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New Theoretical and Numerical Results for the Boundary-Layer Flow of a Nanofluid Past a Stretching Sheet

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Abstract

Nanofluid flow is one of the most important areas of research in the present time due to its wide applications in industry and many other fields. The problem of the boundary layer flow of a nanofluid past a stretching sheet was firstly introduced and studied numerically by Khan and Pop [7]. This important problem is re-investigated analytically. There is no doubt that the exact solutions of any physical model, when available, are of great importance and certainly would lead to a better understanding of the physical aspects of the model. Moreover, the obtained exact solutions play an important role in the validation of any of the numerical methods used in this important growing field of nanofluid flows. The objective of the present paper is not only to search for such exact solutions but also to give exact formulae for the reduced Nusselt number and the reduced Sherwood number which are two quantities of practical interest in such field. The present analytical results have not been reported in the earlier literatures. Moreover, the numerical results are obtained at some moderate and high values of Prandtl and Lewis numbers.

Keywords: Boundary layer; Nanofluid; Stretching sheet; Exact solution

1 Introduction

Nanofluid refers to the fluid with suspended nanoparticles and this term was firstly used by Choi [1]. Choi et al. [2] showed that the addition of small amount (less than 1% by volume) of nanoparticles to conventional heat transfer liquids increased the thermal conductivity of the fluid up to approximately two times. Masuda et al. [3], Lee et al. [4], Xuan and Li [5], and Xuan and Roetzel [6] stated that with low nanoparticles concentrations (1-5 Vol %), the thermal conductivity of the suspensions can increase more than 20%. The boundary layer flow of a nanofluid past a stretching sheet has been firstly investigated by Khan and Pop [7]. Later in the same year, Bachok et al. [8] studied the steady boundary layer flow of a nanofluid past a moving semi-infinite flat plate. In this paper, we focus on the physical model derived by Khan and Pop [7] and given by

$$f''' \quad + f f'' - (f')^2 = 0, \tag{1}$$

$$\frac{1}{Pr}\theta'' \quad + f\theta' + Nb\phi'\theta' + Nt(\theta')^2 = 0, \tag{2}$$

$$\phi'' \quad + Lef\phi' + \frac{Nt}{Nb}\theta'' = 0, \tag{3}$$

subject to the boundary conditions:

$$f(0) = 0, \qquad f'(0) = 1, \quad \theta(0) = 1, \quad \phi(0) = 1, \tag{4}$$

$$f'(\infty) = 0, \qquad \theta(\infty) = 0, \quad \phi(\infty) = 0, \tag{5}$$

where primes denote to differentiation with respect to η and Pr, Le, Nb and Nt are Prandtl number, Lewis number, Brownian motion parameter and thermophoresis parameter, respectively. The exact solution of Eq. (1) with the boundary conditions in (4-5) is already well known and obtained by Crane [9] as $f(\eta) = 1 - e^{-\eta}$. Therefore, the given system reduces to the following set of ordinary differential equations:

$$\theta''(\eta) \quad + Pr\left(1 - e^{-\eta} + Nb\ \phi'(\eta)\right)\theta'(\eta) + PrNt\left[\theta'(\eta)\right]^2 = 0, \tag{6}$$

$$\phi''(\eta) \quad + Le\left(1 - e^{-\eta}\right)\phi'(\eta) + \frac{Nt}{Nb}\ \theta''(\eta) = 0, \tag{7}$$

where the boundary conditions on $\theta(\eta)$ and $\phi(\eta)$ are given in (4-5). According to Kuznetsov and Nield [10], the quantities of practical interest in this study are the reduced Nusselt number Nur and the reduced Sherwood number Shr which are defined respectively by

$$Nur = -\theta'(0), \quad Shr = -\phi'(0). \tag{8}$$

Unlike the numerical study of Khan and Pop [7], we aim in this paper to report new analytical results to the current physical system. There is no doubt that the exact solutions of any physical model, when available, are of great importance and certainly would lead to a better understanding of the physical aspects of the model. Searching for such exact and closed form solutions is the main task of the present paper. Not only, but also obtaining exact formulae for the reduced Nusselt number Nur and the reduced Sherwood number Shr is also a main part of this paper, where only numerical values were reported in the previous studies.

