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New Stability Criterion for Continuous-Time Takagi–Sugeno Fuzzy Systems With Time-Varying Delay

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Abstract—In this technical paper, a new Lyapunov–Krasovskii functional (LKF) is designed to study the stability of continuous-time Takagi–Sugeno fuzzy systems with time-varying delay. The integrand of the LKF depends on integral variable and time t which can help to reduce the number of linear matrix inequalities (LMIs). Then, a new stability criterion is derived by analyzing the sign of the time derivatives of membership functions. Compared with the existing results, larger delay bounds can be obtained by applying the new criterion. In the end, two examples show the effectiveness of the conclusions.

Index Terms—Membership dependent Lyapunov–Krasovskii functional, Takagi–Sugeno’s (T–S) fuzzy model, time-varying delay.

I. INTRODUCTION

In the recent years, great effort has been made to obtain new results for Takagi–Sugeno (T–S) fuzzy systems [19]–[22], especially for T–S fuzzy systems with time-delay and finding the maximum delay bounds has attracted considerable attention. In [1], the technique of introducing free matrices is applied to obtain stability criteria. Then, [1] is improved in [2] by adding some integral inequalities and introducing more free variables. In [3], an augmented LKF which contains triple integral term is utilized to get less conservative results. In [4], the delay-partitioning and reciprocally convex combination are applied to find new delay bounds which are less conservative than [3]. In [5], an input–output approach is used to reduce the conservativeness. Recently, a less conservative stability criterion is proposed in [6] by using partial nonpositive Lyapunov functional with triple integral terms. Note that the results in [6] are based on Jensen inequality which leads to conservatism, in order to overcome this shortcoming, an improved reciprocally convex combination technique is applied in [7] and the obtained results are less conservative than [5] and [6].

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Almost at the same time, new delay bounds are found in [8] where an improved augmented LKF is constructed and some new cross terms are included. In [14], a new LKF which involves lower and upper bound of probabilistic time delay is designed to study the problem of reliable mixed and passivity-based control for a class of stochastic T–S fuzzy systems. For the first time, Zhang and Wang [16] established a novel delay-dependent analysis framework by directly exploiting the sources of augmented Lyapunov functional. Zhang *et al.* [15] proposed a distributed fuzzy optimal control law relied on actual physical meaning which is a new idea of fuzzy multi-agent system. For impulse systems, [17] shows that the delay may contribute to or do harm to the stabilization of delay systems. In [18], the bound of the state-dependent delay is not required but derives from the obtained stability result. It should be noted that the integrand of LKF used in all of the papers mentioned above are independent of the analysis of membership functions which is a very important factor for fuzzy systems. Losing the information of membership functions will lead to conservativeness [13].

In this technical paper, the problem of stability for T–S fuzzy systems with time-varying delay is investigated. First, the possible reason of conservativeness is analyzed. The analysis shows that if we can reduce the number of LMIs without changing the number of variables, we will get a less conservative stability criteria. To do this, a new LKF whose integrand depends not only on the integral variable but also on time t is designed to reduce the LMIs. Then, according to the sign of the time derivative of the membership function, a switching idea is applied to ensure the time derivative of the LKF is negative. In the end, the obtained stability criteria contains less number LMIs and can get less conservative results than the existing ones in the literature.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the T–S fuzzy systems with time-varying delays

Plant rule i ($i = 1, 2, \dots, r$): if θ_1 is μ_{i1} , and \dots and θ_p is μ_{ip} THEN

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + A_{di} x(t - h(t)) \\ x(t) &= \phi(t), t \in [-h(t) \quad 0] \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A_i \in \mathbb{R}^{n \times n}$ and $A_{di} \in \mathbb{R}^{n \times n}$ are known matrices, θ_i are known premise variables, μ_{ij} , $i = 1, 2, \dots, r$, $j = 1, 2, \dots, p$ are fuzzy sets; $h(t)$ is the time-varying delay satisfying

$$0 < h_1 \leq h(t) \leq h_2, d_1 \leq \dot{h}(t) \leq d_2. \quad (2)$$

Applying the center-average defuzzifier, product interferences and singleton fuzzifier, one has the following T-S model:

$$\dot{x}(t) = A_\lambda x(t) + A_{d\lambda} x(t-h(t)) \quad (3)$$

where $A_\lambda = \sum_{i=1}^r \lambda_i(\theta) A_i$, $A_{d\lambda} = \sum_{i=1}^r \lambda_i(\theta) A_{di}$, $\theta = [\theta_1(t), \theta_2(t), \dots, \theta_p(t)]$, and $\lambda_i(\theta)$ are membership functions defined as

