Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat 22:1 (2008), 193—204

NEW SUBCLASS OF GOODMAN-TYPE p-VALENT HARMONIC FUNCTIONS

S. B. Joshi and N. D. Sangle

2000 *Mathematics Subject Classification*. Primary 30C45, Secondary 30C50, 30C55.
Keywords and phrases. Harmonic functions, p-valent functions, Convex combination, Distortion bounds.
Received: December 18, 2007
Communicated by Dragan S. Djordjević

Abstract. In this paper, we have introduced a new subclass of p-valent harmonic functions that are orientation preserving in the open unit disk and are related to Goodman-type analytic uniformly starlike functions. Coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination for the functions belonging to this class are obtained.

1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D, if both u and v are real harmonic in D. There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exit analytic functions U and V so that $u = \mathrm{Re}(U)$ and $v = \mathrm{Im}(V)$. Then

$$f(z) = h(z) + \overline{g(z)},$$

where h and g are respectively, the analytic functions $(U + V/2)$ and $(U - V/2)$. In this case, the Jacobian of $f(z) = h(z) + \overline{g(z)}$ is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 .$$

The mapping $z \to f(z)$ is orientation preserving and locally one to one in D, if and only if $J_f(z) > 0$ in D. The necessity of this condition is a result of Lewy [6]. See also Clunie and Sheil-Small [2].

The function $f(z)=h(z)+\overline{g(z)}$ is said to be harmonic univalent in D, if the mapping $z\to f(z)$ is orientation preserving, harmonic and one to one in D. We call h the analytic part and g the co-analytic part of $f(z)=h(z)+\overline{g(z)}$.

For fixed positive integer $p,$ let $H(p)$ denote the family of functions $f(z)=h(z)+\overline{g(z)}$ that are harmonic, orientation preserving and p-valent in the open unit disk $U=\{z:|z|<1\}$ with the normalization

$$h(z)=z^p+\sum_{n=2}^{\infty}a_{n+p-1}z^{n+p-1}, \quad g(z)=\sum_{n=1}^{\infty}b_{n+p-1}z^{n+p-1}, \quad |b_p|<1. \tag{1.1}$$

Motivated by recent work of Rosy *et al* [9], we define a new subclass as follows:

Let $G_H(p,\gamma)$ denote the subclass of $H(p)$ consisting of functions f in $H(p)$ that satisfy the condition

$$\operatorname{Re}\left\{(1+e^{i\alpha})\frac{z\,f'(z)}{z'\,f(z)}-pe^{i\alpha}\right\}\geq p\gamma, \tag{1.2}$$

where $z'=\frac{\partial}{\partial\theta}(z=re^{i\theta}),\;\; f'(z)=\frac{\partial}{\partial\theta}\left(f(z)=f(re^{i\theta})\right),\;\; p\geq 1,\;\; 0\leq r<1$ and α, θ are real .

We further let $G_{\overline{H}}(p,\gamma)$ denote the subclass of $G_H(p,\gamma),$ consisting of functions $f(z)=h(z)+\overline{g(z)}$ such that h and g are of the form

$$h(z)=z^p-\sum_{n=2}^{\infty}|a_{n+p-1}|\,z^{n+p-1}, \quad g(z)=\sum_{n=1}^{\infty}|b_{n+p-1}|\,z^{n+p-1}. \tag{1.3}$$

For p=1 and $g\equiv 0$ that is, if f is analytic, the family $G_H(1,0)$ is uniformly starlike in U and was first studied by Goodman [3]. In [8], Ronning investigated the uniformly starlike functions of order $\gamma,\ 0\leq\gamma<1$. Later, Jahangiri *et al* [5] constructed a class of harmonic close to convex functions and studied basic properties. Recently, Jahangiri [4], Silverman

[10], Silverman and Silvia [11] studied the harmonic starlike functions. Ahuja and Jahangiri [1] proved that if, $f(z)=h(z)+\overline{g(z)}$ is given by (1.1) and if,

$$\sum_{n=1}^{\infty}(n+m-1)\left(|a_{n+m-1}|+|b_{n+m-1}|\right)\le 2m \tag{1.4}$$

then f is harmonic, p-valent and starlike of order γ in U. This condition is proved to be also necessary if h and g are of the form (1.3). In the present paper we have obtained coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combinations for the class $G_{\overline{H}}(p,\gamma)$.

2. COEFFICIENT BOUNDS

We being with a sufficient coefficient bounds for the class $G_H(p,\gamma)$. These conditions are shown to be necessary for the functions in $G_{\overline{H}}(p,\gamma)$.

