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NEW STUDY ON THE CONVERGENCE OF A FORMAL TRANSFORMATION, II

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Abstract. A system of two nonlinear differential equations with an irregular type singularity not satisfying the Poincaré condition is studied. A two-parameter family of bounded solutions is constructed by the fixed point technique. The domain of holomorphy of the set of functions appearing in the fixed point technique is to be given by a family of the product of two circles over every point in a domain of independent variable. The radius of one circle depends on the argument of the independent variable only, while that of the other essentially depends on the independent variable itself.

1. Introduction.

1°. Assumptions. In a previous paper [6], the author studies a system of two nonlinear differential equations of the form

$$\text{(A)} \qquad x^2\frac{dy}{dx}=(\mu+\alpha x)y+f(x, y, z)\,, \qquad x^2\frac{dz}{dx}=(-\nu+\beta x)z+g(x, y, z)\,,$$

under the assumptions that

(i) x is an independent variable;

(ii) μ and ν are positive numbers and their ratio is irrational;

(iii) α and β are complex constants and there is a positive quantity κ satisfying the inequalities

$$(1.1) \qquad \mu+\kappa\Re\alpha>0\,, \qquad -\nu+\kappa\Re\beta>0\,;$$

(iv) $f(x, y, z)$ and $g(x, y, z)$ are holomorphic and bounded functions of (x, y, z) for

$$(1.2) \qquad |x|<a\,, \qquad |y|<b\,, \qquad |z|<b\,,$$

and their Taylor series expansions in (y, z) contain neither the constant terms nor the linear terms, where a and b are small positive constants.

2°. Review of a previous result. Under these assumptions, the following was proved:

PROPOSITION 1. *Let ε_0 be a preassinged sufficiently small positive number. There exists a formal transformation of the form*

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$$(1.3)\qquad y=u+\sum_{j+k\geqq 2} p_{jk}(x)u^j v^k\,,\qquad z=v+\sum_{j+k\geqq 2} q_{jk}(x)u^j v^k\,,$$

which formally changes the equations (A) *to the linear equations*

$$(\mathrm{A}')\qquad x^2\frac{du}{dx}=(\mu+\alpha x)u\,,\qquad x^2\frac{dv}{dx}=(-\nu+\beta x)v\,.$$

The coefficients $p_{jk}(x)$ *and* $q_{jk}(x)$ *are holomorphic and bounded functions of* x *for a domain of the form*

$$(1.4.\mp)\qquad \left|\arg x\mp\frac{\pi}{2}\right|<\pi-\varepsilon_0\,,\qquad 0<|x|<a'$$

and admit asymptotic expansions in powers of x *as* x *tends to zero through the sector* (1.4.$\mp$). a' *is a small positive constant.*

For the proof of the convergence of the formal transformation (1.3), the following proposition played an important role. Namely the double power series appearing in (1.3) has the particular property which is clarified in the proposition below:

PROPOSITION 2. *For each fixed* j, *the power series in a single variable* $\sum_{k=0}^{\infty} p_{jk}(x)v^k$ *and* $\sum_{k=0}^{\infty} q_{jk}(x)v^k$ *are uniformly convergent for*

$$(1.5.\mp)\qquad \left|\arg x\mp\frac{\pi}{2}\right|<\pi-\varepsilon_0\,,\qquad 0<|x|<a'\,,\quad |v|<b'\,,$$

where a' *and* b' *are small positive constants depending on* j. *Similarily, for each fixed* k, *the power series* $\sum_{j=0}^{\infty} p_{jk}(x)u^j$ *and* $\sum_{j=0}^{\infty} q_{jk}(x)u^j$ *in* u *are uniformly convergent for*

$$(1.5'.\mp)\qquad \left|\arg x\mp\frac{\pi}{2}\right|<\pi-\varepsilon_0\,,\qquad 0<|x|<a'\,,\quad |u|<b'\,,$$

where a' *and* b' *are small positive constants depending on* k.

By the help of this proposition, [6] has introduced truncated differential equations of special type. After proving the existence of a solution for these equations, we have obtained the following main theorem.

THEOREM A. *Let* ε_0 *be a preassigned sufficiently small positive number. The formal transformation* (1.3) *is uniformly convergent for* (x, u, v) *in a domain of the form*

$$(1.6.\mp)\qquad \left|\arg x\mp\frac{\pi}{2}\right|<\pi-\varepsilon_0\,,\qquad 0<|x|<a_0\,,\quad |u|<b_0\,,\quad |v|<b_0\,,$$

where a_0 *and* b_0 *are small positive constants. Namely, there exists a transformation*

$$(1.7)\qquad y=\Phi(x, u, v)\,,\qquad z=\Psi(x, u, v)\,,$$

which changes the equations (A) *to the linear equations* (A′) *in the domain* (1.6.∓). *The Taylor expansions of the functions* $\Phi(x, u, v)$ *and* $\Psi(x, u, v)$ *coincide with the power series expressions* (1.3).

3°. n nonlinear equations. In order to extend Theorem A to the case of n nonlinear differential equations, we consider the case where y and z are vectors and, in particular, the μ, ν, α and β are diagonal matrices. Such equations are written simply in the form

$$\text{(B)} \qquad x^2 \frac{dy}{dx} = (\mathbf{1}_n(\mu) + x\mathbf{1}_n(\alpha))y + f(x, y) .$$

Here, x is a complex independent variable; y is an n-vector; $\mathbf{1}_n(\mu)$ and $\mathbf{1}_n(\alpha)$ are n by n diagonal matrices, respectively, with diagonal entries $\{\mu_j\}$ and $\{\alpha_j\}$ which coincide with the entries of the n-vectors μ and α; $f(x, y)$ is an n-vector with entries holomorphic and bounded in (x, y) for a domain of the form

$$|x| < a , \quad \|y\| = \max_{1 \leqq j \leqq n} |y_j| < b$$

and their Taylor series expansions in powers of y begin with terms of degree at least 2. We assume that

(i) the μ_j are nonzero real numbers independent over the field $\boldsymbol{Q}$ of all rational numbers;

(ii) the α_j are complex numbers and there is a positive quantity κ such that

$$\text{(1.8)} \qquad \mu_j + \kappa \cdot \Re\alpha_j > 0 \qquad \text{for all} \quad j .$$

REMARK. When a factor x appears in the nonlinear term, such a system of nonlinear equations was already studied in [5].

For an arrangement $(p_1, p_2, \cdots, p_n)$ of nonnegative integers p_j and an n-vector z with entries $\{z_j\}$, let $|p| = p_1 + p_2 + \cdots + p_n$ and $z^p = z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}$. In the same way as in Proposition 1, it is not difficult to prove the following:

PROPOSITION 3. *There exists a formal transformation of the form*

$$\text{(1.9)} \qquad y = u + \sum_{|p| \geqq 2} g_p(x) u^p ,$$

which formally changes the equations (B) *to the linear system*

$$\text{(B′)} \qquad x^2 \frac{du}{dx} = (\mathbf{1}_n(\mu) + x\mathbf{1}_n(\alpha))u .$$

The coefficients $g_p(x)$ *are holomorphic and bounded functions of* x *in a domain of the form*

$$\text{(1.10.}\mp\text{)} \qquad \left| \arg x \mp \frac{\pi}{2} \right| < \pi - \varepsilon_0 , \qquad 0 < |x| < a_0 ,$$

and admit asymptotic expansions in powers of x as x tends to zero through the sector (1.10).

In order to prove the convergence of the formal transformation (1.9) by utilizing the method as in the proof of Theorem A, the following problem will have to be solved:

PROBLEM. *For any fixed* p_j, *the power series in the entries of the* $(n-1)$-*vector* $\hat{z}=(z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n)$

$$(1.11) \qquad \sum_{\hat{p}, |p| \geqq 2} g_p(x)\hat{z}^{\hat{p}} \qquad \textit{for} \quad \hat{p}=(p_1, \cdots, p_{j-1}, p_{j+1}, \cdots, p_n)$$

is convergent for

$$(1.12.\mp) \qquad \left|\arg x \mp \frac{\pi}{2}\right| < \pi - \varepsilon_0 , \qquad 0<|x|<a'' , \quad \|\hat{z}\|<b'' .$$

Here a'' *and* b'' *are small positive constants depending on the suffix* j *and the power exponents* p_j.

If we could solve this problem, then the proof of the convergence of the formal transformation (1.9) would be carried out in the same way as for the case of $n=2$. For $n \geqq 3$, however, it seems to be very difficult to solve this problem directly. So, we are forced to study a different method for the proof of the convergence of even the formal transformation (1.3).

In this paper, we prove Theorem A by a different method which will be applicable to the proof of the convergence for the case of $n \geqq 3$.