2 General closed form solutions

Assuming that $\psi(\eta) = \theta'(\eta)$ and rewriting Eq. (6) in terms of ψ, yields

$$\psi'(\eta) + Pr\left(1 - e^{-\eta} + Nb\ \phi'(\eta)\right)\psi(\eta) = -PrNt\,[\psi(\eta)]^2\,, \tag{9}$$

which is the famous Bernoulli nonlinear differential equation in $\psi(\eta)$. This Bernoulli equation reduces to the following first order linear differential equation in the new dependent variable $Z(\eta)$:

$$Z'(\eta) - Pr\left(1 - e^{-\eta} + Nb\ \phi'(\eta)\right)Z(\eta) = PrNt, \tag{10}$$

where $Z(\eta) = 1/\psi(\eta)$. On solving Eq. (10) and using the preceding substitutions we then get $\psi(\eta)$ or $\theta'(\eta)$ as

$$\theta'(\eta) = \frac{e^{-Pr[\eta+e^{-\eta}+Nb\ \phi(\eta)]}}{e^{-Pr(1+Nb)}/\theta'(0) + PrNt\int_0^\eta e^{-Pr[\sigma+e^{-\sigma}+Nb\ \phi(\sigma)]}d\sigma}. \tag{11}$$

Integrating the last equation once again $w.r.t$ η and using the boundary condition ($\theta(0) = 1$), we obtain

$$\theta(\eta) = 1 + \frac{1}{PrNt}\ \ln\left(1 + PrNt\ e^{Pr(1+Nb)}\theta'(0)\int_0^\eta e^{-Pr[\sigma+e^{-\sigma}+Nb\ \phi(\sigma)]}d\sigma\right), \tag{12}$$

where $\theta'(0)$ can be determined by imposing the boundary condition at infinity ($\theta(\infty) = 0$). Therefore

$$\theta'(0) = \frac{(e^{-PrNt} - 1)}{PrNt}\ e^{-Pr(1+Nb)}\left[\int_0^\infty e^{-Pr[\sigma+e^{-\sigma}+Nb\ \phi(\sigma)]}d\sigma\right]^{-1}. \tag{13}$$

Inserting $\theta'(0)$ given above into Eq. (12) leads to

$$\theta(\eta) = 1 + \frac{1}{PrNt}\ \ln\left[1 + (e^{-PrNt} - 1)\frac{\int_0^\eta e^{-Pr[\sigma+e^{-\sigma}+Nb\ \phi(\sigma)]}d\sigma}{\int_0^\infty e^{-Pr[\sigma+e^{-\sigma}+Nb\ \phi(\sigma)]}d\sigma}\right]. \tag{14}$$

Here, an analytical solution is obtained for $\theta(\eta)$ in terms of $\phi(\eta)$, i.e., once $\phi(\eta)$ is given we directly obtain $\theta(\eta)$ by performing the involved integrals. Moreover, this general closed form solution is also useful in deriving either simpler closed form solutions or exact solutions at several particular values of the physical parameters. Details of deriving such exact solutions from Eq. (14) shall be discussed in subsequent sections.

Regarding to the ϕ equation given in (7), it can be also solved as a first order ordinary differential equation in $\phi'(\eta)$. Accordingly, $\phi'(\eta)$ is given by

$$\phi'(\eta) = \gamma e^{-Le(\eta+e^{-\eta})} - \frac{Nt}{Nb} e^{-Le(\eta+e^{-\eta})} \int_0^\eta e^{Le(\xi+e^{-\xi})}\theta''(\xi)d\xi, \tag{15}$$

where γ is a constant defined by $\gamma = e^{Le}\phi'(0)$ and to be determined later. Integrating Eq. (15) once again $w.r.t$ η and using the boundary condition ($\phi(0) = 1$), it then follows

$$\phi(\eta) = 1 + \gamma \int_0^\eta e^{-Le(\sigma+e^{-\sigma})}d\sigma - \frac{Nt}{Nb}\int_0^\eta e^{-Le(\sigma+e^{-\sigma})}\int_0^\sigma e^{Le(\xi+e^{-\xi})}\theta''(\xi)d\xi d\sigma. \tag{16}$$