$$\lambda_i(\theta) = \frac{w_i(t)}{\sum_{i=1}^r w_i(t)}, w_i(t) = \prod_{j=1}^p \mu_{ij}(\theta_j(t))$$

with $w_i(t) \geq 0$ and $\mu_{ij}(\theta_j(t))$ representing the grade of membership of $\theta_j(t)$ in μ_{ij} . For simplicity, single sum is written as $X_\lambda = \sum_{i=1}^r \lambda_i X_i$ and $\dot{X}_\lambda = (dX_\lambda/dt)$ is the time derivative of membership function dependent matrix. The time t is dropped for variables in the following analysis, for example, $x(t)$ is presented as x and $h(t)$ is presented as h . For any matrix X , $\text{He}(X) = X + X^T$.

The aim of this paper is to establish a less conservative stability criterion for T-S fuzzy system (3). Before presenting the main result, let us make some discussions as follows.

A. Possible Reason of Conservativeness

For simplicity, in the following, we only analyze single integral such as $\int_{t-h}^t x(s)^T Q x(s) ds$ and the conclusion is also applicable for double integral.

In most of the existing results, the chosen LKF candidate contains $\int_{t-h}^t x(s)^T Q x(s) ds$ where Q is only one positive matrix. Naturally, it is improved as $\int_{t-h}^t x(s)^T Q(s) x(s) ds$ in [10] to introduce more variables, however, since $Q(s)$ is dependent on the integral variable s , the number of LMIs increases at the same time. For example, for the simplest case (delay independent), choosing the LKF as $V(x) = x^T P x + \int_{t-h}^t x(s)^T Q x(s) ds$ yields the delay-independent stability condition [12]

$$\begin{bmatrix} PA_i + A_i^T P + Q & PA_{di} \\ * & Q \end{bmatrix} < 0, i = 1, 2 \dots r. \quad (4)$$

If Q is replaced by $Q(s) = \sum_{i=1}^r \lambda_i(\theta(s)) Q_i$, we have

$$\begin{bmatrix} PA_i + A_i^T P + Q_j & PA_{di} \\ * & Q_i \end{bmatrix} < 0, i, j = 1, 2 \dots r. \quad (5)$$

The number of free variables increases from 2 to $1+r$, while the number of LMIs increases from r to r^2 at the same time. Equation (5) is no better than (4) because more LMIs means more constraints but (5) is better than (4) if Q_j can be replaced by Q_i . Finding a method to reduce the number of LMIs without changing the number of variables is important. One possible choice is replacing Q with $Q(t) = \sum_{i=1}^r \lambda_i(\theta(t)) Q_i$ instead of $Q(s)$. The following lemmas are useful to deal with the time-derivative of the LKF.

Lemma 1 (see [11]): For a function $F(t) = \int_{\alpha_1}^{\alpha_2} f(s, t) ds$, if α_1, α_2 are differentiable for t and $f(s, t)$ is continuous for s and partially derivative for t , we have

$$\frac{dF(t)}{dt} = \dot{\alpha}_2 f(\alpha_2, t) - \dot{\alpha}_1 f(\alpha_1, t) + \int_{\alpha_1}^{\alpha_2} \frac{\partial f(s, t)}{\partial t} ds. \quad (6)$$

Lemma 2: For a function $F(t) = \int_{\theta_1}^{\theta_2} \int_{s_1(\theta, t)}^{s_2(\theta, t)} f(\theta, s, t) ds d\theta$ where θ_1, θ_2 are differentiable for t ; $s_1(\theta, t), s_2(\theta, t)$ are continuous for θ and partially derivative for t ; $f(s, \theta, t)$ is continuous for s, θ and partially derivative for t , we have

$$\begin{aligned} \frac{dF(t)}{dt} &= \dot{\theta}_2 \int_{s_1(\theta_2, t)}^{s_2(\theta_2, t)} f(\theta_2, s, t) ds - \dot{\theta}_1 \int_{s_1(\theta_1, t)}^{s_2(\theta_1, t)} f(\theta_1, s, t) ds \\ &+ \int_{\theta_1}^{\theta_2} \frac{\partial s_2(\theta, t)}{\partial t} f(\theta, s_2(\theta, t), t) d\theta \\ &+ \int_{\theta_1}^{\theta_2} \int_{s_1(\theta, t)}^{s_2(\theta, t)} \frac{\partial f(\theta, s, t)}{\partial t} ds d\theta \\ &- \int_{\theta_1}^{\theta_2} \frac{\partial s_1(\theta, t)}{\partial t} f(\theta, s_1(\theta, t), t) d\theta. \end{aligned}$$