Theorem 1. Let $f=h+\overline{g}$ with h and g are given by (1.1). If

$$\sum_{n=1}^{\infty}\left[\frac{2n+p-2-\gamma}{p-\gamma}|a_{n+p-1}|+\frac{2n+3p-2+\gamma}{p-\gamma}|b_{n+p-1}|\right]\le 2\ , \tag{2.1}$$

where $|a_1|=1,\ 0\le\gamma<1$. Then f is harmonic p-valent in U and $f\in G_H(p,\gamma)$.

Proof. Suppose that (2.1) holds. Then we have

$$\operatorname{Re}\left\{\frac{(1+e^{i\alpha})\left(zh'(z)-\overline{zg'(z)}\right)-pe^{i\alpha}\left(h(z)+\overline{g(z)}\right)}{h(z)+\overline{g(z)}}\right\}=\operatorname{Re}\frac{A(z)}{B(z)}\ge p\gamma, \tag{2.2}$$

as $f(z)\in H(p),\ \ h(z)+\overline{g(z)}\ne 0.$

where $z=re^{i\theta},\ \ 0\le r<1,\ 0\le\gamma<1,\ 0\le\theta<2\pi.$

Here, we let

$$A(z)=\left(1+e^{i\alpha}\right)\left(z\,h'(z)-\overline{z\,g'(z)}\right)-p\,e^{i\alpha}\left(h(z)+\overline{g(z)}\right)$$ and

$$B(z)=h(z)+\overline{g(z)}.$$

Using the fact that $\operatorname{Re}\omega\geq p\gamma,$ if and only if $\left|p-\gamma+\omega\right|\geq\left|p+\gamma-\omega\right|,$ it suffices to show that

$$\left|A(z)+\left(p-\gamma\right)B(z)\right|-\left|A(z)-\left(p+\gamma\right)B(z)\right|\geq 0\ . \qquad (2.3)$$

Substituting for A (z) and B (z) in (2.3), we obtain

$$\left|\left(p-\gamma\right)h(z)+(1+e^{i\alpha})\,z\,h'(z)-p\,e^{i\alpha}h(z)+\overline{\left(p-\gamma\right)g(z)-(1+e^{i\alpha})\,z\,g'h(z)-p\,e^{i\alpha}g(z)}\right|$$

$$-\left|\left(p+\gamma\right)h(z)-(1+e^{i\alpha})\,z\,h'(z)+p\,e^{i\alpha}h(z)+\overline{\left(p+\gamma\right)g(z)+(1+e^{i\alpha})\,z\,g'h(z)+p\,e^{i\alpha}g(z)}\right|$$

$$=\left|(2p-\gamma)\,z^{p}+\sum_{n=2}^{\infty}\left[\left(n+2p-1-\gamma\right)+e^{i\alpha}\left(n-1\right)\right]a_{n+p-1}z^{n+p-1}-\overline{\sum_{n=1}^{\infty}\left[\left(n-1+\gamma\right)+e^{i\alpha}\left(n+2p-1\right)\right]b_{n+p-1}z^{n+p-1}}\right|$$

$$-\left|\gamma\,z^{p}-\sum_{n=2}^{\infty}\left[\left(n-1-\gamma\right)+e^{i\alpha}\left(n-1\right)\right]a_{n+p-1}z^{n+p-1}+\overline{\sum_{n=1}^{\infty}\left[\left(n+2p-1+\gamma\right)+e^{i\alpha}\left(n+2p-1\right)\right]b_{n+p-1}z^{n+p-1}}\right|$$

$$\geq 2\left(p-\gamma\right)|z|^{p}-\sum_{n=2}^{\infty}\left[\left(4n+2p-4-2\gamma\right)\right]\left|a_{n+p-1}\right||z|^{n+p-1}-\sum_{n=1}^{\infty}\left[\left(4n+6p-4-2\gamma\right)\right]\left|b_{n+p-1}\right||z|^{n+p-1}$$

$$=2\left(p-\gamma\right)|z|^{p}\left\{1-\sum_{n=2}^{\infty}\frac{2n+p-2-\gamma}{p-\gamma}\left|a_{n+p-1}\right||z|^{n-1}+\sum_{n=1}^{\infty}\frac{2n+3p-2+\gamma}{p-\gamma}\left|b_{n+p-1}\right||z|^{n-1}\right\}$$

$$\geq 2\left(p-\gamma\right)|z|^{p}\left\{1-\left[\sum_{n=2}^{\infty}\frac{2n+p-2-\gamma}{p-\gamma}\left|a_{n+p-1}\right|+\sum_{n=1}^{\infty}\frac{2n+3p-2+\gamma}{p-\gamma}\left|b_{n+p-1}\right|\right]\right\}\geq 0, \text{ by (2.1).}$$