Here, the constant γ is determined through applying the boundary condition at infinity, i.e., $\phi(\infty) = 0$ and this gives

$$\gamma = \left(-1 + \frac{Nt}{Nb}\int_0^\infty e^{-Le(\sigma+e^{-\sigma})}\int_0^\sigma e^{Le(\xi+e^{-\xi})}\theta''(\xi)d\xi d\sigma\right) / \left(\int_0^\infty e^{-Le(\sigma+e^{-\sigma})}d\sigma\right). \tag{17}$$

Returning again to Eq. (15) by inserting the expression obtained above for γ, we obtain $\phi(\eta)$ in a closed form as

$$\begin{aligned}\phi(\eta) \;=\; & 1 + \frac{\int_0^\eta e^{-Le(\sigma+e^{-\sigma})}d\sigma}{\int_0^\infty e^{-Le(\sigma+e^{-\sigma})}d\sigma}\left(-1 + \frac{Nt}{Nb}\int_0^\infty e^{-Le(\sigma+e^{-\sigma})}\int_0^\sigma e^{Le(\xi+e^{-\xi})}\theta''(\xi)d\xi d\sigma\right) \\ & -\frac{Nt}{Nb}\int_0^\eta e^{-Le(\sigma+e^{-\sigma})}\int_0^\sigma e^{Le(\xi+e^{-\xi})}\theta''(\xi)d\xi d\sigma. \end{aligned} \tag{18}$$

Although the expression in (18) seems to be rather complex, it can be used to derive exact solutions at specified values of the physical parameters, this is the subject of the subsequent sections.

3 Exact solutions at special cases

The possibility of obtaining exact solutions for the governing equations at several particular values of the physical parameters are introduced in this section. In order to do that, let us begin by discussing the case in which $Nt = 0$ and $Nb \neq 0$.

3.1 Case 1: at $Nt = 0$ and $Nb \neq 0$

Taking the limit of Eq. (14) as $Nt \to 0$, a simpler closed form expression for $\theta(\eta)$ can be obtained as

$$\theta(\eta) = 1 - \frac{\int_0^\eta e^{-Pr[\sigma+e^{-\sigma}+Nb\ \phi(\sigma)]}d\sigma}{\int_0^\infty e^{-Pr[\sigma+e^{-\sigma}+Nb\ \phi(\sigma)]}d\sigma}. \tag{19}$$

However, Eq. (19) requires determining $\phi(\eta)$ in order to perform the integrations. This task can be achieved by directly setting $Nt = 0$ in Eq. (18) and this gives $\phi(\eta)$ as

$$\phi(\eta) = 1 - \frac{\int_0^\eta e^{-Le(\sigma+e^{-\sigma})}d\sigma}{\int_0^\infty e^{-Le(\sigma+e^{-\sigma})}d\sigma}. \qquad (20)$$

Fortunately, the integrations arise in (20) can be evaluated analytically in terms of the well known special functions as follows. On using the assumption $\mu = Le\ e^{-\sigma}$, it can be easily seen that

$$\begin{aligned} \int_0^\eta e^{-Le[\sigma+e^{-\sigma}]}d\sigma &= (Le)^{-Le}\int_{Le\ e^{-\eta}}^{Le} \mu^{Le-1}e^{-\mu}d\mu \\ &= (Le)^{-Le}\ \Gamma(Le, Le\ e^{-\eta}, Le), \end{aligned} \qquad (21)$$

where $\Gamma(a, z_0, z_1) = \int_{z_0}^{z_1} \mu^{a-1}e^{-\mu}d\mu$ is the generalized Gamma function which can be expressed in terms of the incomplete Gamma function as $\Gamma(a, z_0, z_1) = \Gamma(a, z_0) - \Gamma(a, z_1)$, where $\Gamma(a, z) = \int_z^\infty \mu^{a-1}e^{-\mu}d\mu$. On using the result of (21) as $\eta \to \infty$, we have

$$\int_0^\infty e^{-Le[\sigma+e^{-\sigma}]}d\sigma = (Le)^{-Le}\ \Gamma(Le, 0, Le). \qquad (22)$$

