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A New Two-Metric Theory of Gravity with Prior Geometry*

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ABSTRACT

We present a Lagrangian-based metric theory of gravity with three adjustable constants and two tensor fields, one of which is a nondynamical "flat-space metric" η. With a suitable cosmological model and a particular choice of the constants, the "Post-Newtonian limit" of the theory agrees, in the current epoch, with that of General Relativity (GRT); consequently our theory is consistent with current gravitation experiments. Because of the role of η, the gravitational "constant" G is time dependent and gravitational waves travel null geodesics of η rather than the physical metric g. Gravitational waves possess six degrees of freedom. The general exact static spherically symmetric solution is a four parameter family and one of these solutions is investigated in detail. Future experimental tests of the theory are discussed.

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I. INTRODUCTION AND SUMMARY

Within the past few years an elegant theoretical formalism, the "Parametrized Post-Newtonian" (PPN) framework, has been developed[1] to analyze metric[2] theories of gravity. The PPN framework is structured around the "weak gravitational fields" and low velocities of the gravitational matter which characterize typical solar-system tests of gravity. It classifies each gravitation theory as to its form "in the Post-Newtonian (PN) limit." At first it was hoped, and indeed seemed to be true, that the PN limit of each theory of gravity is unique — thus by solar-system experiments alone, one could, in principle, determine the "correct PN limit," which would then correspond to one and only one "correct theory of gravity." In addition, it was hoped and is hoped, that the "correct PN limit" is that of General Relativity (GRT) (although we try not to let this fact prejudice our investigations). To play devil's advocate, a program was initiated to attempt to formulate theories of gravity with the same PN limit (and hence PPN parameters[1]) as GRT. The aims of such a program are two-fold, as one can ask the following questions: (i) If such theories exist, how complex and contrived are their formulations? (ii) Do such theories have anything in common and in what respect do they differ from GRT outside of the PN limit? The first question is primarily only of aesthetic interest. But the second has the possibility of identifying powerful new theoretical and experimental tools for testing relativistic gravity — indeed that has been the case (see Sec. VI and Refs. 3 and 4).

In this paper we present and analyze a new theory of gravity — one which has the same PN limit (for the current epoch) as GRT, given a suitable cosmological model and a particular choice of the adjustable constants.

Analysis of our new theory provides partial answers to questions (i) and (ii) above.

A further motivation for study of this particular theory is to analyze in detail the role of prior geometry[2] in gravitation theories, a role which will be investigated in more general terms in another paper.[5]

To date the authors are aware of three other new metric theories which are candidates for sharing the property of having the same PN limit as GRT (candidates in the sense of contingency upon the existence of special but acceptable cosmological solutions and certain choices of the available adjustable constants). These theories are the Hellings-Nordtvedt theory,[6] Ni's theory,[7] and the Will-Nordtvedt theory.[8] Of these three, Ni's theory contains prior geometric elements like our own.

A. The Lagrangian Formulation

The equations of the theory are obtained, in the usual way, by varying the dynamical variables in the Lagrangian:

$$L = \int \mathcal{L}_G(\boldsymbol{\eta}, \boldsymbol{h})\, d^4x + \int \mathcal{L}_{NG}(\boldsymbol{g}, q_\lambda)\, d^4x \quad , \tag{1a}$$

$$\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{\eta}, \boldsymbol{h}) \quad , \tag{1b}$$

$$\mathbf{Riem}(\boldsymbol{\eta}) = 0 \quad , \tag{1c}$$

where $\boldsymbol{\eta}, \boldsymbol{h}, \boldsymbol{g}$ are second-rank symmetric tensor fields: $\boldsymbol{\eta}$ is an absolute variable[2] (not varied in L), $\boldsymbol{h}$ is dynamical, and $\boldsymbol{g}$ is constructed algebraically from $\boldsymbol{\eta}$ and $\boldsymbol{h}$. The Riemann tensor constructed out of $\boldsymbol{\eta}$ is denoted by $\mathbf{Riem}(\boldsymbol{\eta})$, and consequently Eq. (1c) states that $\boldsymbol{\eta}$ is a "flat-space metric." It is Eq. (1c), the "field equation" for $\boldsymbol{\eta}$, that introduces geometrical structure into the theory which is independent of the matter distribution

— thus the "prior geometry." The gravitational Lagrangian density is denoted by $\mathcal{L}_G$ while the nongravitational Lagrangian density,[2] $\mathcal{L}_{NG}$, is the same as the corresponding quantity in other metric theories, (e.g., GRT), with q_λ representing the matter fields. The "physical metric," governing the response of matter to gravity, is denoted by $\underset{\sim}{g}$.

Explicitly, $\mathcal{L}_G$ and $\underset{\sim}{g}$ are defined by the following:

$$\mathcal{L}_G = - (16\pi)^{-1} \eta^{\alpha\beta}\eta^{\lambda\mu}\eta^{\rho\sigma}(a h_{\lambda\rho|\alpha} h_{\mu\sigma|\beta} + f h_{\lambda\mu|\alpha} h_{\rho\sigma|\beta})(-\eta)^{1/2}, \quad (2)$$

$$g_{\mu\nu} = (1 - Kh)^2 \Delta_\mu{}^\tau \Delta_{\tau\nu} , \quad (3a)$$

$$\Delta^\mu{}_\nu (\delta_\mu{}^\alpha - \frac{1}{2} h_\mu{}^\alpha) = \delta^\alpha{}_\nu \quad . \quad (3b)$$

Conventions and definitions for the above are the following:

(i) Greek indices run 0-3, Latin 1-3.

(ii) units chosen such that G = c = 1 (gravitational constant today and speed of light) (see Sec. VI).

(iii) slashes "|" and semicolons ";" denote covarient differentiation with respect to the flat space-metric $\eta_{\alpha\beta}$ and the curved-space metric $g_{\alpha\beta}$ respectively. Comma "," denotes a partial coordinate derivative.

(iv) η is the determinant of $\eta_{\alpha\beta}$.

(v) $\delta^\alpha{}_\nu$ is the Kronecker delta.

(vi) $\Delta_\mu{}^\nu$ is defined by Eq. (3b).

(vii) indices on $\Delta_{\alpha\beta}$ and $h_{\alpha\beta}$ <u>only</u> are raised and lowered with $\eta_{\mu\nu}$, i.e., $h^\alpha{}_\alpha = h^{\alpha\beta}\eta_{\alpha\beta} \equiv h$, and $\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^\alpha{}_\gamma$; indices on all other tensors will be raised and lowered with $g_{\alpha\beta}$.

(viii) signatures of $\underset{\sim}{\eta}$ and $\underset{\sim}{g}$ are + 2.

(ix) a, f, K are adjustable constants.

Motivation for the rather ungainly expression for the metric [Eqs. (3)] comes from an analysis[9] of the Belinfante-Swihart theory of gravity[10] — a theory which can be reformulated, at lowest order, into a metric theory with "effective metric" of the form of Eqs. (3). From that suggested algebraic form for the metric we have constructed the present full metric theory.

B. Summary

Section II includes a discussion of the field equations and a calculation of the PN limit of the theory. It is shown that there are mathematically ten degrees of freedom in the initial value problem for $h_{\mu\nu}$ (compared with two for $g_{\mu\nu}$ in GRT). In the PN limit there are, in general, "preferred frame effects";[1] such effects are, however, functions of only the cosmological boundary values of $h_{\mu\nu}$. By a certain choice of the cosmological model one can make these effects vanish for the current epoch. We suspect that such time-dependent preferred-frame effects are a common property of prior geometric gravitation theories. At any rate, the observed absence of preferred-frame effects can only place upper limits on the cosmological boundary values of $h_{\mu\nu}$.

Section III derives and discusses the equations of stellar structure for static, spherically symmetric stars. The equations are much more complicated than the corresponding ones in GRT (see Table I) and there is probably no analytic solution even for a star of constant density. In addition, a stellar model is not uniquely specified by giving its equation of state and central pressure, as is the case in most other theories. The exact exterior, static spherically symmetric solution is obtained and is found to be a 4-parameter family.

Section IV includes an analysis of a special exterior spherically symmetric solution. For this special solution, the effective potentials for particles and photons are similar to the corresponding quantities in the Schwarzschild geometry of GRT, outside of a couple of "Schwarzschild radii." However, the physical manifold extends only to $\rho = 1.5$ m,[11] which is a "point at infinity" (not reachable in finite affine parameter by any geodesic).

There are no singularities or horizons (i.e., no black hole) in the physical manifold in this exact solution, but a peculiar geometrical effect in which the proper surface areas of concentric spheres centered on $\rho = 0$ pass through a minimum and then increase as one moves radially inward (decreasing ρ and increasing proper time for radially falling observer). The minimum of areas is approximately $97\pi m^2$ and occurs near $\rho = 2.7$ m. The areas then increase to infinity at $\rho = 1.5$ m, although space is not flat there.

It is also found that one cannot embed the entire constant time, equatorial geometry in a Euclidean 3-space, but that a pseudo-Euclidean space is necessary for $1.5\text{ m} < \rho \lesssim 2.1$ m.

Section V discusses time-dependent solutions, conservation laws, and gravitational waves. Birkhoff's theorem[12] does not hold in this theory, i.e., the exterior geometry of a spherically symmetric and asymptotically flat spacetime need not be static — collapsing stars can radiate monopole gravitational waves. The general plane gravitational wave has six physical degrees of freedom, the maximum number possible in a metric theory of gravity.[3,4]

As the theory is Lagrangian-based, conservation laws follow and one can construct a gravitational stress-energy complex. Appropriately defined, the stress energy-density of this object is positive definite for all possible polarizations of plane waves. In addition there is a purely gravitational quantity conserved all by itself, probably of only mathematical interest.

Section VI discusses the time dependence of the gravitational "constant" and further possible experimental tests of the theory. In particular, a search for time delays between reception of gravitational and electromagnetic bursts and a search for "non-GRT" type polarizations of gravitational waves promise to be important future experimental tests of the theory. Such tests would also be crucial in the theories of Refs. 6, 7, 8; and their identification represents an important success in our program of "devil's advocate."

II. FIELD EQUATIONS AND POST-NEWTONIAN LIMIT

Variation of Eq. (1) with respect to the dynamical field variable $h_{\mu\nu}$ yields the following gravitational field equations:

$$(-\eta)^{1/2}(a\Box h^{\nu\mu} + f\,\eta^{\mu\nu}\Box h) = -4\pi T^{\alpha\beta}(-g)^{1/2}(\partial g_{\alpha\beta}/\partial h_{\mu\nu}) , \tag{4a}$$

where

$$\Box h^{\mu\nu} \equiv \eta^{\alpha\beta}\, h^{\mu\nu}{}_{|\alpha|\beta} , \tag{4b}$$

$$T^{\alpha\beta} \equiv 2(-g)^{-1/2}\,(\delta \mathcal{L}_{NG}/\delta g_{\alpha\beta}) , \tag{4c}$$

and δ is the variational derivative.

From the matter equations, obtained by variation of q_λ in Eq. (1), one can show in the usual manner (see, e.g., Ref. 13)

$$T^{\alpha\beta}{}_{;\beta} = 0 . \tag{5}$$

Equation (5) is the typical "matter response equation" in metric theories.

Contraction of Eq. (4a) with $\eta_{\mu\nu}$ yields an equation for h alone, which can be substituted back into Eq. (4a) to yield

$$\Box h^{\mu\nu} = - (4\pi/a)(- g)^{1/2}(- \eta)^{-1/2} T^{\alpha\beta}[\Theta^{\mu\nu}_{\alpha\beta} - f(a + 4f)^{-1} \Theta^{\gamma\tau}_{\alpha\beta} \eta_{\gamma\tau} \eta^{\mu\nu}] \quad , \tag{6a}$$

where

$$\Theta^{\mu\nu}_{\alpha\beta} \equiv \partial g_{\alpha\beta}/\partial h_{\mu\nu} \quad . \tag{6b}$$

The linearized limit of Eq. (6a) is

$$\Box h^{\mu\nu} = - (4\pi/a) \; T^{\alpha\beta}[\delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} - \eta_{\alpha\beta} \eta^{\mu\nu}(f + 2Ka)(a + 4f)^{-1}] \quad . \tag{7}$$

Unlike metric theories without prior geometry, the four Eqs. (5) do not follow from the gravitational field equations; they are additional equations.[5] However, there is no problem of overdetermination because all of the 10 components of $h^{\mu\nu}$ are now dynamical variables; i.e., if all of the essential coordinate freedom is used up in choosing a frame in which $\eta_{\alpha\beta}$ has a particular set of components, [usually diag(-1, 1, 1, 1)], then there is no coordinate freedom left to adjust the components of $h_{\mu\nu}$.

For example, for a perfect fluid $T^{\alpha\beta}$ is described by four matter variables once an equation of state is given (3 components of four velocity and energy density, for example). Thus Eqs. (5) and (6a) comprise a system of fourteen independent equations for the fourteen unknowns.

We also note that all of the ten Eqs. (6a) involve second time derivatives of $h_{\mu\nu}$. Thus in the Cauchy problem all of the $h_{\mu\nu}$ are to be regarded as dynamical variables and there are ten degrees of freedom. Once $g_{\alpha\beta}$ has been constructed from $\eta_{\alpha\beta}$ and $h_{\alpha\beta}$, however, coordinate transformations can be performed and so there can only be six "physical" degrees of freedom. This is to be contrasted with GRT in which not only can four of the $g_{\alpha\beta}$ be chosen arbitrarily by coordinate conditions, but also four of the field equations involve only first time derivatives. Thus in the corresponding Cauchy problem, the Einstein gravitational field has only two physical degrees of freedom.

