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NEWTON FLOWS FOR REAL EQUATIONS

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1. Introduction. Let $G : \mathbf{R}^n \to \mathbf{R}^n$ be a smooth mapping with Jacobian matrix $DG(x)$. In this paper we shall discuss the dynamical system

$$N(x) = x - DG(x)^{-1}G(x) \tag{1}$$

provided by Newton's method for the system of equations

$$G(x) = 0. \tag{2}$$

If $n = 2$ and G is a rational mapping R of the complex plane $\mathbf{C}$, then the dynamics of (1), though possibly very complicated and delicate, is understood in terms of the classical and recent theory of *Julia sets* [**3**, **4**, **1**]. In particular, since ∞ is typically a repelling fixed point of N one has that

$$J_N = \text{ closure } \{x \in \overline{\mathbf{C}} : N^k(x) = \infty, \text{ for some } k \in \mathbf{N}\} \tag{3}$$

is the Julia set of $N(x) = x - R(x)/R'(x)$ (here $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and $N^k = N \circ \cdots \circ N$ k-times). Moreover, if $\overline{x} \in \mathbf{C}$ is a simple zero of R, i.e., $R'(\overline{x}) \neq 0$, then $\overline{x}$ is an attractive fixed point of N; if

$$A(\overline{x}) = \{x \in \mathbf{C} : N^k(x) \to \overline{x} \text{ as } k \to \infty\}, \tag{4}$$

is its basin of attraction, then

$$\partial A(\overline{x}) = J_N. \tag{5}$$

Since (5) is true for any attractive fixed point of N (or even cycles), J_N is typically a *fractal* set which in addition has the interesting property that Newton's method clearly will diverge for initial values in J_N. On the other hand, if n is not restricted to be 1 or 2 and G is simply

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smooth, the dynamics of (2) is much more delicate and far from being understood. For example:

(a) N may allow strange attractors (see [**5**]) which is not possible in the complex case.

(b) What is the appropriate analogue to a Julia set? Is there a result similar to (5)?

Associated with the dynamical system (1) there is the system of ordinary differential equations

$$\dot{x}(t) = -DG(x(t))^{-1}G(x(t)). \tag{6}$$

Knowledge of the flow defined by this system contributes much to the understanding of the orbit structure of (1). We observe that (1) is simply a particular case ($h = 1$) of an Euler method

$$N_h(x) = x - hDG(x)^{-1}G(x) \tag{7}$$

for (6).

The boundary of the domain of definition of (6) is the singular set

$$S = \{x \in \mathbf{R}^n : \det DG(x) = 0\} \tag{8}$$

(typically (i.e., if 0 is a regular value of $\det DG : \mathbf{R}^n \to \mathbf{R}$) a collection of smooth $n-1$ manifolds); this set plays an important role in relating the systems (6) and (7) (see [**5**] for details).

Our objective here is to give some evidence for an interesting conjecture (which is true for Newton's method for rational mappings of $\mathbf{C}$) for the mapping N.

Define the *Julia-like* set of N by

$$J_N = \text{ closure } \{x \in \mathbf{R}^n : N^k(x) \in S, \text{ some } k \in \mathbf{N} \cup \{0\}\}, \tag{9}$$

generated by the preimages of S. Define the *exploding* set of N by

$$E_N = \text{ closure } \{x \in \mathbf{R}^n : ||N^k(x)|| \to \infty \text{ as } k \to \infty\}, \tag{10}$$

(where $||\cdot||$ is some norm for $\mathbf{R}^n$). While it is apparent that $J_N \neq \emptyset$, it is by no means clear or obvious that $E_N \neq \emptyset$. We then have the following conjecture.

