}$, and $\delta \in R$. These two sets of solutions are the traveling wave solutions of Equation (3) in complex space.

$$u_{211}(\xi) = (-2\sqrt{2}\delta + 4\sqrt{2})k + \sqrt{2}(2\delta^2 - 8)k\frac{sinh(\xi)cosh(\xi)\left[sinh^2(\xi) + \frac{1}{2}\right]}{sinh^2(\xi)cosh^2(\xi) + \delta sinh(\xi)cosh(\xi)\left[sinh^2(\xi) + \frac{1}{2}\right] + \left[sinh^2(\xi) + \frac{1}{2}\right]^2} \tag{48}$$

where $\xi = kx + \lambda t$, $\lambda = \frac{24k^2}{\eta^2}$, $k = \pm\frac{A}{8}$, and $\delta \in R$.

$$u_{212}(\xi) = (-2\sqrt{2}\delta - 4\sqrt{2})k + \sqrt{2}(2\delta^2 - 8)k\frac{sinh(\xi)cosh(\xi)\left[sinh^2(\xi) + \frac{1}{2}\right]}{sinh^2(\xi)cosh^2(\xi) + \delta sinh(\xi)cosh(\xi)\left[sinh^2(\xi) + \frac{1}{2}\right] + \left[sinh^2(\xi) + \frac{1}{2}\right]^2} \tag{49}$$

where $\xi = kx + \lambda t$, $\lambda = -\frac{24k^2}{\eta^2}$, $k = \pm\frac{A}{8}$, and $\delta \in R$. These two sets of solutions also represent different kink/anti-kink solitary-wave solutions and singular solitary-wave solutions under different δ values and different $\pm$ sign choices (see Figure 6).

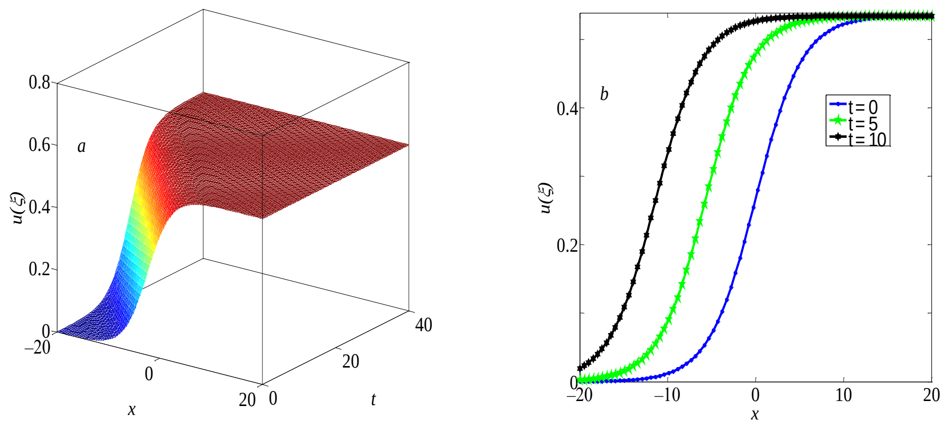

Figure 1. (**a**) Three-dimensional and (**b**) two-dimensional plots represent the kink solitary-wave solution of Equation (8) when $\eta = 1$, $\varepsilon = 1$, $r = 1$, and $k = -0.38$.