Accordingly, an exact solution for $\phi(\eta)$ is obtained as

$$\begin{aligned} \phi(\eta) &= 1 - \frac{\Gamma(Le, Le\ e^{-\eta}, Le)}{\Gamma(Le, 0, Le)} \\ &= \frac{\Gamma(Le, 0, Le\ e^{-\eta})}{\Gamma(Le, 0, Le)}. \end{aligned} \qquad (23)$$

Here, an exact solution is obtained for $\phi(\eta)$ while an analytical solution in closed form is obtained for $\theta(\eta)$ at any values of Pr and Le. It is also important here to mention that even by inserting $\phi(\eta)$ given by Eq. (23) into Eq. (19) the exact form for $\theta(\eta)$ is not reachable. However, the exact form for $\theta(\eta)$ can be obtained at certain values of Prandtl and Lewis numbers, mainely at $Pr = Le = 1$, this point is discussed in the next subsection. Inserting (23) into (19) we obtain $\theta(\eta)$ in closed analytical form as

$$\theta(\eta) = 1 - \frac{I(\eta)}{I(\infty)}, \quad I(\eta) = \int_0^\eta e^{-Pr\left[\sigma+e^{-\sigma}+Nb\ \left(\frac{\Gamma(Le,0,Le\ e^{-\sigma})}{\Gamma(Le,0,Le)}\right)\right]}d\sigma. \qquad (24)$$

It may be suitable now to indicate the advantages of the analytical results presented above to obtain some new numerical results. Since most nanofluids examined to date have large values for the Lewis number, we are interested mainly in the case $Le > 1$. Moreover, Khan and Pop [7] and Rana and Bhargava [13] as well as Makinde and Aziz [14] practically studied Nb and Nt in the range of $0.1 - 0.5$ and Le in the range of $1-25$ for the nanofluid boundary layer over the stretching sheets. Hence, the variation of non-dimensional parameters is considered here to vary in wider ranges than those mentioned above.

The dependent similarity variables $\phi(\eta)$ is plotted for a variation of Lewis number Le in Fig. 1 in the range $10-50$ and in Fig. 2 for high values in the range $100-500$, respectively. In addition, Tables 1-3 presents some of the numerical values of $\theta(\eta)$ at different values of Nb, Pr, and Le.

3.2 Case 2: at $Nt = 0$, $Nb \neq 0$, $Pr = 1$ and $Le = 1$

At $Le = 1$, $\phi(t)$ in (23) reduces to the following simple exact form

$$\phi(\eta) = \alpha\left(1 - e^{-e^{-\eta}}\right), \quad \alpha = \frac{1}{1-e^{-1}}, \tag{25}$$

Substituting (25) into (19) at $Pr = 1$, yields

$$\theta(\eta) = 1 - \frac{\int_0^\eta e^{-[\sigma+e^{-\sigma}+\alpha Nb(1-e^{-e^{-\sigma}})]}d\sigma}{\int_0^\infty e^{-[\sigma+e^{-\sigma}+\alpha Nb(1-e^{-e^{-\sigma}})]}d\sigma}. \tag{26}$$

The integrations arise in the last formula can be also evaluated analytically as follows.

$$\begin{aligned}
\int_0^\eta e^{-[\sigma+e^{-\sigma}+\alpha Nb(1-e^{-e^{-\sigma}})]}d\sigma &= \int_0^\eta e^{-(\sigma+e^{-\sigma})} \times e^{-\alpha Nb(1-e^{-e^{-\sigma}})}d\sigma \\
&= \frac{1}{\alpha Nb}\left[e^{-\alpha Nb(1-e^{-e^{-\sigma}})}\right]_0^\eta, \; Nb \neq 0 \\
&= \frac{1}{\alpha Nb}\left(e^{-\alpha Nb(1-e^{-e^{-\eta}})} - e^{-Nb}\right). \qquad (27)
\end{aligned}$$

