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Newman-Penrose Approach to Twisting Degenerate Metrics*

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Abstract. The well known method of NEWMAN and PENROSE is used to find solutions of the Einstein empty space field equations, which are algebraically special and where the degenerate principal null vectors are not hypersurface orthogonal. As is to be expected the method systematically yields the results obtained by KERR. An explanation is given of the complex coordinate transformation technique of generating new metrics from Schwarzschild's; also a generalisation of Kerr and Schild type metrics is investigated.

1. Introduction

This paper shows how to find solutions of Einstein's field equations in empty space which contain shear-free, diverging, and twisting geodesic rays.

Such solutions were first dealt with in a general manner by KERR [1], although he only summarised his results. As is well known, he found an explicit solution which is a rotating generalisation of the Schwarzschild solution. Another explicit solution was found by NEWMAN, UNTI, and TAMBURINO [2], also a generalisation of the Schwarzschild case, though apparently of less physical interest. KERR and SCHILD [3, 4] found a whole class of solutions, including the above Kerr rotating metric, all having a metric tensor of the form:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\,H l_\mu l_\nu\,,$$

where $\eta_{\mu\nu}$ is the metric tensor of flat space and l_μ is a null vector. Recently ROBINSON, ROBINSON, and ZUND [5] have given more details of KERR's approach, and, as well as making several important simplifications, have found a large class of explicit solutions.

In the following we will repeat the above work using the Newman and Penrose approach [6]. Just as NEWMAN and TAMBURINO [7] showed how the Robinson-Trautman solutions [8] (the twist-free case) could be found in a straightforward way by this method, we can do the same for the above class — again with not too much difficulty.

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In addition there are two sections which are believed to contain essentially new results: (a) we explain why the complex coordinate "trick" of NEWMAN et al. [9, 10] works and (b) we generalise KERR and SCHILD's work, investigating metric tensors of the form:

$$h_{\mu\nu} = g_{\mu\nu} + 2\,H l_\mu l_\nu ,$$

both $h_{\mu\nu}$ and $g_{\mu\nu}$ being solutions of the Einstein vacuum field equations. The latter stems from a paper of EDELEN's [11].

2. Notation

Following NEWMAN and PENROSE [6], we introduce a null tetrad $\underset{m}{z}^{\mu} = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, where Latin (letters from the middle of the alphabet) and Greek indices run from 1 to 4, giving tetrad and tensor components, respectively. The metric tensor is related to its tetrad components as follows:

$$\begin{aligned} g^{\mu\nu} &= 2\,l^{(\mu}n^{\nu)} - 2\,m^{(\mu}\bar{m}^{\nu)} \\ &= \eta^{mn}\,\underset{m}{z}^{\mu}\,\underset{n}{z}^{\nu} \\ \eta_{mn} &= g_{\mu\nu}\,\underset{m}{z}^{\mu}\,\underset{n}{z}^{\nu} \\ \text{and } \eta_{mn}\,\eta^{np} &= \delta^p_m\,. \end{aligned}$$

Tetrad indices are raised and lowered by η^{mn} and η_{mn}. Ricci rotation coefficients are defined by $\gamma_{mnp} = z_{m\,\mu;\nu}\,\underset{n}{z}^{\mu}\,\underset{p}{z}^{\nu}$, and the twelve complex Newman-Penrose spin coefficients are expressed in terms of these as follows:

$$\begin{array}{lll} \varkappa = \gamma_{131} & \pi = -\gamma_{241} & \varepsilon = \tfrac{1}{2}(\gamma_{121} - \gamma_{341}) \\ \varrho = \gamma_{134} & \lambda = -\gamma_{244} & \alpha = \tfrac{1}{2}(\gamma_{124} - \gamma_{344}) \\ \sigma = \gamma_{133} & \mu = -\gamma_{243} & \beta = \tfrac{1}{2}(\gamma_{123} - \gamma_{343}) \\ \tau = \gamma_{132} & \nu = -\gamma_{242} & \gamma = \tfrac{1}{2}(\gamma_{122} - \gamma_{342})\,. \end{array} \tag{2.1}$$

Tetrad components of the Riemann tensor are expressed in terms of the five complex scalars:

$$\begin{aligned} \Psi_0 &= -R_{\mu\nu\varrho\sigma} l^\mu m^\nu l^\varrho m^\sigma \\ \Psi_1 &= -R_{\mu\nu\varrho\sigma} l^\mu n^\nu l^\varrho m^\sigma \\ \Psi_2 &= -\tfrac{1}{2} R_{\mu\nu\varrho\sigma}(l^\mu n^\nu l^\varrho n^\sigma - l^\mu n^\nu m^\varrho \bar{m}^\sigma) \\ \Psi_3 &= -R_{\mu\nu\varrho\sigma} l^\mu n^\nu \bar{m}^\varrho n^\sigma \\ \Psi_4 &= -R_{\mu\nu\varrho\sigma} n^\mu \bar{m}^\nu n^\varrho \bar{m}^\sigma\,. \end{aligned} \tag{2.2}$$

The usual operators are employed to denote intrinsic derivatives:

$$\begin{array}{ll} D\phi = \phi_{;\mu} l^\mu & \Delta\phi = \phi_{;\mu} n^\mu \\ \delta\phi = \phi_{;\mu} m^\mu & \bar{\delta}\phi = \phi_{;\mu} \bar{m}^\mu\,. \end{array} \tag{2.3}$$

3. Procedure

We may choose the real null vector l^μ so that it is tangent to the congruence of geodesic rays. This is equivalent to saying that $\varkappa = 0$ and $\Psi_0 = 0$; by the latter relation we mean that l^μ is a principal null vector of the Riemann tensor. The fact that the rays are shearfree, diverging and twisting means respectively $\sigma = 0$, $\varrho \neq 0$, $\varrho \neq \bar{\varrho}$ (although the twist-free, or hypersurface orthogonal case [7, 8] where $\varrho = \bar{\varrho}$ is automatically included as a sub-case). By the Goldberg-Sachs theorem [6] we must also have $\Psi_1 = 0$ and hence in the sense of the Petrov-Penrose classification our solutions are algebraically special (type II if $\Psi_2 \neq 0$, unless $2\,\Psi_3^2 = 3\,\Psi_2\Psi_4$ i.e. type D; type III if $\Psi_2 = 0$, $\Psi_3 \neq 0$; or type N if $\Psi_2 = \Psi_3 = 0$, $\Psi_4 \neq 0$).