2. New arrangement of a formal solution.

The equations (A) are given by

$$(2.1) \qquad x^2 \frac{dy}{dx} = (\mu + \alpha x)y + f(x, y, z) , \qquad x^2 \frac{dz}{dx} = (-\nu + \beta x)z + g(x, y, z) .$$

We have already proved the following:

THEOREM 1. *There exists a formal transformation of the form*

$$(2.2) \qquad y = u + \sum_{j+k \geqq 2} p_{jk}(x)u^j v^k , \qquad z = v + \sum_{j+k \geqq 2} q_{jk}(x)u^j v^k ,$$

which formally changes the equations (2.1) *to the linear equations*

$$(2.3) \qquad x^2 \frac{du}{dx} = (\mu + \alpha x)u , \qquad x^2 \frac{dv}{dx} = (-\nu + \beta x)v .$$

The coefficients $p_{jk}(x)$ *and* $q_{jk}(x)$ *are holomorphic and bounded functions in x for a domain of the form*

$$\left|\arg x \mp \frac{\pi}{2}\right| < \pi - \varepsilon_0 , \qquad 0 < |x| < r' \tag{2.4.$\mp$}$$

and admit asymptotic expansions in powers of x as x tends to the origin through the sector (2.4.$\mp$), *r' being a small constant.*

We rearrange (2.2) in powers of u:

$$y = u + \sum_{j=0}^{\infty} P_j(x, v)u^j , \qquad z = v + \sum_{j=0}^{\infty} Q_j(x, v)u^j , \tag{2.5}$$

where the coefficients $P_j(x, v)$ and $Q_j(x, v)$ *are expressed as power series in* v:

$$P_j(x, v) = \sum_{k, j+k \geqq 2} p_{jk}(x)v^k , \qquad Q_j(x, v) = \sum_{k, j+k \geqq 2} q_{jk}(x)v^k . \tag{2.6-j}$$

We prove the following:

THEOREM 2. *The coefficients $P_j(x, v)$ and $Q_j(x, v)$ are holomorphic and bounded functions of (x, v) for a domain of the form*

$$\left|\arg x \mp \frac{\pi}{2}\right| < \pi - \varepsilon_0 , \qquad 0 < |x| < r_0 , \quad |v| < r_1 , \tag{2.7.$\mp$}$$

so that the power series (2.6-j) *are uniformly convergent. The r_0 and r_1 are independent of j.*

To prove this, regard (u, v) as a holomorphic general solution $(U(x), V(x))$ of the equations (2.3) such that $(U(x_0), V(x_0)) = (u_0, v_0)$, where x_0 has to be restricted to the domain (2.4.$\mp$). Substitute (2.5) for $\{y, z\}$ into the equations (2.1) and rearrange both sides of the resulting equations in powers of u. Then one can find the differential equations which determine those coefficients. The equations for the pair $\{P_0(x, V(x)), Q_0(x, V(x))\}$ are nonlinear, while the pairs $\{P_j(x, V(x)), Q_j(x, V(x))\}$ for $j \geqq 1$ are linear equations.

In order to derive those equations, observe that

$$\begin{aligned} x^2 \frac{dy}{dx} &= x^2 \frac{dU}{dx} + \sum_{j=0}^{\infty} \left(x^2 \frac{dP_j(x, V)}{dx} + P_j(x, V) \cdot \frac{j}{U} x^2 \frac{dU}{dx} \right) U^j \\ &= (\mu + \alpha x)U + \sum_{j=0}^{\infty} \left(x^2 \frac{dP_j(x, V)}{dx} + j(\mu + \alpha x)P_j(x, V) \right) U^j . \end{aligned} \tag{2.8}$$

$$\begin{aligned} (\mu + \alpha x)y + f(x, y, z) &= (\mu + \alpha x)U \\ &\quad + \sum_{j=0}^{\infty} (\mu + \alpha x)P_j(x, V)U^j + f(x, P_0(x, V), V + Q_0(x, V)) \end{aligned} \tag{2.9}$$

$$+\sum_{j=1}^{\infty}(C(x, V)P_j(x, V)+D(x, V)Q_j(x, V)+G_j(x, V))U^j .$$

Here,

$$(2.10)\qquad \begin{aligned} C(x, v)&=\frac{\partial f}{\partial y}(x, P_0(x, v), v+Q_0(x, v)) ,\\ D(x, v)&=\frac{\partial f}{\partial z}(x, P_0(x, v), v+Q_0(x, v)) \end{aligned}$$

and, in particular,

$$(2.11)\qquad C(x, 0)=0 , \qquad D(x, 0)=0 .$$

The $G_j(x, v)$ are linear forms of the functions $(\partial^{a+b}f/\partial y^a\partial z^b)(x, P_0(x, v), v+Q_0(x, v))$ for $a+b\leqq j$ whose coefficients are polynomials in $\{Q_1(x, v), \cdots, Q_{j-1}(x, v), P_1(x, v), \cdots, P_{j-1}(x, v)\}$. In quite a similar way, we can derive similar equations, by differentiating the second power series expression of (2.5), from the second equation of (2.1) by defining the functions $E(x, v)$, $F(x, v)$ and $H_j(x, v)$, which are respectively similar to the functions $C(x, v)$, $D(x, v)$ and $G_j(x, v)$.

Hence, the pair $\{P_0(x, V(x)), Q_0(x, V(x))\}$ has to satisfy the nonlinear differential equations

$$(2.12)\qquad \begin{cases} x^2\dfrac{dP_0}{dx}=(\mu+\alpha x)P_0+f(x, P_0, V(x)+Q_0) ,\\ x^2\dfrac{dQ_0}{dx}=(-\nu+\beta x)Q_0+g(x, P_0, V(x)+Q_0) . \end{cases}$$

For $j\geqq 1$, the pairs $\{P_j(x, V(x)), Q_j(x, V(x))\}$ are solutions of the linear equations

$$(2.13\text{-}j)\qquad \begin{cases} x^2\dfrac{dP_j}{dx}=(1-j)(\mu+\alpha x)P_j\\ \qquad\qquad +C(x, V(x))P_j+D(x, V(x))Q_j+G_j(x, V(x)) ,\\ x^2\dfrac{dQ_j}{dx}=(-\nu-j\mu+(\beta-j\alpha)x)Q_j\\ \qquad\qquad +E(x, V(x))P_j+F(x, V(x))Q_j+H_j(x, V(x)) . \end{cases}$$

The following facts should be noted:

(i) The equations (2.12) are nonlinear, while the equations (2.13-j) are linear;

(ii) $x=0$ is an irregular singular point;

(iii) The equations (2.12) and (2.13-j) possess formal solutions which are expressed as the power series (2.6-j) with $v=V(x)$.

A theorem due to Malmquist [7] or Iwano [3] implies that *the power series* (2.6-0) *with* $v=V(x)$ *are uniformly convergent whenever the values of* $(x, V(x))$, *considered as points in the* (x, v)*-space, belong to a domain of the form* (2.7.∓), *so that the pair* $\{P_0(x, V(x)), Q_0(x, V(x))\}$ *of the sums becomes a solution of the equations* (2.12) *for the values of* $(x, V(x))$ *in the domain* (2.7.∓). r_0 *and* r_1 *are small positive constants.*

Obviously,

$$P_0(x, V(x))=O(V(x)^2), \qquad Q_0(x, V(x))=O(V(x)^2). \tag{2.14}$$

When we consider $(x, V(x))$ as independent variables, the $P_0(x, v)$ and $Q_0(x, v)$ are holomorphic and bounded functions of (x, v) for the domain (2.7.∓). By virtue of (2.14), these functions vanish at $v=0$. Hence, the condition (2.11) is satisfied.

The coefficients appearing in the equations (2.13-1) become known holomorphic and bounded functions of $(x, V(x))$, considered as independent variables, for the domain (2.7.∓). In order to prove the convergence of the formal solution (2.6-1) with $v=V(x)$, again apply the theorem mentioned above.

In this manner, we can prove that *the pairs* $\{P_j(x, V(x)), Q_j(x, V(x))\}$ *are successively and uniquely determined as solutions of the linear equations* (2.13-j) *in such a way that* $P_j(x, v)$ *and* $Q_j(x, v)$ *are holomorphic and bounded functions of* (x, v) *in the domain* (2.7.∓) *and admit the power series* (2.6-j) *as their Taylor series expansions in powers of* v. This proves Theorem 2.

