axioms

Article

New Type of Degenerate Changhee–Genocchi Polynomials

Maryam Salem Alatawi [1] and Waseem Ahmad Khan [2,*]

1 Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia; msoalatawi@ut.edu.sa
2 Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia
* Correspondence: wkhan1@pmu.edu.sa

Abstract: A remarkably large number of polynomials and their extensions have been presented and studied. In this paper, we consider a new type of degenerate Changhee–Genocchi numbers and polynomials which are different from those previously introduced by Kim. We investigate some properties of these numbers and polynomials. We also introduce a higher-order new type of degenerate Changhee–Genocchi numbers and polynomials which can be represented in terms of the degenerate logarithm function. Finally, we derive their summation formulae.

Keywords: degenerate Genocchi polynomials and numbers; degenerate Changhee–Genocchi polynomials; higher-order degenerate Changhee–Genocchi polynomials and numbers; Stirling numbers

MSC: 11B83; 11B73; 05A19

check for updates

Citation: Alatawi, M.S.; Khan, W.A. New Type of Degenerate Changhee–Genocchi Polynomials. *Axioms* **2022**, *11*, 355. https://doi.org/10.3390/axioms11080355

Academic Editor: Clemente Cesarano

Received: 28 June 2022
Accepted: 21 July 2022
Published: 23 July 2022

1. Introduction

Carlitz first proposed the idea of degenerate numbers and polynomials which are associated with Bernoulli and Euler numbers and polynomials (see [1,2]). After Carlitz introduced the degenerate polynomials, many researchers studied the degenerate polynomials related to unique polynomials in diverse regions (see [3]). Recently, Kim et al. [4–6], Sharma et al. [7,8], Muhiuddin et al. [9,10] gave same new and thrilling identities of degenerate special numbers and polynomials which are derived from the non-differential equation. These identities and technical approach are very useful for reading some issues which can be associated with mathematical physics. This paper aims to introduce a new type of degenerate version of the Changhee–Genocchi polynomials and numbers, the so-called new type of degenerate Changhee–Genocchi polynomials and numbers, constructed from the degenerate logarithm function. We derive some explicit expressions and identities for those numbers and polynomials. Additionally, we introduce a new type of higher-order degenerate Changhee–Genocchi polynomials and establish some properties of these polynomials.

The ordinary Euler and Genocchi polynomials are defined by (see [3,11–15])

$$\frac{2}{e^{\tau}+1}e^{\xi\tau}=\sum_{\omega=0}^{\infty}\mathbb{E}_{\omega}(\xi)\frac{\tau^{\omega}}{\omega!}\quad |\tau|<\pi, \tag{1}$$

and

$$\frac{2\tau}{e^{\tau}+1}e^{\xi\tau}=\sum_{\omega=0}^{\infty}\mathbb{G}_{\omega}(\xi)\frac{\tau^{\omega}}{\omega!}\quad |\tau|<\pi, \tag{2}$$

respectively.

In the case when $\xi=0$, $\mathbb{E}_{\omega}=\mathbb{E}_{\omega}(0)$ and $\mathbb{G}_{\omega}=\mathbb{G}_{\omega}(0)$ are called the Euler and Genocchi numbers, respectively.

We note that
$$\mathbb{G}_0(\xi)=0,\quad \mathbb{E}_\omega(\xi)=\frac{\mathbb{G}_{\omega+1}(\xi)}{\omega+1}\ \ (\omega\geq 0).$$

For any non-zero $\lambda\in\mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by (see [14,15])
$$e_\lambda^\xi(\tau)=(1+\lambda\tau)^{\frac{\xi}{\lambda}},\ \ e_\lambda^1(\tau)=(1+\lambda\tau)^{\frac{1}{\lambda}}. \tag{3}$$

By binomial expansion, we obtain
$$e_\lambda^\xi(\tau)=\sum_{\omega=0}^{\infty}(\xi)_{\omega,\lambda}\frac{\tau^\omega}{\omega!}, \tag{4}$$
where $(\xi)_{0,\lambda}=1$, $(\xi)_{\omega,\lambda}=(\xi-\lambda)(\xi-2\lambda)\cdots(\xi-(\omega-1)\lambda)\ \ (\omega\geq 1)$.

Note that
$$\lim_{\lambda\to 0}e_\lambda^\xi(\tau)=\sum_{\omega=0}^{\infty}\xi^\omega\frac{\tau^\omega}{\omega!}=e^{\xi\tau}.$$

In [1], Carlitz considered the degenerate Euler polynomials given by
$$\frac{2}{(1+\lambda\tau)^{\frac{1}{\lambda}}+1}(1+\lambda\tau)^{\frac{\xi}{\lambda}}=\sum_{\omega=0}^{\infty}\mathbb{E}_{\omega,\lambda}(\xi)\frac{\tau^\omega}{\omega!}\ \ (\lambda\in\mathbb{R}). \tag{5}$$

When $\xi=0,\mathbb{E}_{\omega,\lambda}=\mathbb{E}_{\omega,\lambda}(0)$ are called degenerate Euler numbers. The falling factorial sequence is given by
$$(\xi)_0=1,(\xi)_\omega=\xi(\xi-1)...(\xi-\omega+1)\ \ \ (\omega\geq 1). \tag{6}$$

As is well known, the higher-order degenerate Euler polynomials are considered by L. Carlitz as follows (see [2]):
$$\left(\frac{2}{(1+\lambda\tau)^{\frac{1}{\lambda}}+1}\right)^r(1+\lambda\tau)^{\frac{\xi}{\lambda}}=\sum_{\omega=0}^{\infty}\mathbb{E}_{\omega,\lambda}^{(r)}(\xi)\frac{\tau^\omega}{\omega!}. \tag{7}$$
At the point $\xi=0$, $\mathbb{E}_{\omega,\lambda}^{(r)}=\mathbb{E}_{\omega,\lambda}^{(r)}(0)$ are called the higher-order degenerate Euler numbers. Note that $\lim_{\lambda\to 0}\mathbb{E}_{\omega,\lambda}^{(r)}(\xi)=\mathbb{E}_\omega^{(r)}(\xi)\ \ (\omega\geq 0)$.

