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New stress and velocity fields for highly frictional granular materials

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The idealised theory for the quasi-static flow of granular materials which satisfy the Coulomb-Mohr hypothesis is considered. This theory arises in the limit that the angle of internal friction approaches $\pi/2$, and accordingly these materials may be referred to as being 'highly frictional'. In this limit, the stress field for both two-dimensional and axially symmetric flows may be formulated in terms of a single nonlinear second order partial differential equation for the stress angle. To obtain an accompanying velocity field, a flow rule must be employed. Assuming the non-dilatant double-shearing flow rule, a further partial differential equation may be derived in each case, this time for the stream-function. Using Lie symmetry methods, a complete set of group-invariant solutions is derived for both systems, and through this process new exact solutions are constructed. Only a limited number of exact solutions for gravity driven granular flows are known, so these results are potentially important in many practical applications. The problem of mass flow through a two-dimensional wedge hopper is examined as an illustration.

Keywords: granular materials, exact solutions, Lie symmetries, double-shearing theory, highly frictional materials

1 Introduction

The statics and dynamics of granular materials have been studied extensively through the use of continuum mechanics modelling. With this approach, governing equations for the stress and velocity fields are formulated by coupling the conservation of mass and momentum equations with appropriate constitutive laws and flow rules. For rapid granular flow, the behaviour of particles is determined by inelastic collisions with one another, in a way analogous to dense gases. Accordingly, it is appropriate in this case to use constitutive laws which are similar to those employed in the field of fluid mechanics (see Campbell 1990 for example). For slow granular flows, the dominating mechanics involved is quite different. Here particles are continually sliding and rolling past each other, and it is friction between these particles which is the dominant force. This behaviour is often modelled with theories which involve a yield condition similar to that used in metal plasticity; one such condition, the Coulomb-Mohr hypothesis, is discussed below, and is used as the basis for the present study. The regime somewhere between rapid and slow granular flow may be referred to as the intermediate flow regime. Here the challenge is to determine constitutive equations which also describe rapid and slow flows in the appropriate asymptotic limits. Such a recent study is given in Tarbos, McNamara & Talu (2003). We also note that there are other continuum mechanics models which are designed to model the flow of granular materials in either the slow or intermediate regimes. One sophisticated theory is hypoplasticity. The details of this theory can be very complicated, and so while it may have the capacity to accurately describe granular deformations under many circumstances, it is often intractable in terms of mathematical analysis. Discussions on hypoplasticity, and illustrations of its use, may be found in the collection of papers edited by Kolymbas (2003), and references therein.

As mentioned above, we consider the flow of granular materials that satisfy the Coulomb-Mohr yield condition. This condition postulates that

$$\tau_n \leq \sigma_n \tan\phi + c \tag{1}$$

at each point and on every surface within the material, where σ_n is the normal compressive component of the traction vector on the surface, τ_n is the magnitude of the tangential component, and ϕ and c are two mechanical properties of the material. The angle of internal friction ϕ is a measure of the friction between granular particles as they slide across each other, while the cohesion c can be thought of as a measure of the "stickiness" between particles. Here we consider only slowly flowing materials, and we assume that the inertial terms in the momentum equations can be ignored, the flow referred to as being quasi-static. With this assumption, the equations for the stresses (the equilibrium equations plus the yield condition) decouple from those which describe the velocity field. We note that real-life slow granular flows rarely reach a precise steady-state. Instead there is usually some oscillation in the both the stress and velocity fields, the magnitude of which can be significant at very low velocities. This phenomena is sometimes referred to as 'slip-stick'. However, in many situations, such as mass-flow in industrial hoppers, the magnitude of the oscillations is small, and a steady-state flow field proves to be an accurate description of the time-averaged situation.

The equations for quasi-static flow of Coulomb-Mohr materials have attracted much interest, although much of the work to date has neglected the effects of gravity. Perhaps the most studied gravity-driven flow is that through a vertical wedge or cone, which is primarily used to model flow from a hopper. In the neighbourhood of the hopper outlet, such flows are described by similarity solutions, which are often referred to as describing the 'radial stress field'. These solutions were first considered by both Jenike (1964) and Sokolovskii (1965), and examples of further study are contained in Johanson (1964), Jenike (1965), Spencer & Bradley (1996), and many more. After substitution of the functional forms for the similarity solutions, the governing equations reduce to nonlinear ordinary differential equations, which must be solved numerically. Some exact solutions to various simple problems which include gravitational effects are given in Sokolovskii (1965), O'Mahony and Spencer (1985) and Spencer & Bradley (1992,2002). We emphasise that even with the assumptions behind the quasi-static flow of Coulomb-Mohr materials, analytical solutions to gravity-driven problems are still difficult to determine.

The equations for the stresses in quasi-static flow are quite widely accepted, and predict stress levels which are well in accord with experimental results. However, the correct formulation of the accompanying velocity equations is still controversial. Initially, the coaxial flow rule was used, which was originally proposed by R. Hill (1950) for metal plasticity and then adopted by Drucker & Prager (1952) for granular materials. This approach assumes the principal axes of stress and strain-rate coincide. An alternative is the double-shearing theory originally proposed by Spencer (1964,1982). In this theory the characteristic curves for the stresses and velocities coincide, and every deformation is assumed to consist of simultaneous shears along the two families of stress characteristics.

Here, we are particularly concerned with the idealised theory which arises by considering materials which are characterised by the limiting value for the angle of internal friction $\phi = \pi/2$. Materials with this property may be referred to as being 'highly frictional'. Of course care must be taken when interpreting results obtained from such an idealised theory. This caution notwithstanding, there are many reasons to pursue studies of highly frictional materials. Firstly, the governing equations for the quasi-static flow of a Coulomb-Mohr material depend explicitly on $\beta = \sin\phi$ (see equations (14)-(17) for plane strain, and equations (25)-(28) for axially symmetric strain). So even for moderately large values of the angle of internal friction, the value of β is close to unity, and the approximation $\beta = 1$ is quite reasonable. We therefore use this idealised theory to provide approximate or limiting behaviour of real granular materials which have moderately large values of the angle of internal friction. Alternately, we may view the equations derived for $\phi = \pi/2$ as describing the first term in a regular perturbation, where the correction terms (not considered here) are of order $1 - \beta$. For specific granular materials with moderately high values of the angle of internal friction, the reader is referred to Sture (1999), for example.

Secondly, the theory for highly frictional granular materials happens to be far more tractable than that for materials with general internal friction values. A consequence is that we are able to determine exact solutions to highly nonlinear granular flow problems, which we are unable to find otherwise. These exact solutions can therefore be used as a benchmark for numerical schemes devised to solve more general problems. For example, Hill & Cox (2001) and Cox & Hill (2004) revisit the quasi-static flow in a converging wedge in an attempt to model flow out of a hopper (the 'radial stress field'). Here the numerical scheme (which was used for various values of ϕ) was successfully tested against an exact solution for $\phi = \pi/2$. Furthermore, the fact that we are able to compute exact solutions to gravity-driven granular flow problems is important, because generally such solutions are rare.

There has been some recent interest in the quasi-static flow of highly frictional materials, with particular attention given to constructing exact solutions. In this special limit the problem for the stress field reduces to solving a single second-order nonlinear partial differential equation, which for plane flow is given by

$$h_{xx} - 2hh_{xy} + h^2 h_{yy} = 0, \tag{2}$$

while for axially symmetric flow is given by

$$h_{rr} - 2hh_{rz} + h^2 h_{zz} - \frac{1}{r}(h_r - hh_z) = 0. \tag{3}$$

Here the dependent variable is $h = \cot\psi$, where ψ is the stress angle (see Section 2 for details). The first exact solution to (2), found by Hill & Cox (2001), is a special case of the 'radial stress field' solutions discussed above (for values of $\phi \neq \pi/2$ there are no exact solutions). This similarity solution was extended to axially symmetric deformations in Cox & Hill (2003), and both solutions were used to model sandpiles and flow through hoppers in Cox & Hill (2003) and Hill & Cox (2002). Other families of exact group-invariant solutions have since been derived by Thamwattana & Hill (2003a,b). Using the double-shearing theory, the velocity fields for highly frictional materials are described by

$$\chi_{xx} - 2h\chi_{xy} + h^2\chi_{yy} = h_x\chi_y - h_y\chi_x \tag{4}$$

in two dimensions, and

$$\chi_{rr} - 2h\chi_{rz} + h^2\chi_{zz} - \frac{1}{r}(\chi_r - h\chi_z) = h_r\chi_z - h_r\chi_z \tag{5}$$

for axially-symmetric flows, where χ is a streamfunction, defined in equations (18) and (29). Exact solutions to these equations have been published by Cox & Hill (2004) for hopper flow and by Thamwattana & Hill (2003a) for other special functional forms. The purpose of the current study to use Lie point symmetry methods to determine the complete set of group-invariant solutions to both (2),(4) and (3),(5). This set includes the solutions just mentioned, as well as a number of exact solutions not previously considered.

We mention that for the case of plane strain, a related study has recently been undertaken by the authors in Johnpillai, McCue & Hill (2004), where an alternate mathematical formulation for the stresses is employed, which makes use of the Airy stress function. A complete set of group-invariant solutions for two-dimensional stress fields is derived using Lie symmetry methods, and it can be shown that these solutions are equivalent to ones derived here. In this sense the set of solutions presented here for equation (2) are not new, although many of them are expressed in terms of the stress angle ψ for the first time. We emphasise that when deriving the complete set of group-invariant solutions for the system (2),(4), the results equivalent to Johnpillai *et al.* (2004), which are for stress fields only, are found as part of the analysis. Furthermore, the approach undertaken by Johnpillai *et al.* works for two-dimensional deformations only, and does not contain any results relevant to the system (3),(5).

As mentioned above, in this study we employ the incompressible (or non-dilatant) double-shearing theory (Spencer 1964,1982) to determine velocity fields. Of course, granular materials are compressible, and initial granular deformations are usually accompanied by dilatation or compaction. However, in fully developed slow granular flows, the magnitude of the dilatation or compaction is often small, in which case the assumption of incompressibility is reasonable. In any event, Spencer's theory has been extended to include compressible flows by Mehrabadi & Cowin (1978), whose theory can be referred to as being the dilatant double-shearing theory. A similar study could be undertaken for this theory, at least in principle.