3. New truncated differential equations. For any positive integer N, set

$$\hat{P}_{(N)}(x, u, v)=u+\sum_{j=0}^{2N} P_j(x, v)u^j, \qquad \hat{Q}_{(N)}(x, u, v)=v+\sum_{j=0}^{2N} Q_j(x, v)u^j. \tag{3.1}$$

Apply the change of variables

$$y=\hat{P}_{(N)}(x, \eta, \zeta), \qquad z=\hat{Q}_{(N)}(x, \eta, \zeta). \tag{3.2}$$

When the pair $\{y, z\}$ is expressed as the power series (2.5), it is easy to verify that the pair $\{\eta, \zeta\}$ defined by the equations (3.2) is expressed as power series in u of the form

$$\eta=u+\sum_{j=2N+1}^{\infty} \phi_j(x, v)u^j, \qquad \zeta=v+\sum_{j=2N}^{\infty} \psi_j(x, v)u^j, \tag{3.3}$$

where $\phi_j(x, v)$ and $\psi_j(x, v)$ are holomorphic and bounded functions in (x, v) for a domain of the form

$$\left|\arg x\mp\frac{\pi}{2}\right|<\pi-\varepsilon_0, \qquad 0<|x|<r'_0, \quad |v|<r'_1, \tag{3.4.∓}$$

r'_0 and r'_1 being sufficiently small positive constants.

By noticing this fact, the equations satisfied by the pair $\{\eta, \zeta\}$ are written as

$$(3.5)\qquad \begin{cases} x^2 \dfrac{d\eta}{dx} = (\mu + \alpha x)\eta + \eta^{2N+1}\hat{f}_1(x, \eta, \zeta)\,, \\ x^2 \dfrac{d\zeta}{dx} = (-\nu + \beta x)\zeta + \eta^{2N}\hat{g}_1(x, \eta, \zeta)\,. \end{cases}$$

Here the $\hat{f}_1(x, \eta, \zeta)$ and $\hat{g}_1(x, \eta, \zeta)$ are holomorphic and bounded functions of (x, η, ζ) in a domain of the form

$$(3.6.\mp)\qquad \left|\arg x \mp \frac{\pi}{2}\right| < \pi - \varepsilon_0\,, \qquad 0 < |x| < r_0^{(N)}\,, \quad |\eta| < r_1^{(N)}\,, \quad |\zeta| < r_1^{(N)}\,,$$

where $r_0^{(N)}$ and $r_1^{(N)}$ are sufficiently small positive constants depending on N.

To see this, observe that, when the pair $\{u, v\}$ is considered as solutions of the equations (2.3), the power series (3.3) form a formal solution of the equations (3.5). Insert (3.3) for $\{\eta, \zeta\}$ in the equations (3.5). Then the expression $x^2(d\eta/dx) - (\mu + \alpha x)\eta$ satisfies the order condition $O(u^{N+1})$ and does not involve any term with negative powers in v. If the $\hat{f}_1(x, \eta, \zeta)$ were not holomorphic in (η, ζ), it would involve a term with either negative power in η or in ζ. Hence, the $\eta^{2N+1}\hat{f}_1$ will contain either a term with degree less than $2N+1$ in u, or a term of negative degree in v will appear in the expression $\eta^{2N+1}\hat{f}_1(x, \eta, \zeta)$. This is a contradiction. The same argument can be applied to the second one in (3.5) for the proof of the holomorphy of the $\hat{g}_1(x, \eta, \zeta)$.

Put

$$(3.7)\qquad \Lambda(x) = \frac{\nu}{x} + \beta \log x\,, \qquad \log 1 = 0$$

and let $\{U(x), V(x)\}$ be a holomorphic general solution of the equations (2.3). Make the change of variables

$$(3.8)\qquad \eta = \frac{U(x)}{1 - Y}\,, \qquad \zeta = V(x) + e^{\Lambda(x)}Z\,.$$

A simple calculation gives

$$x^2 \frac{d\eta}{dx} = (\mu + \alpha x)\eta + \frac{U(x)}{(1 - Y)^2}\, x^2 \frac{dY}{dx}\,,$$

$$x^2 \frac{d\zeta}{dx} = (-\nu + \beta x)(V(x) + e^{\Lambda(x)}Z) + e^{\Lambda(x)}x^2 \frac{dZ}{dx}\,.$$

Hence, the equations which the pair $\{Y, Z\}$ satisfies become

$$
(3.9)\qquad \begin{cases} x^2 \dfrac{dY}{dx} = \dfrac{U(x)^{2N}}{(1-Y)^{2N-1}} \hat{f}_1\left(x, \dfrac{U(x)}{1-Y}, V(x)+e^{A(x)}Z\right), \\ x^2 \dfrac{dZ}{dx} = \dfrac{e^{-A(x)}U(x)^{2N}}{(1-Y)^{2N}} \hat{g}_1\left(x, \dfrac{U(x)}{1-Y}, V(x)+e^{A(x)}Z\right). \end{cases}
$$

This system of equations can be written as

$$
(3.10)\qquad \begin{cases} x^2 \dfrac{dY}{dx} = U(x)^{2N} F_N(x, U(x), V(x), Y, e^{A(x)}Z), \\ x^2 \dfrac{dZ}{dx} = e^{-A(x)} U(x)^{2N} G_N(x, U(x), V(x), Y, e^{A(x)}Z). \end{cases}
$$

When the variables x, $U(x)$, $V(x)$, Y, $e^{A(x)}Z$ are considered as independent variables, the $F_N(x, u, v, Y, S)$ and $G_N(x, u, v, Y, S)$ are holomorphic and bounded functions of (x, u, v, Y, S) in a domain of the form

$$
(3.11.\mp)\qquad \begin{cases} \left|\arg x \mp \dfrac{\pi}{2}\right| < \pi - \varepsilon_0, \qquad 0<|x|<r_0^{(N)}, \quad |u|<r_1^{(N)}, \quad |v|<r_1^{(N)}, \\ |Y|<r_2^{(N)}, \quad |S|<r_2^{(N)}, \end{cases}
$$

$r_0^{(N)}$, $r_1^{(N)}$ and $r_2^{(N)}$ being sufficiently small constants depending on N. These functions satisfy inequalities of the form

$$
(3.12)\qquad |F_N(x, u, v, Y, S)| \leqq L_N, \qquad |G_N(x, u, v, Y, S)| \leqq L_N
$$

and

$$
(3.13)\qquad \begin{cases} |F_N(x, u, v, Y_1, S_1) - F_N(x, u, v, Y_2, S_2)| \leqq L_N(|Y_1 - Y_2| + |S_1 - S_2|), \\ |G_N(x, u, v, Y_1, S_1) - G_N(x, u, v, Y_2, S_2)| \leqq L_N(|Y_1 - Y_2| + |S_1 - S_2|) \end{cases}
$$

for the arguments belonging to the domain (3.11.$\mp$), where L_N is a constant depending on N.

4. Stable domains. In order to simplify the description, we utilize some results which were already obtained in Iwano [6]. The pair $\{U(x), V(x)\}$ is a holomorphic general solution of the equations (2.3), namely,

$$
(4.1)\qquad x^2 \frac{du}{dx} = (\mu + \alpha x)u, \qquad x^2 \frac{dv}{dx} = (-\nu + \beta x)v.
$$

It is assumed that there is a positive constant κ satisfying the inequalities

$$
(4.2)\qquad \nu_1(\kappa) \equiv \mu + \kappa \Re\alpha > 0, \qquad \nu_2(\kappa) \equiv -\nu + \kappa \Re\beta > 0.
$$

Let M_0 be the least integer such that

$$(4.3) \qquad \nu_3(\kappa) \equiv M_0 \nu_1(\kappa) - \nu_2(\kappa) > 0 \,.$$

The function $\Xi_0(x) \equiv e^{-\Lambda(x)} U(x)^{M_0}$ satisfies the linear equation

$$(4.4) \qquad x^2 \frac{d\xi_0}{dx} = (M_0\mu + \nu + (M_0\alpha - \beta)x)\xi_0 \,,$$

which satisfies the condition (4.3).

According to the discussion which was developed in Section 4 in Iwano [6], we put

$$(4.5) \qquad \begin{cases} \|\Lambda\| = \max\{\mu, \nu, M_0\mu + \nu\} = M_0\mu + \nu \,, \\ \|\delta\| = \max\{|\Im\alpha|, |\Im\beta|, |M_0\Im\alpha - \Im\beta|\} \,, \\ \|\upsilon\| = \max\{|\Re\alpha|, |\Re\beta|, |M_0\Re\alpha - \Re\beta|\} \,, \\ \|\nu(\kappa)\| = \max\{\nu_1(\kappa), \nu_2(\kappa), M_0\nu_1(\kappa) - \nu_2(\kappa)\} \,, \\ \|\nu(\kappa)\|' = \min\{\nu_1(\kappa), \nu_2(\kappa), M_0\nu_1(\kappa) - \nu_2(\kappa)\} \end{cases}$$

and define the angle Ω by the formula

$$(4.6) \qquad \tan\Omega = \frac{8(\kappa+1)\|\delta\| + 8(\|\Lambda\| + \|\upsilon\|) + 6\|\nu(\kappa)\|'}{\|\nu(\kappa)\|'} \,.$$