The degenerate Genocchi polynomials $\mathbb{G}_\omega(\xi;\lambda)$ are defined by (see [16,17])
$$\frac{2\tau}{e_\lambda(\tau)+1}e_\lambda^\xi(\tau)=\sum_{\omega=0}^{\infty}\mathbb{G}_\omega(\xi,\lambda)\frac{\tau^\omega}{\omega!}. \tag{8}$$

In the case when $\xi=0$, $\mathbb{G}_\omega(\lambda)=\mathbb{G}_\omega(0,\lambda)$ are called degenerate Genocchi numbers.

For $\lambda\in\mathbb{R}$, the degenerate logarithm function $\log_\lambda(1+\tau)$, which is the inverse of the degenerate exponential function $e_\lambda(\tau)$, is defined by (see [6])
$$\log_\lambda(1+\tau)=\sum_{\omega=1}^{\infty}\lambda^{\omega-1}(1)_{\omega,1/\lambda}\frac{\tau^\omega}{\omega!}. \tag{9}$$

It is easy to show that
$$\lim_{\lambda\to 0}\log_\lambda(1+\tau)=\sum_{\omega=1}^{\infty}(-1)^{\omega-1}\frac{\tau^\omega}{\omega!}=\log(1+\tau).$$

Note that $e_\lambda(\log_\lambda(1+\tau))=\log_\lambda(e_\lambda(1+\tau))=1+\tau$.

The degenerate Stirling numbers of the first kind are defined by (see [5,6,18])
$$\frac{1}{\nu!}(\log_\lambda(1+\tau))^\nu=\sum_{\omega=\nu}^{\infty}S_{1,\lambda}(\omega,\nu)\frac{\tau^\omega}{\omega!}\ \ (\nu\geq 0). \tag{10}$$

Note here that $\lim_{\lambda\to 0} S_{1,\lambda}(\omega,\nu) = S_1(\omega,\nu)$, where $S_1(\omega,\nu)$ are called the Stirling numbers of the first kind given by

$$\frac{1}{\nu!}(\log(1+\tau))^{\nu} = \sum_{\omega=\nu}^{\infty} S_1(\omega,\nu)\frac{\tau^{\omega}}{\omega!} \ \ (\nu \geq 0).$$

The degenerate Stirling numbers of the second kind (see [19]) are given by

$$\frac{1}{\nu!}(e_{\lambda}(\tau)-1)^{\nu} = \sum_{\omega=\nu}^{\infty} S_{2,\lambda}(\omega,\nu)\frac{\tau^{\omega}}{\omega!} \ \ (\nu \geq 0). \tag{11}$$

It is clear that $\lim_{\lambda\to 0} S_{2,\lambda}(\omega,\nu) = S_2(\omega,\nu)$, where $S_2(\omega,\nu)$ are called the Stirling numbers of the second kind given by

$$\frac{1}{\nu!}(e^{\tau}-1)^{\nu} = \sum_{\omega=\nu}^{\infty} S_2(\omega,\nu)\frac{\tau^{\omega}}{\omega!} \ \ (\nu \geq 0).$$

The Daehee polynomials are defined by (see [13])

$$\frac{\log(1+\tau)}{\tau}(1+\tau)^{\xi} = \sum_{\omega=0}^{\infty} D_{\omega}(\xi)\frac{\tau^{\omega}}{\omega!}. \tag{12}$$

When $\xi = 0$, $D_{\omega} = D_{\omega}(0)$ are called the Daehee numbers.

Recently, Kim et al. [5] introduced the new type degenerate Daehee polynomials defined by

$$\frac{\log_{\lambda}(1+\tau)}{\tau}(1+\tau)^{\xi} = \sum_{\omega=0}^{\infty} D_{\omega,\lambda}(\xi)\frac{\tau^{\omega}}{\omega!}. \tag{13}$$

When $\xi = 0$, $D_{\omega,\lambda} = D_{\omega,\lambda}(0)$ are called the degenerate Daehee numbers.

The Changhee polynomials are defined by (see [4])

$$\frac{2}{2+\tau}(1+\tau)^{\xi} = \sum_{\omega=0}^{\infty} Ch_{\omega}(\xi)\frac{\tau^{\omega}}{\omega!}. \tag{14}$$

When $\xi = 0$, $Ch_{\omega} = Ch_{\omega}(0)$ are called the Changhee numbers.

The higher-order Changhee polynomials are defined by (see [4])

$$\left(\frac{2}{2+\tau}\right)^{k}(1+\tau)^{\xi} = \sum_{\omega=0}^{\infty} Ch_{\omega}^{(k)}(\xi)\frac{\tau^{\omega}}{\omega!}. \tag{15}$$

When $\xi = 0$, $Ch_{\omega}^{(k)} = Ch_{\omega}^{(k)}(0)$ are called the higher-order Changhee numbers.