The plan of the paper is as follows. In the following section we present the governing equations for the incompressible quasi-static flow of a Coulomb-Mohr material, subject to the double-shearing flow rule. For the special case in which the angle of internal friction becomes $\phi = \pi/2$, we derive the partial differential equations (2)-(5). In Section 3 we use the Lie point symmetries admitted by the systems (2), (4) and (3), (5) to derive optimal systems of group-invariant solutions for each case. As noted above, some of these solutions have been considered previously, but for the new functional forms describing stress fields we determine the corresponding exact solutions in Sections 4. There has been some success in finding exact solutions for the velocity fields, and we give some examples in Section 5. In Section 6 we illustrate the results with reference to the problem of mass flow through a two-dimensional hopper, and show how the high friction angle results compare favourably to those with more realistic angles of internal friction. Finally, we close the paper in Section 7 with a brief discussion.

2 Governing equations

In this section we derive the partial differential equations (2)-(5) which govern the quasi-static flow of a highly frictional Coulomb-Mohr granular material for both plane strain and axially symmetric strain, given the double-shearing flow rule.

2.1 Equations for plane strain

We consider here two-dimensional granular flow in the (x, y)-plane with gravity acting in the negative y-direction. The components of the stress tensor in Cartesian coordinates are denoted by σ_{xx}, σ_{xy} and σ_{yy}, and the principal components of stress by σ_I, σ_{II} and σ_{III}. These principal stresses are ordered so that $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$, and we adopt the convention that stresses are assumed positive in tension, so the maximum principal stress σ_I is (most often) the one which is smallest in magnitude (since granular materials are rarely in tension).

The stress angle ψ is defined by the relationship

$$\tan 2\psi = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}},$$

and is the angle between the positive x-axis and the axis corresponding to σ_I. It follows that the stress components can be written as

$$\sigma_{xx} = -p + q\cos 2\psi, \quad \sigma_{xy} = q\sin 2\psi, \quad \sigma_{yy} = -p - q\cos 2\psi, \tag{6}$$

where p and q are stress invariants, given by

$$p = -\tfrac{1}{2}(\sigma_I + \sigma_{III}) = -\tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}), \tag{7}$$

$$q = \tfrac{1}{2}(\sigma_I - \sigma_{III}) = \tfrac{1}{2}\left\{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\right\}^{1/2}. \tag{8}$$

Physically, p represents an average pressure, while q is the maximum shear stress.

Consider an arbitrary surface whose unit normal $\mathbf{n}$ makes the angle δ with the positive x-axis, and thus is given by $\mathbf{n} = \cos\delta\mathbf{i} + \sin\delta\mathbf{j}$. The normal compressive component and the magnitude of the tangential component of the traction vector are given by

$$\sigma_n = p - q\cos 2(\delta - \psi), \quad \tau_n = q|\sin 2(\delta - \psi)|,$$

so that the quantity $\tau_n - \sigma_n \tan\phi$ attains is maximum when

$$\delta = \psi \pm (\tfrac{1}{4}\pi - \tfrac{1}{2}\phi). \tag{9}$$

For these special values of δ, we denote σ_n by σ and τ_n by τ, so that

$$\sigma = p - q\sin\phi, \quad \tau = q\cos\phi. \tag{10}$$

These results are now used to interpret the yield condition (1).

The Coulomb-Mohr yield hypothesis states that slip may occur on the arbitrary surface element if the condition (1) holds with equality. In this case δ must be the special angle (9), so that

$$\tau = \sigma\tan\phi + c, \tag{11}$$

or equivalently

$$q = p\sin\phi + c\cos\phi. \tag{12}$$

We recall that $0 \leq \phi \leq \pi/2$ is the angle of internal friction, and $c \geq 0$ is the coefficient of cohesion, and assume that both of these the quantities are constants.

As discussed in the Introduction, for quasi-static flow the stresses are completely specified by the yield condition (12) and the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho g, \tag{13}$$

where ρ is the bulk density and g is the acceleration due to gravity. The latter are the (steady) momentum equations without the inertial terms, which may be ignored since we are only interested in slow granular flows. By assuming the density ρ is a constant, and substituting (6) and (12) into (13), we may eliminate p to give

$$\frac{\partial q}{\partial x} = \frac{\beta}{\beta^2 - 1}\{2q\psi_x \sin 2\psi - 2q\psi_y(\beta + \cos 2\psi) + \rho g \beta \sin 2\psi\}, \tag{14}$$

$$\frac{\partial q}{\partial y} = \frac{\beta}{\beta^2 - 1}\{2q\psi_x(\beta - \cos 2\psi) - 2q\psi_y \sin 2\psi + \rho g(1 - \beta \cos 2\psi)\}, \tag{15}$$

which are the governing partial differential equations for arbitrary values of the angle of internal friction ϕ. For convenience we have introduce the parameter β, which is simply $\beta = \sin\phi$.

For quasi-steady granular flow, the equations for the stresses decouple from those which describe the velocity field. The choice of the correct flow rule for the accompanying velocity field is rather controversial; here we use Spencer's double-shearing theory (1964,1982). The velocity components acting in the x- and y-direction are denoted by $u(x, y)$ and $v(x, y)$ respectively, and it follows that the conservation of mass equation for incompressible flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{16}$$

The system is closed by the double-shearing equation

$$\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos 2\psi - \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\sin 2\psi + \beta\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + 2\Omega\right) = 0, \tag{17}$$

where for steady flow Ω is given by

$$\Omega = u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y}.$$

The two equations (16) and (17) can be reduced to one by introducing the streamfunction $\chi(x, y)$ defined by

$$u = \frac{\partial \chi}{\partial y}, \quad v = -\frac{\partial \chi}{\partial x}. \tag{18}$$

Equation (16) is now identically satisfied, and (17) becomes

$$\chi_{xx}(\beta - \cos 2\psi) - 2\chi_{xy}\sin 2\psi + \chi_{yy}(\beta + \cos 2\psi) + 2\beta(\psi_x \chi_y - \psi_y \chi_x) = 0, \qquad (19)$$

where here subscripts denote partial differentiation. We are left to solve the three equations (14), (15) and (19) for the three unknowns q, ψ and χ.

2.2 Plane strain equations for $\phi = \pi/2$

The above formulation is standard, and follows closely the argument given by Spencer (1982). We now wish to consider the limiting case of $\phi = \pi/2$ (or equivalently, $\beta = 1$). In this limit the yield condition (12) becomes $q = p$, or alternately

$$\sigma_{xy}^2 = \sigma_{xx}\sigma_{yy}.$$

At first it appears that (11) implies the shear τ becomes infinite as $\phi \to \pi/2$, however from (10) it is seen that this is untrue. In fact, both τ and σ vanish in this limit in such a way that (11) holds. That is to say that through every point in the material there is a particular surface upon which both the normal and shear stresses are zero. This surface is the one whose normal vector points in the direction of the maximum principal stress (see (9)). It follows from this argument and also from (7)-(8) with $q = p$ that the maximum principal stress σ_I is zero in this limit.

By rewriting (14)-(15) as

$$(\beta - 1)\left(\cos\psi \frac{\partial q}{\partial x} + \sin\psi \frac{\partial q}{\partial y}\right) = \rho g \beta \sin\psi + 2\beta q(\psi_x \sin\psi - \psi_y \cos\psi), \qquad (20)$$

$$(\beta + 1)\left(\sin\psi \frac{\partial q}{\partial x} - \cos\psi \frac{\partial q}{\partial y}\right) = \rho g \beta \cos\psi - 2\beta q(\psi_x \cos\psi + \psi_y \sin\psi), \qquad (21)$$

it is clear that for the special case of $\beta = 1$ (that is, for highly frictional materials with $\phi = \pi/2$) we have

$$q = -\frac{\rho g}{2}\frac{1}{(\psi_x - \psi_y \cot\psi)}, \quad 2(q\sin\psi)_x = \rho g \cos\psi + 2(q\cos\psi)_y.$$

These two equations may be combined to give (2), where $h = \cot\psi$. The stresses may be recovered from h with the use of

$$q = p = -\frac{\rho g}{2}\frac{1+h^2}{hh_y - h_x}, \quad \sigma_{xx} = \rho g \frac{1}{hh_y - h_x},$$

$$\sigma_{xy} = -\rho g \frac{h}{hh_y - h_x}, \quad \sigma_{yy} = \rho g \frac{h^2}{hh_y - h_x}.$$

By substituting the value $\beta = 1$ into (19), we find

$$\chi_{xx} - 2\chi_{xy}\cot\psi + \chi_{yy}\cot^2\psi + \operatorname{cosec}^2\psi(\psi_x\chi_y - \psi_y\chi_x) = 0,$$

which, with the further substitution $h = \cot\psi$, simplifies to (4).

2.3 Equations for axially symmetric strain

For axially symmetric granular flow it is appropriate to use cylindrical polar coordinates (r, φ, z), with gravity acting in the negative z-direction, and all quantities independent of the variable φ. We define the stress angle to be

$$\tan 2\psi = \frac{2\sigma_{rz}}{\sigma_{rr} - \sigma_{zz}};$$

physically ψ is the angle between the maximum principal stress axis and the r-direction. It follows that three of the stress components may be written as

$$\sigma_{rr} = -p + q\cos 2\psi, \quad \sigma_{rz} = q\sin 2\psi, \quad \sigma_{zz} = -p - q\cos 2\psi, \tag{22}$$

where p and q are given by

$$p = -\tfrac{1}{2}(\sigma_{rr} + \sigma_{zz}), \quad q = \tfrac{1}{2}\left\{(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right\}^{1/2}. \tag{23}$$

We also require the Haar-von Karmann hypothesis, which states that the hoop stress, which is a principal stress, be equal to one of the other two principal stresses. We choose $\sigma_I = \sigma_{\varphi\varphi} = \sigma_{II} \geq \sigma_{III}$, and it follows that

$$\sigma_{\varphi\varphi} = -p + q. \tag{24}$$

As with the plane strain case considered previously, the governing equations consist of the equilibrium equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = \rho g,$$

together with the Coulomb-Mohr yield condition (12), where this time p and q are defined in (23). For axially symmetric flow we need the extra condition (24), and we may combine these equations, with the use of (22), to give the two coupled equations

$$\frac{\partial q}{\partial r} = \frac{\beta}{\beta^2 - 1}\left\{2q\psi_r \sin 2\psi - 2q\psi_z(\beta + \cos 2\psi) + \rho g \beta \sin 2\psi + \frac{q(\beta - 1)(\cos 2\psi - 1)}{r}\right\}, \quad (25)$$

$$\frac{\partial q}{\partial z} = \frac{\beta}{\beta^2 - 1}\left\{2q\psi_r(\beta - \cos 2\psi) - 2q\psi_z \sin 2\psi + \rho g(1 - \beta \cos 2\psi) + \frac{q(\beta - 1)\sin 2\psi}{r}\right\}, \quad (26)$$

for the two dependent variables q and ψ. Again, this formulation is standard for arbitrary values of the angle of internal friction, and the reader is referred to Spencer (1982) for further details.