The PPN framework of Nordtvedt, Will, and others can be used to analyze the predictions of all metric theories with respect to solar-system experiments (e.g., light bending, perihelion shift, gravimeter data, earth-moon separation, etc.). The reader is referred to Ref. 1 for a complete summary of the PPN framework. Briefly, this formalism involves expanding the metric, in the manner of Chandrasekhar,[14] in the small dimensionless quantities which occur in the solar system stress energy tensor, e.g.,

$$v^2 \sim U \sim (P/\rho) \sim \Pi \sim O(\epsilon^2) \approx 10^{-7} \quad , \tag{8}$$

where v^2 is the squared velocity of a typical fluid element, U is the Newtonian potential, P/ρ is the pressure divided by energy density (specific pressure) and Π is the specific internal energy. It is found that, in a particular coordinate gauge, and for most metric theories — including ours — there are only nine different functionals which can occur in the metric at PN order and only nine independent parameters multiplying these functionals. Almost all twentieth century gravitation experiments to date can be summarized by their constraints on these nine parameters, the "PPN parameters."

We now calculate in our theory the PN limit, which will involve a perturbation solution of Eq. (6a). For calculational ease we assume a coordinate system in which $\eta_{\alpha\beta}$ takes on Minkowski values. Before we begin, a crucial point must be recognized.[15] The metric $g_{\alpha\beta}$ has the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + O(h) \quad ,$$

and we know that far away from the solar system there is some coordinate system in which $g_{\alpha\beta}$ takes on Minkowski values. However, this coordinate system will, in general, not be the same frame in which $\eta_{\alpha\beta}$ takes on

Minkowski values; there is no a priori reason why the boundary values of $h_{\mu\nu}$ should be zero in this coordinate system. Thus in solving Eq. (6a) we are not at liberty to set equal to zero for all time the "arbitrary constant" which may be added to $h_{\mu\nu}$; this complicates considerably the construction of the PN limit of our theory. However, we feel that this complication and its origin are of sufficient educational value to warrant a detailed discussion.

Denote the nearly constant boundary values of $h_{\mu\nu}$ by $\omega_{\mu\nu}$ ($\omega_{\mu\nu}$ can only change on a cosmological time scale by definition) and the part tied directly to the solar system by $h^*_{\mu\nu}$; i.e.,

$$h_{\mu\nu} = h^*_{\mu\nu} + \omega_{\mu\nu} \quad . \tag{9}$$

Now use the six-parameter invariance group of the Minkowski metric to pick a coordinate system in which $\omega_{\mu\nu}$ is diagonal, reducing $\omega_{\mu\nu}$ to four components. Without justification, but for simplicity, we now assume that the three spatial components of $\omega_{\mu\nu}$ are equal. Such an assumption does not effect the qualitative conclusions of this section. Further assume that

$$|\omega_{\mu\nu}| \ll 1 \quad , \tag{10}$$

although $\omega_{\mu\nu}$ does not have to be as small as the $O(\epsilon)$ indicated in Eq. (8). Equation (10) will turn out to be an assumption consistent with the ultimate experimental limits on the $\omega_{\mu\nu}$.

Next expand Eqs. (3a) and (3b) in a power series in $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} - 2Kh\eta_{\mu\nu} + h_{\mu\nu} + K^2h^2\eta_{\mu\nu} - 2Khh_{\mu\nu} + \frac{3}{4}\, h_{\mu\tau}h^{\tau}{}_{\nu} + \ldots \quad . \tag{11}$$

When Eq. (9) is substituted into Eq. (11) one obtains

$$g_{00} = -D_0 + E_0 h_{00}^{*} - F_0 h^{*} - K^2 h^2 - 2Kh_{00}^{*} h^{*} - \frac{3}{4} h_{00}^{*\,2} , \tag{12a}$$

$$g_{ij} = D\delta_{ij} + Eh_{ij}^{*} + F\delta_{ij} h^{*} - 2Kh^{*} h_{ij}^{*} + K^2 h^{*2} \delta_{ij} + \frac{3}{4} h_{ij}^{*\,2} , \tag{12b}$$

$$g_{0k} = Hh_{0k}^{*} , \tag{12c}$$

where all of the constants appearing in Eqs. (12) have the form: $D_0 = 1 + O(\omega)$, etc., and are given explicitly to $O(\omega^2)$ in Appendix A, along with other constants appearing below. Using Eqs. (12) and a perfect fluid for the matter stress-energy tensor, one obtains from Eq. (6a)

$$\begin{aligned} \Box h^{*\mu\nu} = -(4\pi/a)\, I^{-1} \rho v^{\alpha} v^{\beta} (1 + I_1 h_{00}^{*} + I_2 h^{*} + I_3 v^2) \Big[(1 - 2K\omega)\, \delta^{\mu}{}_{\alpha} \delta^{\nu}{}_{\beta} \\ + L\eta_{\alpha\beta}\eta^{\mu\nu} + \frac{3}{2} \delta^{\mu}{}_{\alpha} \omega^{\nu}{}_{\beta} + M\eta^{\mu\nu}\omega_{\alpha\beta} + N\eta^{\mu\nu}\eta_{\alpha\beta} h^{*} \\ + \frac{3}{2} \delta^{\mu}{}_{\alpha} h^{*\nu}{}_{\beta} - 2Kh^{*} \delta^{\mu}{}_{\alpha} \delta^{\nu}{}_{\beta} + M\eta^{\mu\nu} h^{*}{}_{\alpha\beta} \Big] . \end{aligned} \tag{13}$$

In Eq. (13) I, I_1, I_2, I_3, M, N are all functions of a, f, K, $\omega_{\mu\nu}$ (see Appendix A) and

$$\omega \equiv 3\omega_{11} - \omega_{00} , \tag{14a}$$

$$v^{\alpha} \equiv dx^{\alpha}/dt , \tag{14b}$$

$\rho \equiv$ proper mass-energy density measured in the rest-frame of the fluid. (14c)

To simplify an already complex presentation, we have omitted the pressure from the perfect fluid stress energy tensor and included the internal energy in the total proper energy density ρ. (Such terms are not omitted in quoting the final PPN parameters.) We now write

$$h^{*\mu\nu} = {}^{(1)}h^{*\mu\nu} + {}^{(2)}h^{*\mu\nu} + \dots , \tag{15}$$

in a perturbation expansion and obtain (see Appendix A for notation)

$$\nabla^2\, {}^{(1)}h^{*00} = -\, 4\pi\tau\rho\left[(1 - 2K\omega) + L - \omega_0\left(\frac{3}{2} + M\right)\right] \equiv -\, 4\pi\rho\, C_0 \quad , \tag{16a}$$

$$\nabla^2\, {}^{(1)}h^{*ij} = -\, 4\pi\rho\tau(M\omega_0 - L)\, \delta^{ij} \equiv -\, 4\pi\rho\, C_1\delta^{ij} \quad , \tag{16b}$$

$$\nabla^2\, {}^{(1)}h^{*0k} = -\, 4\pi\tau\rho\left[v^k(1 - 2K\omega) + \frac{3}{2}\,\omega_1 v^k\right] \equiv -\, 4\pi\rho\, C_2 v^k \quad , \tag{16c}$$

$$\nabla^2\, {}^{(2)}h^{*00} = -\, 4\pi\tau\rho(S_0\, {}^{(1)}h^{*00} + S_1\, {}^{(1)}h^{*} + B_0 v^2) + {}^{(1)}h^{*00}_{\ ,00} \quad , \tag{16d}$$

$$\nabla^2\, {}^{(2)}h^{*ij} = -\, 4\pi\tau\rho\left[R_0 v^i v^j + \delta^{ij}(R_1\, {}^{(1)}h^{*00} + R_2\, {}^{(1)}h^{*} + B_1 v^2)\right] + {}^{(1)}h^{*ij}_{\ ,00} \quad , \tag{16e}$$

where

$$\tau \equiv (aI)^{-1} \quad . \tag{17}$$

Solutions of the equations are

$${}^{(1)}h^{*00} = C_0 U \quad , \tag{18a}$$

$${}^{(1)}h^{*ij} = \delta^{ij} C_1 U \quad , \tag{18b}$$

$${}^{(1)}h^{*0k} = C_2 V_k \quad , \tag{18c}$$

$${}^{(2)}h^{*00} = \tau\left[S_0 C_0 + S_1(3C_1 - C_0)\right]\Phi_2 + \tau B_0 \Phi_1 + C_0 \chi_{,00} \quad , \tag{18d}$$

$$\begin{aligned} {}^{(2)}h^{*ij} = {} & \tau R_0 \mathcal{J}_3^{ij} + \tau\delta^{ij}\left[R_1 C_0 + R_2(3C_1 - C_0)\right]\Phi_2 \\ & + \tau B_1 \delta^{ij}\Phi_1 + C_1 \delta^{ij} \chi_{,00} \quad , \end{aligned} \tag{18e}$$

where we have defined the five "potentials" U, V_k, Φ_1, Φ_2, $\mathcal{J}_3^{ij}$, and the "superpotential" χ aa follows:

$$U(\underset{\sim}{x}, t) \equiv \int \rho(\underset{\sim}{x}', t)\, |\underset{\sim}{x} - \underset{\sim}{x}'|^{-1}\, d^3x' \quad , \tag{19a}$$

$$V_k(\underset{\sim}{x}, t) \equiv \int \rho(\underset{\sim}{x}', t)\, |\underset{\sim}{x} - \underset{\sim}{x}'|^{-1}\, v^k d^3x' \quad , \tag{19b}$$

$$\Phi_1(\underset{\sim}{x}, t) \equiv \int \rho(\underset{\sim}{x}', t)\, v^2\, |\underset{\sim}{x} - \underset{\sim}{x}'|^{-1}\, d^3x' \quad , \tag{19c}$$

$$\Phi_2(\underset{\sim}{x}, t) \equiv \int \rho(\underset{\sim}{x}', t)|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1} U(\underset{\sim}{x}', t)\, d^3x' \quad , \tag{19d}$$

$$\mathcal{J}_3^{ij}(\underset{\sim}{x}, t) \equiv \int \rho(\underset{\sim}{x}', t)|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1} v^i v^j\, d^3x' \quad , \tag{19e}$$

$$\nabla^2\chi = U \quad . \tag{19f}$$

Using Eqs. (12) and our solutions, Eqs. (18), we now compute the metric:

$$g_{00} = -D_0 + K_1U + K_2U^2 + K_3\Phi_2 + K_4\Phi_1 + K_1\chi_{,00} \quad , \tag{20a}$$

$$g_{ij} = \delta_{ij}(D + K_5U) \quad , \tag{20b}$$

$$g_{0k} = -HC_2V_k \quad . \tag{20c}$$

Notice that the metric does not approach the standard Minkowski tensor far away from the solar system (when the potentials U, Φ_1, Φ_2, V_k, $\chi \to 0$) because of the leading constants D_0 and D_1. We must therefore make a "scaling" transformation:

$$t = D_0^{-1/2}\,\bar{t} \quad , \tag{21a}$$

$$\underset{\sim}{x} = D^{-1/2}\,\bar{\underset{\sim}{x}} \quad . \tag{21b}$$

In the tensor transformation law for the metric

$$\bar{g}_{\mu\nu}(\bar{x}) = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu} = g_{\alpha\beta}[U(\underset{\sim}{x}, t),\ \Phi_1(\underset{\sim}{x}, t),\ \ldots]\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu} \quad , \tag{22}$$

we also need to express the potentials as functions of the new (barred) coordinates. An example of the procedure is the following: since ρ is a scalar

$$\bar{\rho}(\bar{\underset{\sim}{x}}, t) = \rho(\underset{\sim}{x}, t) \quad , \tag{23a}$$

$$U(\underset{\sim}{x}, t) = \int \rho(\underset{\sim}{x}', t)|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1} d^3x' = \int \bar{\rho}(\bar{\underset{\sim}{x}}', \bar{t})|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1} d^3x' \quad ,$$

$$= D^{-1}\int \bar{\rho}(\bar{\underset{\sim}{x}}', \bar{t})|\bar{\underset{\sim}{x}} - \bar{\underset{\sim}{x}}'|^{-1} d^3\bar{x}' = D^{-1}\,\bar{U}(\bar{\underset{\sim}{x}}, \bar{t}) \quad . \tag{23b}$$

In a similar manner one finds

$$\Phi_2(\underset{\sim}{x}, t) = D^{-2}\,\overline{\Phi}_2(\overline{\underset{\sim}{x}}, \overline{t}) \;, \tag{23c}$$

$$\Phi_1(\underset{\sim}{x}, t) = D_0 D^{-2}\,\overline{\Phi}_1(\overline{\underset{\sim}{x}}, \overline{t}) \;, \tag{23d}$$

$$V_k(\underset{\sim}{x}, t) = D_0^{1/2} D^{-3/2}\,\overline{V}_k(\overline{\underset{\sim}{x}}, \overline{t}) \;, \tag{23e}$$

$$\chi_{,00} = D^{-2} D_0 \overline{\chi}_{,\overline{0}\overline{0}} \;. \tag{23f}$$