Let r_0, r_1 and Δ_N be positive constants depending on N. Then, the stable domain is given by one of the following two domains in the (x, u, v)-space:

$$(4.7.\mp) \qquad \begin{cases} 0 < |x| < r_0\omega(\arg x) \,, & \left|\arg x \mp \dfrac{\pi}{2}\right| < \pi - \varepsilon_0 \,, \\ |u|^N < \Delta_N |x| \chi^{(N)}(\arg x) \,, & |v| < r_N \chi_\beta(\arg x) \,, \end{cases}$$

where the $\omega(\tau)$, $\chi_\beta(\tau)$ and $\chi^{(N)}(\tau)$ are strictly positive valued and continuous functions defined in the τ-interval $[-\pi/2+\varepsilon_0, 3\pi/2-\varepsilon_0]$ or $[-3\pi/2+\varepsilon_0, \pi/2-\varepsilon_0]$. The $\omega(\tau)$ is given by

$$(4.8) \qquad \omega(\tau) = \begin{cases} \dfrac{\cos\Omega}{\sin\varepsilon_0} \,, & \text{for } \left|\tau \mp \dfrac{\pi}{2}\right| \leqq \dfrac{\pi}{2} - \Omega \,, \\ \dfrac{|\cos\tau|}{\sin\varepsilon_0} \,, & \text{for } \dfrac{\pi}{2} - \Omega < \left|\tau \mp \dfrac{\pi}{2}\right| \leqq \pi - \varepsilon_0 \,. \end{cases}$$

The $\chi_\beta(\tau)$ and $\chi^{(N)}(\tau)$ are expressed as

$$(4.9) \qquad \chi_\beta(\tau) = \begin{cases} (\cos\Omega)^{\Re\beta} \,, & \text{for } \left|\tau - \dfrac{\pi}{2}\right| \leqq \dfrac{\pi}{2} - \Omega \,, \\ (\cos\tau)^{\Re\beta} e^{(-\tau+\Omega)\Im\beta} \,, & \text{for } -\dfrac{\pi}{2} + \varepsilon_0 \leqq \tau \leqq \Omega \,, \\ |\cos\tau|^{\Re\beta} e^{(-\tau+\pi-\Omega)\Im\beta} \,, & \text{for } \pi - \Omega \leqq \tau \leqq \dfrac{3\pi}{2} - \varepsilon_0 \end{cases}$$

and

$$(4.10)\quad \chi^{(N)}(\tau)=\begin{cases} 1\,, & \text{for}\quad \left|\tau-\dfrac{\pi}{2}\right|\leqq\dfrac{\pi}{2}-\Omega\,,\\ \left(\dfrac{\cos\tau}{\cos\Omega}\right)^{N\Re\alpha-1}e^{(-\tau+\Omega)N\Im\alpha}\,, & \text{for}\quad -\dfrac{\pi}{2}+\varepsilon_0\leqq\tau\leqq\Omega\,,\\ \left(\dfrac{|\cos\tau|}{\cos\Omega}\right)^{N\Re\alpha-1}e^{(-\tau+\pi-\Omega)N\Im\alpha}\,, & \text{for}\quad \pi-\Omega\leqq\tau\leqq\dfrac{3\pi}{2}-\varepsilon_0\,. \end{cases}$$

For the τ-interval $[-3\pi/2+\varepsilon_0,\ \pi/2-\varepsilon_0]$, the $\chi_\beta(\tau)$ and $\chi^{(N)}(\tau)$ are to be defined in a similar manner. The constant Δ_N appearing in the domain (4.7.$\mp$) must be so chosen that the minimum of the function $\Delta_N\chi^{(N)}(\tau)$ for $|\tau\mp\pi/2|\leqq\pi-\varepsilon_0$ is not less than the unity.

5. The curve $\Gamma(x_0)$ and the stability theorem. A curve $\Gamma(x_0)$ consists generally of two parts Γ' and Γ''.

If $|\arg x_0\mp\pi/2|\leqq\pi/2-\Omega$, the curve $\Gamma(x_0)$ consists of part Γ' only. Let $x_0=A_0+iB_0$, $i=\sqrt{-1}$. Put

$$(5.1)\qquad A=\frac{A_0}{|x_0|^2}\,,\qquad B=\frac{B_0}{|x_0|^2}\,.$$

The variable point $x=x_1(\sigma)$ on Γ' is expressed by the formula

$$(5.2)\qquad \frac{1}{x}=A+\sigma-iBe^{\kappa\sigma}\,,\qquad \text{for}\quad 0\leqq\sigma<\infty\,.$$

If $|\arg x_0\mp\pi/2|>\pi/2-\Omega$, the curve $\Gamma(x_0)$ consists of two parts Γ' and Γ''. The variable point $x=x_2(\tau)$ on the Γ'' is expressed by

$$(5.3)\qquad x=\left(\frac{|x_0|\cos\tau}{\cos\theta_0}\right)\cdot e^{i\tau}\,,\qquad (\theta_0=\arg x_0)\,,$$

in either case of $\theta_0\leqq\tau\leqq\Omega$ or $\pi-\Omega\leqq\tau\leqq\theta_0$.

At the ending point of Γ'', namely at either $\tau=\Omega$ or $\tau=\pi-\Omega$, this curve must be switched to a curve of the form (5.2), where the starting point of the curve Γ' is given by

$$(5.4)\qquad A_0=\frac{|x_0|\cos^2\Omega}{\cos\theta_0}\,,\qquad B_0=\frac{|x_0|\cos\Omega\sin\Omega}{|\cos\theta_0|}\,.$$

The properties of the curve $\Gamma(x_0)$ are studied in Theorem 5 in Iwano [4]. The meaning of stable domain will be clarified in the following theorem:

THEOREM 3. *Let (x_0, u_0, v_0) be an arbitrary point in the domain* (4.7.$\mp$). *Denote by $(U(x), V(x))$ a holomorphic general solution of the equations* (4.1) *satisfying an initial condition $(U, V)=(u_0, v_0)$ at $x=x_0$, where x_0 belongs to the sector $|\arg x\mp\pi/2|<\pi-\varepsilon_0$.*

Then, as x travels on the curve $\Gamma(x_0)$, the values of the functions $(x, U(x), V(x))$, considered as points in the (x, u, v)-space, stay in the domain $(4.7.\mp)$.

PROOF. The case when $|\arg x_0 \mp \pi/2| \leqq \pi/2 - \Omega$. Then curve $\Gamma(x_0)$ is made of the curve Γ' only. As was already shown in Section 4 in Iwano [6], we have the inequalities

$$(5.5)\qquad \begin{cases} \dfrac{1}{|U(x)|}\dfrac{d|U(x)|}{ds_x} \geqq \dfrac{3\nu_1(\kappa)}{5\kappa}\cdot\dfrac{1}{|x|}, \\ \dfrac{1}{|V(x)|}\dfrac{d|V(x)|}{ds_x} \geqq \dfrac{3\nu_2(\kappa)}{5\kappa}\cdot\dfrac{1}{|x|}, \\ \dfrac{1}{|\Xi_0(x)|}\dfrac{d|\Xi_0(x)|}{ds_x} \geqq \dfrac{3\nu_3(\kappa)}{5\kappa}\cdot\dfrac{1}{|x|} \end{cases}$$

as x moves on the curve Γ'. s_x denotes the arclength of this curve measured from the origin to the variable point x. Moreover, it was shown that

$$(5.6)\qquad \frac{2}{5\sqrt{2}} < \frac{d|x|}{ds_x} < \frac{3}{2} \qquad \text{for} \quad x \in \Gamma'.$$

From the second inequality in (5.5) we see that the function $|V(x)|$ is steadily increasing in s_x on the part Γ'. Hence, the inequality $|V(x)| < r_1\chi_\beta(\arg x)$ continues to hold as long as this one does at the starting point x_0.

By utilizing the first inequality in (5.5) and inequality (5.6), we have

$$(5.7)\qquad \begin{aligned} \frac{d}{ds_x}\left\{\frac{|U(x)|^N}{|x|}\right\} &\geqq \frac{3N\nu_1(\kappa)}{5\kappa}\cdot\frac{|U(x)|^N}{|x|^2} - \frac{3}{2}\,\frac{|U(x)|^N}{|x|^2} \\ &= \frac{3(2N\nu_1(\kappa) - 5\kappa)}{10\kappa}\cdot\frac{|U(x)|^N}{|x|^2}. \end{aligned}$$

If N satisfies the inequality

$$(5.8)\qquad N > \frac{5\kappa}{2\nu_1(\kappa)},$$

we see that the function $|U(x)|^N/|x|$ is steadily increasing in s_x. Hence, if the inequality $|U(x)|^N < \Delta_N |x| \chi^{(N)}(\arg x)$ is satisfied at the starting point x_0, then this inequality is true on the curve Γ', because the function $\chi^{(N)}(\arg x)$ is constant for $|\arg x \mp \pi/2| \leqq \pi/2 - \Omega$.