For axially-symmetric flow, we denote the velocity components in the r- and z-directions by $u(r, z)$ and $v(r, z)$ respectively. The equation of mass is written as

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0, \quad (27)$$

while the double-shearing equation is

$$\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)\cos 2\psi - \left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right)\sin 2\psi + \beta\left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} + 2\Omega\right) = 0, \quad (28)$$

where

$$\Omega = u\frac{\partial \psi}{\partial r} + v\frac{\partial \psi}{\partial z}.$$

As in the plane-strain case, we introduce a streamfunction, this time defined by

$$u = \frac{1}{r}\frac{\partial \chi}{\partial z}, \quad v = -\frac{1}{r}\frac{\partial \chi}{\partial r}. \quad (29)$$

Conservation of mass (27) is now automatically satisfied, and the double-shearing equation (28) becomes

$$\chi_{rr}(\beta - \cos 2\psi) - 2\chi_{rz}\sin 2\psi + \chi_{zz}(\beta + \cos 2\psi) - \frac{1}{r}\chi_r(\beta - \cos 2\psi)$$
$$+\frac{1}{r}\chi_z \sin 2\psi + 2\beta(\psi_r\chi_z - \psi_z\chi_r) = 0. \quad (30)$$

2.4 Axially symmetric strain equations for $\phi = \pi/2$

We can rearrange (25) and (26) in an analogous way to (20)-(21), so that for the special case of $\beta = 1$ we may solve for q in terms of ψ

$$q = -\frac{\rho g}{2} \frac{1}{(\psi_r - \psi_z \cot \psi)}, \tag{31}$$

with $h = \cot \psi$ left to satisfy the second order nonlinear partial differential equation (3). With h determined, the stresses may be recovered with

$$q = p = -\frac{\rho g}{2} \frac{1 + h^2}{h h_z - h_r}, \quad \sigma_{rr} = \rho g \frac{1}{h h_z - h_r},$$
$$\sigma_{rz} = -\rho g \frac{h}{h h_z - h_r}, \quad \sigma_{zz} = \rho g \frac{h^2}{h h_z - h_r}.$$

For highly frictional materials with $\beta = 1$, the double-shearing equation (30) becomes

$$\chi_{rr} - 2\chi_{rz} \cot \psi + \chi_{zz} \cot^2 \psi - \frac{1}{r}\chi_r + \frac{1}{r}\chi_z \cot \psi + \mathrm{cosec}^2\psi(\psi_r \chi_z - \psi_z \chi_r) = 0.$$

By substituting $h = \cot \psi$ into this equation we arrive at (5).

3 Lie symmetry analysis

In this section we present Lie symmetries for the systems (2), (4) and (3), (5) and for each case derive the optimal system of group-invariant solutions.

3.1 Plane strain

We are concerned here with solutions to the system of equations (2), (4), which for convenience we rewrite:

$$\begin{aligned} h_{xx} - 2h h_{xy} + h^2 h_{yy} &= 0, \\ \chi_{xx} - 2h\chi_{xy} + h^2 \chi_{yy} &= h_x \chi_y - h_y \chi_x. \end{aligned}$$

It can be shown using classical Lie group analysis that this system admits an 8-parameter Lie transformation group with the 8 associated linearly independent operators

$$\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial y}, \quad \Gamma_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad \Gamma_4 = y\frac{\partial}{\partial y} + h\frac{\partial}{\partial h},$$

$$\Gamma_5 = x\frac{\partial}{\partial y} - \frac{\partial}{\partial h}, \quad \Gamma_6 = \frac{\partial}{\partial \chi}, \quad \Gamma_7 = h\frac{\partial}{\partial \chi}, \quad \Gamma_8 = \chi\frac{\partial}{\partial \chi} \tag{32}$$

(these generators were derived with the algebraic package DIMSYM). For any linear combination of these operators we may determine three invariants of the particular transformation group and hence derive functional forms for h and χ. There are infinitely many of these linear combinations, so we seek to classify the set of all functional forms into families whose members are all equivalent to each other.

We define a relation between two invariant solutions to hold true if the first one can be mapped to the other by applying a transformation group generated by a linear combination of the operators in (32). Since these mappings are reflexive, symmetric and transitive, the relation is an equivalence relation, which induces a natural partition on the set of all group-invariant solutions into equivalence classes. We need only present one solution from each equivalence class (as the rest may be found by applying appropriate group symmetries); a complete set of such solutions is referred to as an 'optimal system' of group-invariant solutions.

The problem of deriving an optimal system of group-invariant solutions is equivalent to finding an optimal system of generators (or subalgebras spanned by these operators). The method used here is that given by Olver (1986), which basically consists of taking linear combinations of the generators in (32), and reducing them to their simplest equivalent form by applying carefully chosen adjoint transformations

$$\mathrm{Ad}(\exp(\epsilon\Gamma_i))\Gamma_j = \Gamma_j - \epsilon[\Gamma_i, \Gamma_j] + \frac{1}{2}\epsilon^2[\Gamma_i, [\Gamma_i, \Gamma_j]] - \ldots.$$

Here $[\Gamma_i, \Gamma_j]$ is the usual commutator, given by

$$[\Gamma_i, \Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i;$$

the list of adjoint operators is shown in Table 1. For brevity we omit the details, and just state the result that an optimal system of generators is

$\{\Gamma_1 + b\Gamma_6, \quad \Gamma_1 + b\Gamma_6 \pm \Gamma_7, \quad \Gamma_1 \pm \Gamma_8, \quad \Gamma_2 + b\Gamma_6, \quad \Gamma_2 \pm \Gamma_7, \quad \Gamma_2 \pm \Gamma_8, \quad \Gamma_3 + a\Gamma_4 + b\Gamma_8,$

$\Gamma_3 + a\Gamma_4 \pm \Gamma_6, \quad \Gamma_3 + a(\Gamma_4 + \Gamma_8) \pm \Gamma_7, \quad \Gamma_4 + a\Gamma_1 + b\Gamma_8, \quad \Gamma_4 + a\Gamma_1 \pm \Gamma_6, \quad \Gamma_4 + a\Gamma_1 + \Gamma_8 \pm \Gamma_7, \quad \Gamma_5,$

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
Γ_1	Γ_1	Γ_2	$\Gamma_3 - \epsilon\Gamma_1$	Γ_4	$\Gamma_5 - \epsilon\Gamma_2$	Γ_6	Γ_7	Γ_8
Γ_2	Γ_1	Γ_2	$\Gamma_3 - \epsilon\Gamma_2$	$\Gamma_4 - \epsilon\Gamma_2$	Γ_5	Γ_6	Γ_7	Γ_8
Γ_3	$e^{\epsilon}\Gamma_1$	$e^{\epsilon}\Gamma_2$	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
Γ_4	Γ_1	$e^{\epsilon}\Gamma_2$	Γ_3	Γ_4	$e^{\epsilon}\Gamma_5$	Γ_6	$e^{-\epsilon}\Gamma_7$	Γ_8
Γ_5	$\Gamma_1 + \epsilon\Gamma_2$	Γ_2	Γ_3	$\Gamma_4 - \epsilon\Gamma_5$	Γ_5	Γ_6	$\Gamma_7 + \epsilon\Gamma_6$	Γ_8
Γ_6	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	$\Gamma_8 - \epsilon\Gamma_6$
Γ_7	Γ_1	Γ_2	Γ_3	$\Gamma_4 + \epsilon\Gamma_7$	$\Gamma_5 - \epsilon\Gamma_6$	Γ_6	Γ_7	$\Gamma_8 - \epsilon\Gamma_7$
Γ_8	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	$e^{\epsilon}\Gamma_6$	$e^{\epsilon}\Gamma_7$	Γ_8

Table 1: Table of adjoint operators. The (i, j)th entry is $\mathrm{Ad}(\exp(\epsilon\Gamma_i))\Gamma_j$, where the Γ_i are given by (32).

$$\Gamma_5 \pm \Gamma_8, \quad \Gamma_5 \pm \Gamma_7, \quad \Gamma_3 - \Gamma_4 \pm \Gamma_2 + b\Gamma_8, \quad \Gamma_3 - \Gamma_4 \pm \Gamma_2 \pm \Gamma_6, \quad \Gamma_3 - \Gamma_4 \pm \Gamma_2 \pm \Gamma_7 - \Gamma_8,$$

$$\Gamma_3 \pm \Gamma_5 + b\Gamma_8, \quad \Gamma_3 \pm \Gamma_5 \pm \Gamma_7, \quad \Gamma_5 \pm \Gamma_1, \quad \Gamma_5 + \Gamma_1 \pm \Gamma_8, \quad \Gamma_5 \pm \Gamma_1 \pm \Gamma_7, \quad \Gamma_6, \quad \Gamma_7, \quad \Gamma_8\}. \tag{33}$$

It is straightforward to derive the corresponding functional forms (again, the reader is referred to Olver (1986), or any standard text on Lie symmetry methods applied to differential equations). These functional forms are listed in Table 2. Note that for each symmetry which contains a plus-or-minus sign, we have taken the plus sign when computing the functional forms. The functional forms corresponding to the minus sign can be obtained via appropriate discrete symmetries. We also note that the symmetries Γ_6, Γ_7 and Γ_8 do not correspond to group-invariant solutions. With the unknown functions f and g determined, the list of functional forms for $h(x, y)$ and $\chi(x, y)$ in Table 2 represents an optimal system of group-invariant solutions to the system (2), (4).