Proof: Let

$$F(t) = \int_{\theta_1}^{\theta_2} F_1(\theta, t) d\theta, F_1(\theta, t) = \int_{s_1(\theta, t)}^{s_2(\theta, t)} f(\theta, s, t) ds$$

then applying Lemma 1 we have

$$\frac{dF(t)}{dt} = \dot{\theta}_2 F_1(\theta_2, t) - \dot{\theta}_1 F_1(\theta_1, t) + \int_{\theta_1}^{\theta_2} \frac{\partial F_1(\theta, t)}{\partial t} d\theta. \quad (7)$$

Since

$$\begin{aligned} \frac{\partial F_1(\theta, t)}{\partial t} &= \frac{\partial s_2(\theta, t)}{\partial t} f(\theta, s_2(\theta, t), t) \\ &- \frac{\partial s_1(\theta, t)}{\partial t} f(\theta, s_1(\theta, t), t) + \int_{s_1(\theta, t)}^{s_2(\theta, t)} \frac{\partial f(\theta, s, t)}{\partial t} ds \end{aligned} \quad (8)$$

substituting (8) into (7), we get the conclusion. ■

B. Discussion of the Time Derivative of Membership Function

In the following, we will discuss how to ensure $\dot{X}_\lambda \leq 0$, $\dot{Y}_\lambda \leq 0$, $\dot{Z}_\lambda \leq 0$, $\dot{U}_\lambda \leq 0$ where $X_i > 0$, $Y_i > 0$, $Z_i > 0$, and $U_i > 0$. Note

$$\dot{X}_\lambda = \sum_{i=1}^r \dot{\lambda}_i X_i = \sum_{k=1}^{r-1} \dot{\lambda}_k (X_k - X_r) \quad (9)$$

$$\dot{Y}_\lambda = \sum_{i=1}^r \dot{\lambda}_i Y_i = \sum_{k=1}^{r-1} \dot{\lambda}_k (Y_k - Y_r) \quad (10)$$

$$\dot{Z}_\lambda = \sum_{i=1}^r \dot{\lambda}_i Z_i = \sum_{k=1}^{r-1} \dot{\lambda}_k (Z_k - Z_r) \quad (11)$$

$$\dot{U}_\lambda = \sum_{i=1}^r \dot{\lambda}_i U_i = \sum_{k=1}^{r-1} \dot{\lambda}_k (U_k - U_r) \quad (12)$$

where $\dot{\lambda}_k$ are the time-derivative of membership functions and are negative or positive as time goes by. Since X_i , Y_i , Z_i , and U_i are variables to be designed, we can use a switching idea to ensure $\dot{X}_\lambda \leq 0$, $\dot{Y}_\lambda \leq 0$, $\dot{Z}_\lambda \leq 0$, and $\dot{U}_\lambda \leq 0$ as follows:

$$\text{if } H_l, \text{ then } C_l \quad (13)$$

where H_l , $l = 1, 2, \dots, 2^{r-1}$ is the set that contains the possible permutations of $\dot{\lambda}_k$ and C_l is the set that contains the constraints of X_i , Y_i , Z_i , and U_i . For example, if $r = 3$, we have

$$\begin{aligned} \dot{X}_\lambda &= \dot{\lambda}_1(X_1 - X_3) + \dot{\lambda}_2(X_2 - X_3) \\ \dot{Y}_\lambda &= \dot{\lambda}_1(Y_1 - Y_3) + \dot{\lambda}_2(Y_2 - Y_3) \\ \dot{Z}_\lambda &= \dot{\lambda}_1(Z_1 - Z_3) + \dot{\lambda}_2(Z_2 - Z_3) \\ \dot{U}_\lambda &= \dot{\lambda}_1(U_1 - U_3) + \dot{\lambda}_2(U_2 - U_3). \end{aligned}$$

