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Newton polygons, curves on torus surfaces, and the converse Weil theorem

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Introduction

For two monic polynomials depending on a single variable, the following identity holds: the product of the values of the first polynomial at the roots of the second polynomial is equal to the product of the values of the second polynomial at the roots of the first one apart from the sign.

André Weil found a far-reaching generalization of this identity. It can be used for any pair of non-zero meromorphic functions on a compact complex curve.

We present definitions necessary for the formulation of Weil's theorem. Let

$$f = c_1 u^{b_1} + \cdots, \qquad g = c_2 u^{b_2} + \cdots, \tag{$*$}$$

where $c_1 \neq 0$ and $c_2 \neq 0$ are the highest-order terms of the Laurent series of meromorphic functions f and g in a neighbourhood of a point a, and u is a local parameter such that $u(a) = 0$.

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For a vector-function (f,g), we call a non-cancellable integer-valued vector $\vec{n}=(n_1,n_2)$ proportional to its *exponent vector* $\vec{b}=(b_1,b_2)$ with a positive integer coefficient k, $\vec{b}=k\vec{n}$, the *type* of the germ at the point a. The coefficient k is called the *multiplicity* of the germ of the vector-function. We call the number $c_2^{n_1}c_1^{-n_2}$ the *reduced Weil number* of the germ $(*)$, where (n_1,n_2) are the components of the type $\vec{n}$ of this germ.

Starting from a compact complex curve Γ and a meromorphic function (f,g) on it, we define a function $\mathrm{Mul}_{\Gamma fg}$ mapping the product $\mathbb{Z}^2_{\mathrm{ir}}\times\mathbb{C}^*$, where $\mathbb{Z}^2_{\mathrm{ir}}$ is a set of non-cancellable integer-valued vectors on the plane, to the set $\mathbb{C}^*$ of non-zero complex numbers (the name of the function is derived from the word 'multiplicity'). The function $\mathrm{Mul}=\mathrm{Mul}_{\Gamma fg}$ has non-negative integer values and is equal to zero everywhere except at a finite number of points. By definition, its value on a pair $(\vec{n},c)$ is equal to the total multiplicity of points on the complex curve Γ at which the germ (f,g) has the type $\vec{n}$ and the reduced Weil number c.

In these terms, Weil's theorem is formulated as follows.

Weil's theorem.

$$\prod(-c)^{\mathrm{Mul}(\vec{n},c)}=1. \tag{1}$$

The degrees of the divisors f and g vanish on a compact curve Γ. In terms of the function $\mathrm{Mul}=\mathrm{Mul}_{\Gamma fg}$, these relations are of the form

$$\sum\mathrm{Mul}(\vec{n},c)\vec{n}=0. \tag{2}$$

The present results are based on the following simple observation: the Weil numbers of the germ (f,g) and of the germ (F,G), where $F=f^{a_{11}}g^{a_{12}}$, $G=f^{a_{21}}g^{a_{22}}$ and $A=\{a_{ij}\}$ is an integer-valued matrix with determinant 1, are equal to one another. This observation suggests that Weil's theorem must be related to the two-dimensional torus geometry (the matrix A specifies an automorphism of the two-dimensional torus $\mathbb{C}^{*2}$) and to the Newton polygon theory. We show that this is indeed so. First, the Weil numbers simplify and refine the classical theorem on Newton polygons (see §2). On the other hand, by using Newton polygons, one can offer a very simple proof of Weil's theorem: it is reduced to the Vieta formula for the product of roots of a polynomial (see §§1–2 and §9).

Is Weil's theorem invertible? That is, is it true that for any function Mul satisfying conditions (1)–(2), there exists a triple Γ,f,g such that $\mathrm{Mul}=\mathrm{Mul}_{\Gamma fg}$? In our paper we show that if the function Mul has more than two characteristic vectors (that is, vectors $\vec{n}\in\mathbb{Z}^2_{\mathrm{ir}}$ such that the function $\mathrm{Mul}(\vec{n},\cdot)$ does not vanish identically on $\mathbb{C}^*$), then we can answer this question affirmatively. We offer a complete description of triples Γ,f,g such that $\mathrm{Mul}=\mathrm{Mul}_{\Gamma fg}$.

In the exceptional case when the function $\mathrm{Mul}_{\Gamma fg}$ has two characteristic vectors, Weil's theorem can be refined. In this case, the function $\mathrm{Mul}=\mathrm{Mul}_{\Gamma fg}$ has the reflexivity property

$$\mathrm{Mul}(\vec{n},c)=\mathrm{Mul}(-\vec{n},c^{-1}).$$

In §9 a simple independent proof of this property is given. The refined Weil theorem proves to be also invertible in this exceptional case.

With a triple Γ, f, g we associate a polygon called the Newton polygon. This polygon can be recovered from the function $\mathrm{Mul} = \mathrm{Mul}_{\Gamma f g}$. We show that it plays the same role as the usual Newton polygon (see §11).

Another theme of our paper is the following one. Let D be a divisor lying on the union M_∞ of one-dimensional orbits of a torus surface M. Is there a curve $\Gamma \subset M$ that does not cross null-dimensional orbits of the surface M for which the divisor D is a divisor of the intersection of the curve Γ with the union of one-dimensional orbits M_∞? We denote the space of such curves Γ by $\mathcal{R}(D)$, and call the divisor D *admissible* if the space $\mathcal{R}(D)$ is not empty. We encode divisors D with some function $\mathrm{Mul} = \mathrm{Mul}_D$ on $\mathbb{Z}^2_{\mathrm{ir}} \times \mathbb{C}^*$. The admissibility condition for the divisor D proves to be identical to the condition for the existence of a triple Γ, f, g such that $\mathrm{Mul} = \mathrm{Mul}_{\Gamma f g}$.