A coordinate system (x^μ) is introduced, adapted to the congruence, with $x^2 = r$ taken as an affine parameter ($r_{,\mu} l^\mu = 1$) whereas x^a ($a = 1,3,4$) are constant along the rays ($x^a_{\ ,\mu} l^\mu = 0$). (Latin letters from the beginning of the alphabet will always range over 1, 3 and 4.) The tetrad vectors then have the following form:

$$
\begin{aligned}
l^\mu &= \delta^\mu_2 \\
n^\mu &= U\,\delta^\mu_2 + X^a\,\delta^\mu_a \\
m^\mu &= \omega\,\delta^\mu_2 + \xi^a\,\delta^\mu_a\,.
\end{aligned}
$$

And thus the operators (2.3) become:

$$
\begin{aligned}
D &= \partial/\partial r \\
\Delta &= U\partial/\partial r + X^a\partial/\partial x^a \\
\delta &= \omega\partial/\partial r + \xi^a\partial/\partial x^a\,.
\end{aligned}
$$

If the l^μ direction is fixed as above, the freedom allowed in the choice of null tetrad is given by the following transformations:

$$
\left.
\begin{aligned}
l^{\mu *} &= A\,l^\mu \qquad & m^{\mu *} &= m^\mu \\
n^{\mu *} &= A^{-1} n^\mu \qquad & A &\text{ is real}
\end{aligned}
\right\} \tag{3.1}
$$

$$
\left.
\begin{aligned}
l^{\mu *} &= l^\mu \qquad & n^{\mu *} &= n^\mu \\
m^{\mu *} &= e^{iC} m^\mu \qquad & C &\text{ is real}
\end{aligned}
\right\} \tag{3.2}
$$

$$
\left.
\begin{aligned}
l^{\mu *} &= l^\mu \\
n^{\mu *} &= n^\mu + \bar{B} m^\mu + B\bar{m}^\mu + B\bar{B} l^\mu \\
m^{\mu *} &= m^\mu + B l^\mu\,.
\end{aligned}
\right\} \tag{3.3}
$$

(We use an asterisk or dash to denote the results of tetrad or coordinate transformations respectively, unless tetrad and coordinate transformations are combined when a dash suffices.) (3.1) and (3.2) may be used to put $\varepsilon = 0$ and (3.3) to put $\tau + \bar{\pi} = 0$. Further transformations are limited to $A = A^\circ$, $C = C^\circ$, $B = B^\circ/\varrho$. (As always a degree sign means 'independent of r'. It is omitted on quantities which appear frequently, e.g. P, Q, etc.) The restriction on B is deduced by using Newman-Penrose equation (4.1).

Permissible coordinate changes are as follows:

$$r' = r \qquad x^{a'} = x^{a'}(x^a) \tag{3.4}$$

$$r' = r + R_0(x^a) \qquad x^{a'} = x^a \tag{3.5}$$

$$r' = R(x^a)\, r \qquad x^{a'} = x^a. \tag{3.6a}$$

(3.6a) must be combined with the remaining tetrad transformation (3.1) to keep $D = \partial/\partial r$:

$$\left.\begin{array}{ll} l^{\mu *} = R^{-1} l^\mu & m^{\mu *} = m^\mu \\ n^{\mu *} = R n^\mu \,. & \end{array}\right\} \tag{3.6b}$$

The Newman-Penrose equations [6, 12 Chap. 4] for empty space, which are equivalent to the Einstein vacuum field equations, are given in three groups of first order partial differential equations. The first two groups are derived from the empty space Ricci and Bianchi identities respectively, and relate the spin coefficients (2.1), the Riemann tensor components (2.2) and their derivatives with respect to the operators (2.3). The third group are obtained from the commutator relations for the operators. Applying the latter to each of the coordinates r, x^a in turn gives a set of equations relating the tetrad components U, ω, X^a, ξ^a to the spin coefficients. All the equations take a simpler form because

$$\varkappa = \sigma = \varepsilon = \tau + \bar{\pi} = 0 = \Psi_0 = \Psi_1 \,. \tag{3.7}$$

Only a few of them are written out explicitly in the next section where they are used to put $\tau = \lambda = 0$. In this simpler form the Newman-Penrose equations are given in the literature. The above three groups each contain equations denoted "radial" or "non-radial", the former being distinguished from the latter because they contain the operator $D = \partial/\partial r$. Radial equations can be directly integrated with respect to r as ordinary differential equations, giving "constants" of integration, independent of r. The results are then substituted into the non-radial equations which, on equating the coefficients of different powers of r to zero, give relations between the "constants".

The principal result of the next two sections is the demonstration that metrics of the class under consideration are all of the form:

$$\begin{aligned} ds^2 = &- \tfrac{1}{2} P^{-2} (r^2 + \Sigma^2)\, d\zeta\, d\bar{\zeta} \\ &+ 2\, (l_\mu\, dx^\mu)\, (dr + \mathrm{Re}\{P^{-1}(r + i\Sigma)\, \bar{\omega} d\zeta\} - U l_\nu\, dx^\nu) \end{aligned}$$

where

$$l_\mu dx^\mu = du - \mathrm{Re}\{\bar{Q} d\zeta\},$$
$$\omega = -\omega^0 (r - i\Sigma)^{-1} + P\partial Q/\partial u$$
$$U = U^0 + r\partial \log P/\partial u + \mathrm{Re}\{\Psi_2^0 (r + i\Sigma)^{-1}\}.$$
$$u = x^1, \zeta = x^3 + i x^4;$$
$$P, Q, \Psi_2^0, \omega^0, U^0, \Sigma$$

are all independent of r; the last three quantities are expressible in terms of P and Q and their derivatives; P, Q and Ψ_2^0 satisfy three differential equations which, if solved, would completely determine the metric.

4. Preliminary Simplifications

Several empty space Newman-Penrose equations with (3.7) holding are now presented.