Making the transformation indicated in Eqs. (22) and (23) and then dropping the bars, $g_{\mu\nu}$ becomes

$$g_{00} = -1 + D_0^{-1}D^{-1}K_1 U + D_0^{-1}D^{-2}K_2 U^2 + D_0^{-1}D^{-2}K_3\Phi_2 + D^{-2}K_4\Phi_1 + D^{-2}K_1\chi_{,00} \;, \tag{24a}$$

$$g_{ij} = \delta_{ij}(1 + D^{-2}K_5 U) \;, \tag{24b}$$

$$g_{0k} = -\,HC_2 D^{-2} V_k \;. \tag{24c}$$

A final coordinate transformation must be made to remove the $\chi_{,00}$ term from g_{00} and reduce the metric to "standard PPN form." However, additional transformations of the form of Eqs. (23) are now negligible corrections and no distinction need be made between functions of new and old coordinates. The result of the final transformation, $t \rightarrow t + 1/2\, D^{-2}K_1\chi_{,0}$, is

$$g_{00} \rightarrow g_{00} - K_1 D^{-2}\chi_{,00} \;, \tag{25a}$$

$$g_{ij} \rightarrow g_{ij} \;, \tag{25b}$$

$$g_{0k} \rightarrow g_{0k} + \frac{1}{4} K_1 D^{-2}(V_k - W_k) \;, \tag{25c}$$

where W_k is a new potential defined by

$$W_k \equiv \int \rho[\underset{\sim}{v} \cdot (\underset{\sim}{x} - \underset{\sim}{x}')]\,|\underset{\sim}{x} - \underset{\sim}{x}'|^{-1}\,(\underset{\sim}{x} - \underset{\sim}{x}')_k\, d^3x' \;. \tag{26}$$

We now demand the proper Newtonian limit, i.e.,

$$g_{00} \approx 1 - 2U + \dots \quad ,$$

which requires

$$K_1 D_0^{-1} D^{-1} = 2 \quad \underline{\text{today}} \quad ; \tag{27}$$

(a consequence of our choosing units in which the gravitational constant is unity today). Equation (27) expresses a constraint between the three adjustable constants a, f, and K for a given set of $\omega_{\mu\nu}$. Comparing Eqs. (24)-(25) with the definitions of the PPN parameters[1] and using Eq. (27) to simplify, one finds

$$\gamma = \frac{1}{2} D^{-2} K_5 \equiv \bar{\gamma}(a, f, K) + O(\omega) \quad , \tag{28a}$$

$$\beta = -\frac{1}{2} D_0^{-1} D^{-2} K_2 \equiv \bar{\beta}(a, f, K) + O(\omega) \quad , \tag{28b}$$

$$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0 \quad , \tag{28c}$$

$$\alpha_1 = 2HC_2 D^{-2} - 4\gamma - 4 = O(\omega) \quad , \tag{28d}$$

$$\alpha_2 = D_0 D^{-1} - 1 = O(\omega) \quad . \tag{28e}$$

where $\bar{\gamma}$ and $\bar{\beta}$ are defined implicitly by the relations

$$a = (2\bar{\gamma} + 2)^{-1} \quad , \tag{29a}$$

$$f = (10\bar{\beta} + 6\bar{\gamma}\,\bar{\beta} - 7\bar{\gamma}^2 - 8\bar{\gamma} - 6)[2(\gamma + 1)(3\bar{\gamma} - 5 - 4\bar{\beta})^2]^{-1} \quad . \tag{29b}$$

In GRT, $\gamma = \beta = 1$ and the other seven parameters vanish. In our theory it is clear that the two adjustable constants, a and f, may be so chosen to give any value to γ and β. For example, if the $\omega_{\mu\nu}$ are all zero, one can satisfy Eq. (27) and have $\gamma = \beta = 1$ with the choice

$$(a, f, K) = (\tfrac{1}{4}, -\tfrac{5}{64}, \tfrac{1}{16}) \quad . \tag{30}$$

It has been shown[16] that the nonvanishing of α_1, α_2, or α_3 leads to noninvariance of the functional form of the metric of Eqs. (24)-(25) under post-Galilean transformations[17] (curved-space versions of Lorentz transformations). New terms, involving the velocity of the Lorentz boost with respect to the current "preferred frame" and multiplied by combinations of α_1, α_2, α_3, appear in the metric. Nordtvedt and Will[18] have calculated the experimental consequences of the resulting "preferred-frame effects" and find that they lead to periodic anomalies in such phenomena as the solid earth tides, secular perihelion shifts, etc. The reader is referred to their paper for further details and we quote here only the current experimental limits on α_1 and α_2:

$$\alpha_1 \leq 0.1 \quad , \tag{31a}$$

$$\alpha_2 \leq 0.02 \quad . \tag{31b}$$

We have calculated explicitly the quite complicated functions $\alpha_1(\omega_{\mu\nu})$, $\alpha_2(\omega_{\mu\nu})$ and have examined their numerical values over a large range of constants a and f (consistent with the experimental limits on γ and β). We find that the experimental constraints indicated in Eqs. (31) require approximately

$$|\omega_0| + |\omega_1| \lesssim .015 \quad . \tag{32}$$

Even if we had not made the simplifying assumptions about the form of $\omega_{\mu\nu}$, its individual elements presumably would still be required to satisfy roughly the constraint of Eq. (32).

Since the $\omega_{\mu\nu}$ are cosmological boundary values of $h_{\mu\nu}$, one must solve the cosmological problem for a particular cosmological model to obtain the theoretical values of the $\omega_{\mu\nu}$. Because of the absolute nature of $\eta_{\alpha\beta}$, it should be possible to construct cosmologies such that, during the current epoch, the curved and flat-space metrics approach Minkowski form, far from the solar system, in the same coordinate system. Such a cosmology would guarantee that the $\omega_{\mu\nu}$ vanish at present, although a time dependent cosmology would certainly cause nonzero values of $\omega_{\mu\nu}$ to occur over cosmological time scales. Indeed, perliminary results from a cosmological solution[19] indicate that it is possible to make all of the $\omega_{\mu\nu}$ arbitrarily small for the current epoch — and still have a reasonable cosmological model. Thus, a consistent solution exists for which the PN limit of our theory is arbitrarily close to that of GRT in the current epoch.

Further details regarding the time dependence of the $\omega_{\mu\nu}$ are given in Sec. VI.

III. THE GENERAL STATIC SPHERICALLY SYMMETRIC SOLUTION AND EQUATIONS OF STELLAR STRUCTURE

A. The General Exterior Static Spherically Symmetric Solution

Before writing down the equations of stellar structure for a static spherically symmetric star, let us construct the general static spherically symmetric exterior solution (which must then be joined onto the solution inside the star).

First of all, choose a coordinate system in which

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} . \tag{33}$$

The most general form of $h_{\mu\nu}$ in this coordinate system which satisfies the symmetry requirements is[20]

$$h_{\mu\nu} = \begin{pmatrix} \varphi(r) & \mu(r) & 0 & 0 \\ \mu(r) & \psi(r) & 0 & 0 \\ 0 & 0 & r^2\lambda(r) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta\,\lambda(r) \end{pmatrix} . \tag{34}$$

The homogeneous field equations for $h_{\mu\nu}$ are simply

$$\eta^{\alpha\beta} h^{\mu\nu}{}_{|\alpha|\beta} = 0 . \tag{35}$$

The solutions to Eqs. (35) which are well behaved at infinity are[21]

$$h_{\mu\nu} = \begin{pmatrix} a_1/r & -2a_4/r^2 & 0 & 0 \\ -2a_4/r^2 & a_2/r - 2a_3/r^3 & 0 & 0 \\ 0 & 0 & r^2(a_2/r + a_3/r^3) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta(a_2/r + a_3/r^3) \end{pmatrix} , \tag{36}$$

where a_1, a_2, a_3, and a_4 are arbitrary constants. We remind the reader that the r coordinate in Eq. (36) has, at this point, no interpretation other than its relation to the group — theoretically defined assumption of spherical symmetry. Construction of $g_{\mu\nu}$ from $h_{\mu\nu}$ is purely algebraic [see Eqs. (3)], and the details will not be given here. Since $h_{\mu\nu}$ has off-diagonal terms, so will $g_{\mu\nu}$. However, having obtained $g_{\mu\nu}$, we can make the coordinate transformation

$$t \to t + \int \frac{g_{0r}}{g_{00}}\, dr , \tag{37}$$

which then diagonalizes the metric, and we finally obtain

$$g_{00} = (1 - Kh)^2\, \gamma^2 \left[\frac{a_4^{\,2}}{r^4} - \left(1 - \frac{1}{2}\,\frac{a_2}{r} + \frac{a_3}{r^3}\right)^2 \right] , \tag{38a}$$

$$g_{rr} = (1 - kh)^2 \gamma^2 \left\{ \left(1 + \frac{1}{2} \frac{a_1}{r}\right)^2 - \frac{a_4^{\,2}}{r^4} + \frac{\left(\frac{a_4^{\,2}}{r^4}\right)\left[2 + \frac{1}{2}(a_1 - a_2)\, r^{-1} + a_3 r^{-3}\right]^2}{\frac{a_4^{\,2}}{r^4} - \left(1 - \frac{1}{2} \frac{a_2}{r} + \frac{a_3}{r^3}\right)^2} \right\}, \tag{38b}$$

$$g_{\theta\theta} = (1 - kh)^2 r^2 \left(1 - \frac{1}{2} \frac{a_2}{r} - \frac{1}{2} \frac{a_3}{r^3}\right)^{-2} , \tag{38c}$$

$$g_{\varphi\varphi} = \sin^2\theta \, g_{\theta\theta} , \tag{38d}$$

$$h \equiv r^{-1}(3a_2 - a_1) , \tag{38e}$$

$$\gamma \equiv \left[1 + \frac{1}{2}(a_1 - a_2)\, r^{-1} - \frac{1}{4} a_1 a_2 r^{-2} + a_3 r^{-3} + \left(a_4^{\,2} + \frac{1}{2} a_1 a_3\right) r^{-4}\right]^{-1} , \tag{38f}$$

$$ds^2 = g_{00} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2 . \tag{39}$$

Equations (38) for the metric indicate a 4-parameter family of solutions for the general static spherically symmetric exterior metric. One can convince himself that all four of the parameters are physical (not removable by coordinate transformations) by transforming to curvature coordinates and verifying that four arbitrary parameters remain.[22] In Sec. IV we will investigate more closely a particular member of the 4-parameter family.

B. Stellar Models

We idealize a star as a spherically symmetric, static mass of perfect fluid and assume a temperature-independent equation of state

$$p = p(\rho) , \tag{40}$$

where p is the pressure and ρ the energy density. We work in the coordinate system in which $\eta_{\mu\nu}$ has the form of Eq. (33).[23] For mathematical simplicity we seek solutions for $h_{\mu\nu}$ which are diagonal, i.e., with $\mu(r) = 0$ in Eq. (34).

Such solutions represent a subclass of all possible solutions and corresponds to the condition $a_4 = 0$ in the exterior metric [cf. Eqs. (36) and (38)].[24]

The metric now has the form

$$g_{\mu\nu} = (1 - Kh)^2 \begin{pmatrix} -(1+\frac{1}{2}\varphi)^{-2} & & & \\ & (1-\frac{1}{2}\psi)^{-2} & & \\ & & r^2(1-\frac{1}{2}\lambda)^{-2} & \\ & & & r^2\sin^2\theta(1-\frac{1}{2}\lambda)^{-2} \end{pmatrix}, \quad (41a)$$

where

$$h \equiv -\varphi + \psi + 2\lambda \quad . \qquad (41b)$$

Equations (5) and (6a) together with Eq. (40) are the necessary set for computing the structure of our stellar model. With the usual fluid stress energy tensor

$$T^{\alpha\beta} = (\rho + p)\, u^\alpha u^\beta + p g^{\alpha\beta} \quad , \qquad (42)$$

one finds that the only nonvacuous equation resulting from Eqs. (5) is

$$dp/dr = \tfrac{1}{2}(\rho + p)\left[2K(1 - Kh)^{-1}\, dh/dr + (1 + \tfrac{1}{2}\varphi)^{-1}\, d\varphi/dr\right] . \qquad (43)$$

Using the Christoffel symbols for $\underset{\sim}{\eta}$, one finds that Eqs. (6a) yield the following:

$$\nabla^2 h = -4\pi\Gamma(a + 4f)^{-1}\left\{8K(\rho - 3p) + (1 - Kh)\left[p(1 - \tfrac{1}{2}\psi)^{-1} + 2p(1 - \tfrac{1}{2}\lambda)^{-1} - \rho(1 + \tfrac{1}{2}\varphi)^{-1}\right]\right\}, \qquad (44a)$$

$$\nabla^2\varphi = (f/a)\nabla^2 h - (4\pi/a)\Gamma\left[2K(3p - \rho) + (1 + \tfrac{1}{2}\varphi)^{-1}(1 - Kh)\rho\right] \quad , \qquad (44b)$$

$$\nabla^2\psi = 4(\psi - \lambda)\, r^{-2} - (f/a)\, \nabla^2 h - (4\pi/a)\Gamma\left[2K(\rho - 3p) + (1 - \tfrac{1}{2}\psi)^{-1}(1 - Kh)p\right] , \qquad (44c)$$

where

$$\nabla^2 \equiv d^2/dr^2 + 2r^{-1}\, d/dr \quad , \tag{45a}$$

$$\Gamma \equiv (1 - Kh)^3 \left(1 + \tfrac{1}{2}\varphi\right)^{-1} \left(1 - \tfrac{1}{2}\psi\right)^{-1} \left(1 - \tfrac{1}{2}\lambda\right)^{-2} \quad . \tag{45b}$$

Equation (44a) follows from taking the trace (with respect to $\underset{\sim}{\eta}$) of Eq. (6a). Equations (44b) and (44c) are the 0-0 and r-r components respectively of Eq. (6a). Altogether, Eqs. (40), (43)-(44) are five highly nonlinear coupled equations for the five unknowns p, ρ, φ, λ, and ψ. Linear combinations of Eqs. (44) can be taken to yield

$$\nabla^2(\psi - \lambda) = \left[\tfrac{1}{2}(\Gamma/a)(1 - Kh)\left(1 - \tfrac{1}{2}\psi\right)^{-1}\left(1 - \tfrac{1}{2}\lambda\right)^{-1} + 6r^{-2}\right](\psi - \lambda) \quad , \tag{46}$$

which is an equation we will later discuss.