The functions

$$f(z)=|z|^p+\sum_{n=2}^{\infty}\frac{p-\gamma}{2n+p-2-\gamma}x_{n+p-1}z^{n+p-1}+\sum_{n=1}^{\infty}\frac{p-\gamma}{2n+3p-2+\gamma}\overline{y}_{n+p-1}\overline{z}^{n+p-1} \tag{2.4}$$

where $$\sum_{n=2}^{\infty}|x_{n+p-1}|+\sum_{n=2}^{\infty}|y_{n+p-1}|=1,$$

show that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.4) are in $G_H(p,\gamma)$ because

$$\sum_{n=1}^{\infty}\left(\frac{2n+p-2-\gamma}{p-\gamma}|a_{n+p-1}|+\frac{2n+3p-2+\gamma}{p-\gamma}|b_{n+p-1}|\right)=1+\sum_{n=2}^{\infty}|x_{n+p-1}|+\sum_{n=2}^{\infty}|y_{n+p-1}|=2.$$

We next show that the condition (2.1) is also necessary for the function in $G_{\overline{H}}(p,\gamma)$.

Theorem 2. Let $f=h+\overline{g}$ be so that h and g are given by (1.3). Then $f(z)\in G_{\overline{H}}(p,\gamma),$ if and only if the inequality (2.1) holds for the coefficient of $f=h+\overline{g}.$

Proof. In view of Theorem 1, we need only show that $f(z)\notin G_{\overline{H}}(p,\gamma)$ if the condition (2.1) does not holds. We note that a necessary condition for $f=h+\overline{g}$ given by (1.3) to be in $G_H(p,\gamma)$ is that

$$\operatorname{Re}\left\{(1+e^{i\alpha})\frac{z\,f'(z)}{z'\,f(z)}-pe^{i\alpha}\right\}\geq p\gamma.$$

This is equivalent to

$$\operatorname{Re}\left\{\frac{\left(1+e^{i\alpha}\right)\left(z\,h'(z)-\overline{z\,g'(z)}\right)-p\,e^{i\alpha}\left(h(z)+\overline{g(z)}\right)}{h(z)+\overline{g(z)}}-p\,\gamma\right\}$$

$$=\operatorname{Re}\left\{\frac{2(p-\gamma)|z|^p-\sum_{n=2}^{\infty}2n+p-2-\gamma|a_{n+p-1}||z|^{n+p-1}-\sum_{n=1}^{\infty}2n+3p-2+\gamma|b_{n+p-1}||\overline{z}|^{n+p-1}}{|z|^p-\sum_{n=2}^{\infty}|a_{n+p-1}||z|^{n+p-1}+\sum_{n=1}^{\infty}|b_{n+p-1}||\overline{z}|^{n+p-1}}\right\}\geq 0$$

.

The above condition must hold for all values of z, $|z| = r < 1$.

Upon choosing the values of z on the positive real axis, we must have

$$\frac{2(p-\gamma)-\sum_{n=2}^{\infty}2n+p-2-\gamma\left|a_{n+p-1}\right|r^{n+p-2}-\sum_{n=1}^{\infty}2n+3p-2+\gamma\left|b_{n+p-1}\right|r^{n+p-2}}{1-\sum_{n=2}^{\infty}\left|a_{n+p-1}\right|n^{n+p-2}+\sum_{n=1}^{\infty}\left|b_{n+p-1}\right|r^{n+p-2}}\geq 0. \qquad (2.5)$$

If the condition (2.1) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exits a $z_0 = r_0 > 1$, for which the quotient in (2.5) is negative. This contradicts the condition for $f(z) \in G_{\overline{H}}(p,\gamma)$ and so the proof is complete.

3. DISTORTION BOUNDS AND EXTREME POINTS

In this section, we shall obtain distortion bounds for functions in $G_{\overline{H}}(p,\gamma)$ and also we determine the extreme points of the closed convex hulls of denoted by $clco\, G_{\overline{H}}(p,\gamma)$.