Radial equations:

$$D\varrho = \varrho^2 \tag{4.1}$$
$$D\tau = 0 \tag{4.2}$$
$$D\alpha = \varrho(\alpha - \bar{\tau}) \tag{4.3}$$
$$D\beta = \bar{\varrho}\beta \tag{4.4}$$
$$D\gamma = -\tau\bar{\tau} + \Psi_2 \tag{4.5}$$
$$D\lambda + \bar{\delta}\bar{\tau} = \varrho\lambda + \bar{\tau}^2 - (\alpha - \bar{\beta})\bar{\tau} \tag{4.6}$$
$$D\mu + \delta\bar{\tau} = \bar{\varrho}\mu + \tau\bar{\tau} + (\bar{\alpha} - \beta)\bar{\tau} + \Psi_2 \tag{4.7}$$
$$D\Psi_2 = 3\varrho\Psi_2 \tag{4.8}$$
$$DU = -(\gamma + \bar{\gamma}) \tag{4.9}$$
$$D\omega = \bar{\varrho}\omega - (\bar{\alpha} + \beta + \tau) \tag{4.10}$$
$$DX^a = 0 \tag{4.11}$$
$$D\xi^a = \bar{\varrho}\xi^a. \tag{4.12}$$

Non-radial equations:

$$\delta\tau = \varrho\bar{\lambda} + (\tau + \beta - \bar{\alpha})\tau \tag{4.13}$$
$$\Delta\varrho - \bar{\delta}\tau = -\varrho\bar{\mu} + (\bar{\beta} - \alpha - \bar{\tau})\tau + \varrho(\gamma + \bar{\gamma}) - \Psi_2 \tag{4.14}$$
$$\delta X^a - \Delta\xi^a = (\tau - \bar{\alpha} - \beta)X^a + \bar{\lambda}\bar{\xi}^a + (\mu - \gamma + \bar{\gamma})\xi^a \tag{4.15}$$
$$\bar{\delta}\xi^a - \delta\bar{\xi}^a = (\bar{\varrho} - \varrho)X^a - (\bar{\alpha} - \beta)\bar{\xi}^a - (\bar{\beta} - \alpha)\xi^a. \tag{4.16}$$

The complex conjugates of (4.13) and (4.14) together with (4.6) and (4.7) respectively yield:

$$D\lambda = \lambda(\varrho - \bar{\varrho}), \tag{4.17}$$
$$D\mu + \Delta\varrho = (\gamma + \bar{\gamma})\bar{\varrho} + \Psi_2 - \bar{\Psi}_2. \tag{4.18}$$

We now integrate radial Eqs. (4.1)–(4.5), (4.8)–(4.12), (4.17) and (4.18):

$$\varrho = -(r + \varrho^0)^{-1} \tag{4.19}$$

$$\tau = \tau^0 \tag{4.20}$$

$$\alpha = \bar{\tau}^0 + \alpha^0 \varrho \tag{4.21}$$

$$\beta = \beta^0 \bar{\varrho} \tag{4.22}$$

$$\gamma = \gamma^0 - \tau^0 \bar{\tau}^0 r + \tfrac{1}{2} \Psi_2^0 \varrho^2 \tag{4.23}$$

$$\Psi_2 = \Psi_2^0 \varrho^3 \tag{4.24}$$

$$U = U^0 - (\gamma^0 + \bar{\gamma}^0) r + \tau^0 \bar{\tau}^0 r^2 - \tfrac{1}{2} \Psi_2^0 \varrho - \tfrac{1}{2} \bar{\Psi}_2^0 \bar{\varrho} \tag{4.25}$$

$$\omega = \omega^0 \bar{\varrho} + (\bar{\alpha}^0 + \beta^0) + \tau^0/\bar{\varrho} \tag{4.26}$$

$$X^a = X^{0a} \tag{4.27}$$

$$\xi^a = \xi^{0a} \bar{\varrho} \tag{4.28}$$

$$\lambda = \lambda^0 \varrho/\bar{\varrho} \tag{4.29}$$

$$\mu = \mu^0 + \tau^0 \bar{\tau}^0 r + \Lambda^0 \bar{\varrho} + \tfrac{1}{2} \Psi_2^0 \varrho \bar{\varrho} + \tfrac{1}{2} \Psi_2^0 \varrho^2 \tag{4.30}$$

where $\Lambda^0 = -U^0 + (\gamma^0 + \bar{\gamma}^0) i\Sigma + \tau^0 \bar{\tau}^0 \Sigma^2 + X^{0a} i\Sigma_{,a}$. Σ is defined as follows: using coordinate transformation (3.5) it is possible to set the real part of ϱ^0 equal to zero. Thus we write:

$$\varrho^0 = i\Sigma\,, \ \Sigma \text{ real}\,. \tag{4.31}$$

Under coordinate transformation (3.4),

$$X^{0a'} = (\partial x^{a'}/\partial x^a)\, X^{0a}$$

$$\xi^{0a'} = (\partial x^{a'}/\partial x^a)\, \xi^{0a}$$

and thus, by the existence theorem for systems of partial differential equations [13], from now on we may take

$$\begin{aligned} X^{01} &= 1 & \xi^{03} &= P \\ \xi^{01} &= PQ & \xi^{04} &= iP\,. \end{aligned} \tag{4.32}$$

Put $x^1 = u$, $x^3 + ix^4 = \zeta$. To keep ξ^{03} and ξ^{04} in the form of (4.32) further coordinate transformations

$$\zeta' = \zeta'(\zeta, \bar{\zeta}, u) \tag{4.33a}$$

must be restricted by:

$$\xi^{0a} \zeta'_{,a} = 0\,. \tag{4.33b}$$

It is convenient to introduce the notation:

$$D_1 \phi = X^{0a} \phi_{,a}$$

$$D_3 \phi = Q \partial\phi/\partial u + \nabla \phi$$

$$D_4 \phi = \bar{Q} \partial\phi/\partial u + \bar{\nabla} \phi\,.$$

As usual

$$\nabla = \partial/\partial x^3 + i\partial/\partial x^4 = 2\, \partial/\partial \bar{\zeta}\,.$$

(4.33 b) now becomes $D_3\zeta' = 0$ and under (4.33) we have:

$$P' = \tfrac{1}{2} P D_3\zeta' \qquad \bar{P}' = \tfrac{1}{2} \bar{P} D_4\zeta' \tag{4.34}$$

$$X^{01'} = X^{01} = 1$$

$$X' \equiv X^{03'} + i X^{04'} = D_1\zeta' . \tag{4.35}$$

Under the remaining null rotations (3.3):

$$X^* = X + 2\, B^0 \bar{P}$$

and hence it is possible to put $X^* = 0$, so that D_1 becomes $\partial/\partial u$ (which greatly facilitates later work). But under (4.33) X transforms as in (4.35), so on using this coordinate transformation in future, to keep $X = 0$, we must combine it with a further null rotation, in which

$$2\, B^0 \bar{P}' = -\, D_1\zeta' . \tag{4.33c}$$

Note the following commutator relations:

$$\begin{aligned} D_1 D_3 - D_3 D_1 &= (D_1 Q)\, D_1 \\ D_1 D_4 - D_4 D_1 &= (D_1 \bar{Q})\, D_1 \\ D_3 D_4 - D_4 D_3 &= (D_3 \bar{Q} - D_4 Q)\, D_1 . \end{aligned} \tag{4.36}$$