Operator	$h(x,y)$	$\chi(x,y)$	ξ
$\Gamma_1 + b\Gamma_6$	$f(y)$	$bx + g(y)$	y
$\Gamma_1 + b\Gamma_6 \pm \Gamma_7$		$(b + f(y))x + g(y)$	
$\Gamma_1 \pm \Gamma_8$		$e^x g(y)$	
$\Gamma_2 + b\Gamma_6$	$f(x)$	$by + g(x)$	x
$\Gamma_2 \pm \Gamma_7$		$yf(x) + g(x)$	
$\Gamma_2 \pm \Gamma_8$		$e^y g(x)$	
$\Gamma_3 + a\Gamma_4 + b\Gamma_8$	$x^a f(\xi)$	$x^b g(\xi)$	y/x^{a+1}
$\Gamma_3 + a\Gamma_4 \pm \Gamma_6$		$\log x + g(\xi)$	
$\Gamma_3 + a(\Gamma_4 + \Gamma_8) \pm \Gamma_7$		$x^a g(\xi)$, $a \neq -1$	
$\Gamma_3 - (\Gamma_4 + \Gamma_8) \pm \Gamma_7$	$x^{-1} f(y)$	$\frac{1}{x} \log x f(y) + \frac{1}{x} g(y)$	y
$\Gamma_4 + a\Gamma_1 + b\Gamma_8$	$yf(\xi)$	$y^b g(\xi)$	$x - a \log y$
$\Gamma_4 + a\Gamma_1 \pm \Gamma_6$		$\log y + g(\xi)$	
$\Gamma_4 + a\Gamma_1 + \Gamma_8 \pm \Gamma_7$		$y \log y f(\xi) + yg(\xi)$	
Γ_5	$-y/x + f(x)$	$g(x)$	x
$\Gamma_5 \pm \Gamma_8$		$e^{y/x} g(x)$	
$\Gamma_5 \pm \Gamma_7$		$\frac{y}{x} f(x) - \frac{y^2}{2x^2} + g(x)$	
$\Gamma_3 - \Gamma_4 \pm \Gamma_2 + b\Gamma_8$	$\frac{1}{x} f(\xi)$	$x^b g(\xi)$	$y - \log x$
$\Gamma_3 - \Gamma_4 \pm \Gamma_2 \pm \Gamma_6$		$\log x + g(\xi)$	
$\Gamma_3 - \Gamma_4 \pm \Gamma_2 \pm \Gamma_7 - \Gamma_8$		$\frac{1}{x} g(\xi)$	
$\Gamma_3 \pm \Gamma_5 + b\Gamma_8$	$-\log x + f(\xi)$	$x^b g(\xi)$	$y/x - \log x$
$\Gamma_3 \pm \Gamma_5 \pm \Gamma_7$		$-\frac{1}{2} \log^2 x + \int \frac{1}{x} f(\xi)\, dx + g(\xi)$	
$\Gamma_5 \pm \Gamma_1$	$-x + f(\xi)$	$g(\xi)$	$y - x^2/2$
$\Gamma_5 + \Gamma_1 \pm \Gamma_8$		$e^x g(\xi)$	
$\Gamma_5 \pm \Gamma_1 \pm \Gamma_7$		$-\frac{1}{2} x^2 + \int f(\xi)\, dx + g(\xi)$	
Γ_6, Γ_7, Γ_8	N/A	N/A	N/A

Table 2: Optimal system of operators and functional forms for plane strain.

3.2 Axially symmetric strain

We consider the system (3), (5) and reproduce it here for convenience,

$$\begin{aligned} h_{rr} - 2hh_{rz} + h^2 h_{zz} - \frac{1}{r}(h_r - hh_z) &= 0, \\ \chi_{rr} - 2h\chi_{rz} + h^2 \chi_{zz} - \frac{1}{r}(\chi_r - h\chi_z) &= h_r \chi_z - h_z \chi_r. \end{aligned}$$

It can be shown that the Lie point symmetries of this system are spanned by the seven basis vectors

$$\begin{aligned} &\Gamma_1 = \frac{\partial}{\partial z}, \quad \Gamma_2 = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}, \quad \Gamma_3 = z\frac{\partial}{\partial z} + h\frac{\partial}{\partial h}, \\ &\Gamma_4 = r\frac{\partial}{\partial z} - \frac{\partial}{\partial h}, \quad \Gamma_5 = \frac{\partial}{\partial \chi}, \quad \Gamma_6 = h\frac{\partial}{\partial \chi}, \quad \Gamma_7 = \chi\frac{\partial}{\partial \chi}, \end{aligned} \tag{34}$$

for which the corresponding adjoint operators are shown in Table 3.2. Employing the

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
Γ_1	Γ_1	$\Gamma_2 - \epsilon\Gamma_1$	$\Gamma_3 - \epsilon\Gamma_1$	Γ_4	Γ_5	Γ_6	Γ_7
Γ_2	$e^{\epsilon}\Gamma_1$	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
Γ_3	$e^{\epsilon}\Gamma_1$	Γ_2	Γ_3	$e^{\epsilon}\Gamma_4$	Γ_5	$e^{-\epsilon}\Gamma_6$	Γ_7
Γ_4	Γ_1	Γ_2	$\Gamma_3 - \epsilon\Gamma_4$	Γ_4	Γ_5	$\Gamma_6 + \epsilon\Gamma_5$	Γ_7
Γ_5	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	$\Gamma_7 - \epsilon\Gamma_5$
Γ_6	Γ_1	Γ_2	$\Gamma_3 + \epsilon\Gamma_6$	$\Gamma_4 - \epsilon\Gamma_5$	Γ_5	Γ_6	$\Gamma_7 - \epsilon\Gamma_6$
Γ_7	Γ_1	Γ_2	Γ_3	Γ_4	$e^{\epsilon}\Gamma_5$	$e^{\epsilon}\Gamma_6$	Γ_7

Table 3: Table of adjoint operators. The (i, j)th entry is $\mathrm{Ad}(\exp(\epsilon\Gamma_i))\Gamma_j$, where the Γ_i are given by (34).

method used by Olver (1986), an optimal system of one-dimensional subalgebras of (34) is found to be generated by

$$\{\Gamma_1 + b\Gamma_5, \quad \Gamma_1 \pm \Gamma_6, \quad \Gamma_1 \pm \Gamma_7, \quad \Gamma_2 + a\Gamma_3 + b\Gamma_7, \quad \Gamma_2 + a\Gamma_3 \pm \Gamma_5, \quad \Gamma_2 + a(\Gamma_3 + \Gamma_7) \pm \Gamma_6,$$

$$\Gamma_3 + b\Gamma_7, \quad \Gamma_3 \pm \Gamma_5, \quad \Gamma_3 + \Gamma_7 \pm \Gamma_6, \quad \Gamma_4 + a\Gamma_1, \quad \Gamma_4 \pm \Gamma_7 + a\Gamma_1, \quad \Gamma_4 + a\Gamma_1 \pm \Gamma_6,$$

$$\Gamma_2 - \Gamma_3 \pm \Gamma_1 + a\Gamma_7, \quad \Gamma_2 - \Gamma_3 \pm \Gamma_1 \pm \Gamma_5, \quad \Gamma_2 - \Gamma_3 \pm \Gamma_1 - \Gamma_7 \pm \Gamma_6, \quad \Gamma_2 \pm \Gamma_4 + a\Gamma_7,$$
$$\Gamma_2 \pm \Gamma_4 \pm \Gamma_6, \quad \Gamma_5, \quad \Gamma_6, \quad \Gamma_7\}, \tag{35}$$

where a and b are arbitrary constants. The corresponding functional forms are presented in Table 4, where it is noted that, as with the plane strain case, we only take the plus option whenever there is a plus-or-minus sign. The other functional forms may be found by applying appropriate discrete symmetries.

4 Exact solutions for the stress fields

As mentioned earlier, for quasi-static flow the equations for the stress field decouple from those which describe the velocity field. As a consequence, (2) and (3) may be solved without reference to the velocity equations. In this section we consider the optimal system of group-invariant solutions to each of (2) and (3) separately, and derive exact solutions which have not been considered previously in the literature.

4.1 Plane strain

It is clear from Table 2 that the optimal system of group-invariant solutions to (2) have the corresponding functional forms

$$1. \quad h = f(y), \quad 2. \quad h = f(x), \quad 3. \quad h = x^a f\left(\frac{y}{x^{a+1}}\right), \quad 4. \quad h = yf(x - a\log y)$$
$$5. \quad h = -\frac{y}{x} + f(x), \quad 6. \quad h = \frac{1}{x}f(y - \log x), \quad 7. \quad h = -\log x + f\left(\frac{y}{x} - \log x\right),$$
$$8. \quad h = -x + f(y - \tfrac{1}{2}x^2). \tag{36}$$

Some of the exact solutions for the functional forms in families 1-5 have been determined previously and for these we simply state the results. The rest are yet to be explicitly considered (as noted in the Introduction, families equivalent to 1-8 have been examined by Johnpillai *et al.* 2004 using a different formulation), and we do so below.

Family 1. The solution here is $f(y) = C_1 y + C_2$, where C_1 and C_2 are constants (Thamwattana & Hill, 2003a).

Operator	$h(r,z)$	$\chi(r,z)$	ξ
$\Gamma_1 + b\Gamma_5$	$f(r)$	$bz + g(r)$	r
$\Gamma_1 \pm \Gamma_6$		$zf(r) + g(r)$	
$\Gamma_1 \pm \Gamma_7$		$e^z g(r)$	
$\Gamma_2 + a\Gamma_3 + b\Gamma_7$	$r^a f(\xi)$	$r^b g(\xi)$	z/r^{a+1}
$\Gamma_2 + a\Gamma_3 \pm \Gamma_5$		$\log r + g(\xi)$	
$\Gamma_2 + a(\Gamma_3 + \Gamma_7) \pm \Gamma_6$		$r^a g(\xi)$, $a \neq -1$	
$\Gamma_2 - (\Gamma_3 + \Gamma_7) \pm \Gamma_6$	$r^{-1} f(z)$	$\frac{1}{r}\log r f(z) + \frac{1}{r} g(z)$	z
$\Gamma_3 + b\Gamma_7$	$zf(r)$	$z^b g(r)$	r
$\Gamma_3 \pm \Gamma_5$		$\log z + g(r)$	
$\Gamma_3 + \Gamma_7 \pm \Gamma_6$		$z \log z f(r) + z g(r)$	
$\Gamma_4 + a\Gamma_1$	$-z/(r+a) + f(r)$	$g(r)$	r
$\Gamma_4 + a\Gamma_1 \pm \Gamma_7$		$e^{z/(r+a)} g(r)$	
$\Gamma_4 + a\Gamma_1 \pm \Gamma_6$		$\frac{z}{r+a} f(r) - \frac{z^2}{2(r+a)^2} + g(r)$	
$\Gamma_2 - \Gamma_3 \pm \Gamma_1 + b\Gamma_7$	$\frac{1}{r} f(\xi)$	$r^b g(\xi)$	$z - \log r$
$\Gamma_2 - \Gamma_3 \pm \Gamma_1 \pm \Gamma_5$		$\log r + g(\xi)$	
$\Gamma_2 - \Gamma_3 \pm \Gamma_1 - \Gamma_7 \pm \Gamma_6$		$\frac{1}{r} g(\xi)$	
$\Gamma_2 \pm \Gamma_4 + b\Gamma_7$	$-\log r + f(\xi)$	$r^b g(\xi)$	$z/r - \log r$
$\Gamma_2 \pm \Gamma_4 \pm \Gamma_6$		$-\frac{1}{2}\log^2 r + \int \frac{1}{r} f(\xi)\, dr + g(\xi)$	
Γ_5, Γ_6, Γ_7	N/A	N/A	N/A

Table 4: Optimal system of operators and functional forms for axial symmetry.