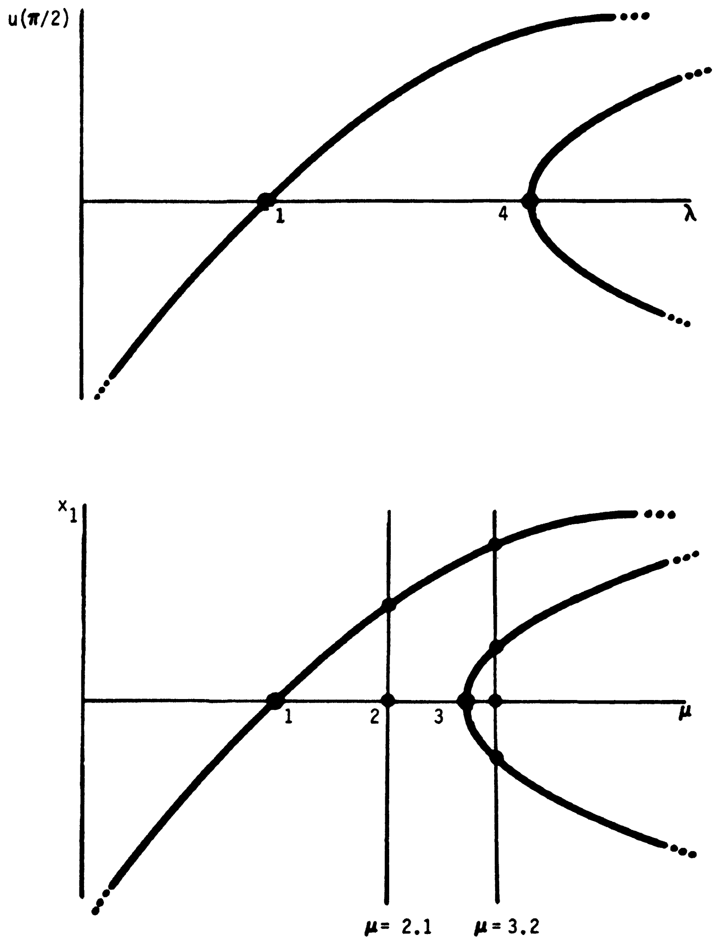


Figure 1. Bifurcation diagrams for (11) and (12).

CONJECTURE. $J_N = E_N$.

(Observe that in the complex case ∞ typically has a dense inverse orbit in J_N (see (3)).)

2. A special case. In this section we shall discuss the above conjecture for a particular model problem in $\mathbf{R}^2$.

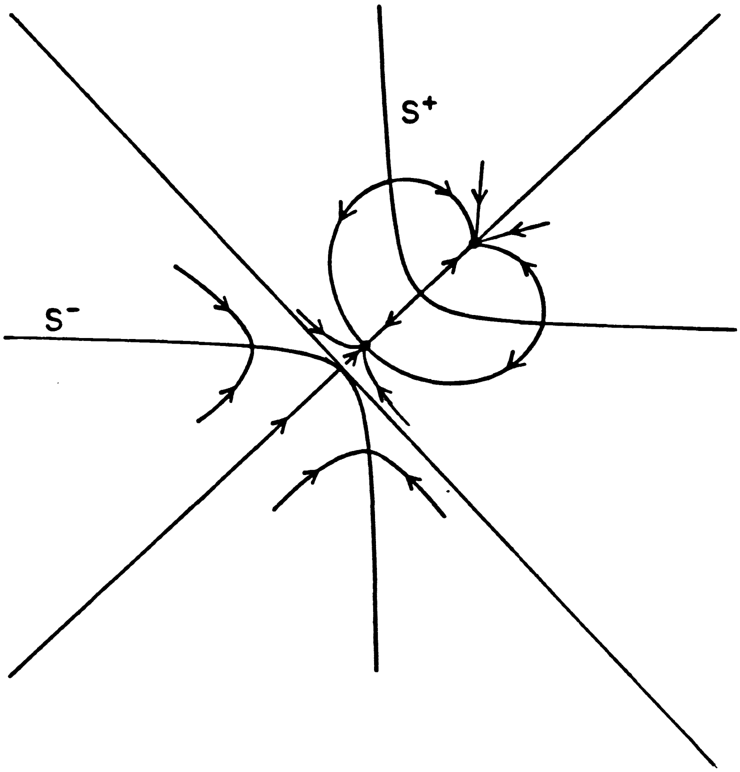


Figure 2(a). Phase Portrait of (6) with G as in (11), $\mu = 2.1$, two sinks.

Let

$$G(x) = Ax - \mu F(x), \tag{11}$$

where

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$F(x) = \begin{pmatrix} f(x_1) \\ f(x_2) \end{pmatrix}$$