Outside of the star the physically acceptable solutions to the homogeneous forms of Eqs. (44) are [cf. Eq. (36)]

$$\varphi = a_1/r \quad , \tag{47a}$$

$$\psi = a_2/r - 2a_3/r^3 \quad , \tag{47b}$$

$$\lambda = a_2/r + a_3/r^3 \quad . \tag{47c}$$

The constants a_1, a_2, and a_3 are to be determined by matching conditions at the surface of the star. The general procedure in constructing stellar models is to choose various central values for the variables, integrate the equations outward from the center until the pressure vanishes, and thus establish the surface of the star. Various boundary conditions must typically be satisfied, but in the case of GRT, for example, the conditions can be satisfied in a trivial manner without multiple trial integrations. The situation here, as we shall see, is vastly more complicated.

As long as the denominators do not vanish (see discussion below), Eqs. (44) are regular at the stellar surface and hence require that $r\varphi$, $r\psi$, $r\lambda$ and their first derivatives be continuous across the surface. Using Eqs. (47) and denoting quantities evaluated at the surface by a subscript s, one obtains the six matching conditions:

$$\varphi_s = a_1/R, \quad (\varphi_{,r})_s = - a_1/R^2 \quad , \tag{48a}$$

$$\psi_s = a_2/R - 2a_3/R^3, \quad (\psi_{,r})_s = - a_2/R^2 + 6a_3/R^4 \quad , \tag{48b}$$

$$\lambda_s = a_2/R + a_3/R^3, \quad (\lambda_{,r})_s = - a_2/R^2 - 3a_3/R^4 \quad , \tag{48c}$$

where $r = R$ is the surface of the star.

What are the appropriate central quantities to be specified? Suppose we regard $(\psi - \lambda)$, and φ as the three independent gravitational potentials. Then a possible but nonunique solution to Eq. (46) is $\psi - \lambda = 0$ everywhere, corresponding to considering r an isotropic radial coordinate. However, forgetting this special case for the moment, the regular solution of Eq. (46) near the origin is

$$\psi - \lambda \sim \text{const. } r^2 \quad .$$

Thus one central condition to be specified is

$$[(\psi - \lambda)/r^2]_c \quad ,$$

where we denote by c quantities at the center, analogously to the quantities at the surface discussed above. The equations for h and φ are regular at the origin as long as the potentials are sufficiently small and therefore, in analogy with the corresponding electrostatic equations, the derivatives, at the center, of φ and λ must vanish. However, the central values of the potentials themselves must be specified, and hence the two other central parameters are φ_c and λ_c. Thus in general we have six parameters to adjust, e.g., a_1, a_2, a_3, φ_c, λ_c, $[(\psi - \lambda)/r^2]_c$ in order to satisfy the six matching constraints given in Eqs. (48), for a given equation of state and central pressure. One way of viewing the boundary conditions is that φ_c, λ_c, $[(\psi - \lambda)/r^2]_c$ must be so chosen as to match onto a regular exterior solution at the star's surface — such a two-point boundary value problem in general has a discrete set of solutions, i.e., for a given p_c and equation of state there may be no $\{\varphi_c, \lambda_c, [(\psi - \lambda)/r^2]_c\}$ such that there is a solution, or there may be many different sets. Thus the central pressure and equation of state do not uniquely specify the stellar model in general. However, we do know that for a weakly gravitating star $(p_c/\langle\rho\rangle \ll 1,\ \varphi,\ \lambda,\ \psi \ll 1)$. Equations (44) become linear and do indeed have unique and well behaved solutions for each central pressure (Newtonian, and post-Newtonian regimes, see Sec. II). However, we can expect that as the models become more and more relativistic, a point is reached where each p_c and equation of state branches into a discrete spectrum of stellar models.

If one trys as a solution to Eq. (46) $\psi = \lambda$, then a more convenient form of the boundary condition is

$$[\varphi/(r\varphi_{,r})]_s = -1 \quad , \tag{49a}$$

$$[(3\lambda + r\lambda_{,r})/3\lambda]_s = \frac{2}{3} \quad . \tag{49b}$$

One then adjust λ_c and φ_c to satisfy Eqs. (49) and defines a_1 and a_2 $(a_3 = 0)$ by

$$a_1 = R\varphi_s \quad , \tag{50a}$$

$$a_2 = \frac{1}{2} R(3\lambda + R\lambda_{,r})_s \quad . \tag{50b}$$

If $\psi \neq \lambda$, then the proper constraints are

$$[\varphi/r\varphi_{,r}]_s = -1 \quad , \tag{51a}$$

$$[(3\lambda + r\lambda_{,r})/(2\lambda + \psi)]_s = \frac{2}{3} \quad , \tag{51b}$$

$$R[(\psi_{,r} - \lambda_{,r})/(\lambda - \psi)]_s = 3 \quad , \tag{51c}$$

and one adjusts λ_c, φ_c and $[(\psi - \lambda)/r^2]_c$ to satisfy these three constraints; defining a_1 and a_2 as in Eqs. (50), and

$$a_3 = \frac{1}{3} R^3(\lambda_s - \psi_s) \quad . \tag{52}$$

As far as the exterior metric is concerned, all of the information about the stellar model is contained in the parameters a_1, a_2, and a_3 (and a_4 in the general case). Each different set of values for these constants corresponds to a different mass and radius of the star. Indeed, the total mass-energy of the star ("gravitating mass") as determined by g_{00} and using Eqs. (41) and (47) is

$$m = \frac{1}{2} a_1 + K(3a_2 - a_1) \quad , \tag{53}$$

(a_1 and a_2 determined by matching conditions at the surface). It is difficult to say what each parameter corresponds to physically (in terms of integrals over the source, etc.) because of the complexity of the inhomogeneous equations [cf. Eqs. (44)]. The only definite statement is that the particular combination of a_1 and a_2 given in Eq. (53) corresponds to the total mass.

A further interesting fact is that, for a given choice of a, f, K, the PPN parameters γ and β — as determined by a $1/r$ expansion of the isotropic version of the metric — are functions of a_1 and a_2 and in general are not equal to their values as determined in the PN limit. (This situation is also true in the Dicke-Brans-Jordan theory.)[25] Only in the case of a weakly gravitating star can one be sure that the two different determinations of γ and β will agree approximately (to within PN precision). In GRT, on the other hand, expansion of the Schwarzschild metric gives $\gamma = \beta = 1$ regardless of stellar model, and in agreement with the γ and β as determined in the PN limit of the theory.

Table I gives a comparison between our stellar-structure equations and those of GRT.

IV. ANALYSIS OF AN EXACT EXTERIOR SOLUTION

A. The Metric

As pointed out in the last section, the general exterior metric of a static spherically symmetric spacetime is a 4-parameter family [cf. Eq. (38)]. Let us analyze a member of that family. First of all, for simplification, we choose $a_3 = a_4 = 0$, which puts the metric of Eq. (38) in isotropic form. Next, using Eq. (53) as a definition of the mass m, we choose a_1, a_2, and K such that a 1/r expansion of the metric indicates that the PPN parameters γ and β are both unity (see Sec. II). In other words, choose a_1 a_2, and K such that[11]

$$g_{00} = -1 + 2m/\rho - 2(m/\rho)^2 + O(\rho^{-3}) \quad , \tag{54a}$$

$$g_{ij} = -\delta_{ij}(1 + 2m/\rho) + O(\rho^{-2}) \quad , \tag{54b}$$

which requires

$$a_1/m = 1 \quad , \tag{55a}$$

$$a_2/m = 3 \quad , \tag{55b}$$

$$K = 1/16 \quad . \tag{55c}$$

It is interesting to note that the value for K given in Eq. (55c) is the same value required for $\gamma = \beta = 1$ in the weak-field PN expansion [cf. Sec. II and Eq. (30)]. Using Eqs. (55) and Eq. (38), one can now write the line element as

$$ds^2 = -\frac{(1-\frac{1}{2}m/\rho)^2}{(1+\frac{1}{2}m/\rho)^2}dt^2 + \frac{(1-\frac{1}{2}m/\rho)^2}{(1-\frac{3}{2}m/\rho)^2}(d\rho^2 + \rho^2 d\Theta^2 + \rho^2 \sin^2\Theta d\varphi^2) \quad . \quad (56)$$

The line element given in Eq. (56) is the simplist static spherically symmetric metric which yields the same light bending and perihelion shift (viz. $\gamma = \beta = 1$) as in GRT. (Note that the value of g_{00} is identical to the corresponding term in the isotropic form[26] of the GRT Schwarzschild geometry.)

B. Geodesic Completeness and Radial Geodesics

A glance at Eq. (56) reveals that $\rho = 1.5$ m is an infinite proper radial distance away from any $\rho > 1.5$ m. To investigate whether this point is removed from the physical manifold we need to look at null and timelike geodesics. Consider equatorial orbits (no loss of generality with spherical symmetry) and consider the first integrals of the motion for particles and photons:

$$u^\alpha u_\alpha = -1 = (u_0)^2 g^{00} + g_{\rho\rho}(u^\rho)^2 + (u_\varphi)^2 g^{\varphi\varphi} \quad , \quad (57a)$$

$$g_{\rho\rho}(P_\rho)^2 + g^{00}(P_0)^2 + g^{\varphi\varphi}(P_\varphi)^2 = 0 \quad , \quad (57b)$$

where $u^\alpha = dx^\alpha/d\tau$ for particles and $P^\alpha = dx^\alpha/d\lambda$ (with λ the affine parameter) for photons. It is well known (see, e.g., Ref. 27) that for a metric of the form of Eq. (26), u_0, u_φ, and (P_φ/P_0) are all constants of the motion, which we shall denote by $-\tilde{E}$, $\tilde{L}$, and ℓ, respectively. Physically, these constants are energy per unit rest mass, angular momentum per unit rest mass, and impact parameter respectively.

Using the above, Eq. (57a) can be written as

$$u^{\rho} = d\rho/d\tau = (\bar{\rho} - \tfrac{1}{2})^{-2}\,(\bar{\rho} + \tfrac{1}{2})(\bar{\rho} - \tfrac{3}{2})[\tilde{E}^2 - \Gamma^2(\bar{\tilde{L}},\bar{\rho})] \quad , \tag{58a}$$

where

$$\bar{\rho} \equiv \rho/m, \quad \bar{\tilde{L}} \equiv \tilde{L}/m, \text{ etc.} \quad , \tag{58b}$$

and

$$\Gamma(\bar{\tilde{L}},\bar{\rho}) \equiv (\bar{\rho} + \tfrac{1}{2})^{-1}[(\bar{\rho} - \tfrac{1}{2})^2 + (\bar{\tilde{L}}/\bar{\rho})^2(\bar{\rho} - \tfrac{3}{2})^2]^{1/2} \quad . \tag{58c}$$

The function Γ plays the role of an effective potential, which we shall discuss later. Equation (57b) can be written as

$$(d\rho/dt)^2 = \frac{1}{\bar{\rho}^2}\left[\frac{\bar{\rho} - \frac{3}{2}}{\bar{\rho} + \frac{1}{2}}\right]^4 (\gamma^2 - \ell^2) \quad , \tag{59a}$$

where

$$\gamma \equiv \bar{\rho}(\bar{\rho} + \tfrac{1}{2})(\bar{\rho} - \tfrac{3}{2})^{-1} \quad . \tag{59b}$$

Consider first radial geodesics ($\tilde{L} = \ell = 0$). Then Eq. (58a) indicates clearly that $\bar{\rho} = 3/2$ is an infinite proper time away from timelike geodesics. If one then uses the fact that $P_0 = g_{00}\, dt/d\lambda$ = constant for the null radial geodesics together with Eq. (59a), then it is also easy to show that $\bar{\rho} = 3/2$ is an infinite affine parameter distance away for null radial geodesics. Equations (58) and (59) indicate that nonradial geodesics between any two values of $\bar{\rho}$ take even longer proper time and affine parameter than do radial geodesics. Thus we have shown that $\bar{\rho} = 3/2$ is really unreachable by particles and photons; in particular, the manifold covered by our coordinate system is maximal.[28] Since one can also show that there are no singularities for $\bar{\rho} \geq 3/2$, our manifold is geodesically complete.[28]

For the special case of radial geodesics, we integrate Eq. (58a) to yield

$$\tau = \pm\left[- b^{-1}\ln\left| \frac{2b(X + 1) + 2c(\bar{\rho} - \frac{3}{2})}{\bar{\rho} - \frac{3}{2}} \right| - d^{-1}X + d^{-3/2}\sin^{-1}\left(\frac{2d\bar{\rho} - 1 - \tilde{E}^2}{2\tilde{E}}\right)\right] \quad (60a)$$

$$+ \text{const. for } \frac{1}{2} < \tilde{E} < 1 \quad ,$$

where

$$b \equiv (4\tilde{E}^2 - 1)^{1/2} \quad , \quad (60b)$$

$$c \equiv 2\tilde{E}^2 - 1 \quad , \quad (60c)$$

$$d \equiv 1 - \tilde{E}^2 \quad , \quad (60d)$$

$$X \equiv [(1 + \tilde{E}^2)\,\bar{\rho} - c(\bar{\rho}^2 + \frac{1}{4})]^{1/2} \quad . \quad (60e)$$

We will not be interested in analytic solutions for values of $\tilde{E}$ other than those indicated in Eq. (60). To obtain the functional relationship between coordinate time t and ρ for $1/2 < \tilde{E} < 1$, add to Eq. (60a) a factor of 4 multiplying the log term and a factor of $(3 - 2\tilde{E}^2)$ multiplying the inverse sine term.