The case where $|\arg x_0 \mp \pi/2| > \pi/2 - \Omega$. Consider the case $-\pi/2 + \varepsilon_0 < \arg x_0 < \Omega$. Observe that

$$V(x) = v_0 \cdot \exp\left(\frac{-\nu}{x_0}\right)\cdot x_0^{-\beta}\cdot \exp\left(\frac{\nu}{x}\right)\cdot x^{\beta}.$$

Assume that

$$(5.9) \qquad |v_0| < r_1\chi_\beta(\arg x_0) = r_1(\cos\theta_0)^{\Re\beta} e^{(-\theta_0+\Omega)\Im\beta}, \qquad (\theta_0 = \arg x_0).$$

Since

$$x = \left(\frac{|x_0|\cos\tau}{\cos\theta_0}\right)e^{i\tau}, \qquad \theta_0 \leqq \tau \leqq \Omega,$$

we have

$$(5.10) \qquad \Re\left(\frac{1}{x}\right) = \Re\left(\frac{1}{x_0}\right) = \frac{\cos\theta_0}{|x_0|}.$$

Hence, on the curve Γ'',

$$|V(x)| < r_1\chi_\beta(\theta_0)e^{(\theta_0-\tau)\Im\beta}\cdot\left|\frac{x}{x_0}\right|^{\Re\beta}$$
$$< r_1(\cos\theta_0)^{\Re\beta}e^{(-\theta_0+\Omega)\Im\beta}e^{(\theta_0-\tau)\Im\beta}\cdot\left|\frac{\cos\tau}{\cos\theta_0}\right|^{\Re\beta}$$
$$= r_1(\cos\tau)^{\Re\beta}\cdot e^{(-\tau+\Omega)\Im\beta} = r_1\chi_\beta(\arg x) \qquad \text{for} \quad \theta_0 \leqq \tau < \Omega.$$

It turns out that, as long as (5.9) holds, we have $|V(x)| < r_1\chi_\beta(\arg x)$ on the curve $\Gamma(x_0)$.

Let us assume that

$$(5.11) \qquad |u_0|^N < \Delta_N |x_0| \chi^{(N)}(\arg x_0).$$

Observe that

$$(5.12) \qquad U(x) = u_0\exp\left(\frac{\mu}{x_0}\right)\cdot x_0^{-\alpha}\exp\left(-\frac{\mu}{x}\right)\cdot x^\alpha.$$

By virtue of (5.11), we have

$$\frac{|U(x)|^N}{|x|} = |u_0|^N\left|\frac{x}{x_0}\right|^{N\Re\alpha} e^{N(\theta_0-\tau)\Re\alpha}\cdot\frac{1}{|x|}$$
$$< \Delta_N\left(\frac{\cos\theta_0}{\cos\Omega}\right)^{N\Re\alpha-1} e^{(-\theta_0+\Omega)N\Im\alpha}\left(\frac{\cos\tau}{\cos\theta_0}\right)^{N\Re\alpha-1} e^{(\theta_0-\tau)N\Im\alpha}$$
$$= \Delta_N\left(\frac{\cos\tau}{\cos\Omega}\right)^{N\Re\alpha-1} e^{(-\tau+\Omega)N\Im\alpha} = \Delta_N\chi^{(N)}(\arg x)$$

on the curve Γ''. Hence, the inequality $|U(x)|^N < \Delta_N|x|\chi^{(N)}(\arg x)$ holds as x moves on the curve $\Gamma(x_0)$ as long as it does at the starting point. q.e.d.

6. Estimation of the integrals of the kernel functions along the curve $\Gamma(x_0)$. When we prove the existence of solutions for the equations (3.10) by utilizing the fixed point technique, it is essential to make a good estimation of the two integrals

$$\int_0^{s_0} \frac{|U(x)|^{2N}}{|x|^2} ds_x \quad \text{and} \quad \int_0^{s_0} e^{-\Re A(x)} \frac{|U(x)|^{2N}}{|x|^2} ds_x, \tag{6.1}$$

where s_0 is the arclength of the curve $\Gamma(x_0)$. The functions $|U(x)|^{2N}/|x|^2$ and $e^{-\Re A(x)}|U(x)|^{2N}/|x|^2$ will play a role of the kernels in the integral equations appearing in Section 7 which are derived from the differential equations (3.10).

THEOREM 4. *We have the estimates*

$$\left\{\begin{aligned} &\int_0^{s_0} \frac{|U(x)|^{2N}}{|x|^2} ds_x \leqq M_1 \frac{|u_0|^{2N}}{|x_0|}, \\ &\int_0^{s_0} e^{-\Re A(x)} \frac{|U(x)|^{2N}}{|x|^2} ds_x \leqq M_2 e^{-\Re A(x_0)} \frac{|u_0|^{2N}}{|x_0|}, \end{aligned}\right. \tag{6.2}$$

where M_1 and M_2 are defined by the formulas (6.28) *and* (6.30).

1°. The estimation of the integrals (6.1) on the curve Γ'. Put

$$\mathscr{A}(x) \equiv \frac{U(x)^{2N}}{x}. \tag{6.3}$$

This function satisfies the linear equation

$$x^2 \frac{d\mathscr{A}}{dx} = (2N\mu + (2N\alpha - 1)x)\mathscr{A}. \tag{6.4}$$

A direct calculation shows

$$\begin{aligned} \frac{d|\mathscr{A}(x)|}{ds_x} &= \frac{d}{ds_x}\left\{\frac{|U(x)|^{2N}}{|x|}\right\} \\ &= 2N \frac{|U(x)|^{2N-1}}{|x|} \frac{d|U(x)|}{ds_x} - \frac{|U(x)|^{2N}}{|x|^2} \frac{d|x|}{ds_x} \\ &\geqq \frac{6N\nu_1(\kappa)}{5\kappa} \frac{|\mathscr{A}(x)|}{|x|} - \frac{3}{2} \frac{|\mathscr{A}(x)|}{|x|} \qquad \text{(from (5.5), (5.6))} \\ &= \frac{12N\nu_1(\kappa) - 15\kappa}{10\kappa} \frac{|\mathscr{A}(x)|}{|x|}. \end{aligned} \tag{6.5}$$

Hence, by integrating this inequality along the curve Γ', we get

$$\int_0^{s_0} \frac{|U(x)|^{2N}}{|x|^2} ds_x \leqq \frac{10\kappa}{12N\nu_1(\kappa) - 15\kappa} \frac{|U(x_0)|^{2N}}{|x_0|}. \tag{6.6}$$

Set

$$\mathscr{B}(x) \equiv e^{-\Lambda(x)} \cdot \frac{U(x)^{2N}}{x}. \tag{6.7}$$

This function is a solution of the linear equation

$$x^2 \frac{d\mathscr{B}}{dx} = (2N\mu + \nu + (2N\alpha - \beta - 1)x)\mathscr{B}. \tag{6.8}$$

Let M_0 be the number defined by the condition (4.3). Write $\mathscr{B}(x)$ as

$$\mathscr{B}(x) = e^{-\Lambda(x)} U(x)^{M_0} \cdot \frac{U(x)^{2N-M_0}}{x} = \Xi_0(x) \cdot \frac{U(x)^{2N-M_0}}{x}.$$

By utilizing the last one in the inequalities (5.5) and the inequality (6.5), (where $2N$ must be replaced by $2N - M_0$), we get at once

$$\begin{aligned} \frac{d|\mathscr{B}(x)|}{ds_x} &= \frac{d|\Xi_0(x)|}{ds_x} \cdot \frac{|U(x)|^{2N-M_0}}{|x|} + |\Xi_0(x)| \cdot \frac{d}{ds_x}\left(\frac{|U(x)|^{2N-M_0}}{|x|}\right) \\ &\geqq \frac{3\nu_3(\kappa)}{5\kappa} \frac{|\Xi_0(x)|}{|x|} \frac{|U(x)|^{2N-M_0}}{|x|} \\ &\quad + \frac{(12N - 6M_0)\nu_1(\kappa) - 15\kappa}{10\kappa} |\Xi_0(x)| \frac{|U(x)|^{2N-M_0}}{|x|^2} \\ &\geqq \left(\frac{3\nu_3(\kappa)}{5\kappa} + \frac{(12N - 6M_0)\nu_1(\kappa) - 15\kappa}{10\kappa}\right) \frac{|\mathscr{B}(x)|}{|x|}. \end{aligned}$$

Thanks to the definition of $\nu_3(\kappa)$ in (4.3), the constant factor of the function $|\mathscr{B}(x)|/|x|$ is equal to

$$\frac{3(M_0\nu_1(\kappa) - \nu_2(\kappa))}{5\kappa} + \frac{(12N - 6M_0)\nu_1(\kappa) - 15\kappa}{10\kappa} = \frac{12N\nu_1(\kappa) - 3\nu_2(\kappa) - 15\kappa}{10\kappa}.$$