In §§6–7 a general curve in the space $\mathcal{R}(D)$ is described. The admissible divisor D is related to a polygon Δ_D. The problem of describing a general curve in the space $\mathcal{R}(D)$ is equivalent to investigating the general equation $P = 0$ with a given Newton polygon $\Delta(P) = \Delta_D$ and fixed coefficients of monomials corresponding to the boundary points of the polygon Δ.

The investigation of the general equation with a given Newton polyhedron is a traditional problem. The initial data in this problem (the Newton polyhedron) are discrete. Besides discrete data (the Newton polygon) we also include continuous data (coefficients on the boundary). In particular, the solution of our problem involves a description of all curves $P = 0$ for which the Newton polygon does not include interior integer-valued points. This can be carried out because there are not many polygons without interior integer points (see §5).

The conditions for admissibility of a divisor D include a discrete and a continuous condition (the Vieta condition, see §4). In §10 a version of Abel's theorem is presented that gives an independent explanation of the Vieta condition.

Let P be a Laurent polynomial on the complex plane. In our paper we define a function $\mathrm{Mul} = \mathrm{Mul}_P$ on the space $\mathbb{Z}^2_{\mathrm{ir}} \times \mathbb{C}^*$. Given the function Mul, does a Laurent polynomial P exist such that $\mathrm{Mul} = \mathrm{Mul}_P$?

This rather obvious problem plays a key role in our paper. Conditions on the function Mul sufficient for the existence of a Laurent polynomial P such that $\mathrm{Mul} = \mathrm{Mul}_P$ prove to be just the same as the conditions described above (that is, the conditions for the existence of a triple Γ, f, g such that $\mathrm{Mul} = \mathrm{Mul}_{\Gamma f g}$, or the condition for admissibility of a divisor D such that $\mathrm{Mul} = \mathrm{Mul}_D$).

The problem of existence of a Laurent polynomial P such that $\mathrm{Mul} = \mathrm{Mul}_P$ can be completely solved using only elementary plane geometry (Pascal's condition for convex polygons) and elementary algebra (Vieta's formula for the product of roots of a polynomial). We start our paper with the solution of this problem.

A remark on the terminology: the words 'polyhedron' and 'polygon' mean a convex polyhedron and a convex polygon.

§1. Newton polygons, the Pascal relation, and the Vieta relation

Let $P(x, y)$ be a Laurent polynomial on the complex plane and $\Delta = \Delta(P)$ its Newton polygon lying in the plane of exponents. The plane of exponents $\mathbb{R}^2$ is supposed to be oriented. (Its orientation is fixed by the order of the variables x, y.) A lattice of integer exponents is fixed in it. Consequently, the dual plane $\mathbb{R}^{2*}$ is also oriented and the dual lattice is defined in it, which we call the lattice of powers. Let $\mathbb{Z}^2_{\text{ir}}$ denote the subset of non-cancellable non-zero vectors in the lattice of powers.

Each vector $\vec{n} \in \mathbb{Z}^2_{\text{ir}}$ determines the orientation of a line in the plane of exponents that is orthogonal to the vector $\vec{n}$. Namely, the line is oriented as the boundary of the region in which the scalar product by the vector $\vec{n}$ is positive. For example, for the usual orientation of the plane, the interior normal $\vec{n}$ to a side of the two-dimensional polygon Δ is oriented so that the motion along the polygon boundary is counter-clockwise.

For each vector $\vec{n} \in \mathbb{Z}^2_{\text{ir}}$, we denote by $\Delta^{\vec{n}}$ a side or a vertex of the polygon Δ where the scalar product with the vector $\vec{n}$ is minimum.

For each integer-valued polygon Δ, a function Len_Δ on $\mathbb{Z}^2_{\text{ir}}$ is defined by calculating the integer length of the face $\Delta^{\vec{n}}$ associated with the vector $\vec{n}$. By definition, the *integer-valued length* of a vertex is equal to zero, and the integer-valued length of a side containing $m+1$ integer-valued points is equal to m.

By the famous Minkowski theorem, a multidimensional polyhedron can be uniquely recovered (up to a parallel translation) from the function $S(\vec{n})$ which takes the normals $\vec{n}$ to the area S of the corresponding faces. The polyhedron exists if and only if the Pascal condition $\sum S(\vec{n})\vec{n} \doteq 0$ holds.

The following simple lemma is a two-dimensional integer version of the Minkowski theorem. Let Len be a function on $\mathbb{Z}^2_{\text{ir}}$ with non-negative integer values.

Lemma (an integer version of the two-dimensional Minkowski theorem). *For a non-zero function* Len *there exists an integer-valued polygon* Δ *such that* $\text{Len} = \text{Len}_\Delta$ *if and only if this function is non-zero only on a finite set and the Pascal condition*

$$\sum \text{Len}_\Delta(\vec{n})\vec{n} = 0$$

holds. The polygon Δ *is uniquely defined by the function* Len *up to a parallel translation.*

Proof. We show how to construct a polygon satisfying the Pascal condition, starting from the function Len. Let us fix the structure of a complex line on the plane $\mathbb{R}^2$ and thus identify $\mathbb{R}^2$ and $\mathbb{R}^{2*}$. Let us consider a finite set of vectors from $\mathbb{Z}^2_{\text{ir}}$ for which the function Len does not vanish. We number the vectors of this set in the order in which they occur in a rotation of a ruler around the origin of coordinates counter-clockwise. We put $l_{\vec{n}_j} = (-i)\,\text{Len}(\vec{n}_j)(\vec{n}_j)$, where i is the imaginary unit. Using the vectors $l_{\vec{n}_j}$, we construct a polygonal line joining the origin of the following vector to the end-point of the previous one. The closure condition of the polygonal line is equivalent to the Pascal condition. The polygonal line thus constructed is the boundary of the desired polygon. The remaining statements of the lemma are proved just as simply as this one.