For radial photon geodesics, Eq. (59a) can be integrated to yield

$$\bar{t} = \pm\,(\bar{\rho} + 2\ln|\bar{\rho} - \frac{3}{2}|) + \text{const.} \quad . \quad (61)$$

Figure 1 illustrates a few of the radial geodesics for photons and particles, the latter released from rest at $\bar{\rho} = 10$ and $\bar{\rho} = 5$. It is interesting to note that the analogous metric in GRT is geodesically incomplete: $\bar{\rho} = 1/2$ can be reached in finite proper time, but requires infinite coordinate time.

It can be shown, from analysis of the metric, that another complete universe exists for $1/2 \leq \bar{\rho} \leq 3/2$. However, if we assume the geometry to be produced by a star which originated in our universe, then its surface lies outside $\bar{\rho} = 3/2$. In the following we consider only the region $\bar{\rho} > 3/2$.

B. Proper Surface Areas and Embedding Diagrams

There are some curious geometrical effects in our manifold, not to be found in the Schwarzschild geometry of GRT. The proper surface area of a sphere described by ρ = const. is

$$A = 4\pi m^2 \,\bar{\rho}^2(\bar{\rho} - \tfrac{1}{2})^2 \,(\bar{\rho} - \tfrac{3}{2})^{-2} \quad . \tag{62}$$

A plot of this area is given in Fig. 2, in which the abcissa is marked off not only by ρ but also by the proper time as measured by a radially falling observer. As can be seen in the figure, the observer sees the sequence of surface areas pass through a minimum, $A_{MIN} = 4\pi m^2\,(49/4 + 5\sqrt{6})$ at $\bar{\rho} = 3/2 + 1/2\sqrt{6}$, and then increase without bound as $\bar{\rho} = 3/2$ is approached.

Another interesting feature arises when we examine the intrinsic geometry of the 2-surface: t = const., $\Theta = \pi/2$ by the use of an embedding diagram. By equating the two-dimensional metric

$$ds^2 = (\bar{\rho} - \tfrac{1}{2})^2 \,(\bar{\rho} - \tfrac{3}{2})^{-2}\,(d\rho^2 + \rho^2 d\Theta^2) \tag{63a}$$

to the metric of a surface of revolution in a Euclidean 3-space

$$ds^2 = dz^2 + dr^2 + r^2 d\varphi^2 = [(dz/dr)^2 + 1]\, dr^2 + r^2 d\varphi^2 \quad , \tag{63b}$$

one can visualize the geometry of Eq. (63a). If we can find $z(r)$, or more easily $z(\rho)$ and $r(\rho)$, then the line element of Eq. (63b) can be drawn. Clearly

$$\bar{r} = \bar{\rho}(\bar{\rho} - \tfrac{1}{2})(\bar{\rho} - \tfrac{3}{2})^{-1} \quad . \tag{64}$$

The function $z(\rho)$ is the solution of the equation

$$\left(\frac{dz}{d\rho}\right)^2 = \left(\frac{\partial s}{\partial \rho}\right)^2 - \left(\frac{dr}{d\rho}\right)^2 = \frac{(\bar{\rho} - \frac{1}{2})^2}{(\bar{\rho} - \frac{3}{2})^2} - \left[\frac{\bar{\rho}^2 - 3\bar{\rho} + \frac{3}{4}}{(\bar{\rho} - \frac{3}{2})^2}\right]^2 \quad , \tag{65a}$$

or

$$d\bar{z}/d\bar{\rho} = \bar{\rho}^{\,1/2}(2\bar{\rho}^2 - 5\bar{\rho} + \tfrac{3}{2})^{1/2}(\bar{\rho} - \tfrac{3}{2})^{-2} \quad . \tag{65b}$$

The right-hand side of Eq. (65b) becomes complex at

$$\bar{\rho} = \tfrac{1}{4}(5 + \sqrt{13}) \approx 2.1 \quad \text{or} \quad \bar{r} \approx 4.8 \quad . \tag{66}$$

This indicates that for $1.5 < \bar{\rho} \lesssim 2.1$ we will have to embed in a pseudo-Euclidean space, i.e.,

$$ds^2 = -\,dz^2 + dr^2 + r^2 d\varphi^2 \quad . \tag{67}$$

The embedding diagram is given in Fig. 3 and includes both the Euclidean part and the pseudo-Euclidean part. The surface is obtained by rotating the curve about the z or iz axis.

C. Particle and Photon Orbits

Analysis of orbits is facilitated by use of the effective potential. Equations (58c) and (59b) give the effective potentials for massive particles and photons. For a given value of $\tilde{L}$, the particle is allowed only in those regions for which $\Gamma(\tilde{L},\bar{\rho}) \leq \tilde{E}$. For photons, γ^2 acts as an "inverse" effective potential; photons are allowed only in regions for which $\gamma \geq \ell$. Figures 4 and 5 illustrate the effective potentials for particles and photons, respectively, with the dots in Fig. 4 indicating extrema of the potential (circular orbits). The closest stable circular orbit for particles occurs for $\tilde{\tilde{L}} \sim 3.88$ at $\bar{\rho} \sim 7$. For particles with larger $\tilde{\tilde{L}}$, the circular orbits with $\bar{\rho} < 7$ are unstable and those with $\bar{\rho} > 7$ are stable. The circular photon orbit occurs at $\bar{\rho} = 1.5 + \sqrt{3}$ or $\bar{r} \sim 5$. This can be compared with the corresponding value of $\bar{r} = 3$ in GRT.

V. GRAVITATIONAL WAVES AND CONSERVATION LAWS

A. Monopole Waves

In the full theory (no linearized approximation) the homogeneous field equations are, as indicated previously,

$$\eta^{\alpha\beta} h^{\mu\nu}{}_{|\alpha|\beta} = 0 \quad , \tag{68}$$

and gravitational waves travel geodesics of $\underset{\sim}{\eta}$ rather than $\underset{\sim}{g}$. The implication of this last fact will be explored later. The simplicity of the vacuum field equations [cf. Eq. (68)] is of great help in constructing solutions.

Consider a time-dependent spherically symmetric solution to Eq. (68), for example

$$h_{00} = r^{-1} e^{i\omega(r-t)} \quad , \tag{69a}$$

$$h_{ij} = \delta_{ij}\, r^{-1} e^{i\omega(r-t)} \quad . \tag{69b}$$

The Riemann tensor constructed from the resulting time-dependent spherically symmetric metric is itself time dependent. From this we conclude the presence of physical monopole waves; thus there is no analogue of Birkhoff's theorem[12] in this theory. The existence of such solutions in our theory and the accompanying monopole radiation complicate the problem of the spherical collapse of a star. As will be shown below, there are other "non-GRT" type gravitation-wave modes in addition to the monopole waves.

B. Linearized Theory and Plane Gravitational Waves

In analyzing weak gravitational waves, one should restrict one's attention to the form and behavior of the Riemann tensor, not only because it is gauge invariant (under infinitesimal coordinate transformations) but also because it is that feature of the gravitational wave which interacts directly with test bodies. Work in a coordinate system in which $\eta_{\mu\nu}$ is Minkowskian and $h_{\mu\nu}$ is small (small deviations from flat space). Then

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - 2Kh\eta_{\mu\nu} + O(h^2) \equiv \eta_{\mu\nu} + h'_{\mu\nu} + O(h^2) \quad , \tag{70}$$

and

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(h'_{\alpha\delta,\beta\gamma} + h'_{\beta\gamma,\alpha\delta} - h'_{\alpha\gamma,\beta\delta} - h'_{\beta\delta,\alpha\gamma}) \quad . \tag{71}$$

Furthermore, restrict one's attention to those solutions of Eq. (68) which represent plane waves travelling in the z direction, i.e.,

$$h'_{\mu\nu} = A_{\mu\nu}e^{ik(z-t)} \quad , \tag{72}$$

where $A_{\mu\nu}$ is a constant amplitude and k a wave number. To analyze the decomposition of $R_{\alpha\beta\gamma\delta}$ into independent "wave modes" in as invariant a manner as possible, one should investigate the transformation properties of $R_{\alpha\beta\gamma\delta}$ under those Lorentz transformations which leave the wave direction fixed. With such transformations in mind one selects a new basis in which the components of $R_{\alpha\beta\gamma\delta}$ are to be computed — the quasi-orthonormal tetrad basis (see, e.g., Ref. 29 for a complete discussion of the "tetrad formalism").

$$\underset{\sim}{k} = 2^{-1/2}(-1,0,0,1) \quad , \tag{73a}$$

$$\underset{\sim}{\ell} = 2^{-1/2}(1,0,0,1) \quad , \tag{73b}$$

$$\underset{\sim}{m} = 2^{-1/2}(0,1,i,0) \quad , \tag{73c}$$

$$\underset{\sim}{\bar{m}} = 2^{-1/2}(0,1,-i,0) \quad . \tag{73d}$$

Note that one of the "tetrad legs" points along the direction of the wave. In such a basis the components of the Riemann tensor are

$$R_{nkm\ell} = R_{\alpha\beta\gamma\delta}n^{\alpha}k^{\beta}m^{\gamma}\ell^{\delta} \text{ , etc. } \quad . \tag{74}$$

Using Eqs. (71)-(74) one finds that the only nonvanishing components of the Riemann tensor are those with two ℓ's — thus there are six possible degrees of freedom. Since there are no restrictions on the Riemann tensor once Eqs. (68) are satisfied, all six tetrad components will in general be nonvanishing and our theory thus has six independent gravitational wave modes. In GRT, as a contrast, the field equations $R_{\alpha\beta} = 0$ imply vanishing of $R_{\ell k \ell k}$, $R_{\ell k \ell m}$, $R_{\ell k \ell \bar{m}}$, and $R_{\ell m \ell \bar{m}}$ so that there are only two degrees of freedom — those represented by $R_{\ell m \ell m}$ and its complex conjugate $R_{\ell \bar{m} \ell \bar{m}}$.

The reader is referred to Refs. 3 and 4 for details of the transformation properties of the objects indicated in Eq. (74). Here we quote only the results: We denote the six wave modes by Ψ_2, Ψ_3, $\bar{\Psi}_3$, Ψ_4, $\bar{\Psi}_4$, Φ_{22} and in terms of the tetrad components of the Riemann tensor and "electric" coordinate components of the Riemann tensor (those which are directly physically measurable) these are

$$\Psi_2 \equiv -\frac{1}{6} R_{\ell k \ell k} = -\frac{1}{6} R_{tztz} \quad , \tag{75a}$$

$$\Psi_3 \equiv -\frac{1}{2} R_{\ell k \ell \bar{m}} = \frac{1}{2}(R_{txtz} - iR_{tytz}) \quad , \tag{75b}$$

$$\bar{\Psi}_3 \equiv -\frac{1}{2} R_{\ell k \ell m} = \frac{1}{2}(R_{txtz} + iR_{tytz}) \quad , \tag{75c}$$

$$\Psi_4 \equiv -R_{\ell \bar{m} \ell \bar{m}} = R_{tyty} - R_{tztz} + 2iR_{txty} \quad , \tag{75d}$$

$$\bar{\Psi}_4 \equiv -R_{\ell m \ell m} = R_{tyty} - R_{tztz} - 2iR_{txty} \quad , \tag{75e}$$

$$\Phi_{22} \equiv \frac{1}{2} R_{\ell m \ell \bar{m}} = -R_{txtx} - R_{tyty} \quad . \tag{75f}$$

The presence or absence of a Ψ_2 component in a gravitational wave is Lorentz invariant. If Ψ_2 is absent in a particular wave, the presence or absence of Ψ_3 (or $\bar{\Psi}_3$) in that wave is also Lorentz invariant. As outlined in Refs.

3 and 4, if either Ψ_2 or Ψ_3 is present in a wave (in many theories they are always absent, but not ours), then it is impossible to decompose the wave into states of definite helicity (spin) in a Lorentz invariant manner: what one observer identifies as "pure spin 0" another observer will identify as "pure spin 0" plus "pure spin 1," etc. . Only waves containing only Φ_{22}, Ψ_4, and $\bar{\Psi}_4$ can be decomposed into pure spins: spin 0 and spin 2. In general, then, there is no unique spin decomposition of waves in our theory and it is of class II_6 (see Refs. 3 and 4 for a complete discussion of the "classification scheme"). The physical imprints of the various modes will be discussed in Sec. VI.