It is now shown that on examining the non-radial equations (4.13) to (4.16) τ is of such a form that under the combined coordinate and tetrad transformation (4.33 a, b and c) it can be equated to zero. We substitute the solutions of the radial Eqs. (4.19)–(4.31) into the non-radial Eqs. (4.13)–(4.16), as described, and obtain respectively the following equations.

$$\xi^{0a}\tau^0_{,a} = \tau^0(\beta^0 - \bar{\alpha}^0) + \bar{\lambda}^0 \tag{4.37}$$

$$-\, \bar{\xi}^{0a}\tau^0_{,a} = -\, \bar{\mu}^0 + i\Sigma\tau^0\bar{\tau}^0 + \tau^0(\bar{\beta}^0 - \alpha^0) \tag{4.38}$$

$$\begin{aligned} \xi^{0b} X^{0a}_{,b} - X^{0b}\xi^{0a}_{,b} = {} & (\mu^0 + 2\,\bar{\gamma}^0 - i\Sigma\tau^0\bar{\tau}^0)\,\xi^{0a} + \bar{\lambda}^0\bar{\xi}^{0a} \\ & -(\bar{\alpha}^0 + \beta^0)\, X^{0a} \end{aligned} \tag{4.39}$$

$$\begin{aligned} \bar{\xi}^{0b}\xi^{0a}_{,b} - \xi^{0b}\bar{\xi}^{0a}_{,b} = {} & -\, 2\, i\Sigma X^{0a} + (2\,\beta^0 + 2\, i\Sigma\tau^0)\,\bar{\xi}^{0a} \\ & -(2\,\bar{\beta}^0 - 2\, i\Sigma\bar{\tau}^0)\,\xi^{0a} . \end{aligned} \tag{4.40}$$

Using (4.32) and $X = 0$, we obtain from (4.39) and (4.40)

$$\lambda^0 = 0\,, \quad \text{hence} \quad \lambda = 0 .$$

$$(\bar{\alpha}^0 + \beta^0) = P D_1 Q$$

$$2\,\beta^0 + 2\, i\Sigma\tau^0 = -\,(P/\bar{P})\, D_3\bar{P}$$

$$2\, i\Sigma = P\bar{P}(D_3\bar{Q} - D_4 Q) .$$

With these results (4.37) becomes

$$D_3(\bar{P}\tau^0) + (D_3\bar{Q} - D_4 Q)\,(\bar{P}\tau^0)^2 = -\, \bar{P} D_1 Q \tau^0 . \tag{4.41}$$

Now under null rotations (3.3):

$$\tau^{0*} = \tau^0 + B^0$$

and thus we show there exists a ζ' as in (4.33a), satisfying (4.33b), and such that B^0 satisfies (4.33c) with $\tau^0 = - B^0$. From (4.33c) and (4.34) it is required first of all that

$$\tau^0 \bar{P} D_4 \zeta' = D_1 \zeta' \,. \tag{4.42}$$

By the existence theorem for first order linear partial differential equations [13], such a ζ' must exist. It is not unique; it can be replaced by

$$\zeta'' = f(\zeta', \bar{\zeta}) \tag{4.43}$$

where f is an arbitrary function of its arguments. Substituting the value of $\bar{P}\tau^0$ obtained from (4.42) into (4.41), and making use of commutator relations (4.36), we obtain

$$\bar{V}\zeta' \, \partial(D_3\zeta')/\partial u - \bar{V}(D_3\zeta') \, \partial\zeta'/\partial u = 0 \,.$$

Thus $D_3\zeta' = g(\zeta', \bar{\zeta})$ for some function g. By a suitable transformation (4.43) it is possible to set $D_3\zeta'' = 0$, and hence the required ζ', satisfying (4.33), with $B^0 = - \tau^0$, exists. With $\tau^0 = 0$, (4.38) implies $\mu^0 = 0$.

Under tetrad transformation (3.2) $P^* = P \exp(iC^\circ)$, so suitable choice of C° makes P real. (4.33) now reduces to

$$\zeta' = \zeta'(\zeta), \zeta' \quad \text{analytic in } \zeta \,. \tag{4.44a}$$

Under (4.44a), $P' = P(d\bar{\zeta}'/d\bar{\zeta})$ and therefore we combine it with a tetrad transformation (3.2) to keep P real, namely:

$$\exp(iC^\circ) = (d\zeta'/d\zeta)^{1/2} (d\bar{\zeta}'/d\bar{\zeta})^{-1/2} \tag{4.44b}$$

so that $P' = P\,|d\zeta'/d\zeta|$.

5. Integration

If we put $\tau = \lambda = 0$, the Newman-Penrose equations have been given in full for the present case [2, Eqs. (2.7) and (2.8)], so we omit them to save space. We can easily integrate the radial equations as follows:

Firstly Eqs. (4.19)–(4.31) only now with $\tau^0 = \mu^0 = \lambda^0 = 0$. Also,

$$\Psi_3 = \Psi_3^0 \varrho^2 + Y_1^0 \varrho^3 + \tfrac{1}{2} Y_2^0 \varrho^4 \,, \tag{5.1}$$

where $Y_1^0 = \xi^{0a} \Psi^0_{2,a} + 3\,\Psi_2^0 (\alpha^0 + \bar{\beta}^0)$,

$Y_2^0 = 3\,\Psi_2^0 (\bar{\omega}^0 + i\bar{\xi}^0 \Sigma_{,a})$.

$$\nu = \nu^0 + \Psi_3^0 \varrho + \tfrac{1}{2} Y_1^0 \varrho^2 + \tfrac{1}{6} Y_2^0 \varrho^3 \tag{5.2}$$

$$\Psi_4 = \Psi_4^0 \varrho + Z_1^0 \varrho^2 + \tfrac{1}{2} Z_2^0 \varrho^3 + \tfrac{1}{3} Z_3^0 \varrho^4 + \tfrac{1}{4} Z_4^0 \varrho^5 \tag{5.3}$$

where $Z_1^0 = 2\,(2\alpha^0 + \bar{\beta}^0)\,\Psi_3^0 + \xi^{0a} \Psi^0_{3,a}$

$Z_2^0 = 2\Psi_3^0 (\bar{\omega}^0 + i\bar{\xi}^{0a} \Sigma_{,a}) + \xi^{0a} Y^0_{1,a} + (5\alpha^0 + 3\bar{\beta}^0)\, Y_1^0$

$Z_3^0 = 3\,Y_1^0 (\bar{\omega}^0 + i\bar{\xi}^{0a} \Sigma_{,a}) + \tfrac{1}{2} \xi^{0a} Y^0_{2,a} + (3\alpha^0 + 2\bar{\beta}^0)\, Y_2^0$

$Z_4^0 = 2\,Y_2^0 (\bar{\omega}^0 + i\bar{\xi}^{0a} \Sigma_{,a})$.