The Changhee–Genocchi polynomials are defined by the generating function (see [20])

$$\frac{2\log(1+\tau)}{2+\tau}(1+\tau)^{\xi} = \sum_{\omega=0}^{\infty} CG_{\omega}(\xi)\frac{\tau^{\omega}}{\omega!}. \tag{16}$$

When $\xi = 0$, $CG_{\omega} = CG_{\omega}(0)$ are called Changhee–Genocchi numbers.

Recently, Kim et al. [20] introduced the modified Changhee–Genocchi polynomials defined by

$$\frac{2\tau}{2+\tau}(1+\tau)^{\xi} = \sum_{\omega=0}^{\infty} CG_{\omega}^{*}(\xi)\frac{\tau^{\omega}}{\omega!}. \tag{17}$$

When $\xi = 0$, $CG_{\omega}^{*} = CG_{\omega}^{*}(0)$ are called the modified Changhee–Genocchi numbers.

From (1) and (17), we see that

$$\frac{2\tau}{2+\tau}(1+\tau)^{\xi} = \frac{2\tau}{e^{\log(1+\tau)}+1}e^{\xi\log(1+\tau)}$$

$$= \tau \sum_{\nu=0}^{\infty} \mathbb{E}_{\nu}(\xi) \frac{1}{\nu!} (\log(1+\tau))^{\nu}$$

$$= \tau \sum_{\omega=0}^{\infty} \left(\sum_{\nu=0}^{\omega} \mathbb{E}_{\nu}(\xi) S_1(\omega, \nu) \right) \frac{\tau^{\omega}}{\omega!}. \tag{18}$$

Thus, from (17) and (18), we obtain

$$\frac{CG^*_{\omega+1}(\xi)}{\omega+1} = \sum_{\nu=0}^{\omega} \mathbb{E}_{\nu}(\xi) S_1(\omega, \nu) \ \ (\omega \geq 0).$$

The λ-Changhee–Genocchi polynomials are defined by (see [21])

$$\frac{2\log(1+\tau)}{(1+\tau)^{\lambda}+1}(1+\tau)^{\lambda\xi} = \sum_{\omega=0}^{\infty} CG_{\omega,\lambda}(\xi) \frac{\tau^{\omega}}{\omega!}. \tag{19}$$

In the case $\xi = 0$, $CG_{\omega,\lambda} = CG_{\omega,\lambda}(0)$ are called the λ-Changhee–Genocchi numbers.

Motivated by the works of Kim et al. [6,20], we first define a new type of degenerate Changhee–Genocchi numbers and polynomials. We investigate some new properties of these numbers and polynomials and derive some new identities and relations between the new type of degenerate Changhee–Genocchi numbers and polynomials and Stirling numbers of the first and second kind. We also define a new type of higher-order Changhee–Genocchi polynomials and investigate some properties of these polynomials.

2. New Type of Degenerate Changhee–Genocchi Polynomials

In this section, we introduce a new type of degenerate Changhee–Genocchi polynomials and investigate some explicit expressions for degenerate Changhee–Genocchi polynomials and numbers. We begin with the following definition as.

For $\lambda \in \mathbb{R}$, we consider the new type of degenerate Changhee–Genocchi polynomials as defined by means of the following generating function

$$\frac{2\log_{\lambda}(1+\tau)}{2+\tau}(1+\tau)^{\xi} = \sum_{\omega=0}^{\infty} CG_{\omega,\lambda}(\xi) \frac{\tau^{\omega}}{\omega!}. \tag{20}$$

At the point $\xi = 0$, $CG_{\omega,\lambda} = CG_{\omega,\lambda}(0)$ are called the new type of degenerate Changhee–Genocchi numbers.

It is clear that

$$\sum_{\omega=0}^{\infty} \lim_{\lambda \to 0} CG_{\omega,\lambda}(\xi) \frac{\tau^{\omega}}{\omega!} = \lim_{\lambda \to 0} \frac{2\log_{\lambda}(1+\tau)}{2+\tau}(1+\tau)^{\xi}$$

$$= \frac{2\log(1+\tau)}{2+\tau}(1+\tau)^{\xi} = \sum_{\omega=0}^{\infty} CG_{\omega}(\xi) \frac{\tau^{\omega}}{\omega!}, \tag{21}$$

where $CG_{\omega}(\xi)$ are called the Changhee–Genocchi polynomials (see Equation (1)).

Theorem 1. *For $\omega \geq 0$, we have*

$$CG_{\omega,\lambda}(\xi) = \sum_{\nu=0}^{\omega} \mathbb{G}_{\nu}(\xi, \lambda) S_{1,\lambda}(\omega, \nu).$$

Proof. Using (8), (10) and (20), we note that

$$\sum_{\omega=0}^{\infty} CG_{\omega,\lambda}(\xi) \frac{\tau^{\omega}}{\omega!} = \frac{2\log_{\lambda}(1+\tau)}{e_{\lambda}(\log_{\lambda}(1+\tau))+1} e_{\lambda}^{\xi \log_{\lambda}(1+\tau)}$$

$$= \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu}(\xi,\lambda) \frac{1}{\nu!} \left(\log_{\lambda}(1+\tau)\right)^{\nu}$$

$$= \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu}(\xi,\lambda) \sum_{\omega=\nu}^{\infty} S_{1,\lambda}(\omega,\nu) \frac{\tau^{\omega}}{\omega!}$$

$$= \sum_{\omega=0}^{\infty} \left(\sum_{\nu=0}^{\omega} \mathbb{G}_{\nu}(\xi,\lambda) S_{1,\lambda}(\omega,\nu) \right) \frac{\tau^{\omega}}{\omega!}. \tag{22}$$