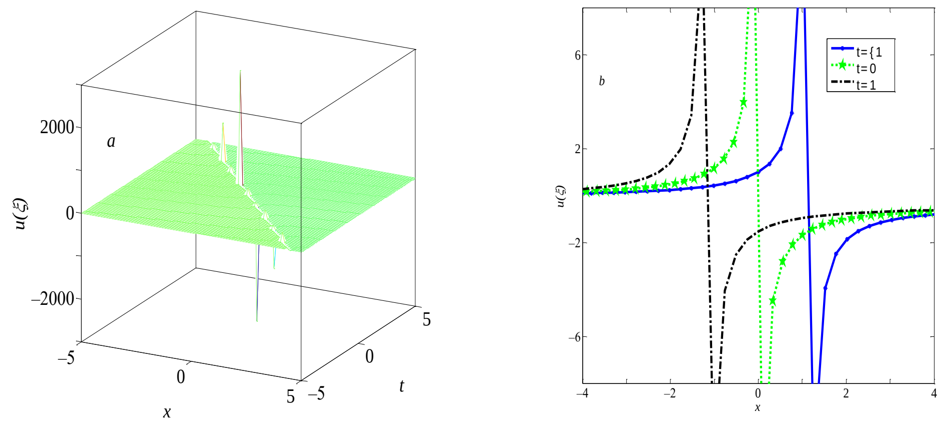

Figure 2. (**a**) Three-dimensional and (**b**) two-dimensional plots represent the singular solitary-wave solution of Equation (10) when $\eta = 1, \varepsilon = 1, r = 1$, and $k = 0.38$.

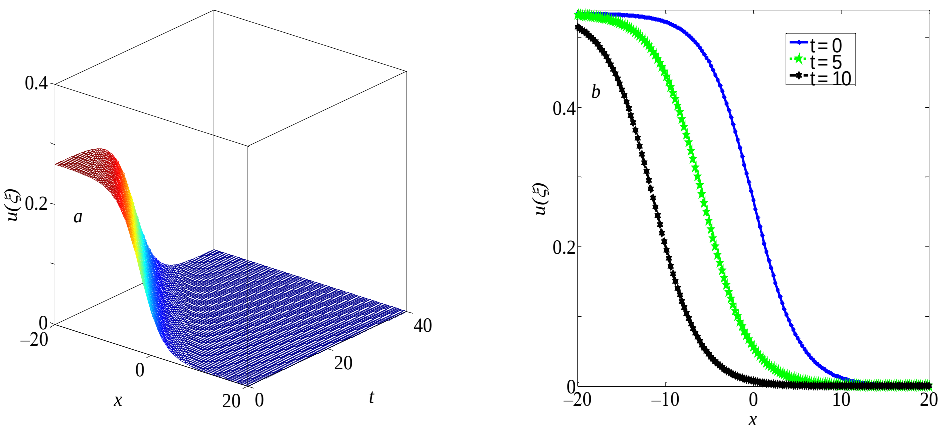

Figure 3. (**a**) Three-dimensional and (**b**) two-dimensional plots represent the anti-kink solitary-wave solution of Equation (20) when $\eta = 1,\ \varepsilon = 1,\ r = 1$, and $k = 0.189$.

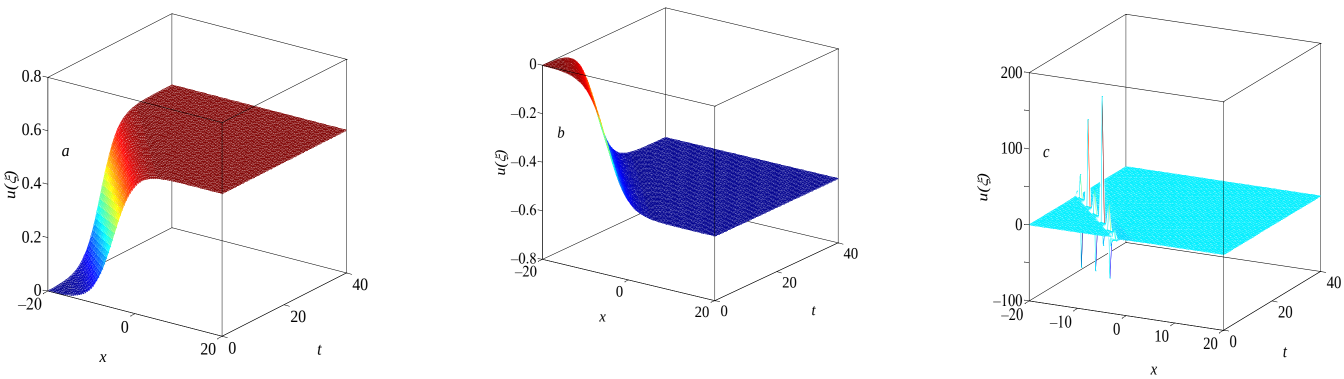

Figure 4. Three-dimensional plots of the solutions of Equation (28) when $\eta = 1$ and $\varepsilon = 1$; (**a**) $k = 0.189$ and $r = 0$; (**b**) $k = -0.189$ and $r = 2$; and (**c**) $k = 0.189$ and $r = 2$.