The non-radial equations are then solved as explained in Section 2. The following list of equations are all that can be obtained, although many of them appear more than once, sometimes in a disguised form, i.e. in linear combinations or differentiated with respect to D_1, D_3 or D_4. Some of the calculations though routine, are very long, and much use is made of the commutator relations (4.36). Because only the method is important, and that has been clearly demonstrated several times [2, 6, 7, 12], we merely quote the results.

$$\omega^0 = -2i\Sigma P D_1 Q - P D_3 i\Sigma \tag{5.4}$$

$$\gamma^0 - \bar{\gamma}^0 = 0 \tag{5.5}$$

$$\gamma^0 + \bar{\gamma}^0 = -P^{-1} D_1 P \tag{5.6}$$

$$\alpha^0 = P D_1 \bar{Q} + \tfrac{1}{2} D_4 P \tag{5.7}$$

$$\beta^0 = -\tfrac{1}{2} D_3 P \tag{5.8}$$

$$2i\Sigma = P^2 (D_3 \bar{Q} - D_4 Q) \tag{5.9}$$

$$U^0 = -\operatorname{Re}\{P^2 D_3 (D_4 \log P + D_1 \bar{Q})\} \tag{5.10}$$

$$\Lambda^0 = -U^0 - i\Sigma D_1 \log P + D_1 i\Sigma \tag{5.11}$$

$$\nu^0 = P D_4 D_1 \log P + D_1 (P D_1 \bar{Q}) \tag{5.12}$$

$$\Psi_3^0 = 2i\Sigma\nu^0 + P D_4 \Lambda^0 + 2P(D_1\bar{Q})\Lambda^0 \tag{5.13}$$

$$\Psi_4^0 = P D_4 \nu^0 + (3 P D_1 \bar{Q} + D_4 P)\nu^0 \tag{5.14}$$

$$D_3 \Psi_2^0 + 3 D_1 Q \Psi_2^0 = 0 \tag{5.15}$$

$$\begin{aligned}\Psi_2^0 - \bar{\Psi}_2^0 - iP^2\{&D_4(D_3\Sigma + \Sigma D_1 Q) + D_1\bar{Q}(D_3\Sigma + \Sigma D_1 Q)\\ &+ D_3(D_4\Sigma + \Sigma D_1\bar{Q}) + D_1 Q(D_4\Sigma + \Sigma D_1\bar{Q})\} + 4i\Sigma U^0 = 0\end{aligned} \tag{5.16}$$

$$D_1\Psi_2^0 - 3\Psi_2^0 D_1 \log P - P D_3 \Psi_3^0 - (2 P D_1 Q - D_3 P)\Psi_3^0 = 0\,. \tag{5.17}$$

Eqs. (5.4)–(5.14) give all the "constants" of integration of the radial equations in terms of P, Q and their derivatives. The last three Eqs. (5.15)–(5.17), relating P, Q and Ψ_2^0 are the ones which must be solved for explicit solutions – KERR calls them "field equations". Having obtained the components of l^μ, n^μ and m^μ it is easy to find the components of their covariant counterparts, and hence the metric given in Section 3, using

$$g_{\mu\nu} = 2 l_{(\mu} n_{\nu)} - 2 m_{(\mu} \bar{m}_{\nu)}\,.$$

Following ROBINSON, ROBINSON and ZUND [5], if we introduce the quantities:

$$N = D_1 \bar{Q} + D_4 \log P$$

and

$$\Xi = D_4 N + N^2$$

we may obtain several simplifications:

$$\Lambda^0 = P^2 D_3 N$$
$$\nu^0 = P D_1 N$$
$$\Psi_3^0 = P^3 D_3 \Xi$$
$$\Psi_4^0 = P^2 D_1 \Xi$$

and thus (5.17) becomes

$$D_1(\Psi_2^0 P^{-3}) - P(D_3 D_3 \Xi + 2\overline{N} D_3 \Xi) = 0 .$$

These last relations all result from repeated application of (4.36).

6. Special Case

Apart from (4.44), the only remaining co-ordinate-tetrad transformation is (3.6). Under the latter, unfortunately, $X^{01^*} = R X^{01}$ and hence it must be combined with a further coordinate transformation in u, of the type (3.4), in order to keep $X^{01} = 1$. The complete transformation is as follows:

$$\begin{aligned} r' &= Rr , & u' &= G(u, \zeta, \bar{\zeta}) \\ \zeta' &= \zeta , & & \\ l^{\mu^*} &= R^{-1} l^\mu , & n^{\mu^*} &= R n^\mu \\ m^{\mu^*} &= m^\mu , & R\partial G/\partial u &= 1 . \end{aligned} \tag{6.1}$$

Under (6.1), the basic quantities transform thus:

$$\Psi_0^{2'} = R^3 \Psi_2^0 , \qquad \Sigma' = R\Sigma$$
$$P' = RP$$
$$Q' = R^{-1} Q + \nabla G = D_3 G$$
$$D_1' Q' = D_1 Q - D_3 \log A .$$

Obviously we could put $P = 1$ or take Q purely imaginary, etc., by means of this transformation.

If $D_1 P$ is put equal to zero, so that (6.1) is restricted to $R = R(\zeta, \bar{\zeta})$, it is easily seen that the conditions

$$D_1 \Sigma = 0 , \quad D_1 D_1 Q = 0 \tag{6.2}$$

(which appear as important conditions in the following) are sufficient to enable one to put

$$D_1 Q = 0 . \tag{6.3}$$

Further transformation of type (6.1) is then restricted to

$$\left.\begin{aligned} r' &= Rr , & u' &= R^{-1} u \\ l^{\mu^*} &= R^{-1} l^\mu , & n^{\mu^*} &= R n^\mu \\ m^{\mu^*} &= m^\mu \text{ where } R \text{ is now constant,} & & \end{aligned}\right\} \tag{6.4a}$$

and

$$u' = u + f(\zeta, \bar{\zeta}) \text{ where } f \text{ is an arbitrary function, } R = 1\,. \tag{6.4b}$$