Therefore, by (20) and (22), we obtain the result. □

Theorem 2. *For $\omega \geq 0$, we have*

$$CG_{\omega,\lambda}(\xi) = \sum_{\sigma=0}^{\omega} \sum_{\nu=0}^{\sigma} \binom{\omega}{\sigma} CG_{\omega-\sigma,\lambda}(\xi)_{\nu,\lambda} S_{1,\lambda}(\sigma,\nu).$$

Proof. By using (4), (10) and (20), we see that

$$\sum_{\omega=0}^{\infty} CG_{\omega,\lambda}(\xi) \frac{\tau^{\omega}}{\omega!} = \frac{2\log_{\lambda}(1+\tau)}{2+t} e_{\lambda}^{\xi \log_{\lambda}(1+\tau)} \tag{23}$$

$$= \sum_{\omega=0}^{\infty} CG_{\omega,\lambda} \frac{\tau^{\omega}}{\omega!} \sum_{\nu=0}^{\infty} (\xi)_{\nu,\lambda} \frac{\left(\log_{\lambda}(1+\tau)\right)^{\nu}}{\nu!}$$

$$= \sum_{\omega=0}^{\infty} GG_{\omega,\lambda} \frac{\tau^{\omega}}{\omega!} \sum_{\sigma=0}^{\infty} \sum_{\omega=0}^{\sigma} (\xi)_{\sigma,\lambda} S_{1,\lambda}(\sigma,\nu) \frac{\tau^{\sigma}}{\sigma!}$$

$$= \sum_{\omega=0}^{\infty} \left(\sum_{\sigma=0}^{\omega} \sum_{\nu=0}^{\sigma} \binom{\omega}{\sigma} CG_{\omega-\sigma,\lambda}(\xi)_{\nu,\lambda} S_{1,\lambda}(\sigma,\nu) \right) \frac{\tau^{\omega}}{\omega!}. \tag{24}$$

Therefore, by (20) and (24), we obtain the result. □

Theorem 3. *For $\omega \geq 0$, we have*

$$\mathbb{G}_{\omega}(\xi,\lambda) = \sum_{\nu=0}^{\omega} CG_{\nu,\lambda}(\xi) S_{2,\lambda}(\omega,\nu).$$

Proof. By replacing τ by $e_{\lambda}(\tau) - 1$ in (20) and using (8) and (11), we obtain

$$\sum_{\nu=0}^{\infty} CG_{\nu,\lambda}(\xi) \frac{1}{\nu!} (e_{\lambda}(\tau) - 1)^{\nu} = \frac{2\tau}{e_{\lambda}(\tau)+1} e_{\lambda}^{\xi}(\tau)$$

$$= \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega}(\xi,\lambda) \frac{\tau^{\omega}}{\omega!}. \tag{25}$$

On the other hand,

$$\sum_{\nu=0}^{\infty} CG_{\nu,\lambda}(\xi) \frac{1}{\nu!} (e_{\lambda}(\tau) - 1)^{\tau} = \sum_{\nu=0}^{\infty} CG_{\nu,\lambda}(\xi) \sum_{\nu=\omega}^{\infty} S_{2,\lambda}(\omega,\nu) \frac{\tau^{\omega}}{\omega!}$$

$$= \sum_{\omega=0}^{\infty} \left(\sum_{\nu=0}^{\omega} CG_{\nu,\lambda}(\xi) S_{2,\lambda}(\omega,\nu) \right) \frac{\tau^{\omega}}{\omega!}. \tag{26}$$

Therefore, by (25) and (26), we obtain the required result. □

Theorem 4. *For $\omega \geq 0$, we have*

$$CG_{\omega,\lambda}(\xi) = \sum_{\nu=0}^{\omega} \mathbb{G}_{\nu}(\xi,\lambda) S_{1,\lambda}(\omega,\nu).$$

Proof. Replacing τ by $\log_{\lambda}(1+\tau)$ in (8) and applying (10), we obtain

$$\frac{2\log_{\lambda}(1+\tau)}{2+\tau}(1+\tau)^{\xi} = \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu}(\xi,\lambda) \frac{1}{\nu!} \left(\log_{\lambda}(1+\tau)\right)^{\nu}$$

$$= \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu}(\xi,\lambda) \sum_{\omega=\nu}^{\infty} S_{1,\lambda}(\omega,\nu) \frac{\tau^{\omega}}{\omega!}$$

$$= \sum_{\omega=0}^{\infty} \left(\sum_{\nu=0}^{\omega} \mathbb{G}_{\nu}(\xi,\lambda) S_{1,\lambda}(\omega,\nu) \right) \frac{\tau^{\omega}}{\omega!}. \tag{27}$$

By using (20) and (27), we acquire the desired result. □

Theorem 5. *For $\omega \geq 0$, we have*

$$CG_{\omega,\lambda}(\xi) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} CG^{*}_{\omega-\nu}(\xi) D_{\nu,\lambda}.$$

Proof. From (13), (17) and (20), we note that

$$\sum_{\omega=0}^{\infty} CG_{\omega,\lambda}(\xi) \frac{\tau^{\omega}}{\omega!} = \frac{2\log_{\lambda}(1+\tau)}{2+\tau}(1+\tau)^{\xi}$$

$$= \frac{2\tau}{2+\tau}(1+\tau)^{\xi} \frac{\log_{\lambda}(1+\tau)}{\tau}$$

$$= \sum_{\omega=0}^{\infty} CG^{*}_{\omega}(\xi) \frac{\tau^{\omega}}{\omega!} \sum_{\nu=0}^{\infty} D_{\nu,\lambda} \frac{\tau^{\nu}}{\nu!}$$

$$= \sum_{\omega=0}^{\infty} \left(\sum_{\nu=0}^{\omega} \binom{\omega}{\nu} CG^{*}_{\omega-\nu}(\xi) D_{\nu,\lambda} \right) \frac{\tau^{\omega}}{\omega!}. \tag{28}$$