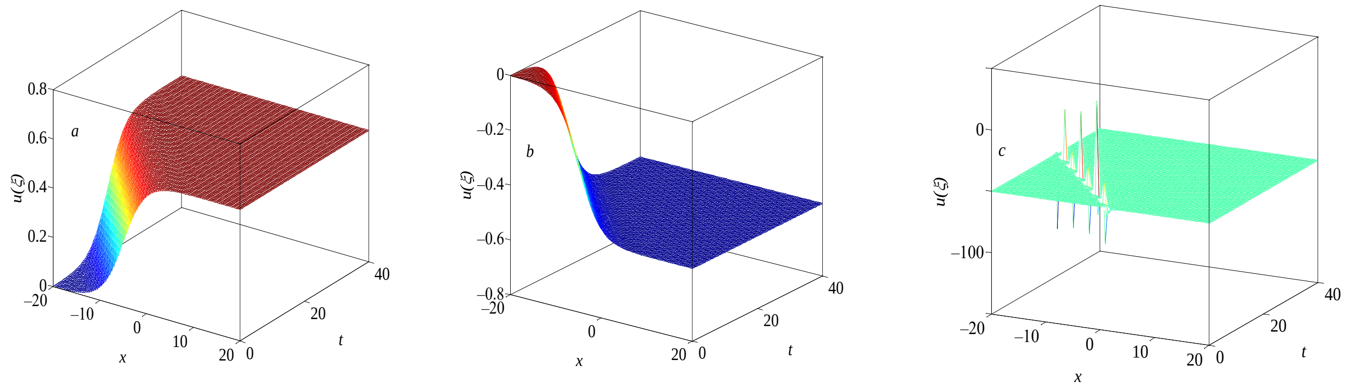

Figure 5. Three-dimensional plots of the solutions of Equation (44) when $\eta = 1$ and $\varepsilon = 1$; (**a**) $\delta = 1$ and $k = -0.189$; (**b**) $\delta = 1$ and $k = 0.189$; and (**c**) $\delta = -3$ and $k = 0.189$.

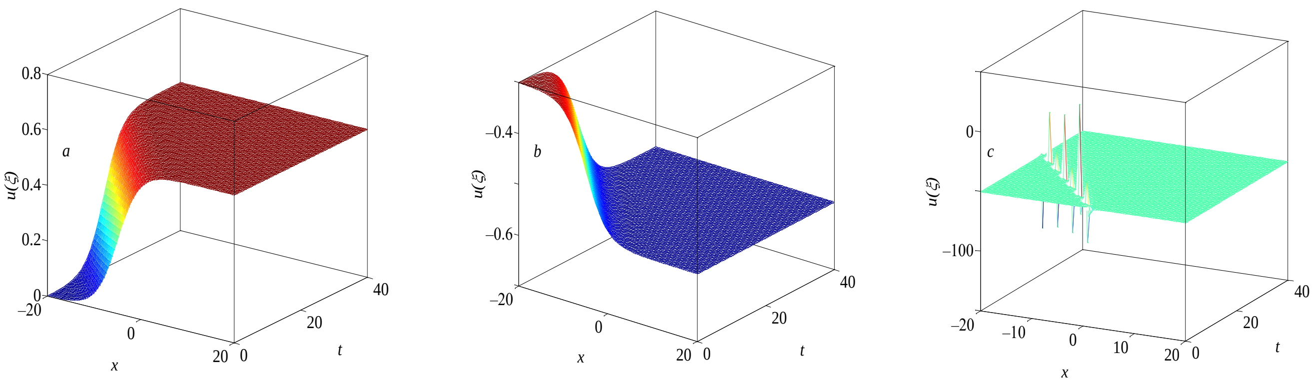

Figure 6. Three-dimensional plots of the solutions of Equation (48) when $\eta = 1$ and $\varepsilon = 1$; (**a**) $\delta = 1$ and $k = -0.0472$; (**b**) $\delta = 1$ and $k = 0.0472$; and (**c**) $\delta = -3$ and $k = 0.0472$.