and

$$f(s) = s - s^2.$$

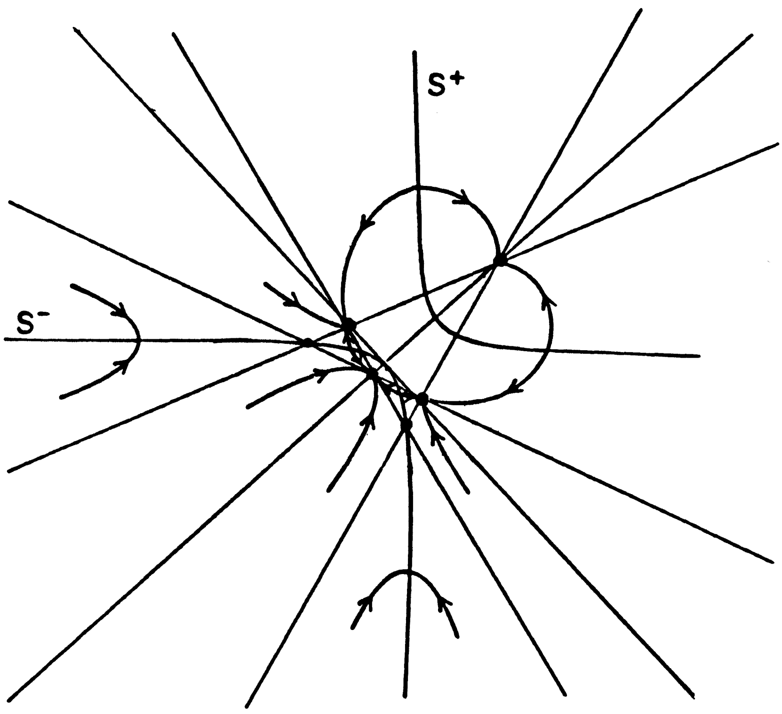


Figure 2(b). Phase Portrait of (6) with G as in (11), $\mu = 3.2$, four sinks.

We note that (11) is a standard two point difference approximation for the boundary value problem

$$\left\{\begin{array}{l} u'' + \lambda f(u) = 0 \\ u(0) = 0 = u(\pi), \end{array}\right. \tag{12}$$

where $\mu = \lambda\delta^2$ and $\delta = \pi/3$. The bifurcation diagrams for (11) and (12) are given by Figure 1, and Figure 2 shows the continuous time flow of (6) for two choices of μ and G as in (11).

In this example the singular set S is given by a pair of hyperbolas S^+ and S^-. One can easily show that S^+ behaves like a global repeller in both cases and S^- like a global attractor for $\mu < 3$. For $\mu > 3$, however, S^- has passed through a bifurcation state (at $\mu = 3$) and as a result decomposes into repelling and attracting components.

Figure 3 and 4 show plots of delicate computer experiments displaying J_N for various choices of h and $\mu = 2.1$ and $\mu = 3.2$.

Apparent from these experiments is the crucial role of the singular set S which generates Cantor sets of curves. In addition, Figure 3 demonstrates the importance of the straight line

$$\mathbf{G}_\mu = \{x = (x_1, x_2) : x_1 + x_2 + (3 - \mu)/\mu = 0\}. \tag{13}$$

One of the results from [**5**] is the following theorem.

THEOREM. *Let* $0 < \mu < 3$ *and* $0 < h < 2$.

(a) $\mathbf{G}_\mu \subset J_N$ *and*

$$J_N = \text{ closure } \{x \in \mathbf{R}^2 : N^k(x) = P_\mu, \text{ some } k \in \mathbf{N}\}$$

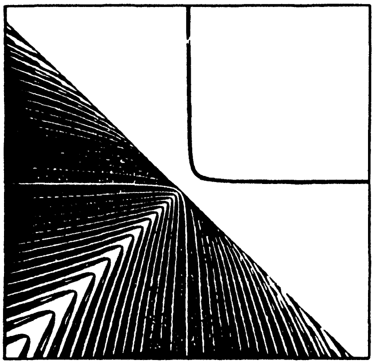
where

$$\{P_\mu\} = S^- \cap \mathbf{G}_\mu.$$

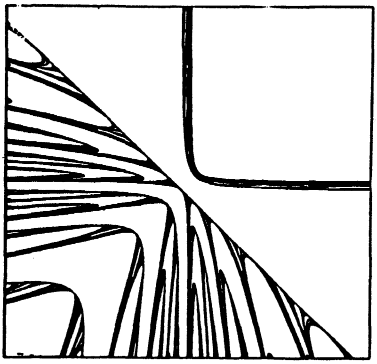
(b) $N|_{\mathbf{G}_\mu}$ *is equivalent to a Newton method on the real line*

$$r(s) = s - hk(s)/k'(s),$$

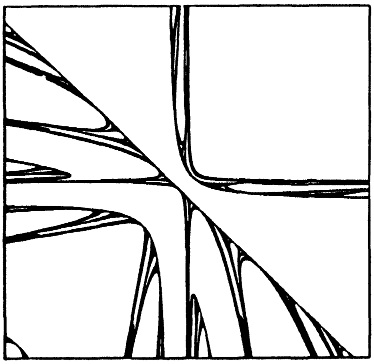
where $k(s) = \mu s^2 - (\mu + 1)(\mu - 3)/4\mu$.