For each vector $\vec{n} \in \mathbb{Z}^2_{\text{ir}}$, using the Laurent polynomial P we define the following polynomial $P_{\vec{n}}$ of one variable. If $\Delta^{\vec{n}}$ is the vertex Q, then the polynomial $P_{\vec{n}}$ is defined as a constant equal to the coefficient of the monomial corresponding to the vertex Q in the Laurent polynomial P. Now let $\Delta^{\vec{n}}$ be a side l of the Newton polygon Δ.

We call vertices A and B on the side l *senior* and *junior*, respectively, if the motion along l from the point A to the point B corresponds to the orientation of the side l related to the vector $\vec{n}$. Let the side l have an integer-valued length m, that is, let it contain $m+1$ integer-valued points. We number the integer-valued points on the side l by numbers from 0 to m, starting from the junior vertex B. Let the coefficient of the monomial in the Laurent polynomial P corresponding to the ith integer-valued point be equal to c_i. We define a polynomial $P_{\vec{n}}$ by the formula $P_{\vec{n}}(\xi) = \sum_{i=0}^{m} c_i \xi^i$. Thus, the degree of the polynomial $P_{\vec{n}}$ is equal to the integer-valued length m of the side l. Its senior coefficient is equal to the coefficient of the vertex A in the Laurent polynomial P. The constant term of the polynomial $P_{\vec{n}}$ is equal to the coefficient of the vertex B in the Laurent polynomial P.

To each Laurent polynomial $P(x,y)$ there corresponds a function Mul_P on $\mathbb{Z}^2_{\mathrm{ir}} \times \mathbb{C}^*$ that associates with a vector $\vec{n}$ and a complex number c the multiplicity of c as a root of the polynomial $P_{\vec{n}}(\xi)$. Each of the polynomials $P_{\vec{n}}$ can be recovered from the function Mul_P up to a multiplier. In particular, the degrees of these polynomials, that is, the function $\mathrm{Len}_{\Delta(P)}$, can be recovered. Therefore, the following question is an algebraic version of the Minkowski question: given the function $\mathrm{Mul}\colon \mathbb{Z}^2_{\mathrm{ir}} \times \mathbb{C}^* \to \mathbb{N}_+$, does a Laurent polynomial P exist such that $\mathrm{Mul} = \mathrm{Mul}_P$? In what follows we completely answer this question.

We say that a vector $\vec{n}_j \in \mathbb{Z}^2_{\mathrm{ir}}$ is characteristic for the function Mul if the restriction $\mathrm{Mul}(\vec{n}, \cdot)$ of this function to $\mathbb{C}^*$ is not identically zero.

Theorem (an algebraic analogue of the two-dimensional Minkowski theorem). *For the non-zero function* Mul *there exists a Laurent polynomial P such that* $\mathrm{Mul} = \mathrm{Mul}_P$ *if and only if the function* Mul *has finitely many characteristic vectors and*

(1) (the one-dimensional case) *if there are at most two characteristic vectors, then the reflexivity condition*

$$\mathrm{Mul}(\vec{n}, c) = \mathrm{Mul}(-\vec{n}, c^{-1})$$

holds;

(2) (the two-dimensional case) *if there are more than two characteristic vectors, then*

(a) *the Pascal condition holds*: $\sum \mathrm{Mul}(\vec{n}, c)\vec{n} = 0$,

(b) *the Vieta condition holds*: $\prod (-c)^{\mathrm{Mul}(\vec{n},c)} = 1$.

Proof. We start with the one-dimensional case.

First we note that the reflexivity condition implies both the Pascal condition and the Vieta condition. Moreover, if the non-zero function Mul has the reflexivity property, then it has exactly two characteristic vectors $\vec{n}_1$ and $\vec{n}_2$ that are related by $\vec{n}_1 + \vec{n}_2 = 0$.

If the Newton polygon of a Laurent polynomial P is a segment l, then the polynomial $P_{\vec{n}}$ does not equal a constant for exactly two vectors, namely, for the vectors $\vec{n}_1$ and $\vec{n}_2$ that are orthogonal to the segment l. In this case we have the obvious relations $\vec{n}_1 = -\vec{n}_2$ and $\xi^m P_{\vec{n}_1}(\xi^{-1}) = P_{\vec{n}_2}(\xi)$, where m is the integer-valued length of the segment l. It follows that the reflexivity condition

$\mathrm{Mul}_P(\vec{n}_1, c) = \mathrm{Mul}_P(-\vec{n}_1, c^{-1})$ holds. Conversely, let the function Mul be reflexive and have two characteristic vectors $\vec{n}_1$ and $\vec{n}_2$, with $\vec{n}_1 + \vec{n}_2 = 0$. A polygon constructed from the function $\mathrm{Len}(\vec{n}) = \sum \mathrm{Mul}(\vec{n}, c)\vec{n}$ is a segment that is orthogonal to the vector $\vec{n}_1$ and has the integer-valued length $\mathrm{Len}(\vec{n}_1) = \sum \mathrm{Mul}(\vec{n}_1, c)$.

We consider the Laurent polynomial P for which the Newton polygon coincides with the constructed segment, and the polynomial $P_{\vec{n}_1}(\xi)$ is equal to $c_0 \prod(\xi - c)^{\mathrm{Mul}(\vec{n}_1, c)}$. These conditions define the Laurent polynomial P.