Therefore, by (20) and (28), we obtain the result. □

Theorem 6. *For $\omega \geq 0$, we have*

$$\frac{CG_{\omega+1,\lambda}(\xi)}{\omega+1} = \sum_{\sigma=0}^{\omega} \sum_{\nu=0}^{\sigma} \binom{\omega}{\sigma} \mathbb{E}_{\nu}(\xi) S_{1}(\sigma,\nu) D_{\omega-\sigma,\lambda}.$$

Proof. From (1), (13) and (20), we note that

$$\sum_{\omega=1}^{\infty} CG_{\omega,\lambda}(\xi) \frac{\tau^{\omega}}{\omega!} = \frac{2\log_{\lambda}(1+\tau)}{2+\tau}(1+\tau)^{\xi}$$

$$= \frac{2\tau}{e^{\log(1+\tau)}+1} e^{\xi\log(1+\tau)} \frac{\log_{\lambda}(1+\tau)}{\tau}$$

$$= \tau \sum_{\nu=0}^{\infty} \mathbb{E}_{\nu}(\xi) \frac{(\log(1+\tau))^{\nu}}{\nu!} \sum_{\omega=0}^{\infty} D_{\omega,\lambda} \frac{\tau^{\omega}}{\omega!}$$

$$= \tau \sum_{\sigma=0}^{\infty} \sum_{\nu=0}^{\sigma} \mathbb{E}_{\nu}(\xi) S_{1}(\sigma,\nu) \frac{\tau^{\sigma}}{\sigma!} \sum_{\omega=0}^{\infty} D_{\omega,\lambda} \frac{\tau^{\omega}}{\omega!}$$

$$= \sum_{\omega=1}^{\infty} \left(\sum_{\sigma=0}^{\omega} \sum_{\nu=0}^{\sigma} \binom{\omega}{\sigma} \mathbb{E}_{\nu}(\xi) S_1(\sigma,\nu) D_{\omega-\sigma,\lambda} \right) \frac{\tau^{\omega}}{\omega!}. \tag{29}$$

By (20) and (29), we obtain the result. □

Theorem 7. *For $\omega \geq 0$, we have*

$$CG_{\omega,\lambda}(\xi) = \sum_{\sigma=0}^{\omega} \sum_{\nu=0}^{\sigma} \binom{\omega}{\sigma} (\nu+1)(\xi)_{\nu,\lambda} \frac{S_{1,\lambda}(\sigma+1,\nu+1)}{\sigma+1} CG^{*}_{\omega-\sigma}.$$

Proof. By using (10), (17) and (20), we see that

$$\begin{aligned}
&\frac{2\log_{\lambda}(1+\tau)}{2+\tau} e_{\lambda}^{\xi \log_{\lambda}(1+\tau)} \\
&= \frac{2\log_{\lambda}(1+\tau)}{2+\tau} \sum_{\nu=0}^{\infty} (\xi)_{\nu,\lambda} \frac{(\log_{\lambda}(1+\tau))^{\nu}}{\nu!} \\
&= \frac{2\tau}{2+\tau} \frac{1}{\tau} \sum_{\nu=0}^{\infty} (\nu+1)(\xi)_{\nu,\lambda} \frac{(\log_{\lambda}(1+\tau))^{\nu+1}}{(\nu+1)!} \\
&= \sum_{\omega=0}^{\infty} CG^{*}_{\omega} \frac{\tau^{\omega}}{\omega!} \frac{1}{\tau} \sum_{\nu=0}^{\infty} (\nu+1)(\xi)_{\nu,\lambda} \sum_{\sigma=\nu+1}^{\infty} S_{1,\lambda}(\sigma,\nu+1) \frac{\tau^{\sigma}}{\sigma!} \\
&= \sum_{\omega=0}^{\infty} CG^{*}_{\omega} \frac{\tau^{\omega}}{\omega!} \sum_{\sigma=0}^{\infty} \sum_{\nu=0}^{\sigma} (\nu+1)(\xi)_{\nu,\lambda} \frac{S_{1,\lambda}(\sigma+1,\nu+1)}{\sigma+1} \frac{\tau^{\sigma}}{\sigma!} \\
&= \sum_{\omega=0}^{\infty} \left(\sum_{\sigma=0}^{\omega} \sum_{\nu=0}^{\sigma} \binom{\omega}{\sigma} (\nu+1)(\xi)_{\nu,\lambda} \frac{S_{1,\lambda}(\sigma+1,\nu+1)}{\sigma+1} CG^{*}_{\omega-\nu} \right) \frac{\tau^{\omega}}{\omega!}.
\end{aligned} \tag{30}$$

Therefore, by (20) and (30), we obtain the result. □

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, the following identity is (see [21])

$$\sum_{a=0}^{d-1} (-1)^{a} (1+\tau)^{a} = \frac{1+(1+\tau)^{d}}{2+\tau}. \tag{31}$$

Theorem 8. *For $d \in \mathbb{N}$ with $d \equiv 1$ (mod 2), we have the following identity*

$$CG_{\omega,\lambda}(\xi) = \sum_{a=0}^{d-1} (-1)^{a} CG_{\omega,\lambda}\left(\frac{a+\xi}{d} \right).$$