3. Discussion and Conclusions

In this paper, we used different methods to solve the general Riccati equation, Equation (5), and obtained a large number of hyperbolic function solutions. Then, we used Equation (5) as an auxiliary equation to solve the Van der Waals normal form. Because the general Riccati equation has a relatively simple form, and the process of seeking the solutions is relatively easy, the solving process was greatly simplified. As a result, abundant new types of solitary-wave solutions were obtained by choosing different coefficients in the Riccati equation. In addition, many of them were not found in other documents, such as Equations (40)–(49). Numerical simulations show that these new solutions are all kink/anti-kink and singular-solitary waves although their forms are different. These solitary waves keep the amplitude and wave velocity constant in the process of propagation or interaction. This method can greatly simplify the calculation process and can construct abundant solitary-wave solutions of NLEEs only by selecting different coefficients, so it is especially suitable for solving complex NLEEs. In the next work, we will use it to solve more complex nonlinear systems. All the calculations, results, and graphics show that it is a simple, effective, and powerful mathematical tool. It is hopeful to construct a wealth of solitary-wave solutions. It can be used as a useful guide for a wide range of nonlinear problems in mathematics and physics.

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Appendix A. Van der Waals Normal Form

The Van der Waals normal form for fluidized granular matter Navier–Stokes equations is one of the most famous models in nature and also industry, which includes conservation equations of mass, momentum, and energy for Newtonian fluids. These equations are used to describe the relationship among velocity, pressure, density, and temperature of moving fluid and to solve the flow with different compression, different velocity, and different viscosity. For Van der Waals normal form for fluidized granular matter Navier–Stokes equations, we consider the following density, momentum, and energy balance equations [17–20]:

$$D_t n + n\nabla\cdot\boldsymbol{u} = 0, \tag{A1}$$

$$D_t\boldsymbol{u} + (mn)^{-1}\nabla\cdot\boldsymbol{p} = 0, \tag{A2}$$

$$nD_t T + \nabla\cdot J + p : \nabla\boldsymbol{u} + \omega = 0, \tag{A3}$$

where $D_t = \frac{\partial}{t} + \boldsymbol{u}\cdot\nabla$ is the material derivative, $n = \int f(v)dv$ is the number density, $T = \frac{m}{dn}\int (v-u)^2 f(v)dv$ is the granular temperature, p is the pressure tensor, u is the flow velocity, ω is the new constitutive, and $J = -K(n,t)\nabla T$, where Kk is the thermal conductivity. The density and horizontal momentum of granular materials satisfy conservation Equations (1)–(3) and the nonlinear term v can be ignored by considering enough small velocity field. $\rho(x,t)$ and $j(x,t)$ are the vertical mean density and horizontal moment, respectively, which obey the continuity equation.

$$\frac{\partial\rho(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial t}, \tag{A4}$$

$$\frac{\partial j(x,t)}{\partial t} = \frac{\partial\varnothing}{\partial t}, \tag{A5}$$

where ϕ is the vertical average of the x-x component. Equations (A4) and (A5) are approximated in the form of Van der Waals in the dominant order

$$\frac{\partial^2 u}{(\partial t)^2} + \frac{\partial^2}{(\partial x)^2}\left[\frac{\partial^2 u}{(\partial x)^2} - \eta\frac{\partial u}{\partial t} - u^3 - \varepsilon u\right] = 0 \tag{A6}$$

where x is the one-dimensional direction of the granular system, η is the effective viscosity, and ε is the bifurcation parameter.