The function $\mathrm{Mul}(P)$ coincides with the function Mul, since the polynomial $P_{\vec{n}_2}(\xi) = \xi^{\mathrm{Len}(\vec{n}_1)} P_{\vec{n}_1}(\xi^{-1})$ has roots inverse to the roots of the polynomial $P_{\vec{n}_1}$. In the one-dimensional case, both assertions of our theorem are proved.

Now we prove it for the two-dimensional case. Let the Laurent polynomial P have a two-dimensional Newton polygon. For a side $l_{\vec{n}}$ of a Newton polygon Δ with interior normal $\vec{n}$, the following relation holds:

$$\prod_c (-c)^{\mathrm{Mul}_P(\vec{n}, c)} = \frac{P_B}{P_A},$$

where A and B are the senior and the junior vertices on the side l, respectively, and P_A, P_B are the coefficients in the Laurent polynomial P corresponding to these vertices. In fact, this relation is the Vieta formula for the product of roots of the polynomial $P_{\vec{n}}$. The product $\prod(-c)^{\mathrm{Mul}_P(\vec{n}, c)}$ is equal to unity, since each vertex of the Newton polygon is the junior vertex for one side and the senior vertex for another.

Now we prove the converse statement. In case 2) the Pascal condition is postulated. Therefore, one can construct a polygon from the function Mul. Obviously, it is two-dimensional. Our knowledge of the function $\mathrm{Mul}(\vec{n}, c)$, the vector $\vec{n}$ being fixed, provides the possibility of recovering all coefficients of the Laurent polynomial P on the side $l_{\vec{n}}$ up to an arbitrary multiplier c_0, $P_{\vec{n}}(\xi) = c_0 \prod(\xi - c)^{\mathrm{Mul}(\vec{n}, c)}$.

We show that the Vieta condition guarantees the possibility of a concordant choice of arbitrary multipliers in the polynomials $P_{\vec{n}}$. Indeed, the sides of a two-dimensional polygon intersect only at vertices. Let us move along the sides $l_1, \dots, l_N$ at the boundary of the polygon Δ and write out the polynomials $P_{\vec{n}_1}, \dots, P_{\vec{n}_N}$ corresponding to the interior normals to these sides. We denote the product $\prod(-c)^{\mathrm{Mul}(\vec{n}_i, c)}$ by V_i. We fix the multiplier $c_0 = Q$ at the senior vertex of the side l_1 and put $P_{\vec{n}_1}(\xi) = Q \prod(\xi - c)^{\mathrm{Mul}(\vec{n}_1, c)}$. Then the coefficient at the junior vertex of this side is equal to QV_1. The junior vertex of the side l_1 is the senior vertex of the side l_2. Therefore, the polynomial $P_{\vec{n}_2}(\xi)$ must be equal to $QV_1 \prod(\xi - c)^{\mathrm{Mul}(\vec{n}_2, c)}$. The coefficient of the junior vertex of the side l_2 is equal to QV_1V_2. Moving along the polygon boundary, we come to the vertex that we have started from. We do not obtain a contradiction, since $QV_1 \cdots V_N = Q$ and the product $\prod(-c)^{\mathrm{Mul}(\vec{n}, c)}$ is equal to unity by virtue of the Vieta condition. Thus, we have recovered the coefficients of the Laurent polynomial P at the boundary of the polygon Δ up to a common factor. Choosing arbitrarily the coefficients of the monomials corresponding to interior points of the polygon Δ, we obtain the polynomial P such that $\mathrm{Mul} = \mathrm{Mul}(P)$. The theorem is proved.

The following corollary has a rather unexpected reformulation in the geometry of torus surfaces (see Theorem 2 in §10).

Corollary. *For each integer-valued polygon Δ there exists a Laurent polynomial Q such that its Newton polygon is Δ and it has the following property: all polynomials $Q_{\vec{n}}$, $\vec{n} \in \mathbb{Z}^2_{\mathrm{ir}}$, have no roots different from -1.*

This corollary follows immediately from the theorem. However, it has an even simpler proof.

Proof of Corollary. If the minimum of the scalar product by the vector $\vec{n} \in \mathbb{Z}^2_{\mathrm{ir}}$ is attained at a side l of the polygon Δ and the integer-valued length of the side l is equal to $m(l)$, then we put $Q_{\vec{n}}(\xi) = (1+\xi)^{m(l)}$. If the minimum is attained at a vertex, then we put $Q_{\vec{n}}(\xi) \equiv 1$. Both the senior and the junior coefficients of the polynomial $(1+\xi)^m$ are equal to 1. Therefore, if the polygon Δ is two-dimensional, then by writing the polynomial $(1+\xi)^{m(l)}$ on each side l of it, we do not get a contradiction: the polygon sides intersect only at vertices, and the coefficient **1** will be written at each vertex. On the other hand, if the polygon is reduced to a single segment, we also do not get a contradiction, since the polynomial $(1+\xi)^m$ is reflexive, $\xi^m(1+\xi^{-1})^m = (\xi+1)^m$. At all interior integer-valued points of the polygon Δ, we can write any coefficients. Thus, we obtain the desired Laurent polynomial Q.

Let us sum up the results. We call a function Mul *admissible* if it satisfies the conditions of the theorem. Given an admissible function Mul, the polygon $\Delta = \Delta(\mathrm{Mul})$ is uniquely constructed (up to a parallel transfer). Given an admissible function Mul, for any Laurent polynomial P such that $\mathrm{Mul}_P = \mathrm{Mul}$, all coefficients of the monomials corresponding to the points on the boundary of the polygon Δ can be uniquely recovered (up to a common non-zero constant multiplier). The coefficients of the interior monomials can be chosen arbitrarily. Therefore, the Laurent polynomials P for which $\mathrm{Mul}_P = \mathrm{Mul}$ are a set of non-zero vectors in the complex linear space whose dimension is equal to $B(\Delta)+1$, where $B(\Delta)$ is the number of interior integer-valued points of the polygon $\Delta = \Delta(\mathrm{Mul})$.