Proof. Thus, for such $d \equiv 1 \pmod 2$, from (19), (20) and (31), we see that

$$\begin{aligned}
\sum_{\omega=0}^{\infty} CG_{\omega,\lambda}(\xi) \frac{\tau^{\omega}}{\omega!} &= \frac{2\log_{\lambda}(1+\tau)}{2+\tau} (1+\tau)^{\xi} \\
&= \sum_{a=0}^{d-1} (-1)^{a} \frac{2\log_{\lambda}(1+\tau)}{(1+\tau)^{d}+1} (1+\tau)^{d(\frac{a+\xi}{d})} \\
&= \sum_{a=0}^{d-1} (-1)^{a} \sum_{\omega=0}^{\infty} CG_{\omega,\lambda}\left(\frac{a+\xi}{d} \right) \frac{\tau^{\omega}}{\omega!} \\
&= \sum_{\omega=0}^{\infty} \left(\sum_{a=0}^{d-1} (-1)^{a} CG_{\omega,\lambda}\left(\frac{a+\xi}{d} \right) \right) \frac{\tau^{\omega}}{\omega!}.
\end{aligned} \tag{32}$$

By (20) and (32), we obtain the result. □

Theorem 9. *For $d \in \mathbb{N}$ with $d \equiv 1$ (mod 2), we have the following identity*

$$2\sum_{a=0}^{d-1}(-1)^a D_{\omega,\lambda}(a) = \frac{CG_{\omega+1,\lambda}}{\omega+1} + \frac{CG_{\omega+1,\lambda}(d)}{\omega+1}.$$

Proof. By using (13), (20) and (31), we see that

$$\begin{aligned}
2\log_\lambda(1+\tau)\sum_{a=0}^{d-1}(-1)^a(1+\tau)^a &= \frac{2\log_\lambda(1+\tau)}{2+\tau} + \frac{2\log_\lambda(1+\tau)}{2+\tau}(1+\tau)^d \\
&= \frac{2\log_\lambda(1+\tau)}{\tau}\left(\sum_{a=0}^{d-1}(-1)^a(1+\tau)^a\right) \\
&= \sum_{\omega=0}^{\infty} CG_{\omega,\lambda}\frac{\tau^{\omega-1}}{\omega!} + \sum_{\omega=0}^{\infty} CG_{\omega,\lambda}(d)\frac{\tau^{\omega-1}}{\omega!} \\
&= \left(2\sum_{a=0}^{d-1}(-1)^a D_{\omega,\lambda}(a)\right)\frac{\tau^\omega}{\omega!} \\
&= \sum_{\omega=0}^{\infty}\left(\frac{CG_{\omega+1,\lambda}}{\omega+1} + \frac{CG_{\omega+1,\lambda}(d)}{\omega+1}\right)\frac{\tau^\omega}{\omega!}. \qquad (33)
\end{aligned}$$

By comparing the coefficients of τ^ω on both sides, we obtain the result. □

Theorem 10. *For $\omega \geq 1$, we have*

$$\omega CG_{\omega-1,\lambda} + 2CG_{\omega,\lambda} = 2(\lambda)^{\omega-1}(1)_{\omega,1/\lambda},$$

with $CG_{0,\lambda} = 0$.

Proof. From (20), we note that

$$\begin{aligned}
2\log_\lambda(1+\tau) &= \sum_{\omega=0}^{\infty} CG_{\omega,\lambda}\frac{\tau^\omega}{\omega!}(\tau+2) \\
&= \sum_{\omega=1}^{\infty} CG_{\omega,\lambda}\frac{\tau^{\omega+1}}{\omega!} + 2\sum_{\omega=0}^{\infty} CG_{\omega,\lambda}\frac{\tau^\omega}{\omega!} \\
&= \sum_{\omega=2}^{\infty} \omega CG_{\omega-1,\lambda}\frac{\tau^\omega}{\omega!} + 2\sum_{\omega=0}^{\infty} CG_{\omega,\lambda}\frac{\tau^\omega}{\omega!} \\
&= 2CG_{1,\lambda}(\tau) + \sum_{\omega=2}^{\infty}(\omega CG_{\omega-1,\lambda} + 2CG_{\omega,\lambda})\frac{\tau^\omega}{\omega!}. \qquad (34)
\end{aligned}$$

On the other hand,

$$2\log_\lambda(1+\tau) = 2\sum_{\omega=1}^{\infty}(\lambda)^{\omega-1}(1)_{\omega,1/\lambda}\frac{\tau^\omega}{\omega!}. \qquad (35)$$

Therefore, by (34) and (35), we obtain the result. □

We now consider a new type of higher-order degenerate Changhee–Genocchi polynomials by the following definition.

Let $r \in \mathbb{N}$, and we consider that a new type of higher-order degenerate Changhee–Genocchi polynomials is given by the following generating function

$$\left(\frac{2\log_\lambda(1+\tau)}{2+\tau}\right)^r(1+\tau)^\xi = \sum_{\omega=0}^{\infty} CG^{(r)}_{\omega,\lambda}(\xi)\frac{\tau^\omega}{\omega!}. \qquad (36)$$

When $\xi = 0, CG^{(r)}_{\omega,\lambda} = CG^{(r)}_{\omega,\lambda}(0)$ are called the new type of higher-order degenerate Changhee–Genocchi numbers.

It is worth noting that

$$\lim_{\lambda\to 0} CG^{(r)}_{\omega,\lambda}(\xi) = CG^{(r)}_{\omega}(\xi),$$

are called higher-order Changhee–Genocchi polynomials.