Appendix B. Construction of Auxiliary Equation and Its Abundant Hyperbolic Function Solutions

In ref. [16], by using the following Riccati equation

$$f'(\xi) = f^2(\xi) + \mu \tag{A7}$$

the following hyperbolic function solutions are obtained:

$$f(\xi) = -\sqrt{-\mu}tanh(\sqrt{-\mu}\xi), (\mu < 0) \tag{A8}$$

$$f(\xi) = -\sqrt{-\mu}coth(\sqrt{-\mu}\xi),\ \mu < 0) \tag{A9}$$

This method is simple and effective and can be used to solve constant coefficient, variable coefficient, and high-order and high-dimensional NLEEs. In this paper, we first consider the Riccati equation in the following general form:

$$f'(\xi) = p_1 f^2(\xi) + q_1 \tag{A10}$$

where p_1 and q_1 are constants to be determined later. In Equations (A8) and (A9), if $\mu = -1$, we can know Equation (A10) has the following simple-form hyperbolic function solutions:

$$f(\xi) = tanh(\xi),\ (p_1 = -1, q_1 = 1) \tag{A11}$$

$$f(\xi) = coth(\xi),\ (p_1 = -1, q_1 = 1) \tag{A12}$$

Next, we construct a new form of auxiliary function $g\ (\xi)$ to solve Equation(A10), which satisfies the following relationship

$$[g'(\xi)]^2 = p_2 g^2(\xi) + q_2 \tag{A13}$$

where p_2 and q_2 are constants to be determined later. It is easy to know that Equation (A13) has the following hyperbolic function solutions:

$$g_1(\xi) = sinh(\xi),\ (p_2 = 1, q_2 = 1) \tag{A14}$$

$$g_2(\xi) = cosh(\xi),\ (p_2 = 1, q_2 = -1) \tag{A15}$$

$$g_3(\xi) = sinh(\xi)\ \pm cosh(\xi),\ (p_2 = 1, q_2 = 0) \tag{A16}$$

$$g_4(\xi) = \frac{1}{sinh(\xi)\ \pm cosh(\xi)},\ (p_2 = 1, q_2 = 0) \tag{A17}$$

If we take p_2 and q_2 in Equations (A13)–(A17) into Equation (A13), we can determine that Equation (A13) is valid. To find the new solutions, we assume that Equation (A10) has the following formal solution:

$$f(\xi) = \frac{g'(\xi)}{g(\xi) + r} \tag{A18}$$

where $r \neq 0$. Substituting Equation (A18) into Equation (A10) and using Equation (A13) and equating each coefficient of $g^i(\xi)$ (i = 0, 1, 2 . . .) to zero, we can obtain

$$\begin{cases} p_1 p_2 + q_1 = 0, \\ p_2 r = 2 q_1 r, \\ -q_2 = p_1 q_2 + q_1 r^2 \end{cases} \tag{A19}$$

Solving them, $p_1 = -\frac{1}{2}$, $q_1 = \frac{p_2}{2}$ and $r = \pm\sqrt{-\frac{q_2}{p_2}}$ can be obtained. Hence, we have the hyperbolic function solutions in the following form:

$$f_1(\xi) = \frac{sinh(\xi)}{cosh(\xi) \pm 1}, (p_1 = -\frac{1}{2}, q_1 = \frac{1}{2}) \tag{A20}$$

$$f_2(\xi) = \frac{cosh(\xi)}{sinh(\xi) \pm i},\ (p_1 = -\frac{1}{2}, q_1 = \frac{1}{2}, i^2 = -1) \tag{A21}$$

Again, suppose Equation (4) has the following formal solution:

$$f(\xi) = \frac{g(\xi) g'(\xi)}{g^2(\xi) + r} \tag{A22}$$

where $r \neq 0$ Substituting Equation (A22) into Equation (A10) and using Equation (A13), we can obtain

$$\begin{cases} p_1 p_2 + q_1 = 0, \\ -q_2 + 2p_2 r = p_1 q_2 + 2q_1 r, \\ r q_2 = q_1 r^2 \end{cases} \tag{A23}$$

Solving them, $p_1 = -2,\ q_1 = 2p_2$ and $r = \frac{q_2}{2p_2}$ can be obtained. Hence, we can obtain the following hyperbolic function solution:

$$f_3(\xi) = \frac{sinh(\xi)cosh(\xi)}{sinh^2(\xi) + \frac{1}{2}} = \frac{sinh(\xi)cosh(\xi)}{cosh^2(\xi) - \frac{1}{2}},\ (p_1 = -2, q_1 = 2) \tag{A24}$$