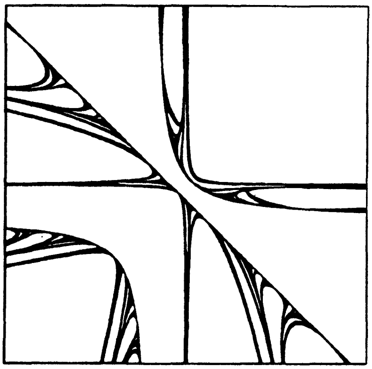
h = 0.3



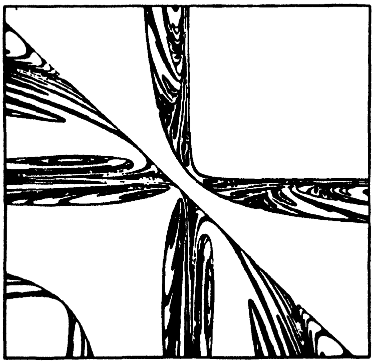
h = 1.0



h = 1.4



h = 1.6



h = 1.7

Figure 3. The Julia like set J_N for (7), G as in (11) and $\mu = 2.1$.

(c) *$N|_{\mathbf{G}_\mu}$ is chaotic, i.e., N restricted to $\mathbf{G}_\mu$ is equivalent to $z \to z^2$ on the unit circle.*

With these observations we are now in a position to discuss the main point of this paper.

3. Theorem and conjecture. *Let G be as in* (11) *and* $0 < \mu < 3$, $0 < h < 2$.

(a) *There is a dense set $\mathbf{H}_\mu \subset \mathbf{G}_\mu$ such that each $Q \in \mathbf{H}_\mu$ is a periodic repeller of N (see (7)).*

(b) *Each $Q \in \mathbf{H}_\mu$ distinguishes a smooth $1-$manifold M_Q which is*

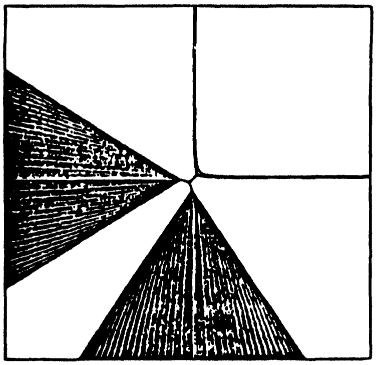
- *diffeomorphic to* $[0, \infty)$
- *invariant under N^p, where Q has period p.*

(c) *For each $x \in M_Q - \{Q\}$, $||N^{kp}(x)|| \to \infty$ as $k \to \infty$.*

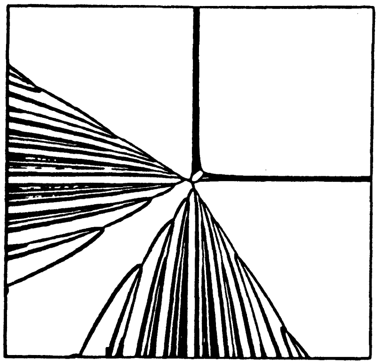
REMARK.

(i) Note that (c) means that E_N is not empty. Computer experiments based on (c) have provided strong evidence that, in the above case, indeed