B. The Stress-Energy Pseudo Tensor for Gravitational Waves

For all Lagrangian-based theories a very general method, with roots going back to Noether,[30] exists for constructing conserved quantities (see Ref. 5 and the references quoted therein for a more complete discussion). Invariance of the gravitational Lagrangian under coordinate transformations leads to the following identities:

$$(t'_{\mu}{}^{\nu} - U_{\mu A}{}^{\nu}\mathcal{L}_G{}^A)_{,\nu} \equiv 0 \quad , \tag{76}$$

where $\mathcal{L}_G$ is the gravitational Lagrangian density, $\mathcal{L}_G{}^A$ is the variational derivative of $\mathcal{L}_G$ with respect to field y_A occurring in $\mathcal{L}_G$,

$$t'_{\mu}{}^{\nu} \equiv -\delta_{\mu}{}^{\nu}\mathcal{L}_G + \frac{\partial\mathcal{L}_G}{\partial y_{A,\nu}}\, y_{A,\mu} \quad , \tag{77a}$$

and $U_{\mu A}{}^{\nu}$ is defined by the functional changes of the y_A, δy_A, under infinitesimal coordinate transformations, i.e.,

$$\bar{x}^{\mu} = x^{\mu} + \xi^{\mu} \quad , \tag{77b}$$

$$\delta y_A = U_{\mu A}{}^{\nu}\xi^{\mu}{}_{,\nu} - y_{A,\mu}\xi^{\mu} \quad . \tag{77c}$$

We have assumed $\mathcal{L}_G$ contains no higher than first derivatives of the y_A; generalization to higher derivatives is straightforward. Equations (76) are of the form of conservation laws and our object is to identify in a physically meaningful way the gravitational portion of the conserved quantity. To facilitate the computation, we assume $\mathcal{L}_G$ has been rewritten in terms of $\eta_{\alpha\beta}$ and $g_{\alpha\beta}$ [which can be done in principle by solving for $h_{\alpha\beta}(g_{\mu\nu}, \eta_{\mu\nu})$]. Using the tensor transformation law for $g_{\alpha\beta}$ and $\eta_{\alpha\beta}$, one easily shows

$$U_{\mu A}{}^{\nu} = -2y_{\mu(\alpha}\delta_{\beta)}{}^{\nu} \quad \text{for} \quad y_A = y_{\alpha\beta} \quad , \tag{78}$$

where parentheses denote symmetrization of indices. Using Eq. (78) we find the relation

$$U_{\mu A}{}^{\nu}\mathcal{L}'^{,A}_G = -2\eta_{\mu(\alpha}\delta_{\beta)}{}^{\nu}(\delta\mathcal{L}'_G/\delta\eta_{\alpha\beta}) - 2g_{\mu(\alpha}\delta_{\beta)}{}^{\nu}(\delta\mathcal{L}'_G/\delta g_{\alpha\beta}) \quad , \tag{79}$$

where

$$\mathcal{L}'_G(\underset{\sim}{\eta}, \underset{\sim}{g}) = \mathcal{L}_G(\underset{\sim}{h}, \underset{\sim}{\eta}) \quad .$$

If we now use the field equations

$$\delta\mathcal{L}'_G/\delta g_{\alpha\beta} = -\delta\mathcal{L}_{NG}/\delta g_{\alpha\beta} \quad , \tag{80}$$

and Eq. (4c), Eq. (79) becomes

$$U_{\mu A}{}^{\nu}\mathcal{L}'^{,A}_G = -2\eta_{\mu(\alpha}\delta_{\beta)}{}^{\nu}(\delta\mathcal{L}'_G/\delta\eta_{\alpha\beta}) + (-g)^{1/2}\, T_\mu{}^\nu \equiv \Lambda_\mu{}^\nu + (-g)^{1/2} T_\mu{}^\nu \quad . \tag{81}$$

We point out that although Eqs. (76) are "strong conservation laws"[31] (identities), one must use Eqs. (80) to get out a physically useful result. Substitution of Eq. (81) into Eq. (76) yields

$$\left(t_\mu{}^\nu - (-g)^{1/2} T_\mu{}^\nu\right)_{,\nu} = 0 \quad , \tag{82a}$$

where

$$t_\mu{}^\nu \equiv t'_\mu{}^\nu - \Lambda_\mu{}^\nu \quad . \tag{82b}$$

The conserved energy momentum vector is then

$$P_\mu = \int \left(t_\mu{}^0 - (-g)^{1/2} T_\mu{}^0 \right) d^3x \quad . \tag{83}$$

Since P_μ in Eq. (83) contains a contribution from the matter stress energy tensor, we know we are on the right track. Problems arise when we notice that the quantity defined in Eq. (82b) is in general not positive definite, as a result of contributions from $\Lambda_\mu{}^\nu$. However, it can be shown from the generalized Bianchi identities of this theory (see Appendix B) that $\Lambda_\mu{}^\nu$ obeys the equation

$$\Lambda_\mu{}^\nu{}_{|\nu} = 0 \quad . \tag{84}$$

Actually, Eddington[32] was the first to point out that conservation laws of the form of Eq. (84) follow from theories with absolute objects.[2] If we now choose to work in the coordinate system in which $\eta_{\alpha\beta}$ is the globally constant Minkowski metric, Eq. (84) becomes

$$\Lambda_\mu{}^\nu{}_{,\nu} = 0 \quad , \tag{85}$$

and we see that $\Lambda_\mu{}^\nu$ is conserved by itself, independently of energy gain or loss from matter ($T_\mu{}^\nu$). Since our usual idea of total energy conservation involves interactions, it is perhaps more useful to omit the separately conserved $\Lambda_\mu{}^\nu$ from consideration and to define, in this frame, the gravitational stress-energy tensor as

$$t_\mu{}^\nu = t_\mu{}'^\nu \quad . \tag{86}$$

Thus $\Lambda_\mu{}^\nu$ represents the energy density of a quantity associated with the absolute field $\eta_{\mu\nu}$; at present we must regard it as a purely mathematical quantity whose noninteraction with matter mirrors the absolute nature of $\eta_{\mu\nu}$. (As an aside, there always exists a $t_\mu{}^\nu$ which is a real tensor and not a pseudo-tensor in prior-geometric theories of gravity.[5])

We point out that in the linearized approximation Eq. (85) is always the expression of Eq. (84) in all frames related to the global Minkowski frame by infinitesimal coordinate (gauge) transformations. We proceed by explicitly calculating $t_\mu{}^\nu$ for the linearized theory. From Eq. (77a)

$$t_\mu{}^\nu = - \delta_\mu{}^\nu \mathcal{L}_G' + \frac{\partial \mathcal{L}_G'}{\partial g_{\alpha\beta,\nu}} g_{\alpha\beta,\mu} = - \delta_\mu{}^\nu \mathcal{L}_G + \frac{\partial \mathcal{L}_G}{\partial h_{\gamma\delta,\omega}} \frac{\partial h_{\gamma\delta,\omega}}{\partial g_{\alpha\beta,\nu}} g_{\alpha\beta,\mu} \ . \tag{87}$$

Inverting the linearized relation between $g_{\alpha\beta}$ and $h_{\alpha\beta}$ [cf. Eq. (70)] and taking the required partial derivatives, we find

$$\partial h_{\gamma\delta,\omega}/\partial g_{\alpha\beta,\nu} = \delta_\gamma{}^{(\alpha}\delta_\delta{}^{\beta)}\delta^\nu{}_\omega + 2K(1 - 8K)^{-1}\ \eta_{\gamma\delta}\eta^{\alpha\beta}\delta^\nu{}_\omega \ . \tag{88}$$

Using Eqs. (87), (88), and Eq. (2) for $\mathcal{L}_G$, we finally obtain

$$t_\mu{}^\nu = (16\pi)^{-1}\ [\delta_\mu{}^\nu(ah^{\gamma\sigma,\beta}h_{\gamma\sigma,\beta} + fh'^{,\alpha}h_{,\alpha}) - 2(ah^{\alpha\beta,\nu}h_{\alpha\beta,\mu} + fh_{,\mu}h'^{,\nu})] . \tag{89}$$

Since $h_{\mu\nu}$ transforms as a tensor, the above expression is gauge invariant. Equation (89) expresses a naturally defined stress-energy complex for the gravitational field.

Consider the energy density in a plane gravitational wave

$$h^{\gamma\sigma} = A^{\gamma\sigma}e^{ik_\alpha x^\alpha} \ ; \quad k_\alpha k^\alpha = 0 \ . \tag{90}$$

Then the first two terms in Eq. (89) do not contribute to $t_\mu{}^\nu$ and one obtains

$$t_\mu{}^\nu \propto k_\mu k^\nu \ , \tag{91a}$$

with

$$t_0^{\;0} = (8\pi)^{-1}[a(h_{\alpha\beta,0})^2 + f(h_{,0})^2] \quad . \tag{91b}$$

With the suggested values for a and f [cf. Eq. (30)], Eq. (91b) indicates a positive definite energy density. It is encouraging to note that for pure spin 2 waves (only ψ_4 present), Eq. (91b) becomes, for $a = 1/4$ [cf. Eq. (30)],

$$t_{0\,\mathrm{spin}\,2}^{\;0} = a(4\pi)^{-1}(h_{xx,0})^2 = (16\pi)^{-1}(h_{xx,0})^2 \quad , \tag{92}$$

which is identical to the corresponding expression in GRT.

VI. THE GRAVITATIONAL CONSTANT AND FURTHER EXPERIMENTAL TESTS

A. A Time-Dependent Gravitational Constant

As discussed in Sec. II, a number of existing solar system experiments place upper limits on the cosmological boundary values of $h_{\mu\nu}$ [cf. Eqs. (31)-(32)]. These constraints can always be satisfied in a given epoch. A more relevant point is the time dependence of the $\omega_{\mu\nu}$, which is directly related to the time dependence of the gravitational constant G. With the choice of adjustable constants given in Eq. (30), and using the explicit functional forms for K_1 D_0, D, one finds from Eq. (27) and Appendix A that

$$1 - \frac{1}{16}\,(19\omega_1 + 7\omega_0) + O(\omega^2) = G \quad . \tag{93a}$$

Thus

$$(\tfrac{1}{G})\; dG/dt \approx -\frac{1}{16}\,(19\omega_1/dt + 7d\omega_0/dt) \quad . \tag{93b}$$

Shapiro et al.[33] have placed limits on the time dependence of the gravitational constant by comparing the periods of planets with the ticking rates of atomic clocks. They find

$$|(\frac{1}{G})\ (dG/dt)| < 4 \times 10^{-10}/\text{year} \quad . \qquad (94)$$

This constitutes an experimental constraint on the magnitude of the time derivatives of $\omega_{\mu\nu}$ occurring in Eq. (93b). Preliminary results from our cosmological solution[19] indicate that the time dependences of ω_0 and ω_1 satisfy Eq. (94), but an improved Shapiro experiment might still prove to be a crucial experimental test of our theory.

B. Gravitational-Wave Experiments

The analysis of the preceding section reveals two crucial new experimental tests of our theory involving gravitational waves — two tests which have blossomed from our current program (see introductory remarks in Sec. I) — two tests which emphasize gravitational wave detection as a powerful new tool for probing metric theories of gravity.[3,4] The two tests are (i) time delay between simultaneously emitted gravitational and electromagnetic waves and (ii) polarizations of gravitational waves.

Since gravitational waves travel along geodesics of the "fast metric" $\eta_{\alpha\beta}$ and electromagnetic waves travel along geodesics of the "slow metric" $g_{\alpha\beta}$, there should be a time delay in reception of the two waves — emitted, for example, in simultaneous bursts by a supernova explosion. For waves emitted at the center of the galaxy, an order of magnitude estimate indicates

$$\text{Time Delay} \sim (m/r)_{\text{galaxy}} \cdot (\text{light travel time})$$

$$\sim (5 \times 10^{-7}) \cdot (3 \times 10^{4}\ \text{light years}) \approx 5\ \text{days} \quad . \qquad (95)$$

Much longer delay times would hold for the Virgo Cluster.

Polarization information is also a crucial experimental test. Equations

(75) indicate a purely longitudinal mode (ψ_2), mixed longitudinal-transverse quadrupole type modes (ψ_3, $\overline{\psi}_3$), a purely transverse "breathing" mode (Φ_{22}), and the familiar transverse quadrupole modes of GRT (ψ_4, $\overline{\psi}_4$). If an observer knows the direction of the wave, he can use Eqs. (75) to unambiguously catalogue the modes. If he does not know the direction of the source, he can still draw some conclusions. For example, if displacements do occur in more than one plane, then either the longitudinal-transverse modes (ψ_3, $\overline{\psi}_3$) are present, or the purely longitudinal mode (ψ_2) is mixed in with one of the purely transverse modes (ψ_4, $\overline{\psi}_4$, Φ_{22}).

It is important to note that until the problem of the generation of the various types of waves by particular sources is solved, our theory can only be verified by the presence of — but not ruled out by the absence of — the various possible modes indicated in Eqs. (75). This is unfortunate. But new doorways have been opened in the area of experimental tests and it is clear that gravitational tests outside of the PPN formalism must be contemplated in the future.