Theorem 11. *For $\omega \geq 0$, we have*

$$CG^{(r+1)}_{\omega,\lambda}(\xi) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} CG_{\nu,\lambda} CG^{(r)}_{\omega-\nu,\lambda}(\xi).$$

Proof. From (20) and (36), we note that

$$\begin{aligned}
\frac{2\log_\lambda(1+\tau)}{2+\tau}\sum_{\omega=0}^{\infty} CG^{(r)}_{\omega,\lambda}(\xi)\frac{\tau^\omega}{\omega!} &= \frac{2\log_\lambda(1+\tau)}{2+\tau}\left(\frac{2\log_\lambda(1+\tau)}{2+\tau}\right)^r (1+\tau)^\xi \\
\left(\sum_{\nu=0}^{\infty} CG_{\nu,\lambda}\frac{\tau^\omega}{\omega!}\right)\left(\sum_{\omega=0}^{\infty} CG^{(r)}_{\omega,\lambda}(\xi)\frac{\tau^\omega}{\omega!}\right) &= \sum_{\omega=0}^{\infty} CG^{(r+1)}_{\omega,\lambda}(\xi)\frac{\tau^\omega}{\omega!} \\
\sum_{\omega=0}^{\infty}\left(\sum_{\nu=0}^{\omega}\binom{\omega}{\nu} CG_{\nu,\lambda} CG^{(r)}_{\omega-\nu,\lambda}(\xi)\right)\frac{\tau^\omega}{\omega!} &= \sum_{\omega=0}^{\infty} CG^{(r+1)}_{\omega,\lambda}(\xi)\frac{\tau^\omega}{\omega!}. \qquad (37)
\end{aligned}$$

Comparing the coefficients of τ in above equation, we obtain the result. □

Theorem 12. *For $r, k \in \mathbb{N}$, with $r > k$, we have*

$$CG^{(r)}_{\omega,\lambda}(\xi) = \sum_{\sigma=0}^{\omega}\binom{\omega}{\sigma} CG^{(r-k)}_{\sigma,\lambda} CG^{(k)}_{\omega-\sigma,\lambda}(\xi) \ \ (\omega \geq 0).$$

Proof. By (36), we see that

$$\begin{aligned}
&\left(\frac{2\log_\lambda(1+\tau)}{2+\tau}\right)^r (1+\tau)^\xi \\
&= \left(\frac{2\log_\lambda(1+\tau)}{2+\tau}\right)^{r-k}\left(\frac{2\log_\lambda(1+\tau)}{2+\tau}\right)^{k} (1+\tau)^\xi \\
&= \left(\sum_{\sigma=0}^{\infty} CG^{(r-k)}_{\sigma,\lambda}\frac{\tau^\sigma}{\sigma!}\right)\left(\sum_{\omega=0}^{\infty} CG^{(k)}_{\omega,\lambda}(\xi)\frac{\tau^\omega}{\omega!}\right) \\
&= \sum_{\omega=0}^{\infty}\left(\sum_{\sigma=0}^{\omega}\binom{\omega}{\sigma} CG^{(r-k)}_{\sigma,\lambda} CG^{(k)}_{\omega-\sigma,\lambda}(\xi)\right)\frac{\tau^\omega}{\omega!}. \qquad (38)
\end{aligned}$$

Therefore, by (36) and (38), we obtain the result. □

Theorem 13. *For $\omega \geq 0$, we have*

$$CG^{(r)}_{\omega,\lambda}(\xi+\eta) = \sum_{\nu=0}^{\omega}\binom{\omega}{\nu} CG^{(r)}_{\omega-\nu,\lambda}(\xi)(\eta)_\nu.$$

Proof. Now, we observe that

$$\sum_{\omega=0}^{\infty} CG^{(r)}_{\omega,\lambda}(\xi+\eta)\frac{\tau^\omega}{\omega!} = \left(\frac{2\log_\lambda(1+\tau)}{2+\tau}\right)^r (1+\tau)^{\xi+\eta}$$

$$= \left(\sum_{\sigma=0}^{\infty} CG_{\sigma,\lambda}^{(r)}(\xi) \frac{\tau^{\sigma}}{\sigma!} \right) \left(\sum_{\nu=0}^{\infty} (\eta)_{\nu} \frac{\tau^{\nu}}{\nu!} \right)$$

$$= \sum_{\omega=0}^{\infty} \left(\sum_{\nu=0}^{\omega} \binom{\omega}{\nu} CG_{\omega-\nu,\lambda}^{(r)}(\xi)(\eta)_{\nu} \right) \frac{\tau^{\omega}}{\omega!}. \quad (39)$$

Equating the coefficients of τ^{ω} on both sides, we obtain the result. □

Theorem 14. *For $\omega \geq 0$, we have*

$$CG_{\omega,\lambda}^{(r)} = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} CG_{\nu}^{(*,r)} D_{\omega-\nu,\lambda}^{(r)}.$$

Proof. By making use of (36), we have

$$\left(\frac{2\log_{\lambda}(1+\tau)}{2+\tau} \right)^{r} = \left(\frac{2t}{2+\tau} \right)^{r} \left(\frac{\log_{\lambda}(1+\tau)}{\tau} \right)^{r}$$

$$= \left(\sum_{\nu=0}^{\infty} CG_{\nu}^{(*,r)} \frac{\tau^{\nu}}{\nu!} \right) \left(\sum_{\omega=0}^{\infty} D_{\omega,\lambda}^{(r)} \frac{\tau^{\omega}}{\omega!} \right)$$

$$= \sum_{\omega=0}^{\infty} \left(\sum_{\nu=0}^{\omega} \binom{\omega}{\nu} CG_{\nu}^{(*,r)} D_{\omega-\nu,\lambda}^{(r)} \right) \frac{\tau^{\omega}}{\omega!}. \quad (40)$$

Therefore, by (36) and (40), we obtain the result. □

3. Conclusions

Motivated by the research work of [6,20,21], we defined a new type of degenerating Changhee–Genocchi polynomials which turned out to be classical ones in the special cases. We also derived their explicit expressions and some identities involving them. Later, we introduced the higher-order degenerate Changhee–Genocchi polynomials and deduced their explicit expressions and some identities by making use of the generating functions method, analytical means and power series expansion.