Clearly the "field equations" (5.15)–(5.17) are not necessarily compatible; following KERR consider them with $P = 1$, when they take the simpler form:

$$D_3 \Psi_2^0 + 3 D_1 Q \Psi_2^0 = 0 \tag{6.5}$$

$$\Psi_2^0 - \bar{\Psi}_2^0 + D_4 D_4 D_3 Q - D_3 D_3 D_4 \bar{Q} = 0 \tag{6.6}$$

$$D_1 \Psi_2^0 - D_1 (D_3 D_3 D_4 \bar{Q}) + |D_1 D_3 Q|^2 = 0\,. \tag{6.7}$$

[In deriving (6.6) and (6.7) extensive use is made of relations (4.36).] These equations are compatible if $\Psi_2^0 = 0$ (Type III or N). Otherwise, using (6.6), (6.5) and (6.7) become of the form:

$$D_1 (\Psi_2^0 + \bar{\Psi}_2^0) + \Theta = 0$$

$$D_3 (\Psi_2^0 + \bar{\Psi}_2^0) + 3 D_1 Q (\Psi_2^0 + \bar{\Psi}_2^0) + \Phi = 0\,.$$

Here Θ and Φ are expressions containing Q and its derivatives only. Using commutator relations (4.36) it is easy to see that $(\Psi_2^0 + \bar{\Psi}_2^0)$ can be obtained in just two ways from these two equations, namely:

$$3 (D_1 D_1 Q) (\Psi_2^0 + \bar{\Psi}_2^0) - 4 D_1 Q \Theta + D_1 \Phi - D_3 \Theta = 0$$

$$6 (D\, i \Sigma) (\Psi_2^0 + \bar{\Psi}_2^0) - 2 i \Sigma \Theta - 3 D_1 \bar{Q} \Phi + 3 D_1 Q \bar{\Phi} + D_3 \bar{\Phi} - D_4 \Phi = 0\,.$$

As KERR pointed out, if one or both of $D_1 D_1 Q$ and $D_1 \Sigma$ is non-zero, the expressions for $\Psi_2^0 + \bar{\Psi}_2^0$ can be substituted back into (6.5)–(6.7) giving further conditions on Q.

Alternatively (6.2) holds. In this special case, allowing P to be such that $D_1 P = 0$ instead of $P = 1$ does not affect (6.2), and thus the above coordinate system may be used with (6.3) and (6.4) holding. The "field equations" (5.15)–(5.17) now take the simple form:

$$\Psi_2^0 = c u + a + i b \tag{6.8}$$

where c is a real constant; a and b are real and independent of u.

$$P^2 \nabla \bar{\nabla} (P^2 \nabla \bar{\nabla} \log P) = c \tag{6.9}$$

$$2 i b = P^2 \nabla \bar{\nabla} (P^2 (\nabla \bar{Q} - \bar{\nabla} Q)) + 2 P^4 (\nabla \bar{Q} - \bar{\nabla} Q) \nabla \bar{\nabla} \log P \tag{6.10}$$

$$\nabla (a + i b) + Q c = 0\,. \tag{6.11}$$

When $c = 0$ obviously all quantities are independent of u, and thus $V^\mu = \delta_1^\mu$ is a Killing vector.

7. Explicit Solutions

All the explicit twisting degenerate solutions found so far belong to the special subcase of Section 6. In fact ROBINSON, ROBINSON and ZUND [5] have skilfully succeeded in bringing all the previous explicit solutions into one larger class which they completely solve. Their method is outlined in the following.

The solutions of (6.8)–(6.11) are those with

$$P^2 \nabla \bar{\nabla} \log P = -U^0 = \text{constant} \tag{7.1}$$

[(6.4a) may be used to put $U^0 = +1$, 0, or -1 if required]. This means that the Gaussian curvature of the 2-space with metric $ds^2 = \frac{1}{2} P^{-2} d\zeta\, d\bar{\zeta}$ is constant, and thus under (4.44) it is possible to put

$$P = 2^{-1/2} (1 - \tfrac{1}{2} U^0 \zeta \bar{\zeta}) \tag{7.2}$$

(6.9) and (6.11) are now equivalent to $c = 0$ and $a + ib = M(\zeta)$ respectively, where M is an analytic function of ζ. It remains to solve (6.10). Since, by (7.2), $\bar{\nabla}\bar{\nabla} P = 0$, (6.10) may be written as

$$\begin{aligned} 2ib &= M(\zeta) - \bar{M}(\bar{\zeta}) \\ &= P^3 \{\bar{\nabla}\bar{\nabla}(P^{-1} \nabla P^2 Q) - \nabla\nabla(P^{-1} \bar{\nabla} P^2 Q)\}\,. \end{aligned} \tag{7.3}$$

The general solution of this equation is $Q = Q_0 + Q_1$, where Q_1 is a particular integral and Q_0 is the general solution of the homogeneous equation. A simple form for Q_1 [5] is:

$$Q_1 = \begin{cases} -(\sqrt{2}\, U^{0\,2} P \bar{\zeta})^{-1} \bar{M}(\bar{\zeta}), & U^0 \neq 0 \\ \frac{1}{4} \zeta^2 \int \bar{M}(\bar{\zeta})\, d\bar{\zeta}, & U^0 = 0\,. \end{cases} \tag{7.4}$$

Under (6.4b), $Q' = Q + \nabla f$ or, regarding Q_1 as fixed,

$$Q_0' = Q_0 + \nabla f\,. \tag{7.5}$$

The homogeneous equation, i.e. (7.3) with $b = 0$, is equivalent to the existence of a real "potential" Π, such that:

$$P^{-1} \nabla P^2 Q_0 = \nabla\nabla \Pi\,.$$

But under (7.5), if we write $f = P^{-1} \Pi$,

$$P^{-1} \nabla P^2 Q_0' = 0\,. \tag{7.6}$$

Further change in f being limited to $\nabla\nabla(Pf) = 0$. Hence without loss of generality take Q_0 as the general solution of (7.6):

$$Q_0 = P^{-2} L(\zeta)$$

where L is an analytic function of ζ.

The NUT solution [2] is now given by $Q_0 = 0$, with $M(\zeta) = a + ib$ $=$ constant ($b \neq 0$). [A transformation (6.4b) with f such that:

$$\nabla f = (\sqrt{2}\, U^{0\,2}\, P\bar{\zeta})^{-1}\, a - (U^{0\,2}\bar{\zeta})^{-1}\, ib$$

takes Q into $(2^{3/2}\, U^0 P)^{-1}\, ib\zeta$ which is of the form given in the reference].