It is obvious that $h(\xi) = \frac{1}{f(\xi)}$ is also the solution of Equation (A10) in the condition of $p_1' = -q_1,\ q_1' = -p_1$. Equations (A11) and (A12) are a pair of solutions satisfying this condition. Therefore, the following three equations are also the solutions of Equation (A10):

$$f_4(\xi) = \frac{cosh(\xi) \pm 1}{sinh(\xi)},\ (p_1 = -\frac{1}{2}, q_1 = \frac{1}{2}) \tag{A25}$$

$$f_5(\xi) = \frac{sinh(\xi) \pm i}{cosh(\xi)},\ (p_1 = -\frac{1}{2}, q_1 = \frac{1}{2},\ i^2 = -1) \tag{A26}$$

$$f_6(\xi) = \frac{sinh^2(\xi) + \frac{1}{2}}{sinh(\xi)cosh(\xi)} = \frac{cosh^2(\xi) - \frac{1}{2}}{sinh(\xi)cosh(\xi)},\ (p_1 = -2, q_1 = 2) \tag{A27}$$

Then, we introduce a more general Riccati equation in the following:

$$z'(\xi) = p_3 z^2(\xi) + q_3 z(\xi) + r_3 \tag{A28}$$

where p_3, q_3, and r_3 are constants to be determined later. We first use Equation (A13) and the following form solution to solve Equation (A28):

Case 1.

$$z(\xi) = \frac{g(\xi)}{g(\xi) + r} \tag{A29}$$

where $r \neq 0$. In this case, only when $q_2 = 0$ and $[g'(\xi)]^2 = p_2 g^2(\xi)$ is Equation (A28) solvable. Substituting Equation (A29) into Equation (A28) and using Equation (A13), we can obtain

$$\begin{cases} p_3 + q_3 + r_3 = 0, \\ q_3 r + 2r_3 r = \pm\sqrt{p_2} r, \\ r_3 r^2 = 0. \end{cases} \tag{A30}$$

Solving them, $p_3 = \mp\sqrt{p_2},\ q_3 = \pm\sqrt{p_2},\ r_3 = 0$, and r is any non-zero constant. Thus, we obtain

$$z_1(\xi) = \frac{sinh(\xi) + cosh(\xi)}{sinh(\xi) + cosh(\xi) + r},\ (p_3 = -1, q_3 = 1, r_3 = 0, r \neq 0) \tag{A31}$$

$$z_2(\xi) = \frac{sinh(\xi) - cosh(\xi)}{sinh(\xi) - cosh(\xi) + r},\ (p_3 = 1, q_3 = -1, r_3 = 0, r \neq 0) \tag{A32}$$

$$z_3(\xi) = \frac{1}{r[sinh(\xi) - cosh(\xi)] + 1},\ (p_3 = -1, q_3 = 1, r_3 = 0, r \neq 0) \tag{A33}$$

$$z_4(\xi) = \frac{1}{r[sinh(\xi) + cosh(\xi)] + 1},\ (p_3 = 1, q_3 = -1, r_3 = 0, r \neq 0) \tag{A34}$$

For the second case, we take the following form:

Case 2.

$$z(\xi)=\frac{g(\xi)+g'(\xi)+r}{g(\xi)+r} \tag{A35}$$

where $r\neq 0$. Substituting Equation (A35) into Equation (A28) and using Equation (A13), we can obtain

$$\begin{cases} p_3p_2+p_3+q_3+r_3=0,\\ 2p_3r+2q_3r+2r_3r=p_2r,\\ p_3q_2+p_3r^2+q_3r^2+r_3r^2=-q_2,\\ 2p_3+q_3=0. \end{cases} \tag{A36}$$