Therefore

$$\int_0^\infty e^{-[\sigma+e^{-\sigma}+\alpha Nb(1-e^{-e^{-\sigma}})]}d\sigma = \frac{1}{\alpha Nb}\left(1-e^{-Nb}\right). \tag{28}$$

On substituting (27) and (28) into (26), we obtain $\theta(\eta)$ in the exact form:

$$\theta(\eta) = \frac{1-e^{-\alpha Nb(1-e^{-e^{-\eta}})}}{1-e^{-Nb}}. \tag{29}$$

3.3 Case 3: at $Nt = 0$ and $Nb = 0$

In this case, the system (6-7) reduces to a single boundary value problem for θ while the boundary value problem for ϕ becomes ill-posed and is of no physical significance. The BVP for θ becomes

$$\theta''(\eta) + Pr(1-e^{-\eta})\theta'(\eta) = 0, \tag{30}$$

with the boundary conditions $\theta(0) = 1$ and $\theta(\infty) = 0$. The exact solution of the current case can be easily obtained by setting $Nb = 0$ in (26), consequently

$$\theta(\eta) = 1 - \frac{\int_0^\eta e^{-Pr[\sigma+e^{-\sigma}]}d\sigma}{\int_0^\infty e^{-Pr[\sigma+e^{-\sigma}]}d\sigma}. \tag{31}$$

The integrations included in the formula (31) were already evaluated in a previous subsection and hence this analytical expression for $\theta(\eta)$ leads to the following exact solution at any Prandtl number Pr:

$$\theta(\eta) = \frac{\Gamma(Pr, 0, Pr\ e^{-\eta})}{\Gamma(Pr, 0, Pr)}. \tag{32}$$

The dependent similarity variables $\theta(\eta)$ is plotted for a variation of Prandtl number Pr in Fig. 3 in the range $10-50$ and in Fig. 4 for high values in the range $100-500$, respectively.

4 General formulae for Nur and Shr numbers

The reduced Nusselt (Nur) and reduced Sherwood (Shr) numbers are two quantities of practical interest in the nanofluid flow problems. The objective of this section is to provide analytical formulae for the mentioned numbers. To do so, we return to the definitions introduced in (8) and use the results obtained in section (2) to obtain the following formulae for Nur and Shr

$$Nur = \left(\frac{(1 - e^{-PrNt})e^{-Pr(1+Nb)}}{PrNt}\right) / \left(\int_0^\infty e^{-Pr\left[\sigma+e^{-\sigma}+Nb\ \phi(\sigma)\right]}d\sigma\right), \tag{33}$$

and

$$Shr = e^{-Le}\left(1 + \frac{Nt}{Nb}\int_0^\infty e^{-Le(\sigma+e^{-\sigma})}\int_0^\sigma e^{Le(\xi+e^{-\xi})}\theta''(\xi)d\xi d\sigma\right) / \left(\int_0^\infty e^{-Le(\sigma+e^{-\sigma})}d\sigma\right). \tag{34}$$

5 Exact formulae at special cases

Here, exact formulae for Nur and Shr are successfully obtained at three different special cases and below is a discussion in this regard.

5.1 Case 1: at $Nt = 0$ and $Nb \neq 0$

On taking the limit of Eqs. (33) and (34) as $Nt \to 0$, we have

$$Nur = \left(e^{-Pr(1+Nb)}\right) / \left(\int_0^\infty e^{-Pr\left[\sigma+e^{-\sigma}+Nb\ \phi(\sigma)\right]}d\sigma\right), \tag{35}$$

$$Shr = \quad e^{-Le} / \left(\int_0^\infty e^{-Le(\sigma + e^{-\sigma})} d\sigma \right). \tag{36}$$

Therefore the exact values of the reduced Nusselt (Nur) and reduced Sherwood (Shr) numbers are given respectively by

$$Nur = \quad \left(e^{-Pr(1+Nb)}\right) / \left(\int_0^\infty e^{-Pr\left[\sigma + e^{-\sigma} + Nb\left(\frac{\Gamma(Le,0,Le\ e^{-\sigma})}{\Gamma(Le,0,Le)}\right)\right]} d\sigma \right), \tag{37}$$

$$Shr = \quad \frac{(Le)^{Le} e^{-Le}}{\Gamma(Le,0,Le)}. \tag{38}$$

Table 4 shows the variation of the reduced Nusselt number (Nur) for different values of Nb at some moderate and high values of Prandtl and Lewis numbers.