We assume, besides (5.8), that N satisfies

$$N > \max\left\{\frac{5\kappa}{2\nu_1(\kappa)}, \frac{\nu_2(\kappa) + 5\kappa}{4\nu_1(\kappa)}\right\}. \tag{6.9}$$

Then,

$$\frac{d|\mathscr{B}(x)|}{ds_x} \geqq \frac{12N\nu_1(\kappa) - 3\nu_2(\kappa) - 15\kappa}{10\kappa} \frac{|\mathscr{B}(x)|}{|x|}. \tag{6.10}$$

By integrating this differential inequality along the curve Γ', we have immediately

$$(6.11)\qquad \int_0^{s_0} e^{-\Re A(x)} \frac{|U(x)|^{2N}}{|x|^2} ds_x \leqq \frac{10\kappa}{12N\nu_1(\kappa)-3\nu_2(\kappa)-15\kappa} e^{-\Re A(x_0)} \frac{|U(x_0)|^{2N}}{|x_0|}.$$

2°. The estimation of the integrals (6.1) on the curve Γ''. Since the variable point x on Γ'' is given by the formula (5.3), we have

$$(6.12)\qquad \frac{dx}{d\tau} = i\left(\frac{|x_0|}{\cos\theta_0}\right)e^{2i\tau}$$

and

$$(6.13)\qquad \frac{ds_x}{d\tau} = \begin{cases} -\dfrac{|x_0|}{\cos\theta_0}, & \text{for } \theta_0 \leqq \tau < \Omega, \\ -\dfrac{|x_0|}{\cos\theta_0}, & \text{for } \pi-\Omega < \tau \leqq \theta_0, \end{cases}$$

where s_x is the arclength of the curve Γ'' measured from its ending point x_0' to the variable point x.

As is shown in (5.10), on the curve Γ'', the function $\Re(\mu/x)$ is unchanged. Since

$$(6.14)\qquad U(x) = u_0 \cdot \exp\left(-\frac{\mu}{x}+\frac{\mu}{x_0}\right)\cdot\left(\frac{x}{x_0}\right)^{\alpha},$$

we have

$$(6.15)\qquad \begin{aligned} \frac{|\mathscr{A}(x)|}{|x|} = \frac{|U(x)|^{2N}}{|x|^2} &= \frac{|u_0|^{2N}}{|x|^2}\exp\left(-2N\Re\left(\frac{\mu}{x}-\frac{\mu}{x_0}\right)\right)\cdot\left|\frac{x}{x_0}\right|^{2N\Re\alpha} e^{2N(\tau-\theta_0)\Im\alpha} \\ &= \frac{|u_0|^{2N}}{|x_0|^2}\left|\frac{\cos\tau}{\cos\theta_0}\right|^{2N\Re\alpha-2} e^{2N(\theta_0-\tau)\Im\alpha} \\ &\leqq \frac{|u_0|^{2N}}{|x_0|^2}\left(\frac{1}{\sin\varepsilon_0}\right)^{2N\Re|\alpha|-2} e^{2N(\pi-\Omega+\varepsilon_0)|\Im\alpha|}, \end{aligned}$$

because of the inequalities $|\cos\theta_0| \geqq \sin\varepsilon_0$ and $\pi-\Omega+\varepsilon_0 \geqq |\theta_0-\tau|$. Since $ds_x = -(|x_0|/\cos\theta_0)d\tau$ and $|\cos\theta_0| \geqq \sin\varepsilon_0$, the integration of the inequality (6.15), excluding the middle terms, gives

$$(6.16)\qquad \int_{\Gamma''} \frac{|U(x)|^{2N}}{|x|^2} ds_x \leqq \frac{|u_0|^{2N}}{|x_0|}\left(\frac{1}{\sin\varepsilon_0}\right)^{2N\Re\alpha-1}\cdot\pi\cdot e^{2N(\pi-\Omega+\varepsilon_0)|\Im\alpha|}.$$

By the notation (4.5), this inequality implies

$$(6.16\text{-bis})\qquad \int_{\Gamma''} \frac{|U(x)|^{2N}}{|x|^2} ds_x \leqq \frac{|u_0|^{2N}}{|x_0|}\left(\frac{1}{\sin\varepsilon_0}\right)^{2N\|\upsilon\|-1}\cdot\pi\cdot e^{2N\pi\|\delta\|}.$$

On the other hand, we have on the curve Γ''

$$
\begin{aligned}
e^{-\Re A(x)} &= \exp\left(-\Re\left(\frac{\nu}{x}\right)\right)\cdot |x|^{-\Re\beta} e^{\tau\Im\beta} \\
&= \exp\left(-\Re\left(\frac{\nu}{x_0}\right)\right)\cdot |x_0|^{-\Re\beta} e^{\theta_0\Im\beta}\left|\frac{x}{x_0}\right|^{-\Re\beta} e^{(\tau-\theta_0)\Im\beta} \\
&= e^{-\Re A(x_0)}\left|\frac{x}{x_0}\right|^{-\Re\beta} e^{(\tau-\theta_0)\Im\beta} \qquad (6.17)\\
&= e^{-\Re A(x_0)}\left|\frac{\cos\tau}{\cos\theta_0}\right|^{-\Re\beta} e^{(\tau-\theta_0)\Im\beta} \\
&\leqq e^{-\Re A(x_0)}\left(\frac{1}{\sin\varepsilon_0}\right)^{|\Re\beta|} e^{(\pi-\Omega+\varepsilon_0)|\Im\beta|},
\end{aligned}
$$

which, by the help of (6.15), implies

$$
\begin{aligned}
\frac{|\mathscr{B}(x)|}{|x|} &= e^{-\Re A(x)}\frac{|U(x)|^{2N}}{|x|^2} \qquad (6.18)\\
&\leqq e^{-\Re A(x_0)}\frac{|u_0|^{2N}}{|x_0|^2}\left(\frac{1}{\sin\varepsilon_0}\right)^{2N|\Re\alpha|+|\Re\beta|-2} e^{(\pi-\Omega+\varepsilon_0)(2N|\Im\alpha|+|\Im\beta|)}.
\end{aligned}
$$

By integrating this inequality along the curve Γ'', namely from θ_0 to Ω or from $\pi-\Omega$ to θ_0, we have at once

$$
\begin{aligned}
&\int_{\Gamma''} e^{-\Re A(x)}\frac{|U(x)|^{2N}}{|x|^2}\,ds_x \qquad (6.19)\\
&\leqq e^{-\Re A(x_0)}\frac{|u_0|^{2N}}{|x_0|}\left(\frac{1}{\sin\varepsilon_0}\right)^{2N|\Re\alpha|+|\Re\beta|-1}\cdot\pi\cdot e^{(\pi-\Omega+\varepsilon_0)(2N|\Im\alpha|+|\Im\beta|)}.
\end{aligned}
$$

By the notation (4.5),

$$
\begin{aligned}
&\int_{\Gamma''} e^{-\Re A(x)}\frac{|U(x)|^{2N}}{|x|^2}\,ds_x \qquad (6.19\text{-bis})\\
&\leqq e^{-\Re A(x_0)}\frac{|u_0|^{2N}}{|x_0|}\left(\frac{1}{\sin\varepsilon_0}\right)^{(2N+1)\|\mathfrak{v}\|-1}\cdot\pi\cdot e^{(2N+1)\pi\|\delta\|}.
\end{aligned}
$$

3°. The estimation of the integrals (6.1) on the curve $\Gamma(x_0)$.

When $|\theta_0\mp\pi/2|\leqq\pi/2-\Omega$, we have $\Gamma(x_0)=\Gamma'$. Hence, by virtue of (6.6) and (6.11), we have at once

$$\int_{\Gamma(x_0)} \frac{|U(x)|^{2N}}{|x|^2} ds_x \leqq \frac{10\kappa}{12N\nu_1(\kappa)-15\kappa} \frac{|U(x_0)|^{2N}}{|x_0|} \tag{6.20}$$

and

$$\int_{\Gamma(x_0)} e^{-\Re A(x)} \frac{|U(x)|^{2N}}{|x|^2} ds_x \leqq \frac{10\kappa}{12N\nu_1(\kappa)-3\nu_2(\kappa)-15\kappa} e^{-\Re A(x_0)} \frac{|U(x_0)|^{2N}}{|x_0|}. \tag{6.21}$$

When $|\theta_0 \mp \pi/2| > \pi/2 - \Omega$, we have $\Gamma(x_0) = \Gamma' \cup \Gamma''$. The starting point x'_0 of the part Γ' is given by

$$x'_0 = \begin{cases} \left(\dfrac{|x_0|\cos\Omega}{\cos\theta_0}\right) e^{i\Omega}, & \theta_0 < \Omega, \\ \left(\dfrac{|x_0|\cos(\pi-\Omega)}{\cos\theta_0}\right) e^{i(\pi-\Omega)}, & \theta_0 > \pi - \Omega. \end{cases} \tag{6.22}$$