After solving Equations (A36), we obtain $p_3=-\frac{1}{2},\ q_3=1,\ r_3=-\frac{1}{2}+\frac{p_2}{2}$, and $r=\pm\sqrt{-\frac{q_2}{p_2}}$. Therefore, the solutions can be expressed as

$$z_5(\xi)=\frac{sinh(\xi)+cosh(\xi)\pm i}{sinh(\xi)\pm i},\ (p_3=-\frac{1}{2},q_3=1,r_3=0,\ i^2=-1) \tag{A37}$$

$$z_6(\xi)=\frac{sinh(\xi)+cosh(\xi)\pm 1}{cosh(\xi)\pm 1},\ (p_3=-\frac{1}{2},q_3=1,r_3=0) \tag{A38}$$

For the third case, we take the following form:

Case 3.

$$z(\xi)=\frac{1}{g^2(\xi)+r} \tag{A39}$$

In this case, only when $q_2=0$ and $[g'(\xi)]^2=p_2g^2(\xi)$ is Equation (A28) solvable. Substituting Equation (A39) into Equation (A28) and using Equation (A13), we can obtain

$$\begin{cases} p_3+q_3r+r_3r^2=0,\\ q_3=\mp 2\sqrt{p_2},\\ r_3=0. \end{cases} \tag{A40}$$

Solving them, $p_3=\pm 2\sqrt{p_2}r,\ \ q_3=\mp 2\sqrt{p_2},\ \ r_3=0$, and r is any constant. Then, we obtain

$$z_7(\xi)=\frac{1}{[sinh(\xi)+cosh(\xi)]^2+r},\ (p_3=2r,q_3=-2,r_3=0,r\in R) \tag{A41}$$

$$z_8(\xi)=\frac{1}{[sinh(\xi)-cosh(\xi)]^2+r},\ (p_3=-2r,q_3=2,r_3=0,r\in R) \tag{A42}$$

$$z_9(\xi)=\frac{[sinh(\xi)-cosh(\xi)]^2}{r[sinh(\xi)-cosh(\xi)]^2+1},\ (p_3=2r,q_3=-2,r_3=0,r\in R) \tag{A43}$$

$$z_{10}(\xi)=\frac{[sinh(\xi)+cosh(\xi)]^2}{r[sinh(\xi)+cosh(\xi)]^2+1},\ (p_3=-2r,q_3=2,r_3=0,r\in R) \tag{A44}$$

For the forth case, we use Equation (A10) and take the following form:

Case 4.

$$z(\xi)=\frac{1}{f(\xi)+r} \tag{A45}$$

Substituting Equation (A45) into Equation (A28) and using Equation (A10), we can obtain

$$\begin{cases} p_3 + q_3 r + r_3 r^2 = -q_1, \\ q_1 + 2r_3 r = 0, \\ r_3 = -p_1. \end{cases} \quad \text{(A46)}$$

Solving them, $p_3 = -q_1 - p_1 r^2$, $q_3 = 2p_1 r$, $r_3 = -p_1$, and r is any constant. Then, we have

$$z_{11}(\xi) = \frac{1}{tanh(\xi) + r} = \frac{cosh(\xi)}{sinh(\xi) + rcosh(\xi)}, \ (p_3 = -1 + r^2, q_3 = -2r, r_3 = 1, r \in R) \quad \text{(A47)}$$

$$z_{12}(\xi) = \frac{1}{coth(\xi) + r} = \frac{sinh(\xi)}{cosh(\xi) + rsinh(\xi)}, \ (p_3 = -1 + r^2, q_3 = -2r, r_3 = 1, r \in R) \quad \text{(A48)}$$

$$z_{13}(\xi) = \frac{cosh(\xi) \pm 1}{sinh(\xi) + r[cosh(\xi) \pm 1]}, \ (p_3 = -\frac{1}{2} + \frac{1}{2}r^2, q_3 = -r, r_3 = \frac{1}{2}, r \in R) \quad \text{(A49)}$$

$$z_{14}(\xi) = \frac{sinh(\xi) \pm i}{cosh(\xi) + r[sinh(\xi) \pm i]}, \ (p_3 = -\frac{1}{2} + \frac{1}{2}r^2, q_3 = -r, r_3 = \frac{1}{2}, \ i^2 = -1, r \in R) \quad \text{(A50)}$$