Theorem 3. If $f(z) \in G_{\overline{H}}(p,\gamma)$, then

$$|f(z)| \leq \left(1+|b_p|\right)r^p + \left(\frac{p-\gamma}{2+p-\gamma}-\frac{3p+\gamma}{2+p-\gamma}|b_p|\right)r^{p+1}, \qquad |z| = r < 1$$

and

$$|f(z)| \geq \left(1-|b_p|\right)r^p - \left(\frac{p-\gamma}{2+p-\gamma}-\frac{3p+\gamma}{2+p-\gamma}|b_p|\right)r^{p+1}, \qquad |z| = r < 1.$$

Proof. We only prove the right hand inequality. The argument for left hand inequality is similar and will be omitted. Let $f(z) \in G_{\overline{H}}(p,\gamma)$. Taking the absolute value of f, we obtain

$$|f(z)| \le \left(1+|b_p|\right) r^p + \sum_{n=2}^{\infty}\left(|a_{n+p-1}|+|b_{n+p-1}|\right) r^{n+p-1}$$

$$\le \left(1+|b_p|\right) r^p + \sum_{n=2}^{\infty}\left(|a_{n+p-1}|+|b_{n+p-1}|\right) r^{p+1}$$

$$= \left(1+|b_p|\right) r^p + \frac{p-\gamma}{2+p-\gamma}\sum_{n=2}^{\infty}\left[\frac{2+p-\gamma}{p-\gamma}|a_{n+p-1}| + \frac{3p+\gamma}{p-\gamma}|b_{n+p-1}|\right] r^{p+1}$$

$$\le \left(1+|b_p|\right) r^p + \frac{p-\gamma}{2+p-\gamma}\sum_{n=2}^{\infty}\left[\frac{2n+p-2-\gamma}{p-\gamma}|a_{n+p-1}| + \frac{2n+3p-2+\gamma}{p-\gamma}|b_{n+p-1}|\right] r^{p+1}$$

$$\le \left(1+|b_p|\right) r^p + \frac{p-\gamma}{2+p-\gamma}\left(1-\frac{3p+\gamma}{p-\gamma}|b_p|\right) r^{p+1} \quad \text{by (2.1)}$$

$$= \left(1+|b_p|\right) r^p + \left(\frac{p-\gamma}{2+p-\gamma}-\frac{3p+\gamma}{2+p-\gamma}|b_p|\right) r^{p+1}.$$

Theorem 4. $f \in clco\, G_{\overline{H}}(p,\gamma)$, if and only if f can be expressed as

$$f(z)=\sum_{n=1}^{\infty} x_{n+p-1}h_{n+p-1}+y_{n+p-1}g_{n+p-1} \tag{3.1}$$

where $z \in U$,

$$h_{p-1}(z)=z^p, \quad h_{n+p-1}(z)=z^p-\frac{p-\gamma}{2n+p-2-\gamma}\, z^{n+p-1}.$$

$$\left(n=2,3,4,...\right), \; g_{n+p-1}(z)=z^p+\frac{p-\gamma}{2n+3p-2+\gamma}\, \overline{z}^{n+p-1}.$$

$$\left(n=1,2,3,4,...\right), \; \sum_{n=1}^{\infty}\left(x_{n+p-1}+y_{n+p-1}\right)=1, \quad x_{n+p-1}\ge 0 \;\; and \;\; y_{n+p-1}\ge 0.$$

Proof. For the functions f given by (3.1), we may write

$$f(z)=\sum_{n=1}^{\infty}\left(x_{n+p-1}h_{n+p-1}(z)+y_{n+p-1}g_{n+p-1}(z)\right)$$

$$=x_{p-1}h_{p-1}(z)+y_{p-1}g_{p-1}(z)+\sum_{n=2}^{\infty}x_{n+p-1}\left(z^{p}+\frac{p-\gamma}{2n+p-2-\gamma}\right)z^{n+p-1}$$

$$+\sum_{n=1}^{\infty}y_{n+p-1}\left(z^{p}-\frac{p-\gamma}{2n+3p-2+\gamma}\right)\overline{z}^{n+p-1}$$

$$=\sum_{n=1}^{\infty}\left(x_{n+p-1}+y_{n+p-1}\right)z^{p}-\sum_{n=2}^{\infty}\frac{p-\gamma}{2n+p-2-\gamma}x_{n+p-1}z^{n+p-1}$$

$$+\sum_{n=1}^{\infty}\frac{p-\gamma}{2n+3p-2+\gamma}y_{n+p-1}\overline{z}^{n+p-1}.$$

Then

$$=\sum_{n=2}^{\infty}\frac{2n+p-2-\gamma}{p-\gamma}\left(\frac{p-\gamma}{2n+p-2-\gamma}x_{n+p-1}\right)+\sum_{n=1}^{\infty}\frac{2n+3p-2+\gamma}{p-\gamma}\left(\frac{p-\gamma}{2n+3p-2+\gamma}y_{n+p-1}\right)$$

$$=\sum_{n=2}^{\infty}x_{n+p-1}+\sum_{n=2}^{\infty}y_{n+p-1}=1-x_{1}\leq 1,$$

and so $f\in clco\,G_{\overline{H}}(p,\gamma)$.