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APPENDIX A

CONSTANTS APPEARING IN PN LIMIT (Sec. II)

$$\omega_0 \equiv \omega_{00}$$

$$\omega_1 \equiv \omega_{11}$$

$$\omega \equiv 3\omega_1 - \omega_0$$

Eq. (12a): $D_0 \equiv 1 - 2K\omega + K^2\omega^2 + 2K\omega\omega_0 + \frac{3}{4}\omega_0{}^2 - \omega_0$

$E_0 \equiv 1 - 2K\omega - \frac{3}{2}\omega_0$

$F_0 \equiv - 2K + 2K^2\omega + 2K\omega_0$

Eq. (12b): $D \equiv 1 - 2K\omega + \omega_1 + K^2\omega^2 - 2K\omega\omega_1 + \frac{3}{4}\omega_1{}^2$

$E \equiv 1 - 2K\omega + \frac{3}{2}\omega_1$

$F \equiv - 2K(1 + \omega_1) + 2K^2\omega$

Eq. (12c): $H \equiv 1 - 2K\omega - \frac{3}{4}\omega_0 + \frac{3}{4}\omega_1$

Eq. (13): $I \equiv D_0{}^{1/2} D^{-3/2}$

$I_1 \equiv \frac{1}{2}\left(\frac{E}{D} + \frac{E_0}{D_0}\right)$

$I_2 \equiv \frac{1}{2}\left(\frac{3F}{D} - \frac{F_0}{D_0} + \frac{E}{D}\right)$

$I_3 \equiv \frac{D}{D_0}$

$L \equiv - (a + 4f)^{-1}\, [f(1 - 2K\omega) + 2Ka(1 - K\omega)]$

$M \equiv - (a + 4f)^{-1}\, (2Ka + \frac{3}{2} f)$

$N \equiv 2k(f + Ka)(a + 4f)^{-1}$

Eq. (16d): $S_0 \equiv I_1(1 - 2K\omega + L - \frac{3}{2}\omega_0 - M\omega_0) - \frac{3}{2} - M$

$S_1 \equiv I_2(1 - 2K\omega + L - \frac{3}{2}\omega_0 - M\omega_0) + N - 2K$

$B_0 \equiv I_3(1 - 2K\omega + L - \frac{3}{2}\omega_0 - M\omega_0) - L - M\omega_1$

Eq. (16e): $R_0 \equiv 1 - 2K\omega + \frac{3}{2}\omega_1$

$R_1 \equiv I_1(M\omega_0 - L) + M$

$R_2 \equiv I_2(M\omega_0 - L) - N$

$B_1 \equiv I_3(M\omega_0 - L) + L + M\omega_1$

Eq. (20a): $K_1 \equiv E_0C_0 - F_0(3C_1 - C_0)$

$K_2 \equiv -[K^2(3C_1 - C_0)^2 + 2KC_0(3C_1 - C_0) + \frac{3}{4}C_0{}^2]$

$K_3 \equiv \tau[S_0C_0 + S_1(3C_1 - C_0)](E_0 + F_0) - 3\tau F_0[RC_0 + R_2(3C_1 - C_0)]$

$K_4 \equiv \tau[E_0B_0 - F_0(R_o + 3B_1 - B_0)]$

Eq. (20b): $K_5 \equiv EC_1 + F(3C_1 - C_0)$

APPENDIX B

RELATIONS FOLLOWING FROM GENERALIZED BIANCHI IDENTITIES

Assume that L_G has been rewritten as a function of $\eta_{\mu\nu}$ and $g_{\mu\nu}$. Since L_G is a scalar, its variation under infinitesimal coordinate transformations must vanish, i.e.,

$$\delta L_G = \int \left(\frac{\delta \mathcal{L}'_G}{\delta \eta_{\alpha\beta}} \delta\eta_{\alpha\beta} + \frac{\delta \mathcal{L}'_G}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} \right) d^4x = 0 \quad . \tag{B1}$$

Under the coordinate transformation

$$\bar{x}^\alpha = x^\alpha + \xi^\alpha \quad , \tag{B2}$$

the functional changes in the tensors $\underset{\sim}{\eta}$ and $\underset{\sim}{g}$ are

$$\begin{aligned} \delta\eta_{\alpha\beta} &= - \eta_{\alpha\beta,\mu}\xi^\nu - \eta_{\alpha\gamma}\xi^\gamma{}_{,\beta} - \eta_{\beta\gamma}\xi^\gamma{}_{,\alpha} \ , \\ &= - 2\tau_{(\alpha|\beta)}, \quad \text{where} \quad \tau_\alpha \equiv \eta_{\alpha\beta}\xi^\beta \ , \end{aligned} \tag{B3}$$

$$\delta g_{\alpha\beta} = - 2\xi_{(\alpha;\beta)} \quad . \tag{B4}$$

Now define

$$\delta\mathcal{L}'_G/\delta\eta_{\alpha\beta} \equiv (-\eta)^{1/2}\gamma^{\alpha\beta} \quad , \tag{B5}$$

and use the field equations to write

$$\delta\mathcal{L}'_G/\delta g_{\alpha\beta} = - \frac{1}{2}(-g)^{1/2}T^{\alpha\beta} \quad . \tag{B6}$$

Using Eqs. (B2)-(B6), Eq. (B1) can be written in the form

$$\begin{aligned} 0 = \int \Big[&(-\eta)^{1/2}(\gamma^{\alpha\beta}\tau_\alpha)_{|\beta} - \frac{1}{2}(-g)^{1/2}(T^{\alpha\beta}\xi_\alpha)_{;\beta} + \frac{1}{2}(-g)^{1/2}T^{\alpha\beta}{}_{;\beta}\,\xi_\alpha \\ &- (-\eta)^{1/2}\gamma^{\alpha\beta}{}_{|\beta}\,\tau_\alpha \Big] d^4x \quad . \end{aligned} \tag{B7}$$

Now if we remember that

$$(\gamma^{\alpha\beta}\tau_\alpha)_{|\beta} = (-\eta)^{-1/2}\left[(-\eta)^{1/2}\gamma^{\alpha\beta}\tau_\alpha\right]_{,\beta} \quad , \tag{B8}$$

and also the corresponding equation for the covariant derivative with respect to $g_{\alpha\beta}$, the first two terms in (B7) vanish with proper boundary conditions on ξ^α. Now use the matter equations, Eqs. (5), and the arbitrariness of ξ^α (and hence τ_α) to get from Eq. (B7)

$$\gamma^{\alpha\beta}{}_{|\beta} = 0 \quad . \tag{B9}$$

Equation (B9) is not an identity; we had to use both the matter and gravitational field equations to obtain it. [We would have obtained an identity in the place of Eq. (B9) had we not enforced the dynamical equations.] Since $\underset{\sim}{\eta}$ is covariantly constant with respect to "slash," Eqs. (B5) and (B9) imply the desired relation

$$\Lambda_\mu{}^\nu{}_{|\nu} \equiv [\,-\,2\eta_{\mu(\alpha}\delta^\nu{}_{\beta)}(\delta\mathcal{L}'_G/\delta\eta_{\alpha\beta})]_{|\nu} = 0 \quad . \tag{B10}$$

TABLE I

Comparison of Construction of Stellar Models

	GRT	Two-Metric Theory
Number of coupled differential equations which must be integrated to find star's surface	2	4
Type of differential equations used in determining metric functions	First-order linear	Second order nonlinear
Number of quantities whose central values must be chosen to satisfy boundary conditions	1	4
Analytic Solutions	Yes	Probably not
Uniqueness of solution for given central pressure and equation of state	Yes	No
Number of parameters in exterior metric	1	4

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20. Note that, having chosen the coordinate system in which $\eta_{\mu\nu}$ has the form of Eq. (33), we are not at liberty to assume $h_{\mu\nu}$ is diagonal.

21. In this section and for the rest of the paper, except Sec. VI, we assume that the cosmological boundary values of $h_{\mu\nu}$ are arbitrarily small for the current epoch. See Sec. II for a discussion and justification of this point.

22. One can argue as follows: Let A be a coordinate system which contains the minimum number of arbitrary parameters. A transformation from A to curvature coordinates C cannot decrease the number of arbitrary parameters, by definition, and cannot increase the number since the transformation is only a function of the parameters occurring in A. Hence C has the same number of arbitrary parameters as A, i.e., the minimum possible number.

23. If $\eta_{\mu\nu}$ were not of the form of Eq. (34), aside from transformations of the type given in Eq. (37), in the same frame in which we assume the star to be static, then the resultant $g_{\mu\nu}$ would have such features as gravitational waves, anisotropies, etc. and thus be physically

inconsistent with the assumed condition of the star.

24. Note that only in such solutions (with $a_4 = 0$) are the gravitational wave cones (based upon $\eta_{\mu\nu}$ — Sec. V) and light cones (based upon $g_{\mu\nu}$) mutually symmetric. When one metric is diagonal and the other not, the two cones are "tilted" with respect to each other.

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FIGURE CAPTIONS

Fig. 1. Radial Geodesics of Particles and Photons. The number along the curves indicate proper time values for massive particles released from rest at $\bar{\rho} = 10, 5$. One of the curves is a photon geodesic and all curves have $\bar{t} \to \infty$ and affine parameter $\to \infty$ as $\bar{\rho} \to 1.5$.

Fig. 2. Proper Surface Area of Sphere $\bar{\rho}$ = Const. The upper abcissa gives the proper time of an observer released from $\bar{\rho} = 10$ as a local coordinate marker.

Fig. 3. Embedding Diagram for Equatorial Geometry. Solid line indicates Euclidean embedding (refers to z ordinate) and dashed line indicates pseudo-Euclidean embedding (refers to iz ordirate). Numbers along curve indicate values of $\bar{\rho}$.

Fig. 4. Effective Potential for Massive Objects. Dots indicate circular orbits.

Fig. 5. Effective Potential for Photons.