$$z_{15}(\xi) = \frac{sinh(\xi)}{rsinh(\xi) + cosh(\xi) \pm 1}, \ (p_3 = -\frac{1}{2} + \frac{1}{2}r^2, q_3 = -r, r_3 = \frac{1}{2}, r \in R) \quad \text{(A51)}$$

$$z_{16}(\xi) = \frac{cosh(\xi)}{rcosh(\xi) + sinh(\xi) \pm i}, \ (p_3 = -\frac{1}{2} + \frac{1}{2}r^2, q_3 = -r, r_3 = \frac{1}{2}, \ i^2 = -1, \ r \in R) \quad \text{(A52)}$$

$$z_{17}(\xi) = \frac{2sinh^2(\xi) + 1}{2sinh(\xi)cosh(\xi) + r[2sinh^2(\xi) + 1]}, \ (p_3 = -2 + 2r^2, q_3 = -4r, r_3 = 2, r \in R) \quad \text{(A53)}$$

$$z_{18}(\xi) = \frac{2sinh(\xi)cosh(\xi)}{2rsinh(\xi)cosh(\xi) + 2sinh^2(\xi) + 1}, \ (p_3 = -2 + 2r^2, q_3 = -4r, r_3 = 2, r \in R) \quad \text{(A54)}$$

For the fifth case, we use the following form:

Case 5.

$$z(\xi) = \frac{f(\xi)}{f^2(\xi) + \delta f(\xi) + r} \quad \text{(A55)}$$

where δ and r are the constants to be determined. Substituting Equation (A55) into Equation (A28) and using Equation (A10), we can obtain

$$\begin{cases} q_1 r = r_3 r^2, \\ -q_1 + p_1 r = r_3 \delta^2 + 2r_3 r + p_3, \\ 2\delta r_3 + q_3 = 0, \\ r_3 = -p_1 \end{cases} \quad \text{(A56)}$$

Solving them, $p_3 = -4q_1 + p_1\delta^2$, $q_3 = 2p_1\delta$, $r_3 = -p_1$, $r = -\frac{q_1}{p_1}$ and δ is any constant. Then we can obtain.

$$z_{19}(\xi) = \frac{sinh(\xi)[cosh(\xi) \pm 1]}{2sinh^2(\xi) + \delta sinh(\xi)cosh(\xi) \pm \delta sinh(\xi) \pm 2cosh(\xi) + 2} \quad \text{(A57)}$$

$(p_3 = -2 + \frac{\delta^2}{2}, q_3 = -\delta, r_3 = \frac{1}{2}, \delta \in R)$.

$$z_{20}(\xi) = \frac{cosh(\xi)[sinh(\xi) \pm i]}{2sinh^2(\xi) + \delta cosh(\xi)(sinh(\xi) \pm i) \pm \delta icosh(\xi) \pm 2isinh(\xi)} \quad \text{(A58)}$$

$(p_3 = -2 + \frac{\delta^2}{2}, q_3 = -\delta, r_3 = \frac{1}{2}, i^2 = -1,\ \delta \in R).$

$$z_{21}(\xi) = \frac{sinh(\xi)cosh(\xi)\left[sinh^2(\xi) + \frac{1}{2}\right]}{sinh^2(\xi)cosh^2(\xi) + \delta sinh(\xi)cosh(\xi)\left[sinh^2(\xi) + \frac{1}{2}\right] + \left[sinh^2(\xi) + \frac{1}{2}\right]^2} \quad \text{(A59)}$$

$(p_3 = -8 + 2\delta^2, q_3 = -4\delta, r_3 = 2, \delta \in R).$

In this case, there are still some solutions that are the same as the previous ones, and we do not give them.

In Riccati Equations (A7), (A10), and (A28), including auxiliary Equation (A13), p_i, q_i, and r_i are all constants. By selecting different values of p_i, q_i, and r_i, these equations have different solutions. Through the construction of different forms of solutions, we have given abundant new hyperbolic function solutions of these equations, and these solutions have been checked by MATLAB 2014a software. In fact, there are still a large number of new solutions to them, which need to be further explored in the future, and these solutions can be added to our current research.

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