§2. Newton polygons and Weil numbers

Let Γ be the germ of an analytical curve at the point a, and let u be a local parameter, $u\colon \Gamma \to C$, such that $u(a) = 0$. We consider the germ of (f,g), a meromorphic function on Γ. Let

$$f = c_1 u^{b_1} + \cdots, \quad g = c_2 u^{b_2} + \cdots, \tag{1}$$

where $c_1 \neq 0$ and $c_2 \neq 0$ are the higher-order terms of the Laurent series of these functions. (Here and in what follows, we assume that neither the germ of f nor the germ of g vanishes identically. We also always assume that the germ of (f,g) is not the germ of a constant vector-function.)

We call the number

$$\{f,g\}_a = (-1)^{b_1+b_2+b_1 b_2} c_2^{b_1} c_1^{-b_2}$$

the *Weil number* of the germ of the vector-function (f,g)

It is easy to check that the number $\{f,g\}_a$ is well defined, that is, it does not depend on the choice of the local parameter u.

The Weil numbers occur in Weil's theorem (see §9), which is closely related to the present investigation.

We present definitions of a number of invariants of the germ of the vector-function (f,g). We call a non-cancellable integer-valued vector $\vec{n}=(n_1,n_2)$ proportional to its *exponent vector* $\vec{b}=(b_1,b_2)$ with integer coefficient k, $\vec{b}=k\vec{n}$, the *type* of the germ of the vector-function (f,g). The coefficient k is called the *multiplicity* of the germ. The *reduced Weil number* $[f,g]_a$ of the germ of the vector-function (f,g) is the number $[f,g]=c_2^{n_2}c_1^{-n_1}$, where (n_1,n_2) are the components of the type $\vec{n}$ of this germ.

The next statement follows from the definitions.

Lemma 1. *The Weil number is expressed in terms of the reduced Weil number and the germ multiplicity as follows*:

$$\{f,g\}_a=(-[f,g]_a)^k.$$

The invariants of the germ of the meromorphic function that we are dealing with are preserved under power transformations. Let $A=\{a_{ij}\}$ be a unimodular 2×2-matrix (that is, a matrix with integer coefficients and unit determinant). Let a power transformation of the vector-function (f,g) be given by the matrix A. The germ of (f,g) is transformed to the germ of (F,G), where

$$F=f^{a_{11}}g^{a_{12}} \quad \text{and} \quad G=f^{a_{21}}g^{a_{22}}.$$

Lemma 2. *For each unimodular matrix A, the germ type of the vector-function (F,G) is equal to $A\vec{n}$, where $\vec{n}$ is the germ type of the vector-function (f,g). The multiplicity, the reduced Weil number and the Weil number for the germ of the vector-function (F,G) are the same as for the vector-function (f,g).*

Lemma 2 can be proved by straightforward calculation. It suggests that the Weil numbers are to be related to the theory of Newton polygons and to the two-dimensional torus geometry.

In the present paper we show that this is indeed so. First of all, by using the Weil numbers we can simplify the formulation of the classical theorem on Newton polygons that will be discussed below.

Let an algebraic curve Γ lie within a torus $\mathbb{C}^{*2}$, x and y being the coordinate functions. On the normalization $\overline{\Gamma}$ of this curve we define two meromorphic functions x and y that meromorphically map the curve $\overline{\Gamma}$ on the torus $\mathbb{C}^{*2}$ onto the curve $\Gamma\subset\mathbb{C}^{*2}$. The following theorem goes back to Newton.

Theorem 1 (on Newton polygons). *Let a curve Γ in $\mathbb{C}^{*2}$ be defined by the equation $P(x,y)=0$, and suppose that it has no multiple components. On the normalization $\overline{\Gamma}$ of this curve there exist points at which the vector-function (x,y) is of type $\vec{n}\neq 0$ if and only if the minimum of the scalar product by the vector $\vec{n}$, on the Newton polygon $\Delta(P)$, is attained on a side of this polygon. Moreover, calculated with regard to multiplicity, the number of points where the vector-function (x,y) is of type $\vec{n}$ and its reduced Weil number is a fixed number ξ_0 is equal to the degree* $\operatorname{ord}_{\xi_0}P_{\vec{n}}$ *of the polynomial $P_{\vec{n}}$ at the point ξ_0.*

Proof. We perform a power transformation of the (x, y)-plane by using a unimodular matrix A that takes a non-cancellable vector $\vec{n}$ to the vector $(1,0)$. This transformation changes neither the reduced Weil numbers nor the multiplicity of the germs of the vector-function (x, y) on the curve $\overline{\Gamma}$ (see Lemma 1). We are interested in points on the curve $\overline{\Gamma}$ in a neighbourhood of which the functions x and y are of the form

$$x = c_1 u^k + \cdots ,$$
$$y = c_2 + \cdots ,$$

where $c_1 c_2 \neq 0$, $k > 0$. We assume that the minimum of the scalar product by the vector $(1,0)$ is reached on a side l of the polygon $\Delta(P)$. Let us perform a parallel transfer of the polygon $\Delta(P)$ in such a way that the junior vertex on the side l coincides with the origin of coordinates. The parallel transfer corresponds to multiplication of the Laurent polynomial P by a monomial and does not change the curve Γ defined on the torus $\mathbb{C}^{*2}$ by the equation $P = 0$.