There are four constraints C_l , $l = 1, 2, 3, 4$ to ensure $\dot{X}_\lambda \leq 0$, $\dot{Y}_\lambda \leq 0$, $\dot{Z}_\lambda \leq 0$, $\dot{U}_\lambda \leq 0$ and (13) is expressed as follows:

$$\begin{aligned} &\text{If } H_1, \text{ then } C_1; \text{ If } H_2, \text{ then } C_2 \\ &\text{If } H_3, \text{ then } C_3; \text{ If } H_4, \text{ then } C_4 \end{aligned}$$

where

$$\begin{aligned} H_1 : & \dot{\lambda}_1 \leq 0, \dot{\lambda}_2 \leq 0; H_2: \dot{\lambda}_1 \leq 0, \dot{\lambda}_2 > 0 \\ H_3 : & \dot{\lambda}_1 > 0, \dot{\lambda}_2 \leq 0; H_4: \dot{\lambda}_1 > 0, \dot{\lambda}_2 > 0 \\ C_1 : & \left\{ \begin{array}{l} X_1 \geq X_3, X_2 \geq X_3, Y_1 \geq Y_3, \\ Y_2 \geq Y_3, Z_1 \geq Z_3, Z_2 \geq Z_3, \\ U_1 \geq U_3, U_2 \geq U_3. \end{array} \right\} \\ C_2 : & \left\{ \begin{array}{l} X_1 \geq X_3, X_2 < X_3, Y_1 \geq Y_3, \\ Y_2 < Y_3, Z_1 \geq Z_3, Z_2 < Z_3, \\ U_1 \geq U_3, U_2 < U_3. \end{array} \right\} \\ C_3 : & \left\{ \begin{array}{l} X_1 < X_3, X_2 \geq X_3, Y_1 < Y_3, \\ Y_2 \geq Y_3, Z_1 < Z_3, Z_2 \geq Z_3, \\ U_1 < U_3, U_2 \geq U_3. \end{array} \right\} \\ C_4 : & \left\{ \begin{array}{l} X_1 < X_3, X_2 < X_3, Y_1 < Y_3, \\ Y_2 < Y_3, Z_1 < Z_3, Z_2 < Z_3, \\ U_1 < U_3, U_2 < U_3. \end{array} \right\}. \end{aligned}$$

Based on the above discussion, we get the following lemma.

Lemma 3: For some membership function dependent matrices X_λ , Y_λ , Z_λ , and U_λ where $X_i > 0$, $Y_i > 0$, $Z_i > 0$, and $U_i > 0$ are free variables, we have $\dot{X}_\lambda \leq 0$, $\dot{Y}_\lambda \leq 0$, $\dot{Z}_\lambda \leq 0$, and $\dot{U}_\lambda \leq 0$, if the switching rules (13) are satisfied for $l = 1, 2, \dots, 2^{r-1}$.

III. MAIN RESULT

Theorem 1: For some given d_q , h_v , v , $q = 1, 2$, if there exist matrices $\mathbb{P}_i > 0$, $R_i > 0$, $Q_i > 0$, $S_i > 0$, X_{1i} , X_{2i} , X_{3i} , X_{4i} such that the inequalities (14)–(17) hold for $i, j, k = 1, 2, \dots, r$, the fuzzy system (3) is asymptotically stable

$$\dot{\mathbb{P}}_\lambda \leq 0, \dot{R}_\lambda \leq 0, \dot{Q}_\lambda \leq 0, \dot{S}_\lambda \leq 0 \quad (14)$$

$$\Phi_{iikvq} < 0 \quad (15)$$

$$\Phi_{ijkvq} + \Phi_{jikvq} \leq 0 \quad (16)$$

$$\begin{bmatrix} R_i & 0 & X_{1i} & X_{2i} \\ * & 3R_i & X_{3i} & X_{4i} \\ * & * & R_i & 0 \\ * & * & * & 3R_i \end{bmatrix} \geq 0 \quad (17)$$