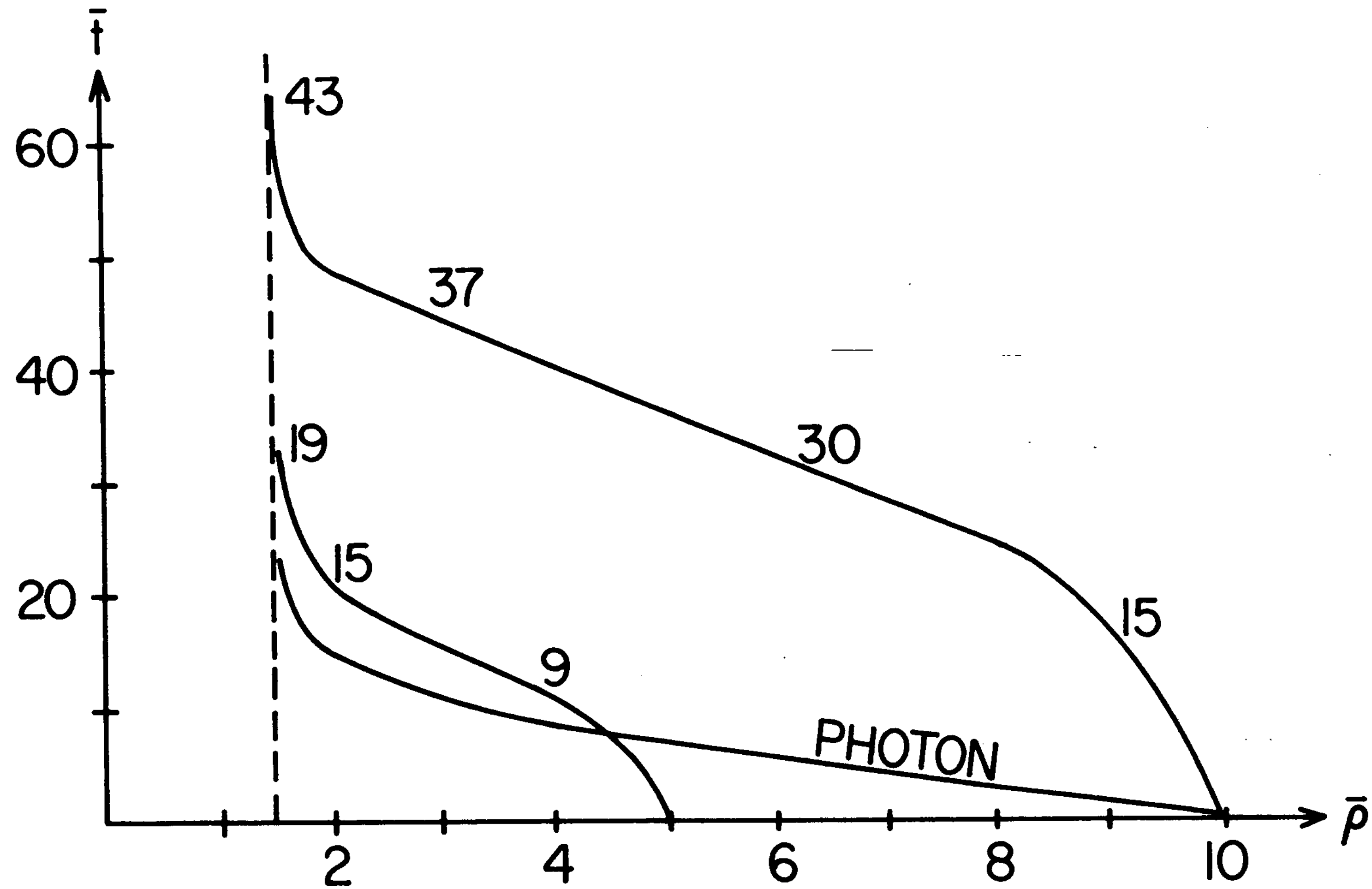
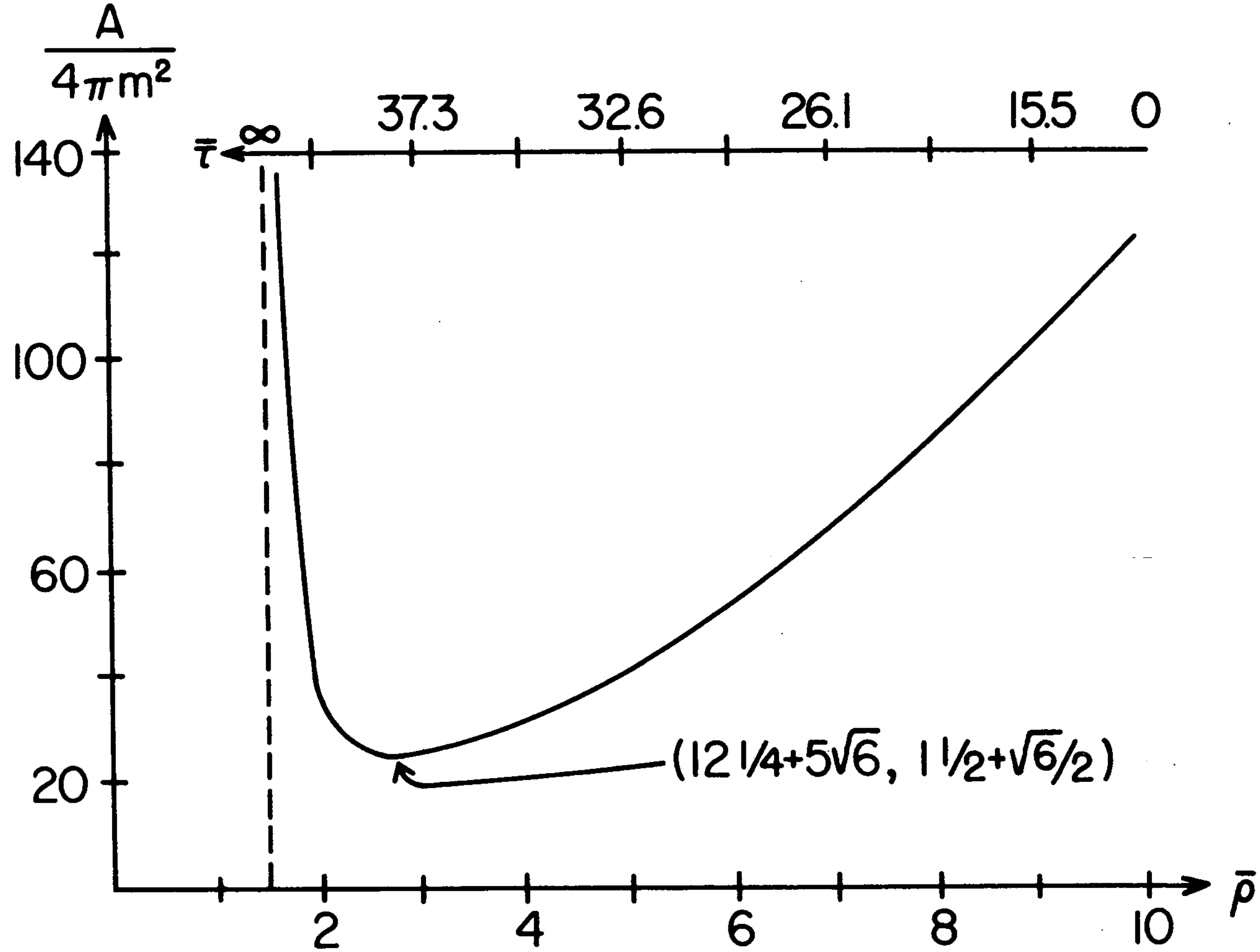
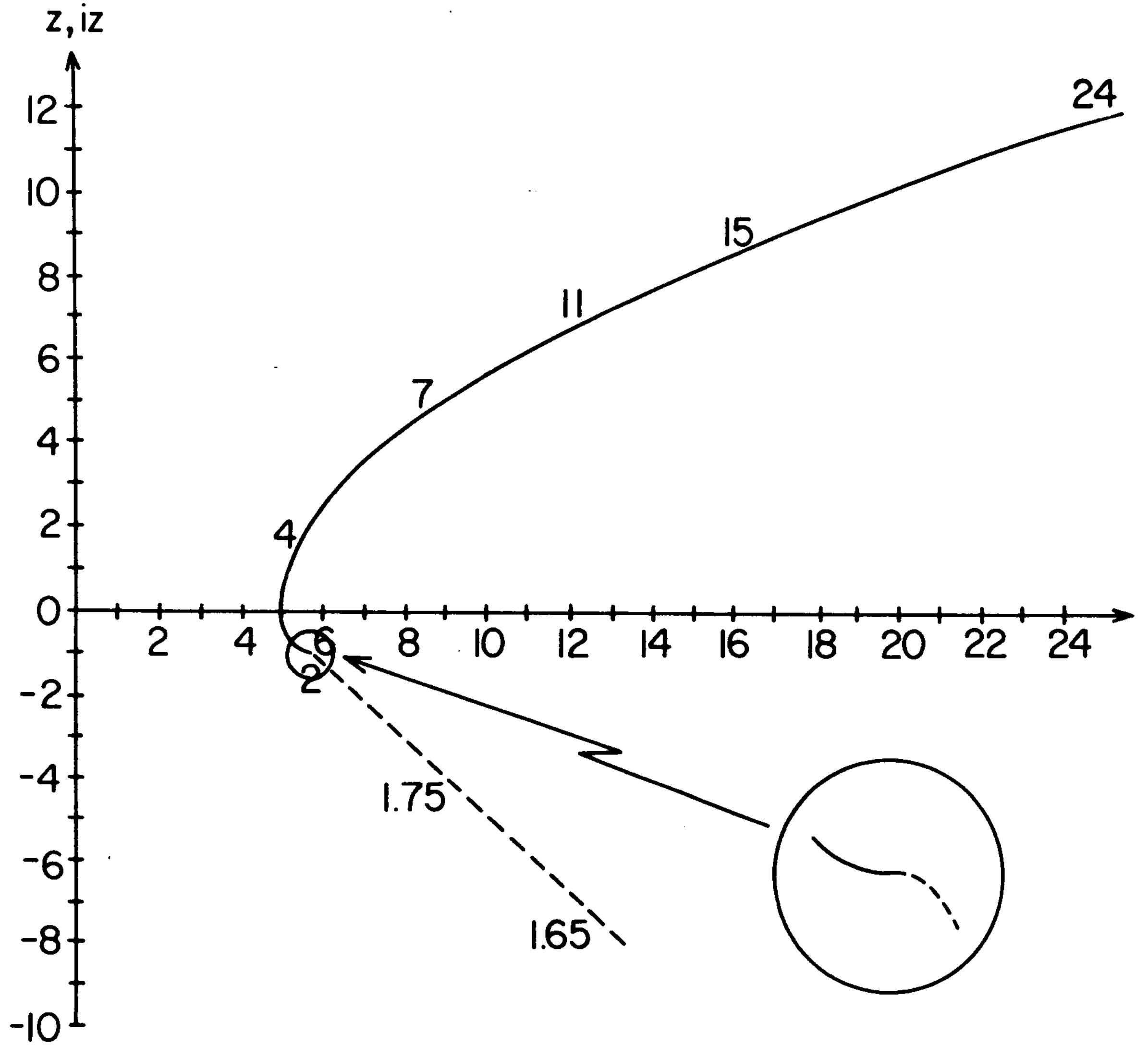


Fig. 1

Fig. 2





z, iz
r
24
15
11
7
4
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1.75
1.65
12
10
8
6
4
2
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22
24

Fig. 3

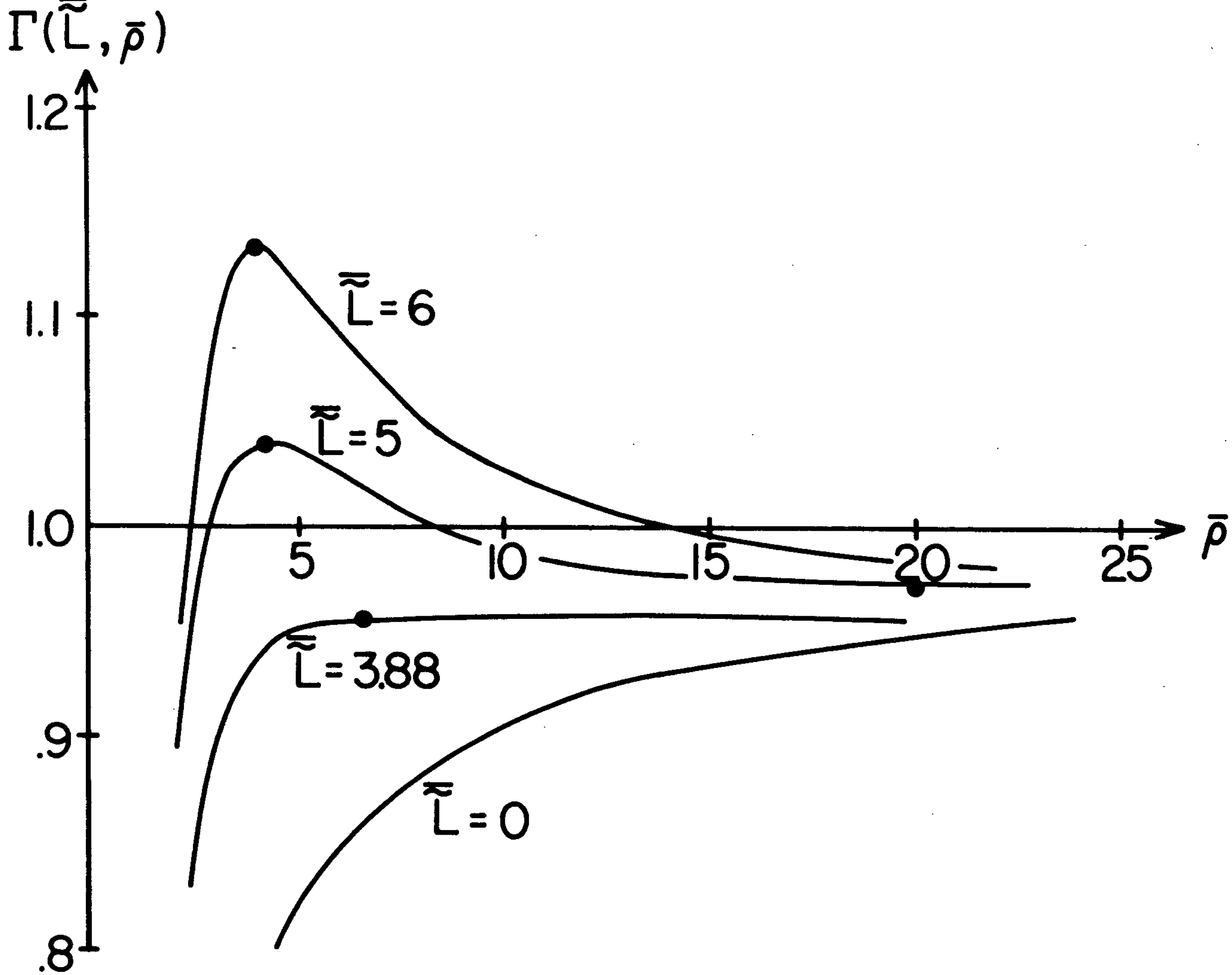

Fig. 4

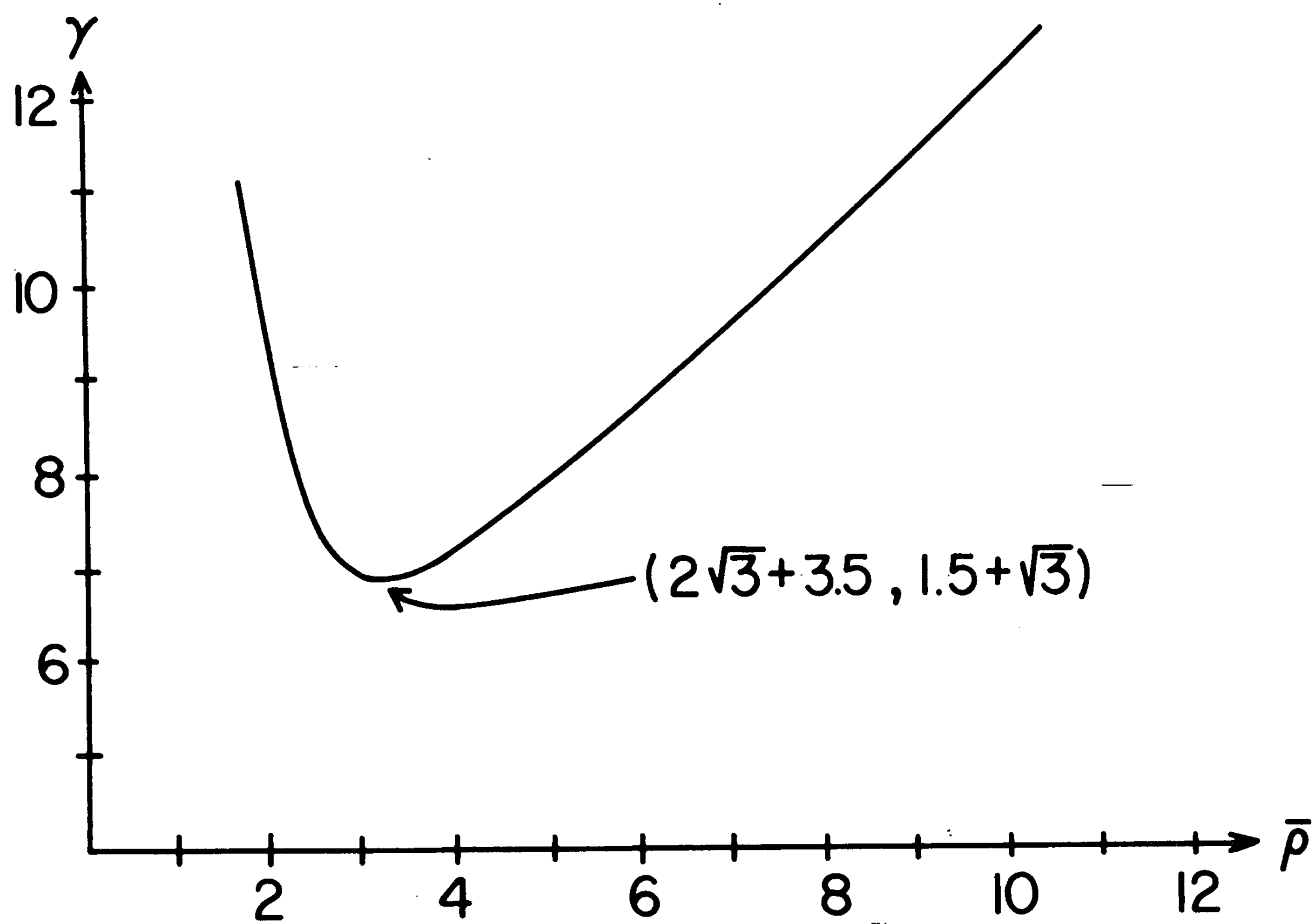

Fig. 5