Author Contributions: Conceptualization, M.S.A. and W.A.K.; methodology, W.A.K.; software, M.S.A.; validation, M.S.A.; formal analysis, W.A.K.; investigation, M.S.A.; resources, W.A.K.; data curation, M.S.A.; writing—original draft preparation, W.A.K; writing—review and editing, W.A.K.; visualization, M.S.A.; supervision, W.A.K. All authors have read and agreed to the published version of the manuscript.

Funding: There is no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors wish to express their appreciation to the reviewers for their helpful suggestions which greatly improved the presentation of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Carlitz, L. Degenerate Stirling Bernoulli and Eulerian numbers. *Util.Math.* **1979**, 15, 51–88.
2. Carlitz, L. A degenerate Staud-Clausen theorem. *Arch. Math.* **1956**, 7, 28–33. [CrossRef]
3. Khan, W.A.; Muhiuddin, G.; Muhyi, A.; Al-Kadi, D. Analytical properties of type 2 degenerate poly-Bernoulli polynomials associated with their applications. *Adv. Diff. Equ.* **2021**, *2021*, 420. [CrossRef]
4. Kim, D.S.; Kim, T. Higher-order Changhee numbers and polynomials. *Adv. Studies Theor. Phys.* **2014**, *8*, 365–373. [CrossRef]

5. Kim, T.; Kim, D.S.; Kim, H.-Y.; Kwon, J. Some results on degenerate Daehee and Bernoulli numbers and polynomials. *Adv. Diff. Eq.* **2020**, *2020*, 311. [CrossRef]
6. Kim, D.S.; Kim, T. A note on a new type of degenerate Bernoulli numbers. *Russ. J. Math. Phys.* **2020**, *27*, 227–235. [CrossRef]
7. Sharma, S.K.; Khan, W.A.; Araci, S.; Ahmed, S.S. New type of degenerate Daehee polynomials of the second kind. *Adv. Differ. Equ.* **2020**, *2020*, 428. [CrossRef]
8. Sharma, S.K.; Khan, W.A.; Araci, S.; Ahmed, S.S. New construction of type 2 of degenerate central Fubini polynomials with their certain properties. *Adv. Differ. Equ.* **2020**, *2020*, 587. [CrossRef]
9. Muhiuddin, G.; Khan, W.A.; Muhyi, A.; Al-Kadi, D. Some results on type 2 degenerate poly-Fubini polynomials and numbers. *Comput. Model. Eng. Sci.* **2021**, *29*, 1051–1073. [CrossRef]
10. Muhiuddin, G.; Khan, W.A.; Al-Kadi, D. Construction on the degenerate poly-Frobenius-Euler polynomials of complex variable. *J. Function Spaces* **2021**, *2021*, 3115424. [CrossRef]
11. Khan, W.A.; Acikgoz, M.; Duran, U. Note on the type 2 degenerate multi-poly-Euler polynomials. *Symmetry* **2020**, *12*, 1691. [CrossRef]
12. Khan, W.A.; Haroon, H. Some symmetric identities for the generalized Bernoulli, Euler and Genocchi polynomials associated with Hermite polynomials. *Springer Plus* **2016**, *5*, 1920. [CrossRef] [PubMed]
13. Khan, W.A.; Nisar, K.S.; Duran, U.; Acikgoz, M.; Araci, S. Multifarious implicit summation formulae of Hermite-based poly-Daehee polynomials. *Appl. Math. Inf. Sci.* **2018**, *12*, 305–310. [CrossRef]
14. Khan, W.A.; Muhyi, A.; Ali, R.; Alzobydi, K.A.H.; Singh, M.; Agarwal, P. A new family of degenerate poly-Bernoulli polynomials of the second kind with its certain related properties. *AIMS Math.* **2021**, *6*, 12680–12697. [CrossRef]
15. Khan, W.A.; Ali, R.; Alzobydi, K.A.H.; Ahmed, N. A new family of degenerate poly-Genocchi polynomials with its certain properties. *J. Funct. Spaces* **2021**, *2021*, 6660517. [CrossRef]
16. Lim, D. Some identities of degenerate Genocchi polynomials. *Bull. Korean Math. Soc.* **2016**, *53*, 569–579. [CrossRef]
17. Sharma, S.K. A note on degenerate poly-Genocchi polynomials. *Int. J. Adv. Appl. Sci.* **2020**, *7*, 1–5. [CrossRef]
18. Kwon, H.-I.; Kim, T.; Park, J.W. A note on degenerate Changhee-Genocchi polynomials and numbers. *Glob. J. Pure Appl. Math.* **2016**, *12*, 4057–4064.
19. Kim, T. A note on degenerate Stirling numbers of the second kind. *Proc. Jangjeon Math. Soc.* **2018**, *21*, 589–598.
20. Kim, B.M.; Jang, L.-C.; Kim, W.; Kwon, H.I. Degenerate Changhee-Genocchi numbers and polynomials. *J. Ineq. Appl.* **2017**, *2017*, 294. [CrossRef]
21. Kim, B.M.; Jeong, J.; Rim, S.H. Some explicit identities on Changhee-Genocchi polynomials and numbers. *Adv. Diff. Equ.* **2016**, *2016*, 202. [CrossRef]