Family 2. The solution here is $f(x) = C_1 x + C_2$, where C_1 and C_2 are constants (Thamwattana & Hill, 2003a).

Family 3. In what follows s is a parameter, $I(s)$ is the integral

$$I(s) = \int^s \frac{e^{s/2}}{s^{1/2}}\, ds + C_1, \tag{37}$$

and C_1 and C_2 are constants of integration. For this family the function $f(\xi)$ satisfies the nonlinear ordinary differential equation

$$[f + (a+1)\xi]^2 f'' + [2f - (a-2)(a+1)\xi] f' + a(a-1) f = 0, \tag{38}$$

where $\xi = y/x^{a+1}$. This equation can be solved exactly for four different values of a. For $a = -1$, $h = f(y)/x$, where f is given by the parametric solution (Thamwattana & Hill, 2003a)

$$f = C_2 s^{1/2} e^{s/2}, \quad y = -\tfrac{1}{2} C_2 I(s). \tag{39}$$

For $a = 0$, we have $h = f(y/x)$, and f is given by (Hill & Cox 2001)

$$f = C_2 I(s), \quad \xi = \frac{y}{x} = C_2 \left(\frac{2e^{s/2}}{s^{1/2}} - I(s) \right), \tag{40}$$

and for $a = 1$, the functional form is $h = x f(y/x^2)$, where f is given parametrically by (Thamwattana & Hill, 2003b)

$$f = C_2 \left(\frac{2e^{s/2}}{s^{1/2}} - I(s) \right), \quad \xi = \frac{y}{x^2} = -\tfrac{1}{2} C_2 \left(\frac{2e^{s/2}}{s^{1/2}} + \frac{1-s}{s} I(s) \right). \tag{41}$$

For $a = 2$, the invariant solutions are of the form $h = x^2 f(y/x^3)$. This family is new, and f satisfies the equation

$$(f + 3\xi)^2 f'' + 2f(1 + f') = 0, \tag{42}$$

which admits the single Lie point symmetry $\xi\, \partial/\partial\xi + f\, \partial/\partial f$. It follows that we may reduce (42) to

$$(T+3)^2 \frac{d^2T}{dS^2} + [(T+3)^2 + 2T] \frac{dT}{dS} + 2T(T+1) = 0,$$

with the use of the canonical variables $T = f/\xi$, $S = \log\xi$. By making the substitution $T'(S) + T(S) + 1 = \eta(T)$, we arrive at

$$(T+3)^2(\eta - T - 1)\frac{d\eta}{dT} + 2T\eta = 0,$$

which, after introducing the new variable κ via $T = -3\kappa/(1+\kappa)$, becomes

$$\frac{d\kappa}{d\eta} = \frac{-\kappa(\eta+2) + 1 - \eta}{2\eta\kappa}.$$

The further substitution $\kappa = -u - \frac{1}{2}\eta$ gives rise to the Bernoulli-type equation

$$\frac{d\eta}{du} - \frac{2u}{2u+1}\eta = \frac{1}{2u+1}\eta^2.$$

This equation is easily integrated, and by working backwards we obtain, after some algebra, the parametric solution

$$f = C_2\left(\frac{2e^{s/2}}{s^{1/2}} + \frac{1-s}{s}I(s)\right), \quad \xi = -\tfrac{1}{3}C_2\left[\frac{2(s-2)e^{s/2}}{s^{3/2}} + \frac{3-s}{s}I(s)\right],$$

where $I(s)$ is given above by (37).

Family 4. These are invariant solutions of the form $h = yf(x - a\log y)$, where f must satisfy the equation

$$f''(1+af)^2 - ff'(2+af) = 0,$$

and primes are used to denote differentiation with respect to $\xi = x - a\log y$. By applying the transformation $u(f) = f'(\xi)$, we arrive at

$$\frac{du}{df} = \frac{(2+af)f}{(1+af)^2},$$

which is integrated to give

$$u = \frac{f^2}{1+af} + C_1^2,$$

where C_1 is an arbitrary constant. Further integration reveals the solution

$$\xi = \tfrac{1}{2}a\log[f^2 + C_1^2(1+af)] + \frac{2 - a^2C_1^2}{C_1\sqrt{4 - a^2C_1^2}}\left\{\arctan\left(\frac{2f + C_1^2 a}{C_1\sqrt{4-a^2C_1^2}}\right) - \frac{\pi}{2}\right\} + C_2, \quad (43)$$

where C_2 is another constant. This solution is valid provided $a^2C_1^2 \leq 4$, however for values of C_1 outside this range we may rewrite (43) accordingly. We note that for $a = 0$ the solution simplifies significantly to

$$f(x) = C_1 \tan(C_1 x - C_1 C_2).$$

Family 5. The solution here is $f(x) = C_1/x + C_2 x^2$, where C_1 and C_2 are constants (Thamwattana & Hill, 2003a).

Family 6. These are solutions to (2) of the form

$$h = \frac{1}{x} f(y - \log x),$$

which correspond to family 6 in (36). Here f satisfies the ordinary differential equation

$$(f+1)^2 f'' + (2f+3) f' + 2f = 0,$$

where the primes denote differentiation with respect to the variable $\xi = y - \log x$. By making the substitution $u(f) = f'(\xi)$ we obtain

$$(f+1)^2 u \frac{du}{df} + (2f+3)u + 2f = 0,$$

which, upon making the further substitution $w = f + 1$, may be rewritten as

$$u\left(\frac{du}{dw} + \frac{1}{w^2}\right) + \frac{2}{w}\left(u + 1 - \frac{1}{w}\right) = 0. \tag{44}$$

Now the quantity within the first set of brackets in (44) is the derivative of the quantity in the second set, so further simplifications can be made by setting $s = u + 1 - 1/w$. The result is the linear equation

$$\frac{dw}{ds} + \frac{s-1}{2s} w = \frac{1}{2s},$$

which has the solution

$$f = w - 1 = -2 + \tfrac{1}{2} s^{1/2} e^{-s/2} I(s), \tag{45}$$

where $I(s)$ is the integral defined in (37). It follows from

$$\frac{d\xi}{ds} = \frac{w}{2s}$$

that the parametric solution is completed by

$$\xi = y - \log x = -\tfrac{1}{2}\log s + \tfrac{1}{4}\int^s \frac{e^{-t/2}}{t^{1/2}} I(t)\,dt + C_2, \tag{46}$$

where C_2 is a constant of integration.

We note there is a particular solution $u + 1 - 1/w = 0$, which corresponds to

$$y + \log x = \log f - f + C_3.$$

Family 7. These are group-invariant solutions of the form

$$h = -\log x + f\left(\frac{y}{x} - \log x\right),$$

where f satisfies the ordinary differential equation

$$(f + \xi + 1)^2 f'' + (2f + 2\xi + 1)f' + 1 = 0,$$

with primes denoting differentiation with respect to $\xi = y/x - \log x$. To simplify this equation we first set $w = f + \xi + 1$, which yields

$$w''w^2 + (2w - 1)w' - 2(w - 1) = 0,$$

and then introduce the function $u(w) = w'(\xi)$, so that

$$u\left(\frac{du}{dw} - \frac{1}{w^2}\right) + \frac{2}{w}\left(u - 1 + \frac{1}{w}\right) = 0. \tag{47}$$

This ordinary differential equation is of the same form as (44), and the solution method is the same. The resulting parametric solution is

$$f = -\xi - 1 + \frac{e^{-s/2}}{2s^{1/2}} I(s), \quad \xi = -\tfrac{1}{4}\int^s \frac{e^{-t/2}}{t^{3/2}} I(t)\,dt + C_2,$$

where C_1 and C_2 are constants, and $I(s)$ is the integral defined in (37).

We note the particular solution $\xi = -f + C_3 e^{-f}$, with C_3 a constant, which corresponds to $u - 1 + 1/w = 0$.

Family 8. We consider here invariant solutions $h = -x + f(y - \frac{1}{2}x^2)$, corresponding to family 8 in (36). The function f satisfies the ordinary differential equation $f''f^2 - f' = 0$, with primes denoting differentiation with respect to $\xi = y - \frac{1}{2}x^2$. This equation is easily solved by setting $u(f) = f'(\xi)$, so that $u'f^2 = 1$. Straight-forward integration leads to the implicit solution

$$\xi = \frac{1}{C_1}(f + \log|C_1 f - 1|) + C_2,$$

where both C_1 and C_2 are constants of integration.

4.2 Axially symmetric strain

The optimal system of group-invariant solutions of (3) consists of the functional forms (see the second column of Table 4)

$$1.\quad h = f(r),\quad 2.\quad h = r^a f\left(\frac{z}{r^{a+1}}\right),\quad 3.\quad h = zf(r),\quad 4.\quad h = -\frac{z}{r+a} + f(r),$$

$$5.\quad h = \frac{1}{r}f(z - \log r),\quad 6.\quad h = -\log r + f\left(\frac{z}{r} - \log r\right). \tag{48}$$

Here the families 1, 2 (with $a = -1, 0, 2$) and 4 (with $a = 0$) have been considered previously. The rest are new, and we consider them in detail below.

Family 1. The solution here is $f(r) = C_1 r^2 + C_2$, where C_1 and C_2 are constants (Thamwattana & Hill, 2003a).

Family 2. For this family the function f satisfies

$$[f + (a+1)\xi]^2 f'' + [3f - (a-3)(a+1)\xi]f' + a(a-2)f = 0, \tag{49}$$

where $\xi = z/r^{a+1}$. In the following solutions for the values $a = -1, 0, 2, 3$ we use the integrals

$$K_1(s) = \int^s \frac{e^{s/3}}{s^{1/3}}\,ds + C_1,\quad K_2(s) = \int^s \frac{e^{s/3}}{s^{2/3}}\,ds + C_1, \tag{50}$$

where C_1 is a constant of integration.