where

$$\begin{aligned} \mathbb{P}_i &= \begin{bmatrix} P_{1i} & P_{2i} & P_{3i} \\ * & P_{4i} & P_{5i} \\ * & * & P_{6i} \end{bmatrix} \\ \Phi_{ijklq} &= \begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} & \bar{\Phi}_{13} & \bar{\Phi}_{14} & \bar{\Phi}_{15} \\ * & \bar{\Phi}_{22} & \bar{\Phi}_{23} & \bar{\Phi}_{24} & \bar{\Phi}_{25} \\ * & * & \bar{\Phi}_{33} & \bar{\Phi}_{34} & \bar{\Phi}_{35} \\ * & * & * & -\frac{12R_i}{h_2} & -\frac{4X_{4i}}{h_2} \\ * & * & * & * & -\frac{12R_i}{h_2} \end{bmatrix} \\ \bar{\Phi}_{11} &= \mathbf{He}(P_{1i}A_j + P_{2i}) + S_i + Q_i + h_2A_i^T R_k A_j - \frac{4}{h_2}R_i \\ \bar{\Phi}_{12} &= -(1-d_q)P_{2i} + (1-d_q)P_{3i} + h_2A_i^T R_k A_{dj} \\ &\quad + P_{1i}A_{dj} - \frac{1}{h_2}(2R_i + X_{1i} + X_{2i} + X_{3i} + X_{4i}) \\ \bar{\Phi}_{13} &= -P_{3i} - \frac{1}{h_2}(X_{2i} + X_{4i} - X_{1i} - X_{3i}) \\ \bar{\Phi}_{14} &= h_v A_j^T P_{2i} + h_v P_{4i} + \frac{1}{h_2}6R_i \\ \bar{\Phi}_{15} &= (h_2 - h_v)A_j^T P_{3i} + (h_2 - h_v)P_{5i} + \frac{2}{h_2}(X_{2i} + X_{4i}) \\ \bar{\Phi}_{22} &= -(1-d_q)Q_i + h_2A_{di}^T R_k A_{dj} \\ &\quad - \frac{1}{h_2}(8R_i + He(X_{3i} + X_{4i} - X_{1i} - X_{2i})) \\ \bar{\Phi}_{23} &= -\frac{1}{h_2}(2R_i + X_{1i} - X_{2i} - X_{3i} + X_{4i}) \\ \bar{\Phi}_{24} &= h_v A_{dj}^T P_{2i} - h_v(1-d_q)P_{4i} + h_v(1-d_q)P_{5i}^T \\ &\quad + \frac{2}{h_2}(3R_i + X_{3i}^T + X_{4i}^T) \\ \bar{\Phi}_{25} &= (h_2 - h_v)A_{dj}^T P_{3i} - (h_2 - h_v)(1-d_q)(P_{5i} - P_{6i}) \\ &\quad + \frac{2}{h_2}(-X_{2i} + X_{4i} + 3R_i) \\ \bar{\Phi}_{33} &= -S_i - \frac{4}{h_2}R_i, \bar{\Phi}_{34} = -h_v P_{5i}^T + \frac{2}{h_2}(-X_{3i}^T + X_{4i}^T) \\ \bar{\Phi}_{35} &= -(h_2 - h_v)P_{6i} + \frac{6}{h_2}R_i. \end{aligned}$$

Proof: Based on [9], we design the LKF as

$$V(x) = V_1(x) + V_2(x) + V_3(x) + V_4(x)$$

where

$$V_1(x) = \tilde{x}^T P_\lambda \tilde{x}, V_2(x) = \int_{t-h_2}^t \int_{\theta}^t \dot{x}(s)^T R_\lambda \dot{x}(s) ds d\theta$$

$$V_3(x) = \int_{t-h}^t x(s)^T Q_\lambda x(s) ds$$

$$V_4(x) = \int_{t-h_2}^t x(s)^T S_\lambda x(s) ds$$

$$\tilde{x} = \begin{bmatrix} x^T & \int_{t-h}^t x^T(s) ds & \int_{t-h_2}^{t-h} x^T(s) ds \end{bmatrix}^T$$

$$\mathbb{P}_\lambda = \begin{bmatrix} P_{1\lambda} & P_{2\lambda} & P_{3\lambda} \\ P_{2\lambda}^T & P_{4\lambda} & P_{5\lambda} \\ P_{3\lambda}^T & P_{5\lambda}^T & P_{6\lambda} \end{bmatrix}.$$

It follows that:

$$\dot{V}_1(x) = 2\dot{\tilde{x}}^T P_\lambda \tilde{x} + \tilde{x}^T \dot{P}_\lambda \tilde{x}.$$

Applying Lemmas 1 and 2, we have

$$\dot{V}_2(x) = h_2 \dot{x}^T R_\lambda \dot{x} - \int_{t-h_2}^t \dot{x}(s)^T R_\lambda \dot{x}(s) ds$$

$$+ \int_{t-h_2}^t \int_{\theta}^t \dot{x}(s)^T \dot{R}_\lambda \dot{x}(s) ds d\theta$$

$$\dot{V}_3(x) = x^T Q_\lambda x - (1 - \dot{h})x(t-h)^T Q_\lambda x(t-h) + \int_{t-h}^t x(s)^T \dot{Q}_\lambda x(s) ds$$

$$\dot{V}_4(x) = x^T S_\lambda x - x(t-h_2)^T S_\lambda x(t-h_2) + \int_{t-h_2}^t x(s)^T \dot{S}_\lambda x(s) ds.$$