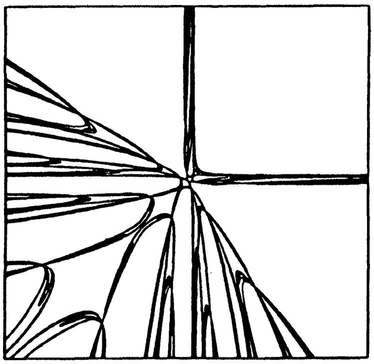
$$J_N = E_N.$$



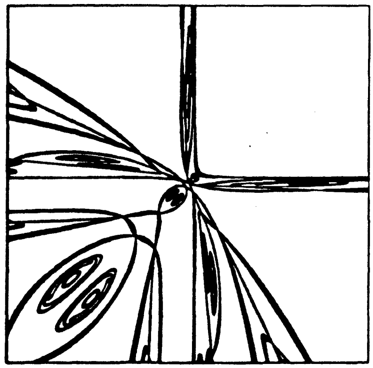
h = 0.3



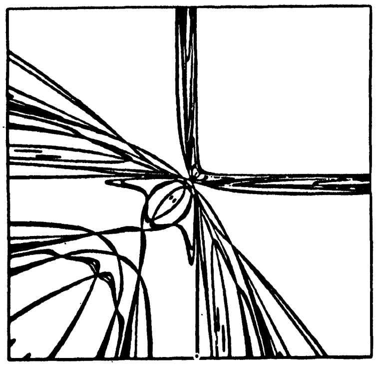
h = 1.0



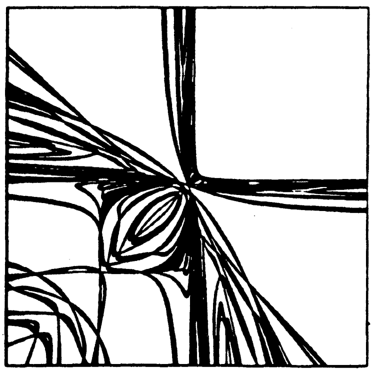
h = 1.4



h = 1.6



h = 1.7



h = 1.8

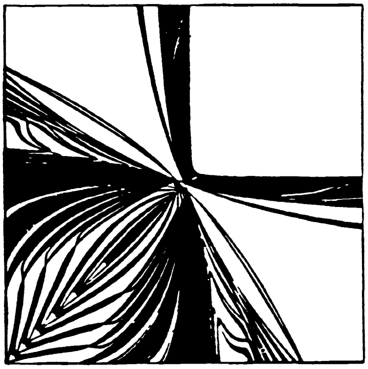


Figure 4. The Julia like set J_N for (7), G as in (11), $\mu = 3.2$.

(ii) The 1-manifolds above are similar to the "hairs" as discussed in [**2**] on Julia sets for the exponential family

$$E_\lambda(x) = \lambda \exp x$$

in **C**.

(iii) It is not difficult to show that the continuous time flow (6) remains bounded for all time. In that regard Euler's method is surprisingly different for all $h > 0$ (see [**5**])!

We shall now give a sketch of a proof of our main result:

Step 1. On shows that the dynamics of N restricted to $\mathbf{G}_\mu$ is equivalent to the dynamics

$$\alpha \to 2\alpha (\mathrm{mod}\, 1)$$
$$\alpha \in [0,1].$$

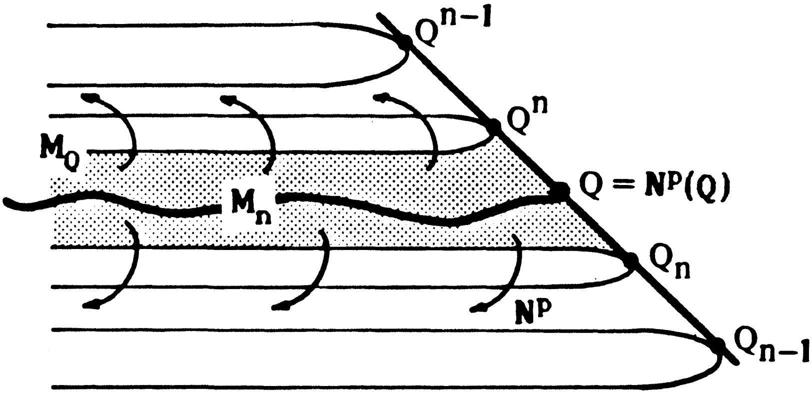


Figure 5.

In this equivalence the point $\{P_\mu\} = S^- \cap \mathbf{G}_\mu$ corresponds to 1/2. As a consequence one obtains the dense subset $\mathbf{H}_\mu \subset \mathbf{G}_\mu$ as asserted in (a).

Step 2. Let $Q \in \mathbf{H}_\mu$ be a point of period p. Using binary operations from step 1, one can find sequences $\{Q_n\}_{n=1}^{\infty}$ and $\{Q^n\}_{n=1}^{\infty}$ in $\mathbf{G}_\mu$ such that

- $\lim Q_n = Q = \lim Q^n$
- $N^p(Q_n) = Q_{n-1}, N^p(Q^n) = Q^{n-1}$
- $\{Q_n\}, \{Q^n\} \subset \{x : N^k(x) = P_\mu,$ some $k \in N\}$.

Step 3. Each of the points Q_n and Q^n is a touch point of a component of the iterated inverse images of S^-. The latter is the set

$$\cup_{k\geq 0} N^{-k}(S^-),$$

where $N^{-k}(X) = \{x : N^k(x) \in X\}$. Using these components one may construct sets M_n as given in Figure 5, and show that $N^p(M_n) \subset M_{n-1}$. Now one defines

$$M_Q = \cap_{n\geq 1} M_n,$$

so that $N^p(M_Q) \subset M_Q$ will follow by construction.

Step 4. On finally must show that

$$||N^{kp}(x)|| \to \infty \text{ as } k \to \infty,$$

whenever $x \in M_Q$. This part of the proof is supported by experimental evidence, except for special choices of h and μ, e.g., this step is not too difficult to establish in case $\mu = 2, h = 1$ and $p = 2$.

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