For $a = -1$ we have $h = f(z)/r$, where f is given parametrically by (Thamwattana & Hill, 2003a)

$$f = C_2 s^{1/3} e^{s/3},\quad z = -\tfrac{1}{3}C_2 K_2(s).$$

For $a = 0$, the functional form is $h = f(z/r)$, and f is given by (Cox & Hill 2003)

$$f = C_2 K_1(s), \quad \xi = C_2 \left(\frac{3e^{s/2}}{s^{1/3}} - K_1(s) \right)$$

The functional form for $a = 2$ is $h = r^2 f(z/r^3)$. Here the solution for f is (Thamwattana & Hill, 2003b)

$$f = C_2 \left(\frac{3e^{s/3}}{s^{2/3}} - K_2(s) \right), \quad \xi = \frac{z}{r^3} = -\tfrac{1}{3} C_2 \left(\frac{3e^{s/3}}{s^{2/3}} + \frac{2-s}{s} K_2(s) \right).$$

The solution for $a = 3$ is new. Here, the corresponding functional form is $h = r^3 f(z/r^4)$ and the ordinary differential equation (49) is just

$$(f + 4\xi)^2 f'' + 3f(1 + f') = 0.$$

This equation is almost identical to (42), and the solution procedure is the same. We omit the details, and just state the parametric solution

$$f = C_2 \left(\frac{3e^{s/3}}{s^{1/3}} + \frac{1-s}{s} K_1(s) \right), \quad \xi = -\tfrac{1}{4} C_2 \left(\frac{3(s-3)e^{s/3}}{s^{4/3}} + \frac{4-s}{s} K_1(s) \right),$$

where $K_1(s)$ is defined in (50).

Family 3. For this family $h = zf(r)$, where f satisfies

$$f'' - (2f + 1/r)f' + f^2/r = 0. \tag{51}$$

We observe that on multiplying equation (51) by $1/r$, the equation may be integrated to yield the Riccatti equation

$$f' = -C_1^3 r + f^2,$$

which may be solved in the usual way to obtain (Polyanin & Zaitsev 1995)

$$f(r) = u'(r)/u(r), \quad u(r) = C_2 \mathrm{Ai}(C_1 r) + \mathrm{Bi}(C_1 r),$$

where $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are Airy functions (Abramowitz & Stegun 1970).

Family 4. The functional form for the operator $\Gamma_4 + a\Gamma_1$ is $h(r, z) = -z/(r + a) + f(r)$. Upon substitution of this functional form into (3), we arrive at the linear equation

$$(r + a)^2 [rf'' - f'] - (3r + a)f = 0. \tag{52}$$

On using MAPLE, the solution of (52) is given by

$$f(r) = \frac{C_1 + C_2(3r^2 + 8ar + 6a^2)r^2}{r + a}, \tag{53}$$

where C_1 and C_2 arbitrary constants of integration.

Family 5. We consider the functional form

$$h(r, z) = \frac{1}{r} f(z + \log r)$$

from family 5 in (48), and after substituting it into the partial differential equation (3), we arrive at

$$(f - 1)^2 f'' + (3f - 4)f' + 3f = 0, \tag{54}$$

where the primes denote differentiation with respect to the variable $\xi = z + \log r$. This example is evidently analogous to family 6 in Section 4.1, and we treat it the same way. Equation (54) may be transformed into

$$w^2 w'' + (3w - 1)w' + 3(w + 1) = 0$$

by making the substitution $w(\xi) = f - 1$, which itself may be reduced to

$$u\left(\frac{du}{dw} - \frac{1}{w^2}\right) + \frac{3}{w}\left(u + 1 + \frac{1}{w}\right) = 0, \tag{55}$$

by setting $u(w) = w'(\xi)$. A further transformation $s = u + 1 + 1/w$ is used to derive the linear equation

$$\frac{dw}{ds} + \frac{s - 1}{3s} w = \frac{1}{3s},$$

which has the solution

$$f = w + 1 = \tfrac{1}{3} s^{1/3} e^{-s/3} K_1(s),$$

where $K_1(s)$ is the integral defined in (50)). By integrating

$$\frac{d\xi}{ds} = \frac{1 - f}{3s},$$

we derive

$$\xi = z + \log r = \tfrac{1}{3} \log s - \tfrac{1}{9} \int^s \frac{e^{-t/3}}{t^{2/3}} K_1(t)\, dt + C_2,$$

where C_2 is a constant of integration.

We note there is a particular solution $u + 1 + 1/w = 0$, which corresponds to

$$z + \log r = \log f - f + C_3.$$

Family 6. Here group-invariant solutions

$$h = -\log r + f\left(\frac{z}{r} - \log r\right),$$

from family 6 in (48) are considered. The function f satisfies the ordinary differential equation

$$(f + \xi + 1)^2 f'' + (3f + 3\xi + 2)f' + 2 = 0,$$

where primes denote differentiation with respect to $\xi = z/r - \log r$. This equation may be transformed into

$$w''w^2 + (3w - 1)w' - 3(w - 1) = 0,$$

with $w = f + \xi + 1$, and then to

$$u\left(\frac{du}{dw} - \frac{1}{w^2}\right) + \frac{3}{w}\left(u - 1 + \frac{1}{w}\right) = 0$$

with $u(w) = w'(\xi)$. Again, this ordinary differential equation is very similar to (44), (47) and (55). By using analogous solution methods, it is found that the parametric solution consists of the two relations

$$f = -\xi - 1 + \frac{e^{-s/3}}{3s^{1/3}} K_2(s), \quad \xi = -\tfrac{1}{9}\int^s \frac{e^{-t/3}}{t^{4/3}} K_2(t)\, dt + C_2,$$

where C_2 is a constant, and $K_2(s)$ is the integral defined in (50).

We note there is a particular solution $\xi = -f + C_3 e^{-f}$, with C_3 a constant, which comes from $u - 1 + 1/w = 0$.

5 Some velocity fields

The previous section was concerned with the optimal system of group-invariant solutions to each of the single nonlinear partial differential equations (2) and (3). These equations govern the stress fields for two-dimensional and axially symmetric granular flows for highly frictional materials. For each of these solutions there are group-invariant solutions for the velocity fields, the functional forms for which are presented in Tables 2 and 4 (assuming the velocities are governed by the double-shearing theory). We may substitute these functional forms for the velocity fields into (4) or (5) (along with the appropriate solution for h) to yield ordinary differential equations, which are linear with non-constant coefficients, and are often too difficult to solve analytically. In this section we study some examples of these velocity fields, indicating typical differential equations which are encountered, and solving them where possible. We emphasise that examples of velocity fields have been chosen here, and that it is possible to solve for some of the functional forms not presented below.

5.1 Plane strain

Here we consider the accompanying velocity field for families 3 and 6 of (36), where the functional forms for these fields are given in table 2. For family 3 we have success in solving the resulting differential equations exactly, while for family 6 we do not.

Family 3 with $\chi = x^b g(\xi)$. First we consider the pair of functional forms $h = x^a f(\xi)$, $\chi = x^b g(\xi)$, where $\xi = y/x^{a+1}$. Here f satisfies (38), while g satisfies the linear ordinary differential equation

$$[f + (a+1)\xi]^2 g'' + (a - 2b + 2)[f + (a+1)\xi]g' + b(f' + b - 1)g = 0, \tag{56}$$

where the dashes denote differentiation with respect to ξ. In the Section 4.1 we presented solutions to (38) for the special values $a = -1, 0, 1, 2$. We list the corresponding solutions to (56) here, which are found with the help of the symbolic manipulation package MAPLE.

For $a = -1$, f is given by (39), and (56) reduces to

$$4s^2 \frac{d^2 g}{ds^2} - 2s(s - 2b)\frac{dg}{ds} - b(s - b + 2)g = 0,$$

which has the solution

$$g = s^{-b/2}(C_3 e^{s/2} + C_4).$$

For $a = 0$, the functional form f is now given by (40). The differential equation (56) becomes

$$4s^2\frac{d^2g}{ds^2} - 2s(s-2b-1)\frac{dg}{ds} - b(s-b+1)g = 0,$$

which can be solved to give

$$g = s^{-b/2}(C_3 I(s) + C_4). \tag{57}$$

For $a = 1$ the ordinary differential equation is

$$4s^2 I(s)\frac{d^2g}{ds^2} + 2s[(2b+1)I(s) - 2e^{s/2}s^{1/2}]\frac{dg}{ds} + b[(b-1)I(s) - 2e^{s/2}s^{1/2}]g = 0,$$

and the solution is given by

$$g = s^{-b/2}[C_3(2e^{s/2} - s^{1/2}I(s)) + C_4].$$

Similarly, for $a = 2$ (56) becomes

$$4s^2[I(s) - 2e^{s/2}s^{-1/2}]\frac{d^2g}{ds^2} + 4s[bI(s) - (2b+1)e^{s/2}s^{-1/2}]\frac{dg}{ds}$$

$$+b[(b-2)I(s) - 2(b-1)e^{s/2}s^{-1/2}]g = 0,$$

and the solution is

$$g = s^{-b/2}[C_3(2s^{1/2}e^{s/2} + (1-s)I(s)) + C_4].$$

The solutions for $a = 1$ and $a = 2$ can be checked by substitution (with MAPLE, for example).

Family 3 with $\chi = \log x + g(\xi)$. Now we consider the pair of functional forms $h = x^a f(\xi)$, $\chi = \log x + g(\xi)$, where $\xi = y/x^{a+1}$ The function f satisfies (38), but in this case g is given by the solution to

$$[f + (a+1)\xi]^2 g'' + (a+2)[f + (a+1)\xi]g' + f' - 1 = 0, \tag{58}$$

where, again, the dashes denote differentiation with respect to ξ. We solve (58) for $a = -1, 0, 1, 2$ by integrating directly, since (58) is first order in g'.