5.2 Case 2: at $Nt = 0$, $Nb \neq 0$, $Pr = 1$ and $Le = 1$

In this case, the reduced Nusselt (Nur) number can be evaluated by differentiating $\theta(\eta)$ in (29) $w.r.t\ \eta$ and then substituting $\eta = 0$, while the reduced Sherwood (Shr) number is computed easily by setting $Le = 1$ in formula (38). Accordingly, we obtain

$$Nur = \quad \frac{\alpha Nb\ e^{-(1+Nb)}}{1 - e^{-Nb}}, \tag{39}$$

$$Shr = \quad \alpha\ e^{-1}, \tag{40}$$

where $\Gamma(1, 0, 1) = 1 - e^{-1}$.

5.3 Case 3: at $Nt = 0$ and $Nb = 0$

Differentiating (32) $w.r.t\ \eta$ at $\eta = 0$, we can get the reduced Nusselt number as

$$Nur = \frac{(Pr)^{Pr} e^{-Pr}}{\Gamma(Pr, 0, Pr)}. \tag{41}$$

Here, it is worth mentioning that only numerical values were available for the reduced Nusselt number and obtained by Khan and Pop [7], Wang [11], and Gorla and Sidawi [12] at different values of Pr. These numerical values are compared with the exact values obtained by using the analytical expression (41) and summarized in Table 5.

6 Conclusions

Closed form solutions have been obtained for a system of ordinary differential equations describing the boundary layer flow of a nanofluid past a stretching sheet. In addition, various exact solutions were derived from the general closed form solutions at particular values of the physical parameters. The current results may not be handled in the earlier literatures for the present problem. Moreover, the results are of great importance for the validation of any of the numerical methods. Furthermore, analytical and exact formulae were obtained for two physical parameter of practical interest, the reduced Nusselt number and the reduced Sherwood number, at several choices of the physical parameters of interest. The numerical results have been also presented in the form of graphs and tables.

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Table 1: Numerical values for $\theta(\eta)$ at $Nb = 0.1$, $Pr = 10$, $Le = 10$.

η	$\theta(\eta)$	η	$\theta(\eta)$	η	$\theta(\eta)$
0.1	0.85162	0.6	0.147787	1.1	0.00679509
0.2	0.683841	0.7	0.0868472	1.2	0.00330928
0.3	0.514391	0.8	0.048667	1.3	0.00156988
0.4	0.361712	0.9	0.0261572	1.4	0.000727263
0.5	0.238279	1.0	0.0135513	1.5	0.000329744

Table 2: Numerical values for $\theta(\eta)$ at $Nb = 0.5$, $Pr = 30$, $Le = 15$.

η	$\theta(\eta)$	η	$\theta(\eta)$	η	$\theta(\eta)$
0.1	0.999874	0.6	0.150526	1.1	3.1189×10^{-5}
0.2	0.995795	0.7	0.0403579	1.2	3.8592×10^{-6}
0.3	0.946871	0.8	0.00849282	1.3	4.3783×10^{-7}
0.4	0.741548	0.9	0.00149342	1.4	4.5926×10^{-8}
0.5	0.406026	1.0	0.000228501	1.5	4.4948×10^{-9}

Table 3: Numerical values for $\theta(\eta)$ at $Nb = 0.3$, $Pr = 40$, $Le = 50$.

η	$\theta(\eta)$	η	$\theta(\eta)$	η	$\theta(\eta)$
0.1	0.998322	0.6	0.00569066	1.1	4.6694×10^{-8}
0.2	0.914775	0.7	0.000765639	1.2	3.9103×10^{-9}
0.3	0.520419	0.8	0.0000856989	1.3	3.0010×10^{-10}
0.4	0.162225	0.9	8.1112×10^{-6}	1.4	1.4720×10^{-10}
0.5	0.0344498	1.0	6.5895×10^{-7}	1.5	1.3772×10^{-10}

Table 4: Variation of Nur with Nb for (a) $Pr = 10$, $Le = 15$, (b) $Pr = 15$, $Le = 25$, and (c) $Pr = 30$, $Le = 50$.