Thus, the time derivative of $V(x_t)$ along the trajectories of (3) is

$$\dot{V}(x) = \zeta^T \Phi \zeta - \int_{t-h_2}^t \dot{x}(s)^T R_\lambda \dot{x}(s) ds + \Psi$$

where

$$\zeta = \begin{bmatrix} x \\ x(t-h) \\ x(t-h_2) \\ \frac{1}{h} \int_{t-h}^t x(s) ds \\ \frac{1}{h_2-h} \int_{t-h_2}^{t-h} x(s) ds \end{bmatrix}^T$$

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & -P_{3\lambda} & \Phi_{14} & \Phi_{15} \\ * & \Phi_{22} & 0 & \Phi_{24} & \Phi_{25} \\ * & * & -S_\lambda & -hP_{5\lambda}^T & \Phi_{35} \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}$$

$$\Phi_{11} = \mathbf{He}(P_{1\lambda} A_\lambda + P_{2\lambda}) + S_\lambda + Q_\lambda + h_2 A_\lambda^T R_\lambda A_\lambda$$

$$\Phi_{12} = P_{1\lambda} A_{d\lambda} - (1 - \dot{h})P_{2\lambda} + (1 - \dot{h})P_{3\lambda} + h_2 A_\lambda^T R_\lambda A_{d\lambda}$$

$$\Phi_{14} = h A_\lambda^T P_{2\lambda} + h P_{4\lambda}$$

$$\Phi_{15} = (h_2 - h) A_\lambda^T P_{3\lambda} + (h_2 - h) P_{5\lambda}$$

$$\Phi_{22} = -(1 - \dot{h})Q_\lambda + h_2 A_{d\lambda}^T R_\lambda A_{d\lambda}$$

$$\Phi_{24} = h A_{d\lambda}^T P_{2\lambda} - h(1 - \dot{h})P_{4\lambda} + h(1 - \dot{h})P_{5\lambda}^T$$

$$\Phi_{25} = (h_2 - h) A_{d\lambda}^T P_{3\lambda} - (h_2 - h)(1 - \dot{h})(P_{5\lambda} - P_{6\lambda})$$

$$\Phi_{35} = -(h_2 - h)P_{6\lambda}$$

$$\Psi = \tilde{x}^T \dot{P}_\lambda \tilde{x} + \int_{t-h}^t x(s)^T \dot{Q}_\lambda x(s) ds$$

$$+ \int_{t-h}^t x(s)^T \dot{S}_\lambda x(s) ds + \int_{t-h_2}^t \int_{\theta}^t \dot{x}(s)^T \dot{R}_\lambda \dot{x}(s) ds d\theta.$$

Because of the constraint (14), we have

$$\dot{V}(x) \leq \zeta^T \Phi \zeta - \int_{t-h_2}^t \dot{x}(s)^T R_\lambda \dot{x}(s) ds.$$

Then, applying the method in [9] to deal with the term $-\int_{t-h_2}^t \dot{x}(s)^T R_\lambda \dot{x}(s) ds$, we get the conclusion. ■

Since the constraints (14) in Theorem 1 are not LMIs, we design the following algorithm to find the maximum delay bound.

Algorithm 1: Based on Lemma 3, applying (15)–(17) with each constraint \mathbf{C}_l , we get a corresponding delay bound denoted as h_{2l} , $l = 1, 2, \dots, 2^{r-1}$ and the final maximum delay bound is $\bar{h}_2 = \min_{1 \leq l \leq 2^{r-1}} (h_{2l})$.

Remark 1: At any time t , there exists a corresponding constraint $\bar{\mathbf{C}}_l$ such that the time derivative of LKF is negative. Since \bar{h}_2 is the minimum of all the h_{2l} , $l = 1, 2, \dots, 2^{r-1}$, the fuzzy system (3) is asymptotically stable for any delay belonging to the interval $[h_1, \bar{h}_2]$.

Remark 2: Different from the existing results [4]–[8], the integrand of the LKF designed in this paper depends not only on the integral variable but also on time t . This design can help to reduce the number of LMIs without changing the number of free variables. In addition, a switching method is applied to deal with the time derivative of the membership functions, although some matrix constraints have to be added, the simulations show that larger delay bound can be obtained by the criterion in this paper than existing results.

Remark 3: Note that the number of decision variables in Theorem 1 is $(10n^2 + 3n)r$, while, the number of decision variables in [8, Th. 1] is $137n^2 + 6n + r(36n^2 + n)$ and in [7, Th. 1] is $42.5n^2 + 8.5n$. The method in this paper is simple but effective and can be combined with other techniques such as the improved reciprocally convex combination in [7] or the improved augmented LKF in [8].