For $a=-1$ the functional form g satisfies

$$4s^2\frac{d^2g}{ds^2}-2s^2\frac{dg}{ds}=s+2,$$

which implies g is given by

$$g=-\tfrac{1}{2}\log s+C_3e^{s/2}+C_4$$

For $a=0$ equation (58) reduces to

$$4s^2\frac{d^2g}{ds^2}-2s(s-1)\frac{dg}{ds}=s+1;$$

here the solution for g is

$$g=-\tfrac{1}{2}\log s+C_3I(s)+C_4$$

For $a=1$ we have

$$4s^2I(s)\frac{d^2g}{ds^2}+2s[I(s)-2e^{s/2}s^{1/2}]\frac{dg}{ds}=I(s)+2e^{s/2}s^{1/2},$$

with

$$g=-\tfrac{1}{2}\log s-C_3(s^{1/2}I(s)-2e^{s/2})+C_4$$

and finally, for $a=2$ (58) becomes

$$2s^2[I(s)-2e^{s/2}s^{-1/2}]\frac{d^2g}{ds^2}-2e^{s/2}s^{1/2}\frac{dg}{ds}=I(s)-e^{s/2}s^{-1/2},$$

which can be solved to give

$$g=-\tfrac{1}{2}\log s+C_3(sI(s)-I(s)-2s^{1/2}e^{s/2})+C_4.$$

Family 3 with $\chi=\log xf(y)/x+g(y)/x$. There is one more functional form $\chi=\log xf(y)/x+g(y)/x$, which corresponds to $a=-1$ (see Table 2). Here g is the solution to

$$f^2g''+3fg'-(f'-2)g-f(f'+3)=0, \tag{59}$$

where f is given by (39). With this solution for f, the linear equation (59) reduces to

$$4s^2\frac{d^2g}{ds^2}-2s(s+2)\frac{dg}{ds}+(s+3)g=-(s-2)s^{1/2}e^{p/2},$$

which has the solution (MAPLE)

$$g = s^{1/2}e^{s/2}(-\tfrac{1}{2}\log s + C_3) + C_4 s^{1/2}.$$

Family 6 with $\chi = x^b g(\xi)$ and $\chi = \log x + g(\xi)$. For family 6 the functional form for h is $h = f(\xi)/x$, where $\xi = y - \log x$, and f is given by (45)-(46). We may investigate solutions for the streamfunction of the form $\chi = x^b g(\xi)$, where g satisfies

$$4s^2[I(s)e^{-s/2}s^{1/2} - 2)\frac{d^2g}{ds^2} + 2s[(2b - s + 2)I(s)e^{-s/2}s^{1/2} + 2(s - 2b - 1)]\frac{dg}{ds}$$
$$+b[(b + s - 2)I(s)e^{-s/2}s^{1/2} - 2(b + s - 1)]g = 0$$

and solutions of the form $\chi = \log x + g(\xi)$, where g satisfies

$$4s^2[I(s)e^{-s/2}s^{1/2} - 2)\frac{d^2g}{ds^2} + 2s[2(s - 1) - (s - 2)I(s)e^{-s/2}s^{1/2}]\frac{dg}{ds}$$
$$= 2(s - 1) - (s - 2)I(s)e^{-s/2}s^{1/2}. \tag{60}$$

In this example the differential equations appear too difficult to solve exactly, due to the complexity of the coefficients, although we may certainly integrate (60) once to give

$$g' = \frac{(s - 2)I(s) - 2s^{1/2}e^{s/2}}{2s^{3/2}(s^{1/2}I(s) - 2e^{s/2})} - \frac{e^{s/2}}{s^{1/2}(s^{1/2}I(s) - 2e^{s/2})}\int^s \frac{I(t)}{t^2}e^{-t/2}\,dt + C_3.$$

5.2 Axially symmetric strain

Here we consider the accompanying velocity field for family 4 of (48). The functional forms for these fields are given in Table 4.

Family 4 with $\chi = g(r)$. The equation for g in this case is simply

$$r(r + a)g'' - (2r + a)g' = 0,$$

which has the exact solution $g = C_3(2r^3 + 3ar^2) + C_4$.

Family 4 with $\chi = e^{z/(r+a)}g(r)$. This functional form is more complex than the previous one. Here, the governing equation for g is

$$r(r + a)^2g'' - (r + a)(2f + r + a + 1)g' + [rf^2 + (3r + a)f - r(r + a)f'] = 0,$$

where f is given by (53). Even with the help of MAPLE we are unable to solve this equation exactly.

Family 4 with $\chi = \frac{z}{r+a} f(r) - \frac{z^2}{2(r+a)^2} + g(r)$**.** For this functional form g satisfies

$$r(r+a)^2 g'' - (r+a)(2r+a)g' - 3r(r+a)ff' + (2r+a)f^2 = 0,$$

which is of a similar form to the last equation. However, this time we are able to solve exactly, with the solution given by

$$\begin{aligned} g &= C_3(2r^3 + 3ar^2) + C_4 - \frac{2C_1C_2a^4 + C_1^2 + a^8C_2^2}{2(r+a)^2} - C_1C_2r(3r+2a) \\ &\quad + \tfrac{1}{6}C_2^2 r(21r^5 + 54ar^4 + 15a^2r^3 - 28a^3r^2 - 15a^4r - 6a^5). \end{aligned}$$

6 Application of solutions - flow through a wedge

In this section we briefly illustrate an application of the group-invariant solutions considered above. The problem is flow through a two-dimensional wedge (see Figure 1), which can be used to model the discharge of material near the outlet of an industrial hopper.

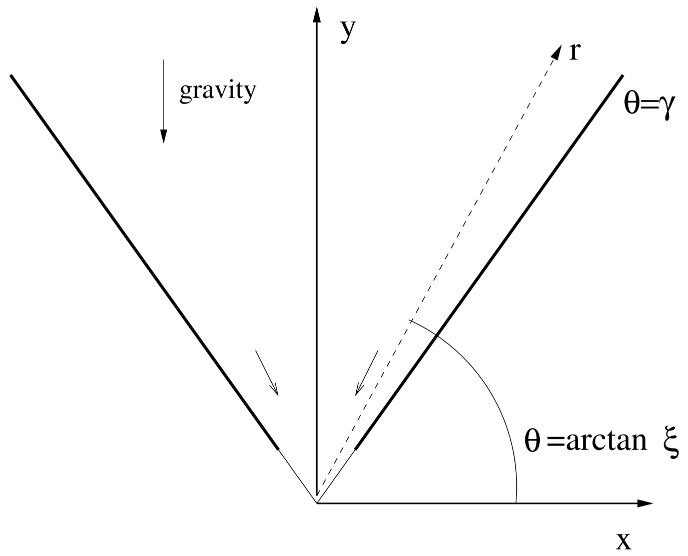

Figure 1: Schematic for mass flow through a wedge-shaped hopper

Suppose the discharge is operating under mass flow so that the entire mass of granules is in motion, as opposed to funnel flow where flow is restricted to a central region. In the neighbourhood of the hopper outlet, the stress field can be accurately approximated by the so-called 'radial stress field' (due to Jenike 1964), as mentioned in the Introduction. The required form of solution is

$$q = \rho g x F(\xi), \quad \psi = \Psi(\xi), \quad \xi = y/x, \tag{61}$$

and after substituting these expressions into (14)-(15), we arrive at a system of two coupled first order ordinary differential equations in F and Ψ. The reader is referred to Jenike (1964) for details, noting that care should be taken with the different coordinate system.

Now suppose that the angle between the hopper wall and the x-axis is γ as shown in Figure 1, and that a Coulomb friction condition holds on the wall, with the angle of wall friction given by μ. It can be shown that the friction boundary condition yields

$$\psi = \gamma - \tfrac{1}{2}\mu - \tfrac{1}{2}\arctan\left(\frac{\sin\mu}{\sin\phi}\right) \quad \text{on} \quad \xi = \tan\gamma, \tag{62}$$

provided that $\mu \le \phi$. For the special limit $\phi = \pi/2$ this condition reduces to

$$\psi = \gamma - \mu. \tag{63}$$

The stress and velocity fields are symmetric in the hopper, and thus we need only consider flow in the range $\tan\gamma \le \xi < \infty$. The condition of symmetry is

$$\psi \to \pi/2 \quad \text{as} \quad \xi \to \infty. \tag{64}$$

From (61) it is clear the required functional form for the special case of $\phi = \pi/2$ is

$$h = f(\xi), \quad \xi = y/x,$$

which is family 3 in (36) with $a = 0$. The exact solution is given by (40) with (37), and is due to Hill & Cox (2001). After applying the boundary conditions (63)-(64) we find that the constants of integration are given by

$$C_1 = 0, \quad C_2 = \frac{\tan\gamma}{2e^{s_0/2}/s_0^{1/2} - I(s_0)},$$

where here s_0 denotes the value of the parameter s corresponding to $\xi = \tan\gamma$, and is the root of the transcendental equation

$$[2e^{s_0/2}s_0^{1/2} - I(s_0)]\cot(\gamma - \mu)\cot\gamma - I(s_0) = 0. \tag{65}$$

Again, the reader is referred to Hill & Cox (2001) for further details.

Shown in Figure 2 is the dependence of the stress angle ψ and the angle $\arctan\xi$ for three different values of the angle of internal friction $\phi = \pi/6$, $\pi/3$ and $\pi/2$. For this figure the chosen values of γ and μ are $5\pi/12$ and $\pi/12$ respectively. The plot for $\phi = \pi/2$ is taken from the exact solution (40) with the constants C_1 and C_2 given by (65). To obtain the other two plots, the system of equations found by substituting (61) into (14)-(15) is solved numerically with a shooting method subject to the boundary conditions (62) and (64). It is seen that even for values of ϕ as low as $\phi = \pi/3$, the exact solution (40) corresponding to $\phi = \pi/2$ provides a good approximation.

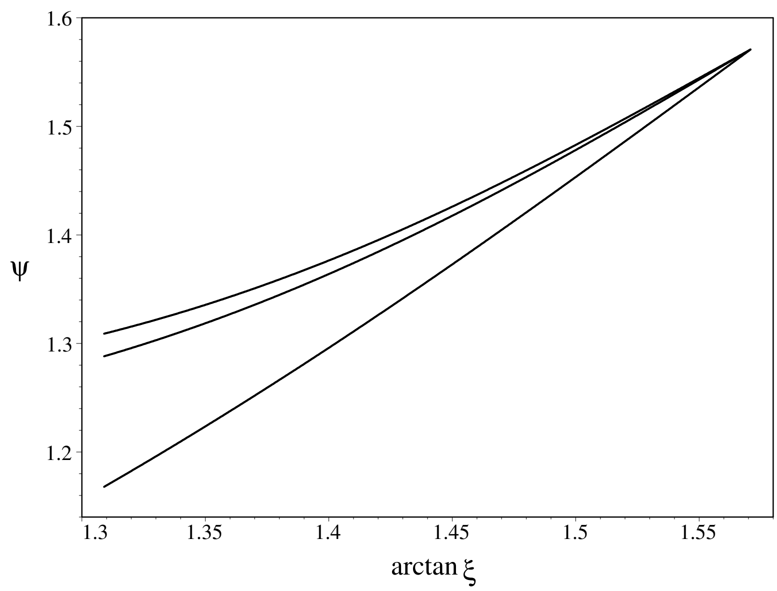

Figure 2: Typical plots of the stress angle ψ versus the angle $\arctan\xi$ for mass flow near the outlet of a wedge-shaped hopper. From top to bottom, the plots are drawn for $\phi = \pi/2$, $\pi/3$ and $\pi/6$ respectively. In all cases the hopper angle $\gamma = 5\pi/12$ and the angle of wall friction $\mu = \pi/12$.