The estimation of the integrals on the part Γ' is immediately obtained from (6.6) and (6.11), where x_0 should be replaced by the point x'_0. We see from (6.14) that

$$U(x'_0) = u_0 \cdot \exp\left(-\frac{\mu}{x'_0} + \frac{\mu}{x_0}\right) \cdot \left(\frac{x'_0}{x_0}\right)^{\alpha}.$$

Hence, on the part Γ'', we have for $\Omega > \theta_0$

$$\begin{aligned} |U(x'_0)| &= |u_0| \cdot \left|\frac{x'_0}{x_0}\right|^{\Re\alpha} e^{(\theta_0-\Omega)\Im\alpha} = |u_0| \cdot \left(\frac{\cos\Omega}{\cos\theta_0}\right)^{\Re\alpha} e^{(\theta_0-\Omega)\Im\alpha} \\ &< |u_0| \cdot \left(\frac{1}{\sin\varepsilon_0}\right)^{\|\upsilon\|} e^{\pi\|\delta\|}. \end{aligned} \tag{6.23}$$

This inequality holds also for $\theta_0 > \pi - \Omega$. Since

$$|x'_0| = \frac{|x_0|\cos\Omega}{|\cos\theta_0|},$$

we have

$$\frac{1}{|x'_0|} = \frac{|\cos\theta_0|}{\cos\Omega} \frac{1}{|x_0|} \leqq \frac{1}{\cos\Omega} \frac{1}{|x_0|} \leqq \frac{1}{\sin\varepsilon_0} \frac{1}{|x_0|} \tag{6.24}$$

and

$$\left|\frac{x'_0}{x_0}\right| \leqq \frac{1}{\sin\varepsilon_0}, \qquad \left|\frac{x_0}{x'_0}\right| \leqq \frac{1}{\sin\varepsilon_0}. \tag{6.25}$$

On the other hand, we see, by utilizing (5.10), that

$$
\begin{aligned}
e^{-\Re A(x_0')} &= \exp\left(-\Re\left(\frac{v}{x_0'}+\beta\log x_0'\right)\right)=\exp\left(-\Re\left(\frac{v}{x_0}+\beta\log x_0'\right)\right)\\
&= \exp\left(-\Re\left(\frac{v}{x_0}+\beta\log x_0\right)\right)\cdot\left|\left(\frac{x_0}{x_0'}\right)^{\beta}\right|\\
&= e^{-\Re A(x_0)}\left|\frac{x_0}{x_0'}\right|^{\Re\beta} e^{(\arg x_0'-\arg x_0)\Im\beta}\\
&\leqq \begin{cases} e^{-\Re A(x_0)}\left(\dfrac{1}{\sin\varepsilon_0}\right)^{|\Re\beta|} e^{(\Omega-\theta_0)|\Im\beta|} & (\theta_0<\Omega)\,,\\ e^{-\Re A(x_0)}\left(\dfrac{1}{\sin\varepsilon_0}\right)^{|\Re\beta|} e^{(\theta_0-\pi+\Omega)|\Im\beta|} & (\theta_0>\pi-\Omega)\,,\end{cases}\\
&\leqq e^{-\Re A(x_0)}\left(\frac{1}{\sin\varepsilon_0}\right)^{\|v\|} e^{\pi\|\delta\|}\,.
\end{aligned} \tag{6.26}
$$

It follows from (6.6) and (6.16-bis) that

$$
\int_{\Gamma(x_0)}\frac{|U(x)|^{2N}}{|x|^2}\,ds_x\leqq M_1\frac{|u_0|^{2N}}{|x_0|}\,, \tag{6.27}
$$

where M_1 has the form

$$
\begin{aligned}
M_1 &= \left(\frac{1}{\sin\varepsilon_0}\right)^{2N\|v\|-1}\cdot\pi\cdot e^{2N\pi\|\delta\|}+\frac{10\kappa}{12N\nu_1(\kappa)-15\kappa}\left(\frac{1}{\sin\varepsilon_0}\right)^{2N\|v\|+1} e^{2N\pi\|\delta\|}\\
&= \left(\frac{1}{\sin\varepsilon_0}\right)^{2N\|v\|-1} e^{2N\pi\|\delta\|}\left(\pi+\frac{10\kappa}{12N\nu_1(\kappa)-15\kappa}\left(\frac{1}{\sin\varepsilon_0}\right)^2\right).
\end{aligned} \tag{6.28}
$$

We have from (6.11) and (6.19-bis)

$$
\int_{\Gamma(x_0)} e^{-\Re A(x)}\frac{|U(x)|^{2N}}{|x|^2}\,ds_x\leqq M_2 e^{-\Re A(x_0)}\frac{|u_0|^{2N}}{|x_0|}\,, \tag{6.29}
$$

where M_2 is given by

$$
\begin{aligned}
M_2 &= \left(\frac{1}{\sin\varepsilon_0}\right)^{(2N+1)\|v\|-1}\cdot\pi\cdot e^{(2N+1)\pi\|\delta\|}\\
&\quad+\frac{10\kappa}{12N\nu_1(\kappa)-3\nu_2(\kappa)-15\kappa}\left(\frac{1}{\sin\varepsilon_0}\right)^{(2N+1)\|v\|+1} e^{(2N+1)\pi\|\delta\|}\\
&= \left(\frac{1}{\sin\varepsilon_0}\right)^{(2N+1)\|v\|-1} e^{(2N+1)\pi\|\delta\|}
\end{aligned} \tag{6.30}
$$

$$\times\left(\pi+\frac{10\kappa}{12N\nu_1(\kappa)-3\nu_2(\kappa)-15\kappa}\left(\frac{1}{\sin\varepsilon_0}\right)^2\right).$$

The inequality (6.27) with (6.28) and the inequality (6.29) with (6.30) prove Theorem 4. q.e.d.

7. Existence of a solution by the fixed point technique. To prove the existence of solutions for the equations (3.10) by the fixed point technique (for example, Hukuhara [1], [2]), we consider a stable domain of the form (4.7.∓), namely,

$$(7.1\text{-}N.\mp)\qquad \begin{cases} 0<|x|<r_0''\omega(\arg x)\,, & \left|\arg x\mp\dfrac{\pi}{2}\right|<\pi-\varepsilon_0\,,\\ |u|^N<\varDelta_N|x|\cdot\chi^{(N)}(\arg x)\,, & |v|<r_1''\chi_\beta(\arg x)\,, \end{cases}$$

where the functions $\omega(\tau)$, $\chi_\beta(\tau)$ and $\chi^{(N)}(\tau)$ are defined by (4.8), (4.9) and (4.10). Here the constants r_0'', r_1'' and $\varDelta_N$, depending on N, are to be so chosen as to satisfy inequalities of the form

$$(7.2)\qquad \begin{cases} r_0''\max\limits_\tau\omega(\tau)<r_0^{(N)}\,,\\ r_1''\max\limits_\tau\chi_\beta(\tau)<r_1^{(N)}\,, \qquad \left(\varDelta_N\max\limits_\tau\chi^{(N)}(\tau)\cdot r_0''\max\limits_\tau\omega(\tau)\right)^{1/N}<r_1^{(N)}\,,\\ \varDelta_N\min\limits_\tau\chi^{(N)}(\tau)>1\,, \end{cases}$$

in the τ-interval $[-\pi/2+\varepsilon_0,\ 3\pi/2-\varepsilon_0]$ or $[-3\pi/2+\varepsilon_0,\ \pi/2-\varepsilon_0]$, where the $r_0^{(N)}$ and $r_1^{(N)}$ are the same as those appearing in (3.11.∓).

We consider a family $\mathscr{F}$ of pairs $\{\phi, \psi\}$ of functions $\phi(x, u, v)$ and $\psi(x, u, v)$, which are holomorphic and bounded in (x, u, v) for the domain (7.1-N.∓) and, moreover, satisfy inequalities of the form

$$(7.3)\qquad |\phi(x,u,v)|\leqq K_N|u|^N\,,\qquad |\psi(x,u,v)|\leqq K_Ne^{-\Re A(x)}|u|^N\,.$$

Here the constant K_N, depending on N, must satisfy inequalities of the form

$$(7.4)\qquad \begin{cases} K_N\cdot r_0''\max\limits_\tau\omega(\tau)<r_2^{(N)}\,, & K_N\cdot r_0''\max\limits_\tau\omega(\tau)\cdot\varDelta_N\max\limits_\tau\chi^{(N)}(\tau)<r_2^{(N)}\,,\\ 4L_N\cdot M_1\cdot\varDelta_N\max\limits_\tau\chi^{(N)}(\tau)<K_N\,, & 4L_N\cdot M_2\cdot\varDelta_N\max\limits_\tau\chi^{(N)}(\tau)<K_N \end{cases}$$

for the τ-interval. These inequalities will be obviously satisfied if we first take the value of K_N sufficiently large and then that of r_0'' sufficiently small. Hence,

$$(7.5)\qquad \begin{cases} |\phi(x, U(x), V(x))| < r_2^{(N)}, & |e^{A(x)}\psi(x, U(x), V(x))| < r_2^{(N)}, \\ L_N M_1 \dfrac{|U(x)|^N}{|x|} \leqq \dfrac{1}{4} K_N, & L_N M_2 \dfrac{|U(x)|^N}{|x|} \leqq \dfrac{1}{4} K_N, \\ K_N |U(x)|^N < r_2^{(N)} < 1 \end{cases}$$

as long as the values of $(x, U(x), V(x))$ belong to the domain $(7.1\text{-}N.\mp)$.