The Kerr solution is given by $Q_1 = 0$, $L = ik\zeta$, where k is a real constant, and $M(\zeta)$ is a real constant. (Transformations of type (4.44) and (6.4a) are needed to bring the Kerr metric into its usual form, with angular coordinates, θ, ϕ [15, p. 191]).

The Demianski metric [10] is given by a superposition of the above Kerr and NUT values for Q. Finally, the Kerr-Schild metrics are given by $Q_1 = 0$, $M =$ real constant, as will be shown in Section 9.

8. Complex Coordinate "Trick"

A method of generating the Kerr and Demianski metrics from Schwarzschild's, by using a complex coordinate transformation, has been demonstrated [9, 10]. Why this method works for solutions of the Eqs. (6.8)–(6.11) is easily seen. Such a complex transformation in general can be written:

$$r' = r + iT\,, \qquad u' = u + iS$$
$$\zeta' = \zeta \tag{8.1}$$

S and T are both functions of ζ, $\bar{\zeta}$ only, and are real. After the transformation, r' and u' are taken as real, with r and u complex. Thus:

$$\varrho = -(r + i\Sigma)^{-1}, \quad \text{whereas} \quad \bar{\varrho} = -(\bar{r} - i\Sigma)^{-1}\,.$$

We see that the effect of (8.1) is to leave P and Ψ_2^0 unchanged (although it may be necessary to add a constant to the latter), with:

$$Q' = Q + \nabla iS$$
$$\Sigma' = \Sigma - T\,. \tag{8.2}$$

If P, Q, Ψ_2^0, Σ *are solutions of* (6.8)–(6.11), *then* P', Q', Σ', $\Psi_2^{0\prime}$ *will also be solutions*, where $P' = P$,

$$\Psi_2^{0\prime} = \Psi_2^0 + \text{const.} = c'u' + a' + ib'\,,$$
$$c' = c, \qquad a' + ib' = (a + a_0) + i(b + b_0) - iSc\,, \tag{8.3}$$

a_0, b_0 are real constants.

T must satisfy

$$b' - b + P^2\,\nabla\bar{\nabla} T + 2\,T P^2 \nabla\bar{\nabla} \log P = 0 \tag{8.4}$$

and since $2i\Sigma' = P^2(\nabla\bar{Q}' - \bar{\nabla} Q')$, for consistency we must have

$$P^2 \nabla\bar{\nabla} S = T\,. \tag{8.5}$$

Note that for a given T, solution of (8.4), $S = S_0 + S_1$, where S_1 is a particular integral of (8.5) and S_0 is a solution of the homogeneous equation $\nabla\bar{\nabla} S_0 = 0$. Therefore, by use of (6.4b), we can make S_0 zero and only S_1 need be considered.

For example consider the cases for which this method was discovered. The Schwarzschild solution is characterised by:

$$c = b = 0\,, \qquad a = \text{constant}$$
$$Q = \Sigma = 0$$
$$P \text{ as in (7.1), (7.2), } U^0 < 0\,.$$

Then by the above method we can easily obtain the Kerr or NUT metrics. For the Kerr metric put $a_0 = b_0 = 0$,

$$T = 2k(1 + \tfrac{1}{2}U^0\zeta\bar{\zeta})(1 - \tfrac{1}{2}U^0\zeta\bar{\zeta})^{-1}$$
$$S = \sqrt{2}k\, U^{0-1}\, P^{-1}\,.$$

For the NUT metric put $a_0 = 0$, $b_0 \neq 0$,

$$T = \text{constant}$$
$$S = -\,T\, U^{0-1} \log P\,.$$

9. Generalised K-S Solutions

By a generalised Kerr Schild (K-S) solution of the Einstein vacuum field equations we mean one which has a metric of the form:

$$h_{\mu\nu} = g_{\mu\nu} + 2Hl_\mu l_\nu \tag{9.1}$$

where $g_{\mu\nu}$ is also a metric which is a solution of the Einstein vacuum field equations and l_μ is a null vector with respect to $g_{\mu\nu}$. Kerr and Schild [3, 4] dealt with the case where $g_{\mu\nu}$ was the metric of flat space, but they approached the calculations with $g_{\mu\nu}$ in the usual Cartesian coordinates whereas we shall consider the case when $g_{\mu\nu}$ need not be flat [11] and use the same kind of null geodesic coordinates as in the previous sections.

Define

$$h^{\mu\nu} = g^{\mu\nu} - 2Hl^\mu l^\nu$$

where indices have been raised with respect to $g_{\mu\nu}$.

It follows that

$$h^{\mu\nu}h_{\nu\varrho} = \delta^\mu_\varrho$$

and

$$h^{\mu\nu}l_\mu l_\nu = 0$$

i.e. $h^{\mu\nu}$ is the inverse of $h_{\mu\nu}$ and l_μ is null with respect to both metrics. Use a semicolon or a ∇ to denote covariant differentiation with respect to $h_{\mu\nu}$ or $g_{\mu\nu}$ respectively. Then from the identities

$$h_{\mu\nu;\varrho} = 0 \quad \text{and} \quad \nabla_\varrho g_{\mu\nu} = 0$$

it follows straight-forwardly [11] that

$$T^\varrho_{\mu\nu} = h^{\varrho\varkappa}\{\nabla_\mu(Hl_\nu l_\varkappa) + \nabla_\nu(Hl_\mu l_\varkappa) - \nabla_\varkappa(Hl_\mu l_\nu)\}$$

where $T^\varrho_{\mu\nu} = \Gamma^\varrho_{\mu\nu} - \gamma^\varrho_{\mu\nu}$. $\Gamma^\varrho_{\mu\nu}$ and $\gamma^\varrho_{\mu\nu}$ are Christoffel symbols for $h_{\mu\nu}$ and $g_{\mu\nu}$ respectively. Further:

$$R_{\sigma\mu} = K_{\sigma\mu} + \nabla_\mu T^\varrho_{\sigma\varrho} - \nabla_\varrho T^\varrho_{\sigma\mu} + T^\varkappa_{\sigma\varrho} T^\varrho_{\varkappa\mu} - T^\varkappa_{\sigma\mu} T^\varrho_{\varkappa\varrho} \tag{9.2}$$

where $R_{\sigma\mu}$ and $K_{\sigma\mu}$ are the Ricci tensors for $h_{\mu\nu}$ and $g_{\mu\nu}$ respectively.