Conversely, suppose that $f\in clco\,G_{\overline{H}}(p,\gamma)$. Set

$$x_{n+p-1}=\frac{2n+p-2-\gamma}{p-\gamma}\left|a_{n+p-1}\right|\quad\left(n=2,3,\ldots\right)$$

and

$$y_{n+p-1}=\frac{2n+3p-2+\gamma}{p-\gamma}\left|b_{n+p-1}\right|\quad\left(n=1,2,3,\ldots\right).$$

Then note that by Theorem 2,

$$0 \le x_{p-1} \le 1 \ and\ y_{p-1} = 1 - x_{p-1} - \sum_{n=1}^{\infty}\left(x_{n+p-1} + y_{n+p-1}\right).$$

Consequently, we obtain $f(z) = \sum_{n=1}^{\infty}\left(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1}\right)$. Using Theorem 2, it is easily seen that $G_{\overline{H}}(p,\gamma)$ is convex and closed, so $clco\, G_{\overline{H}}(p,\gamma) = G_{\overline{H}}(p,\gamma)$.

4. CONVOLUTION AND CONVEX LINEAR COMBINATION

In this section, we show that the class $G_{\overline{H}}(p,\gamma)$ is invariant under convolution and convex combinations of its members.

For harmonic functions

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} b_{n+p-1} \overline{z}^{n+p-1} \ and\ F(z) = z^p - \sum_{n=1}^{\infty} A_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} B_{n+p-1} \overline{z}^{n+p-1}$$

we define the convolution of $f\ and\ F$ as

$$\left(f * F\right)(z) = z^p - \sum_{n=1}^{\infty} a_{n+p-1} A_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} b_{n+p-1} B_{n+p-1} \overline{z}^{n+p-1}. \tag{4.1}$$

Using this definition, we show that the class $G_{\overline{H}}(p,\gamma)$ is closed under convolution.

Theorem 5. For $0 \le \beta \le \gamma < 1$, let $f(z) \in G_{\overline{H}}(p,\gamma)$ and $F(z) \in G_{\overline{H}}(p,\beta)$. Then

$$f * F \in G_{\overline{H}}(p,\gamma) \subset G_{\overline{H}}(p,\beta).$$

Proof. Let

$$f(z) = z^p - \sum_{n=1}^{\infty} \left|a_{n+p-1}\right| z^{n+p-1} + \sum_{n=1}^{\infty} \left|b_{n+p-1}\right| \overline{z}^{n+p-1} \text{ be in } G_{\overline{H}}(p,\gamma)$$

and

$$F(z) = z^p - \sum_{n=1}^{\infty} \left|A_{n+p-1}\right| z^{n+p-1} + \sum_{n=1}^{\infty} \left|B_{n+p-1}\right| \overline{z}^{n+p-1} \text{ be in } G_{\overline{H}}(p,\beta).$$

Note that $A_{n+p-1} \le 1$ and $B_{n+p-1} \le 1$. Obviously, the coefficients of f *and* F must satisfy conditions similar to the inequality (2.1). So for the coefficients of $f * F$ we can write

$$\sum_{n=1}^{\infty}\left[\frac{2n+p-2-\gamma}{p-\gamma}\left|a_{n+p-1}A_{n+p-1}\right| + \frac{2n+3p-2+\gamma}{p-\gamma}\left|b_{n+p-1}B_{n+p-1}\right|\right]$$

$$\le \sum_{n=1}^{\infty}\left[\frac{2n+p-2-\gamma}{p-\gamma}\left|a_{n+p-1}\right| + \frac{2n+3p-2+\gamma}{p-\gamma}\left|b_{n+p-1}\right|\right].$$

This right hand side of the above inequality is bounded by 2 because $f(z) \in G_{\overline{H}}(p,\gamma)$. By the same token, we then conclude that $f * F \in G_{\overline{H}}(p,\gamma) \subset G_{\overline{H}}(p,\beta)$.