After such a transformation, the restriction of the Laurent polynomial P to the y-axis is defined for $y \neq 0$. The polynomial $P(0, y)$ coincides with the polynomial $P_{\vec{n}}(\xi)$ defined above when $\xi = y$. On the one hand, the multiplicity of the intersection of the y-axis with the closure of the curve $\Gamma \subset \mathbb{C}^{*2}$ at the point $(0, \xi_0)$ is equal to the multiplicity of the root ξ_0 of the polynomial $P(0, y) = P_{\vec{n}}(y)$. On the other hand, this multiplicity is equal to the number of points on the normalized curve $\overline{\Gamma}$ in the neighbourhood of which

$$x = c_1 u^k + \cdots ,$$
$$y = \xi_0 + \cdots ,$$

with regard to their multiplicities k. This completes the proof.

If the minimum of the scalar product by the vector $\vec{n}$ is reached at a vertex of the polygon $\Delta(P)$, then the theorem is proved similarly (and even more simply: in this case the polynomial $P_{\vec{n}}$ is constant and has no roots at all).

The theorem on Newton polygons can immediately be generalized to algebraic curves with multiple components. For the germ (1) of a vector-function on a component of an algebraic curve of multiplicity μ, the *total multiplicity* is the number $k\mu$, where k is the multiplicity for the germ (1).

Theorem 1′ (on Newton polygons). *On the normalization $\overline{\Gamma}$ of the curve Γ in $\mathbb{C}^{*2}$ defined by the equation $P(x, y) = 0$ there exist points at which the vector-function (x, y) is of type $\vec{n} \neq 0$ if and only if the minimum of the scalar product by the vector $\vec{n}$ is attained on the Newton polygon $\Delta(P)$ on a side of this polygon. Moreover, the number of points (calculated with regard to total multiplicities) at which the vector-function (x, y) is of type $\vec{n}$ and its reduced Weil number is a fixed number ξ_0 is equal to the degree* $\operatorname{ord}_{\xi_0} P_{\vec{n}}$ *of the polynomial $P_{\vec{n}}$ at the point ξ_0.*

Theorem 1′ follows automatically from Theorem 1. As a matter of fact, the Laurent polynomial can be expressed as the product of irreducible Laurent polynomials for which Theorem 1 is satisfied.

§3. Parametrization of one-dimensional orbits on a torus surface

As is well known, the Newton polyhedra are most closely related to the torus compactifications of a torus [4].

The two-dimensional situation is a special one. The group $\mathbb{C}^*$ has only one non-trivial automorphism (that takes a point t to t^{-1}). The specific character of a two-dimensional torus is the existence of a simultaneous natural choice of parametrization on each of its connected quotient groups of dimension 1. This parametrization depends only on the orientation of the plane of one-parameter groups and is changed to the opposite under change of the orientation. Let us determine this parametrization. We fix the orientation of the plane of one-parameter groups. Let $O_{\vec{b}}$ be a one-parameter subgroup of $\mathbb{C}^{*2}$ corresponding to the vector $\vec{b} = (b_1, b_2)$, $\vec{b} \neq 0$. (Such a group is defined by a homomorphism of the standard group $\mathbb{C}^*$ to the group $\mathbb{C}^{*2}$ taking the point τ to the point $\tau^{\vec{b}} = (\tau^{b_1}, \tau^{b_2})$.) Let $\pi_{\vec{b}}$ denote the projection of the group $\mathbb{C}^{*2}$ onto a quotient group of $\mathbb{C}^{*2}$ with respect to the subgroup $O_{\vec{b}}$.

Definition. The parametrization t of a quotient group $\mathbb{C}^{*2}/O_{\vec{b}}$ is called the parametrization *consistent* with the orientation of the plane of one-parameter groups if for any integer-valued vector $\vec{m}$ such that the pair of vectors $(\vec{b}, \vec{m})$ defines the correct orientation of one-parameter groups, the following equality holds:

$$\lim_{\tau \to 0} t\big(\pi_{\vec{b}}(\tau^{\vec{m}})\big) = 0.$$

It is easy to verify that there exists a parametrization consistent with the orientation of the plane of one-parameter groups. Let us describe this parametrization in coördinates. We consider a standard two-dimensional torus $\mathbb{C}^{*2}$ with coordinate functions (x, y) and with the standard orientation of the plane of one-parameter groups. Let $\vec{b} = (b_1, b_2)$ a non-zero integer-valued vector, $\vec{n} = (n_1, n_2)$ a non-cancellable integer-valued vector such that $\vec{b} = k\vec{n}$, where k is a positive integer, and $\pi_{\vec{b}}$ the projection of $\mathbb{C}^{*2}$ onto the quotient group $\mathbb{C}^{*2}/O_{\vec{b}}$.

It is easy to check the following statement.

Lemma. *A mapping taking a point $c \in \mathbb{C}^{*2}$ to the parameter t of the point $\pi_{\vec{b}}(c)$ is described by the formula $t = c_2^{n_1} c_1^{-n_2}$, where $c = (c_1, c_2)$ and $\vec{n} = (n_1, n_2)$, under the quotient group parametrization described above.*

If we fix an isomorphism between one-dimensional quotient groups of the torus $\mathbb{C}^{*2}$ and the group $\mathbb{C}^*$, we obtain the following corollary.

Assertion 1. *On any torus surface there exists a natural parametrization of each one-dimensional orbit depending only on the orientation of the plane of one-parameter groups.*

Proof. Let $M_\infty^{\vec{n}}$ be a one-dimensional orbit corresponding to the ray generated by the vector $\vec{n}$ in the fan of this surface (see [3], [7]). The points of the torus $\mathbb{C}^{*2}$ tend to this orbit under the action of the one-parameter group $t^{\vec{n}}$ as $t \to 0$. The orbit $M_\infty^{\vec{n}}$ is naturally isomorphic to the quotient group of the torus $\mathbb{C}^{*2}$ with respect to the subgroup $t^{\vec{n}}$. The above described parametrization of this quotient group presents a natural parametrization of the orbit.