For some two-rule fuzzy systems, especially the premise variable is independent of the system states and the time derivative of the membership function is monotone increasing or monotone decreasing (for example, $\lambda_1 = (1/[1 + \exp(-t)])$, $\lambda_2 = (\exp(-t)/[1 + \exp(-t)])$), we have the following corollary.

Corollary 1: For some given $h_v > 0$, d_q , $v, q = 1, 2$, if $\dot{\lambda}_1 > 0$ and there exist matrices $\mathbb{P}_i > 0$, $R_i > 0$, $Q_i > 0$, $S_i > 0$, X_{1i} , X_{2i} , X_{3i} , X_{4i} such that

$$\mathbb{P}_2 > \mathbb{P}_1, R_2 > R_1, Q_2 > Q_1, S_2 > S_1 \quad (18)$$

and the inequalities (15)–(17) hold for $i, j, k = 1, 2$, the two-rule fuzzy system (3) is asymptotically stable for any delay $h(t)$ belonging to $[h_1, h_2]$.

Proof: If $r = 2$, we have

$$\dot{\mathbb{P}}_\lambda = \dot{\lambda}_1(\mathbb{P}_1 - \mathbb{P}_2), \dot{R}_\lambda = \dot{\lambda}_1(R_1 - R_2)$$

$$\dot{Q}_\lambda = \dot{\lambda}_1(Q_1 - Q_2), \dot{S}_\lambda = \dot{\lambda}_1(S_1 - S_2).$$

Since $\dot{\lambda}_1 > 0$, (14) is ensured by (18), and thus the proof is completed. ■

TABLE I
MAXIMUM DELAY BOUND OBTAINED BY
DIFFERENT METHODS WITH $d_2 = 0.1$

Method	Maxium h_2	Method	Maxium h_2
[4]	1.7659		
[5]	1.5253	Theorem 1 with (I)	1.8146
[6]	1.7718	Theorem 1 with (II)	1.8146
[7]	1.8276	Theorem 1 with Algorithm 1	2.7269
[8]	2.3268		

IV. SIMULATION EXAMPLE

Example 1: Consider the following nonlinear system with time-varying delay:

$$\begin{aligned}\dot{x}_1 &= 0.5(1 - \sin^2(\theta))x_2 - x_1(t-h) - (1 + \sin^2(\theta))x_1 \\ \dot{x}_2 &= \text{sgn}\left(|\theta| - \frac{\pi}{2}\right)(0.9 \cos^2(\theta) - 1)x_1(t-h) \\ &\quad - x_2(t-h) - (0.9 + 0.1 \cos^2(\theta))x_2.\end{aligned}$$

This nonlinear system can be modeled as a two-rule T-S fuzzy system with time-varying delay as in [4]–[8] with the membership functions

$$\lambda_1 = \frac{1}{1 + \exp(-2\theta)}, \lambda_2 = \frac{\exp(-2\theta)}{1 + \exp(-2\theta)}$$

and

$$\begin{aligned}A_1 &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix} \\ A_{d1} &= \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}.\end{aligned}$$

According to whether the memberships are dependent on the system states, we have the following two cases.

A. $\theta = x_I$

For this case, the membership functions are dependent on the system states. Let $d_2 = 0.1$, the maximum delay bound $h_2 = 1.7659, h_2 = 1.5253, h_2 = 1.7718$ can be found in [4]–[6], respectively. Recently, the maximum delay bound $h_2 = 1.8276$ is obtained by using the improved reciprocally convex combination technique in [7] as $d_2 = 0.1$, almost at the same time, $h_2 = 2.3268$ is obtained by applying the augmented LKF in [8]. As $d_1 = 0, d_2 = 0.1, h_1 = 1$, applying the inequalities (15)–(17) with the constraint $C_1 : \{\mathbb{P}_1 \geq \mathbb{P}_2, Q_1 \geq Q_2, R_1 \geq R_2, S_1 \geq S_2\}$ and $C_2 : \{\mathbb{P}_1 < \mathbb{P}_2, Q_1 < Q_2, R_1 < R_2, S_1 < S_2\}$, respectively, we get $h_{22} = 2.7269, h_{21} = 2.7725$, so the final maximum delay bound is $\bar{h}_2 = \min_{1 \leq v \leq 2}(h_{2v}) = 2.7269$. Table I shows the maximum delay obtained by different methods. Obviously, the method in this paper is less conservative than the existing ones.

We also consider two special cases of the new LKF.

(I) $\mathbb{P}_\lambda, Q_\lambda, R_\lambda, S_\lambda$ are replaced by \mathbb{P}, Q, R, S .