(a) $Pr = 10$	$Le = 15$	(b) $Pr = 15$	$Le = 25$	(c) $Pr = 30$	$Le = 50$
Nb	Nur	Nb	Nur	Nb	Nur
0.1	1.25014	0.1	1.08397	0.1	0.504407
0.2	0.624821	0.2	0.347643	0.2	0.0392508
0.3	0.292663	0.3	0.100141	0.3	0.0025553
0.4	0.130496	0.4	0.026987	0.4	0.000153622
0.5	0.0561177	0.5	0.00697556	0.5	0.000008843

Table 5: Comparison of results for the reduced Nusselt number for $Nt = Nb = 0$.

Pr	Present results	Khan and Pop [7]	Wang [11]	Gorla and Sidawi [12]
0.07	0.0655625	0.0663	0.0656	0.0656
0.20	0.169089	0.1691	0.1691	0.1691
0.70	0.453916	0.4539	0.4539	0.5349
2.00	0.911358	0.9113	0.9114	0.9114
7.00	1.895403	1.8954	1.8954	1.8905
20.00	3.353904	3.3539	3.3539	3.3539
70.00	6.462199	6.4621	6.4622	6.4622

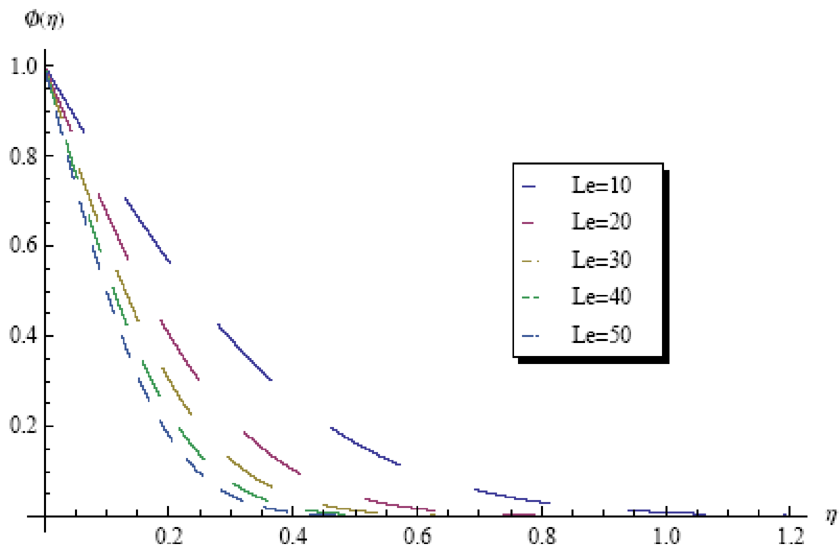

Figure 1: Plots of dimensionless similarity functions $\phi(\eta)$ for specified Lewis numbers.