This torus construction is related to the Weil numbers as follows. Let $f = c_1u^{b_1}+\cdots$, $g = c_2u^{b_2}+\cdots$ be a germ of a meromorphic mapping of the curve (Γ, a) to the torus $\mathbb{C}^{*2}$. Let $\vec{b} = k\vec{n}$, $k > 0$, and let $M^{\vec{n}} \supset \mathbb{C}^{*2}$ be a torus surface whose fan is a ray generated by the vector $\vec{n}$. The surface $M^{\vec{n}}$ contains exactly one one-dimensional orbit $M^{\vec{n}}_{\infty}$. The order of the components f and g in the vector-function (f, g) fixes the orientation of the plane, and consequently fixes the parametrization of the one-dimensional orbit $M^{\vec{n}}_{\infty}$.

Assertion 2. *The germ of a meromorphic mapping $(f, g)\colon \Gamma \to \mathbb{C}^{*2}$ can be continued to the germ of the analytic mapping $(\widetilde{f}, \widetilde{g})\colon \Gamma \to M^{\vec{n}}$. This analytic mapping takes a point a to a point of the one-dimensional orbit $M^{\vec{n}}_{\infty}$ with the parameter t equal to the reduced Weil number $[f, g]_a$. The image of the curve Γ intersects the orbit $M^{\vec{n}}_{\infty}$ with multiplicity k equal to the multiplicity of the germ of (f, g).*

Proof. As $u \to 0$ the point $(c_1u^{b_1} + \cdots, c_2u^{b_2} + \cdots)$ tends to a point on the orbit $M^{\vec{n}}_{\infty}$ that, by the lemma, has parameter equal to $c_2^{n_1}c_1^{-n_2}$. This number is equal to the reduced Weil number. The remaining facts of the statement are easily verified by inspection.

§4. Curves on a torus surface

Let M be a compact torus surface, possibly having singularities at points that are null-dimensional orbits. Let finitely many points with given positive multiplicities be fixed on the union M_∞ of one-dimensional orbits of the surface M. Does there exist a curve on the surface M that does not pass through null-dimensional orbits and intersects one-dimensional orbits at given points with given multiplicities? We present here the complete answer to this question.

In order to formulate it, we first encode the divisors D, whose supports lie in the union of one-dimensional orbits of M, by the function

$$\mathrm{Mul}_D\colon \mathbb{Z}^2_{\mathrm{ir}} \times \mathbb{C}^* \to \mathbb{N}_+$$

to be defined below. We fix the orientation in the plane of one-parameter groups of the torus $\mathbb{C}^{*2}$. Thereby we fix the parametrization of each one-dimensional orbit on any torus compactification M of the torus $\mathbb{C}^{*2}$. Let a vector $\vec{n} \in \mathbb{Z}^2_{\mathrm{ir}}$ correspond to the one-dimensional orbit $M^{\vec{n}}_{\infty}$ on the surface M. In this case we set the function $\mathrm{Mul}_D(\vec{n}, c)$ equal to the multiplicity with which a point having parameter c on the one-dimensional orbit $M^{\vec{n}}_{\infty}$ is included in the divisor D. If the vector $\vec{n} \in \mathbb{Z}^2_{\mathrm{ir}}$ corresponds to a null-dimensional orbit on M, then we set the function $\mathrm{Mul}_D(\vec{n}, c)$ equal to zero for all points $c \in \mathbb{C}^*$.

Theorem. *A divisor D is the divisor of the intersection of a curve not passing through null-dimensional orbits of a torus surface M with the union M_∞ of one-dimensional orbits of M if and only if this curve is the closure of the curve defined in $\mathbb{C}^{*2}$ by the equation $P = 0$, the function Mul_P of the Laurent polynomial P being equal to the function Mul_D of the divisor D: $\mathrm{Mul}_P = \mathrm{Mul}_D$.*

Proof. The torus surface M contains the torus $\mathbb{C}^{*2}$. Any algebraic curve lying on the surface M and including no one-dimensional orbits as components is the closure of a curve lying in the torus $\mathbb{C}^{2*}$. Each curve in $\mathbb{C}^{2*}$ is given by an equation $P = 0$, where P is a Laurent polynomial. Let us consider the function Mul_P constructed from such a Laurent polynomial. If the vector $\vec{n} \in \mathbb{Z}^2_{\mathrm{ir}}$ in the fan of the surface M corresponds to a null-dimensional orbit A, then the function $\mathrm{Mul}_P(\vec{n}, \cdot)$ must vanish identically on $\mathbb{C}^*$. Otherwise, as one can see from Theorem 1′ on Newton polygons, the closure of the curve $P = 0$ in the surface M contains the null-dimensional orbit A. Let a vector $\vec{n} \in \mathbb{Z}^2_{\mathrm{ir}}$ in the fan of the surface M correspond to the one-dimensional orbit $M^{\vec{n}}_{\infty}$. Then, by virtue of Theorem 1′ on Newton polygons and Assertion 2 of §3, the closure of the curve $P = 0$ intersects the orbit $M^{\vec{n}}_{\infty}$ only at points whose parameters are roots of the polynomial $P_{\vec{n}}$. The multiplicity of the intersection point is equal to the multiplicity of the corresponding root of the polynomial $P_{\vec{n}}$. The theorem is proved.