Choose a point (x_0, u_0, v_0) in the domain $(7.1\text{-}N.\mp)$ in an arbitrary manner. Let $\{U(x), V(x)\}$ be the solution of the equations (4.1) such that $\{U(x_0), V(x_0)\} = \{u_0, v_0\}$. By the help of Theorem 2 and by the definition of the K_N, we see that the functions $\mathscr{F}_N(x)$ and $\mathscr{G}_N(x)$ given by the expressions

$$(7.6)\qquad \begin{cases} \mathscr{F}_N(x) \equiv F_N(x, U(x), V(x), \phi(x, U(x), V(x)), e^{A(x)}\psi(x, U(x), V(x))), \\ \mathscr{G}_N(x) \equiv G_N(x, U(x), V(x), \phi(x, U(x), V(x)), e^{A(x)}\psi(x, U(x), V(x))) \end{cases}$$

become holomorphic functions of x on the curve $\Gamma(x_0)$, because of the conditions (7.2). Moreover, by virtue of (3.12), they satisfy the inequalities

$$(7.7)\qquad |\mathscr{F}_N(x)| \leqq L_N, \qquad |\mathscr{G}_N(x)| \leqq L_N.$$

The mapping $\mathscr{T}$ is to be defined by

$$(7.8)\qquad \mathscr{T}: \{\phi(x, u, v), \psi(x, u, v)\} \to \{\Phi(x, u, v), \Psi(x, u, v)\},$$

where the $\Phi(x, u, v)$ and $\Psi(x, u, v)$ are given by the integrals

$$(7.9)\qquad \begin{cases} \Phi(x_0, u_0, v_0) = \displaystyle\int_{\Gamma(x_0)} \frac{U(x)^{2N}}{x^2} \mathscr{F}_N(x)dx, \\ \Psi(x_0, u_0, v_0) = \displaystyle\int_{\Gamma(x_0)} e^{-A(x)} \frac{U(x)^{2N}}{x^2} \mathscr{G}_N(x)dx. \end{cases}$$

As was already proved, the integrals of the kernel functions are bounded. Therefore, the integrals (7.9) are uniformly convergent with respect to (x_0, u_0, v_0). This implies, after a short reasoning, that the functions $\Phi(x, u, v)$ and $\Psi(x, u, v)$ are holomorphic in (x, u, v) at the point (x_0, u_0, v_0) and, consequently, for the domain $(7.1\text{-}N.\mp)$. Moreover, by virtue of (7.5), these functions are bounded by

$$(7.10)\qquad \begin{cases} |\Phi(x_0, u_0, v_0)| \leqq L_N M_1 \cdot \dfrac{|u_0|^{2N}}{|x_0|} \leqq \dfrac{1}{4} K_N |u_0|^N, \\ |\Psi(x_0, u_0, v_0)| \leqq L_N M_2 \cdot e^{-\Re A(x_0)} \dfrac{|u_0|^{2N}}{|x_0|} \leqq \dfrac{1}{4} K_N e^{-\Re A(x_0)} |u_0|^N. \end{cases}$$

Therefore, we see that $\{\Phi, \Psi\} \in \mathscr{F}$. According to our standard analysis (for example, Iwano [5, pp. 124–132]), we can show that *the mapping* $\mathscr{T}$ *possesses a fixed point to which corresponds a solution* $\{\Phi_N(x, U(x), V(x)), \Psi_N(x, U(x), V(x))\}$ *of the equations* (3.10).

By the inequalities (3.13), we can prove that *a solution of the equations* (3.10) *satisfying the order condition*

$$(7.11)\qquad Y=O(U(x)^N), \qquad Z=O(e^{-A(x)}U(x)^N)$$

is unique.

In the proof of this assertion, the factor 1/4 appearing in (7.10) will be useful. To prove this, assume that there exist two solutions. Denote by $\{\hat{\Phi}(x, U(x), V(x)), \hat{\Psi}(x, U(x), V(x))\}$ their difference. Then we want to prove that, for any integer m,

$$(7.12.m)\qquad |\hat{\Phi}(x, u, v)| \leqq \frac{1}{2^m} K_N |u|^N, \qquad |\hat{\Psi}(x, u, v)| \leqq \frac{1}{2^m} K_N e^{-\Re A(x)} |u|^N,$$

which implies $\hat{\Phi}(x, u, v)=\hat{\Psi}(x, u, v)\equiv 0$. However, for $m=1$, the inequalities follow immediately from (7.10). Assume that the inequalities (7.12.m) are satisfied. Then, we see that the last one in the inequalities (7.5) gives

$$|\hat{\Phi}(x, u, v)| \leqq \frac{1}{2^m}, \qquad |\hat{\Psi}(x, u, v)| \leqq \frac{1}{2^m} e^{-\Re A(x)}.$$

It follows from (3.13) and (7.10) that

$$|\hat{\Phi}(x_0, u_0, v_0)| \leqq \frac{1}{2^{m-1}} \cdot L_N \cdot \left| \int_{\Gamma(x_0)} \frac{|U(x)|^{2N}}{|x|^2} dx \right| \leqq \frac{1}{2^{m+1}} K_N |u_0|^N,$$

which proves that the inequalities (7.12.$m+1$) hold.

Taking the transformations (3.2) and (3.8) into account, we see that the pair $\{\mathscr{Y}_N, \mathscr{Z}_N\}$ of the functions $\mathscr{Y}_N(x, U(x), V(x))$ and $\mathscr{Z}_N(x, U(x), V(x))$ defined by

$$(7.13)\qquad \begin{cases} \mathscr{Y}_N(x, u, v) \equiv \hat{P}_{(N)}\left(x, \dfrac{u}{1-\Phi_N(x, u, v)}, v+e^{A(x)}\Psi_N(x, u, v)\right), \\ \mathscr{Z}_N(x, u, v) \equiv \hat{Q}_{(N)}\left(x, \dfrac{u}{1-\Phi_N(x, u, v)}, v+e^{A(x)}\Psi_N(x, u, v)\right), \end{cases}$$

becomes a solution of the equations (2.1), whenever the values of $(x, U(x), V(x))$, considered as points in the (x, u, v)-space, belong to the domain (7.1-$N.\mp$). Hence, the $\mathscr{Y}_N(x, u, v)$ and $\mathscr{Z}_N(x, u, v)$ are considered as holomorphic and bounded functions of (x, u, v) for a domain of the form

$$(7.14.\mp)\qquad \left|\arg x \mp \frac{\pi}{2}\right| < \pi - \varepsilon_0, \qquad 0<|x|<a_N, \quad |u|^N<|x|, \quad |v|<b_N,$$

where a_N and b_N depend on N. However, our standard analysis, as was done in Iwano [6], we have the following:

PROPOSITION 4. *The solution* $\{\mathscr{Y}_N(x, U(x), V(x)), \mathscr{Z}_N(x, U(x), V(x))\}$ *is independent of* N.

Denote this solution by $\{\Phi(x, U(x), V(x)), \Psi(x, U(x), V(x))\}$. Then, we have the following:

PROPOSITION 5. *The functions* $\Phi(x, u, v)$ *and* $\Psi(x, u, v)$ *become holomorphic in* (x, u, v) *for a domain of the form*

$$\left|\arg x \mp \frac{\pi}{2}\right| < \pi - \varepsilon_0, \qquad 0 < |x| < a_0, \quad |u| < b_0, \quad |v| < b_0. \tag{7.15.$\mp$}$$

a_0 *and* b_0 *are small positive constants independent of* N.

Indeed, let (x_0, u_0, v_0) be an arbitrary point in the domain (7.15.$\mp$). Then, choose a large positive integer N such that $|u_0|^N < |x_0|$. By the independence of N, we observe that the relations

$$\Phi(x, u, v) = \mathscr{Y}_N(x, u, v), \qquad \Psi(x, u, v) = \mathscr{Z}_N(x, u, v) \tag{7.16}$$

hold identically in a neighbourhood of the point (x_0, u_0, v_0). This proves our assertion.

Therefore, the functions $\Phi(x, u, v)$ and $\Psi(x, u, v)$ admit Taylor series expansions (in u and consequently) in u and v, which coincide with the power series appearing in the formal transformation (2.2). This completes the proof of Theorem A.

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