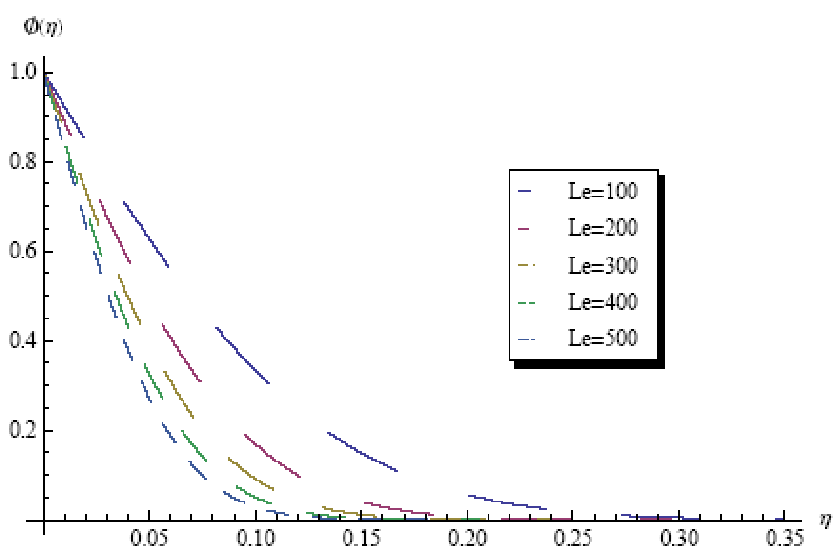

Figure 2: Plots of dimensionless similarity functions $\phi(\eta)$ for high values of Lewis numbers.

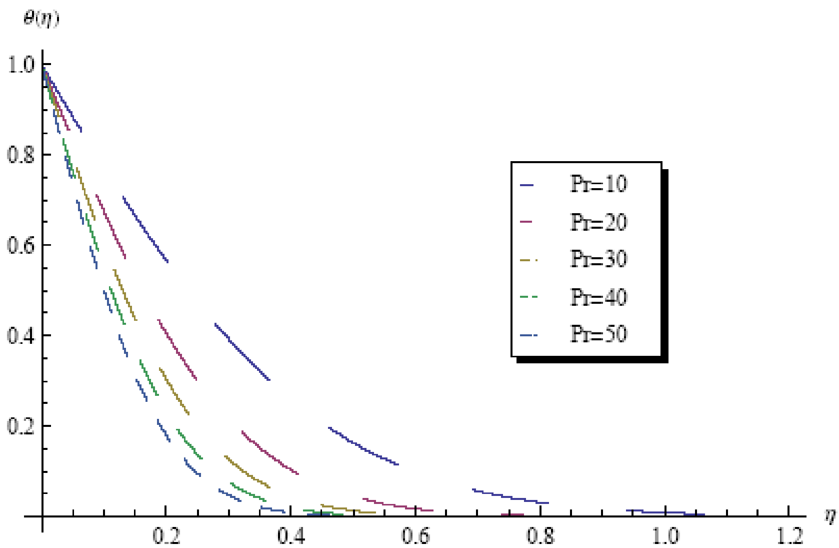

Figure 3: Plots of dimensionless similarity functions $\theta(\eta)$ for specified Prandtl numbers.

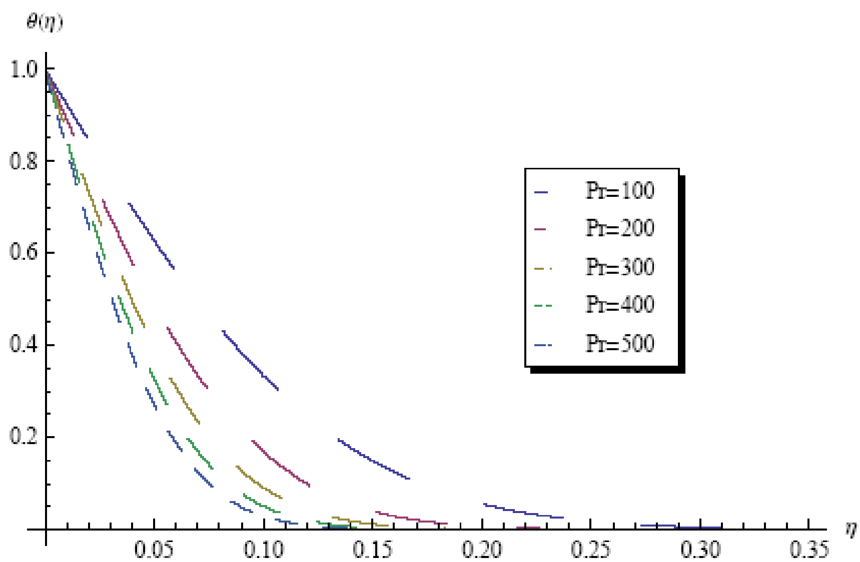

Figure 4: Plots of dimensionless similarity functions $\theta(\eta)$ for specified Prandtl numbers.

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