From Table 2 we see that there are possible velocity fields which have streamfunctions of the form

$$\chi = x^b g(\xi), \quad \xi = y/x. \tag{66}$$

Here g is given exactly by (57), where s is the same parameter used in the stress field above. For convenience we employ polar coordinates (r, θ), defined by

$$r = (x^2 + y^2)^{1/2}, \quad \tan\theta = y/x = \xi,$$

and define the velocity components in the r and θ directions to be v_r and v_θ respectively. With the use of (66) these components can be written as

$$v_r = x^{b-1}\left((1+\xi)^{1/2}g' - \frac{bg\xi}{(1+\xi)^{1/2}}\right), \quad v_\theta = -\frac{bgx^{b-1}}{(1+\xi)^{1/2}}.$$

For boundary conditions we wish to have v_θ vanish both on $\xi = \tan\gamma$ and as $\xi \to \infty$. It turns out the only nontrivial solution with this property is the one with $b = 0$, implying that v_θ is identically zero, and that v_r is given by

$$v_r = \frac{V}{4r}\, s\left\{\frac{1}{C_2^2} + \left(\frac{2e^{s/2}}{s^{1/2}} - I(s)\right)^{1/2}\right\}. \tag{67}$$

Here we have set $C_3 = -V/4C_2$, forcing $rv_r \to V$ as $\xi \to \infty$. The constant V is arbitrary, and cannot be determined within the current quasi-static framework. We note that (67) was first derived in Cox & Hill (2004).

The dependence of rv_r on $\arctan\xi$ for three different values of ϕ is shown in Figure 3. The plot for $\phi = \pi/2$ is drawn using the exact solution (67), while the other two curves are computed via a numerical solution to the full equations. The details of the calculations for general ϕ are straightforward, but are not presented here, and instead we refer to either Spencer & Bradley (1996) or Cox & Hill (2004). Again, we note that the exact solution for $\phi = \pi/2$ is in close agreement with the numerical one for $\phi = \pi/3$, which is a more realistic value for the angle of internal friction.

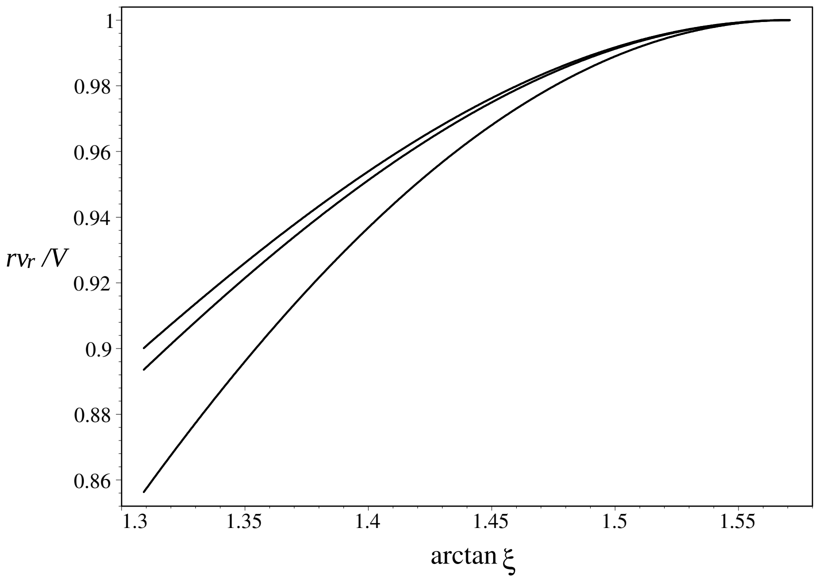

Figure 3: Typical plots of the quantity rv_r/V versus the angle $\arctan\xi$ for mass flow near the outlet of a wedge-shaped hopper. From top to bottom, the plots are drawn for $\phi = \pi/2$, $\pi/3$ and $\pi/6$ respectively. In all cases the hopper angle $\gamma = 5\pi/12$ and the angle of wall friction $\mu = \pi/12$.

7 Discussion

For a granular material which satisfies the Coulomb-Mohr yield condition, the stress field for gravity-driven quasi-static flow is governed by a system of two highly nonlinear coupled partial differential equations for the invariant q and the stress angle ψ (equations (14)-(15) in plane strain and equations (25)-(26) for axially symmetric flow). With the added assumption that the flow conforms to the non-dilatant double-shearing theory, the associated velocity field is governed by a linear partial differential equation (with non-constant coefficients) for the streamfunction χ (equation (19) for plane strain and (30) for axially symmetric flow). Due to the high level of nonlinearity involved in these equations, analytic progress is rare, and exact solutions can only be found in the most simple geometrical cases.

For the limiting case in which the angle of internal friction $\phi = \pi/2$, the two equations for the stress field can be combined into one (equation (2) for plane strain and (3) for axially symmetry flow). This equation is still nonlinear, however there are an infinite number of exact group-invariant solutions which can be found with the use of Lie symmetry methods. For each of these solutions for the stress field, there are exact solutions for the streamfunction. In this paper we systematically classify all these group-invariant solutions into equivalence classes, and present the minimal set or "optimal system" of such solutions. In the process we have been able to identify a number of exact solutions which are new, and to illustrate the potential utility of these solutions, the problem of mass flow through a two-dimensional wedge shaped hopper is examined.

References

Abramowitz, M. & Stegun, I. *Handbook of mathematical functions.* New York: Dover.

Campbell, C. S. (1990) Rapid granular flows. *Ann. Rev. Fluid Mech.*, **22**, 57–92.

Cox, G. M. & Hill, J. M. (2003) Some exact mathematical solutions for granular stock piles and granular flow in hoppers. *Math. Mech. Solids*, **8**, 21–50.

Cox, G. M. & Hill, J. M. (2004) Some exact velocity profiles for granular flow in converging hoppers. *Z. Angew. Math. Phys.*, to appear.

Drucker, D. C. & Prager, W. (1952) Soil mechanics and plastic analysis or limit design. *Q. Appl. Math.*, **10**, 157–165.

Hill, J. M. & Cox, G. M. (2001) An exact parametric solution for granular flow in a converging wedge. *Z. Angew. Math. Phys.*, **52**, 657–668.

Hill, J. M. & Cox, G. M. (2002) On the problem of the determination of force distributions n granular heaps using continuum theory. *Q. J. Mech. Appl. Math.*, **55**, 655–668.

Hill, R. (1950) *The Mathematical Theory of Plasticity.* Oxford: Clarendon Press.

Jenike, A. W. (1964) Steady gravity flow of frictional-cohesive solids in converging channels. *J. Appl. Mech.*, **31**, 5–11.

Jenike, A. W. (1965) Gravity flow of frictional-cohesive solids - Convergence to radial stress fields. *J. Appl. Mech.*, **32**, 205–207.

Johanson, J. R. (1964) Stress and velocity fields in the gravity flow of bulk solids. *J. Appl. Mech.*, **31**, 499–506.

Johnpillai, I. K., McCue, S. W. & Hill, J. M. (2004) Lie group symmetry analysis for granular media stress equations. *J. Math. Anal. Appl.*, submitted.

Kolymbas, D. (ed.) (2000) *Constitutive Modelling of Granular Materials.* Berlin: Springer Verlag.

Mehrabadi, M. M. & Cowin, S. C. (1978) Initial planar deformation of dilatant granular materials. *J. Mech. Phys. Solids*, **26**, 269-284.

Olver, P. J. (1986) *Applications of Lie Groups to Differential Equations.* New York: Springer-Verlag.

O'Mahony, T. C. & Spencer, A. J. M. (1985) Theory of rectilinear flow of granular materials. *Int. J. Eng. Sci.*, **23**, 139–150.

Polyanin, A. D. & Zaitsev, V. F. (1995) *Exact Solutions for Ordinary Differential Equations.* Boca Raton, CRC Press.

Sokolovskii, V. V. (1965) *Statics of Granular Media.* Oxford: Pergamon Press.

Spencer, A. J. M. (1964) A theory of the kinematics of ideal soils under plane strain conditions. *J. Mech. Phy. Solids*, **12**, 337–351.

Spencer, A. J. M. (1982) Deformations of ideal granular materials. *Mechanics of Solids; the Rodney Hill Anniversary Volume* (H. G. Hopkins & M. J. Sewell, eds). Oxford: Pergamon Press, pp. 607–652.

Spencer, A. J. M. & Bradley, N. J. (1992) Gravity flow of a granular material in compression between vertical walls and through a tappering vertical channel. *Q. J. Mech. Appl. Math.*, **45**, 733–746.

Spencer, A. J. M. & Bradley, N. J. (1996) Gravity flow of granular materials in converging wedges and cones. *Proceedings of the 8th International Symposium on Continuum Models of Discrete Systems* (K. Z. Markov, ed.). Singapore: World Scientific, pp. 58–590.

Spencer, A. J. M. & Bradley, N. J. (2002) Gravity flow of granular materials in contracting cylinders and tapered tubes. *Int. J. Eng. Sci.*, **40**, 1529–1552.

Sture, S. (1999) Constitutive issues in soil liquefaction. *Proceedings of the International Workshop on the Physics and Mechanics of Soil Liquefaction* (P. V. Lade & J. A. Yamamuro, eds). Rotterdam: Balkema, pp. 133–143.

Tarbos, G. I., McNamara, S. & Talu, I. (2003) Slow and intermediate flow of a frictional bulk powder in the Couette geometry. *Powder Technology*, **131**, 23–39.

Thamwattana, N. & Hill, J. M. (2003a) Analytical stress and velocity fields for gravity flows of highly frictional granular materials. *Acta Mech.*, **164**, 91–112.

Thamwattana, N. & Hill, J. M. (2003b) Analytical solutions for tapering quadratic and cubic rat-holes in highly frictional granular solids. *Int. J. Solids Structures*, **40**, 5923-5948.