Finally, we show that $G_{\overline{H}}(p,\gamma)$ is closed under convex combination of its members.

Theorem 6. The family $G_{\overline{H}}(p,\gamma)$ is closed under convex combination.

Proof. For $i = 1,2,3,\ldots,$ *let* $f_i \in G_{\overline{H}}(p,\gamma)$ *where* f_i is given by

$$f_i(z) = z^p - \sum_{n=2}^{\infty}\left|a_{i,n+p-1}\right| z^{n+p-1} + \sum_{n=1}^{\infty}\left|b_{i,n+p-1}\right| \overline{z}^{n+p-1}.$$

Then, by (2.1),

$$\sum_{n=1}^{\infty}\frac{2n+p-2-\gamma}{p-\gamma}\left|a_{i,n+p-1}\right| + \frac{2n+3p-2+\gamma}{p-\gamma}\left|b_{i,n+p-1}\right| \le 2, \tag{4.2}$$

for $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1,$ the convex combination of f_i may be written as

$$\sum_{n=1}^{\infty} t_i f_i(z) = z^p - \sum_{n=2}^{\infty}\left(\sum_{n=1}^{\infty} t_i \left|a_{i,n+p-1}\right|\right) z^{n+p-1} + \sum_{n=1}^{\infty}\left(\sum_{n=1}^{\infty} t_i \left|b_{i,n+p-1}\right|\right) \overline{z}^{n+p-1}.$$

Then, by (4.2),

$$\sum_{n=1}^{\infty}\left[\frac{2n+p-2-\gamma}{p-\gamma}\left|\sum_{n=1}^{\infty}\left(t_i\left|a_{i,n+p-1}\right|\right)\right|+\frac{2n+3p-2+\gamma}{p-\gamma}\left|\sum_{n=1}^{\infty}\left(t_i\left|b_{i,n+p-1}\right|\right)\right|\right].$$

$$=\sum_{i=1}^{\infty}t_i\left\{\sum_{n=1}^{\infty}\frac{2n+p-2-\gamma}{p-\gamma}\left|a_{i,n+p-1}\right|+\frac{2n+3p-2+\gamma}{p-\gamma}\left|b_{i,n+p-1}\right|\right\}$$

$$\leq 2\sum_{n=1}^{\infty}t_i=2.$$

This is the condition required by (2.1) and so $\sum_{i=1}^{\infty}t_i f_i \in G_{\overline{H}}(p,\gamma)$.

ACKNOWLEDGMENTS: The authors wish to express their sincere thanks to the referee of this paper for several useful comments and suggestions.

REFERENCES

[1] O. P. Ahuja and J. M. Jahangiri, *Multivalent harmonic starlike functions*, Ann.Univ. Mariae Curie-Sklodowska Sect, LV, (**1**), (2001), 1-13.

[2] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acd. Aci. Fenn. Ser. A. I Math, **9,** (1984), 3-25.

[3] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155,** (1991), 364-370.

[4] J. M. Jahangiri, *Harmonic functions starlike in the unit disc*, J. Math. Anal. Appl, **235,** (1999), 470-477.

[5] J. M. Jahangiri, Y.C. Kim and H. M. Srivastava, *Construction of a certain class of harmonic close to convex functions associated with the Alexander integral Transform*, Integral Transform. Spec. Fun.**14**, (2003), 237-242.

[6] H. Lewy, On *the non-vanishing of the Jacobian in certain one to one mappings*, Bull. Amer. Math. Soc.**42,** (1936), 689-110.

[7] F. Ronning, *Uniformly convex functions and a corresponding class of starlike functions,* Proc. Amer. Math. Soc, **118**, No.1, (1993), 190-196.

[8] F. Ronning, *On uniform starlikeness and related properties of univalent functions*, Complex variables, **24,** (1994), 233-239.

[9] T. Rosy, A. B. Stephen, K. G. Subramanian and J. M. Jahangiri, *Goodman- Ronning Type harmonic univalent functions,* Kyungpook Math, J, **41**(1) (2001), 45-54.

[10] H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. and Appl, **220,** (1998), 283-289.

[11] H. Silverman and E. M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math, **28,** (1999), 275-284.

Address

S. B. Joshi:

Department of Mathematics
Walchand College of Engineering,
Sangli, (M.S) India 416 415.
E-mail: joshisb@hotmail.com

N. D. Sangle:

Department of Mathematics
Annasaheb Dange College of Engineering,
Ashta, Sangli, (M.S) India 416 301.
E-mail: navneet_sangle@rediffmail.com