(II) $\mathbb{P}_\lambda, Q_\lambda, R_\lambda, S_\lambda$ are replaced by $\mathbb{P}, Q_{\lambda(s)} = \sum_{i=1}^r \lambda_i(\theta(s))Q_i, R, S_{\lambda(s)} = \sum_{i=1}^r \lambda_i(\theta(s))S_i$.

Case I means the used LKF is independent of the membership functions. Case II means \mathbb{P}, R are independent of the membership functions and $Q_{\lambda(s)}, S_{\lambda(s)}$ are dependent on the integrand

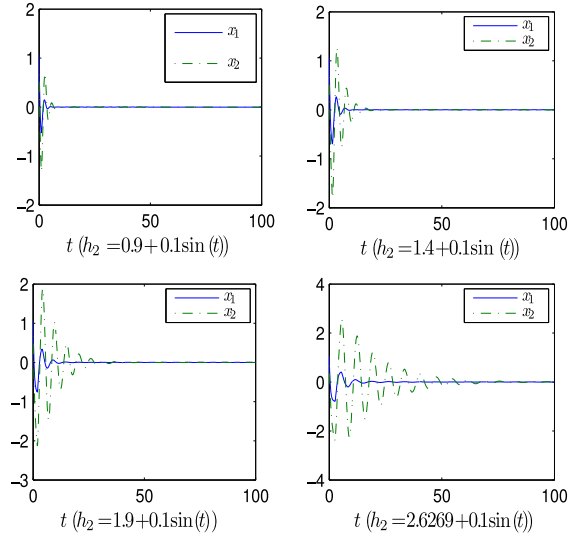


Fig. 1. State responses of the system for different delays.

variable s instead of time t . Applying Theorem 1 with (I), we only get $h_2 = 1.8146$ for $d_2 = 0.1$ which is even more conservative than [7]. Applying Theorem 1 with (II), we get the same result as case (I). This simulation shows that the LKF used in Theorem 1 is very effective. The trajectories of the system with $\phi(t) = [1.5 \ 1]^T, h = 0.9 + 0.1 \sin(t), h = 1.4 + 0.1 \sin(t), h = 1.9 + 0.1 \sin(t)$, and $h = 2.6269 + 0.1 \sin(t)$ are shown in Fig. 1 which shows that the considered system is asymptotically stable for any delays satisfying $h_2 \leq 2.7269$ but the needed time is different (larger delay need more time to be stable).

B. $\theta = t$

For this case, the membership functions are independent of the system states. Since $\hat{\lambda}_1 = ([2 \exp(-2t)]/[1 + \exp(-2t)]^2) > 0$, applying Corollary 1 we get the maximum delay bound $h_2 = 2.7269$ which is the same as case in Section IV-A.

Example 2: Consider the two-rule fuzzy system with the following system matrices:

$$\begin{aligned}A_1 &= \begin{bmatrix} -3.2 & 0.6 \\ 0 & -2.1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} \\ A_{d1} &= \begin{bmatrix} 1 & 0.9 \\ 0 & 2 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix}.\end{aligned}$$

For this example, as $d_1 = 0, d_2 = 0.1, h_1 = 0$, applying the inequalities (15)–(17) with the constraint $C_1 : \{\mathbb{P}_1 \geq \mathbb{P}_2, Q_1 \geq Q_2, R_1 \geq R_2, S_1 \geq S_2\}$ and $C_2 : \{\mathbb{P}_1 < \mathbb{P}_2, Q_1 < Q_2, R_1 < R_2, S_1 < S_2\}$, respectively, we get $h_{21} = 1.0362, h_{22} = 1.031$, so the final maximum delay bound is $\bar{h}_2 = \min_{1 \leq v \leq 2}(h_{2v}) = 1.031$ which is less conservative than the existing results, for example, applying the method in [10] with $\rho = 0.1$ we only get the maximum delay bound $h_2 = 0.4809$.

V. CONCLUSION

In this paper, we designed a new LKF to study the continuous-time T-S fuzzy system with time-varying delay.

A switching method is applied to deal with the time derivative of the membership functions. The simulation shows that the method in this paper is effective and can get less conservative results than the existing ones. Because of the feedback gains, this method cannot be extended directly to stabilization or observer design, but after some changes, for example, changing V_1 as a simple one and using the zero equation we can get the stabilization condition, then using the one-step method we can design the observer, however, for the case that the premise variables depend on the states estimated by the fuzzy observer, the results are complicated and should be considered in the near future.

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