We assume that $g_{\mu\nu}$ is a known metric, with

$$K_{\sigma\mu} = 0 \tag{9.3}$$

and proceed to find the possible solutions for $h_{\mu\nu}$ with

$$R_{\sigma\mu} = 0\,. \tag{9.4}$$

Introduce a null tetrad z^μ_m with respect to $g_{\mu\nu}$; l_μ in (9.1) is taken as one of the tetrad vectors, z^μ_1. Then (9.3) is equivalent to the usual empty space Newman-Penrose equations written in terms of this null tetrad. Using (9.2), (9.4), takes on a tetrad component version, namely:

$$\begin{aligned} 0 = R_{sm} &= R_{\sigma\mu} z^s_\sigma z^\mu_m \\ &= T^q_{sq;m} - T^q_{sm;q} - \gamma_s{}^k{}_m T^q_{kq} - \gamma^q{}_{km} T^k_{sq} - \gamma_q{}^k{}_m T^q_{sk} \\ &\quad + \gamma_s{}^k{}_q T^q_{km} + \gamma_{qk}{}^q T^k_{sm} + \gamma_m{}^k{}_q T^q_{sk} + T^k_{sq} T^q_{km} \\ &\quad - T^k_{sm} T^q_{kq} \end{aligned} \tag{9.5}$$

where $T^k_{mn} = T^\varkappa_{\mu\nu} z^\mu_m z^k_\varkappa z^\nu_n$ and a semicolon denotes the usual intrinsic derivative. The first of the equations gives

$$R_{11} = \varkappa\bar{\varkappa}H = 0\,.$$

Hence $\varkappa = 0$, and l_μ is tangent to a congruence of null geodesics. *We now make the extra assumption that σ is also zero*; $\sigma = 0$ follows from the form of the Riemann tensor for $h_{\mu\nu}$ when $g_{\mu\nu}$ is flat [4] but it does not appear to follow in general (the possibility of $\sigma \neq 0$ is being investigated [14]). The case $\varrho = 0$ (which should present no difficulties) is now omitted and thus $\varkappa = \sigma = 0$ implies that $g_{\mu\nu}$ belongs precisely to that class of metrics considered in Sections 2–5, and all the results obtained there ($\tau = \pi = \lambda = 0$, etc.) are now freely applied here.

The remaining Eqs. (9.5) which are not identically zero are

$$0 = R_{12} = -D^2H + (\varrho + \bar{\varrho})\,DH + (\varrho - \bar{\varrho})^2 H \tag{9.6}$$

$$0 = R_{34} = -(\varrho + \bar{\varrho})\,DH + (\varrho^2 + \bar{\varrho}^2)\,H \tag{9.7}$$

$$\begin{aligned} 0 = R_{22} = {} & (\varrho + \bar{\varrho})\,(\Delta H + 2(\gamma + \bar{\gamma})\,H) - \delta(\bar{\delta} H + 2(\alpha + \bar{\beta})H) \\ & - (\bar{\alpha} + 3\beta)\,(\bar{\delta} H + 2(\alpha + \bar{\beta})\,H) - \bar{\delta}(\delta H + 2(\bar{\alpha} + \beta)\,H) \\ & - (\alpha + 3\bar{\beta})\,(\delta H + 2(\bar{\alpha} + \beta)\,H) + \mu(DH + 2(\varrho - \bar{\varrho})\,H) \\ & + \bar{\mu}(DH - 2(\varrho - \bar{\varrho})\,H) - 2HD(\gamma + \bar{\gamma}), \end{aligned} \tag{9.8}$$

$$\begin{aligned} 0 = R_{23} = {} & -\delta\{(\varrho - \bar{\varrho})\,H\} - (\bar{\alpha} + \beta)\,(\varrho - \bar{\varrho})\,H + \bar{\varrho}\,\delta H \\ & + 2\bar{\varrho}(\bar{\alpha} + \beta)\,H - D\delta H - 2D\{(\bar{\alpha} + \beta)\,H\} \end{aligned} \tag{9.9}$$

(9.6) and (9.7) are radial equations which give

$$H = H^0(\varrho + \bar{\varrho})\,.$$

The non-radial Eqs. (9.8) and (9.9) are solved in the usual way. From (9.8) we obtain several of the Eqs. (5.4)–(5.17) as well as:

$$X^{0a} H^0_{,a} + 3(\gamma^0 + \bar{\gamma}^0)\,H^0 = 0\,, \tag{9.10}$$

$$\mu^0 - \bar{\mu}^0 = 0\,. \tag{9.11}$$

From (9.9):

$$\xi^{0a} H^0_{,a} + 3(\bar{\alpha}^0 + \beta^0)\,H^0 = 0\,. \tag{9.12}$$

We transform coordinates by (6.1) so that $D_1 P = 0$. (9.10)–(9.12) become

$$D_1 H^0 = 0\,, \tag{9.13}$$

$$D_1 \Sigma = 0\,, \tag{9.14}$$

$$D_3 H^0 + 3(D_1 Q)\,H^0 = 0\,. \tag{9.15}$$

(9.13) and (9.15) imply that $D_1 D_1 Q = 0$, and thus together with (9.14) we see that not only does $g_{\mu\nu}$ belong to the general class of 2–5, but also to the special case of Section 6. We are permitted to put $D_1 Q = 0$ and thus (9.13) and (9.15) are equivalent to saying that H^0 is constant. Examining the form of $h^{\mu\nu}$ it is clear that H^0 only enters into the h^{22} term, in fact

$$\Psi_2^{0\prime} = \Psi_2^0 + 2H^0$$

where $\Psi_2^{0\prime}$ relates to $h^{\mu\nu}$ and Ψ_2^0 to $g^{\mu\nu}$. *Thus in general the only difference between $h_{\mu\nu}$ and $g_{\mu\nu}$ is a real constant in Ψ_2^0*, which in (6.8)–(6.11) trivially will always give a new solution. This is not quite so trivial when $g_{\mu\nu}$ is flat, i.e. the Kerr-Schild case. In this case:

$$\Psi_2^0 = 0\,,$$

$$\Psi_3^0 = P\bar{\nabla}(P^2 \nabla\bar{\nabla} \log P) = 0\,.$$

Therefore $P^2 \nabla \bar{\nabla} \log P = - U^0 =$ constant. Hence $\Psi_2^{0\prime}$ is a real constant and $h_{\mu\nu}$ has the form given at the end of Section 7. $V^\mu = \delta_1^\mu$ is a Killing vector with respect to both $h_{\mu\nu}$ and $g_{\mu\nu}$, and is in fact translational with respect to $g_{\mu\nu}$:

$$g_{\mu\nu} V^\mu V^\nu = - 2 U^0 .$$

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