Let us summarize. Suppose that the support of a divisor D lies on at most two orbits. Then the desired curve exists if and only if

(1) there are exactly two such orbits with corresponding opposite vectors $\vec{n}_1$ and $\vec{n}_2$, $\vec{n}_1 + \vec{n}_2 = 0$;
(2) the point on the orbit $M^{\vec{n}_1}_{\infty}$ with parameter c is included in the divisor D with the same multiplicity as the point that is included in D with parameter c^{-1} on the orbit $M^{\vec{n}_2}_{\infty}$.

Let a divisor D be included in the union of at least three one-dimensional orbits. Then the desired curve exists if and only if

(1) the Pascal condition $\sum \mathrm{Mul}_D(\vec{n}, c) = 0$ holds,
(2) the Vieta condition $\prod(-c)^{\mathrm{Mul}_D(\vec{n},c)} = 1$ holds.

If conditions (1) and (2) are satisfied, then the desired curve Γ can be defined as follows. From the function Mul_D we construct a Newton polygon Δ_D such that

$$\mathrm{Len}_{\Delta_D}(\vec{n}) = \sum_c \mathrm{Mul}_D(\vec{n}, c).$$

This polygon is uniquely defined up to a parallel transfer. Further, for the Laurent polynomial P we single out the coefficients of all monomials corresponding to the boundary points of the polygon Δ. These coefficients are uniquely defined up to a common multiplier from the condition that the degree of the polynomial $P_{\vec{n}}$ at the point c is equal to $\mathrm{Mul}_D(\vec{n}, c)$. All the remaining coefficients of the Laurent polynomial P are arbitrary.

We denote by $\mathcal{R}(D)$ the space of all curves on the surface M not passing through null-dimensional orbits and intersecting the union of one-dimensional orbits with respect to the divisor D. Since proportional equations define the same curve, $\mathcal{R}(D)$ is a projectivization of the space of Laurent polynomials P for which $\mathrm{Mul}_P = \mathrm{Mul}_D$.

The space of Mul_P is the complement to the point 0 in a linear space of dimension $B(\Delta) + 1$, where $B(\Delta)$ is the number of interior integer-valued points of the polygon Δ, $\Delta_D = \Delta(P)$. (We identify Laurent polynomials whose quotient is a monomial.) Therefore the space $\mathcal{R}(D)$ is a projective space of dimension $B(\Delta)$.

§5. Polygons without interior integer-valued points

We begin with a list of exceptional polygons, none of which contains interior integer-valued points.

List of exceptional polygons.

1. A segment of integer length m. It is transformed by some unimodular transformation to the interval with vertices $(0,0)$, $(m,0)$.
2. A triangle whose sides have the integer-valued length 2. It is transformed by some unimodular transformation to the simplex with vertices $(0,0)$, $(2,0)$, $(2,0)$.
3. A triangle of integer-valued height 1 whose base is of integer-valued length equal to m. It is transformed by some unimodular transformation to the simplex with vertices $(0,0)$, $(m,0)$, $(1,0)$.
4. A trapezium of integer-valued height 1 whose bases are integer-valued of lengths k and m, where $0 < k \leqslant m$. It is transformed by an integer-valued transformation to the trapezium with the vertices $(0,0)$, $(m,0)$, $(1,0)$, $(1,k)$.

Theorem. *If an integer-valued polygon contains interior integer-valued points, then for each side l of it there exists an integer-valued interior point whose integer-valued height with respect to the side l is equal to* 1. *All integer-valued polygons without interior integer-valued points are included in the list of exceptional polygons.*

Proof. We show that if an integer-valued polygon is not included in the list of exceptional polygons, then for each side l of it there exists an interior integer-valued point whose integer-valued height with respect to l is equal to 1. Obviously, this implies both assertions of the theorem. We perform a unimodular transformation of the plane (u,v) so that the polygon Δ lies in the upper half-plane $v \geqslant 0$, whereas the side l lies on the axis $v = 0$. If the intersection of the polygon Δ with the line $v = 1$ is empty, then the integer-valued polygon Δ is a segment. If this intersection is a point, then Δ is a triangle of integer-valued height 1. If the segment $\Delta \cap (v = 1)$ contains an interior integer-valued point, then either this point is an interior point of the polygon Δ whose integer-valued height with respect to the side l is equal to 1, or the segment $\Delta \cap (v = 1)$ is a side of the polygon Δ. In the latter case, the polygon Δ is an integer-valued trapezium with integer-valued height equal to 1. We show that if the segment $\Delta \cap (v = 1)$ does not contain an interior integer-valued point, then the polygon Δ is an exceptional polygon. Indeed, in this case the segment $\Delta \cap (v = 1)$ is transformed to the band $0 \leqslant u \leqslant 1$ by a unimodular transformation of the plane fixing all points of the horizontal coordinate axis and mapping all horizontal lines to themselves.

Let the segment $\Delta \cap (v = 1)$ coincide with the segment $(0 \leqslant u \leqslant 1, v = 1)$. The following situations are possible. If the side l is of length 1, then the polygon Δ lies in the band $0 \leqslant u \leqslant 1$ and is a trapezium with integer-valued height equal to 1. If the side l is of length 2, then the polygon Δ is a simplex $(u \geqslant 0, v \geqslant 0, u + v \leqslant 2)$ with integer-valued sides equal to 2. In our case, there are no integer-valued polygons Δ with sides l of length larger than 2.

If the length of the segment $\Delta \cap (v = 1)$ is less than 1, then the only possible cases are those in which the length of the side l is equal to 1, whereas the polygon Δ

is a triangle with integer-valued height equal to 1 (with vertex on the line $u = 0$ or on the line $u = 1$). The theorem is proved.

§6. Polygon Δ_D with an interior integer-valued point

In this section we describe a general curve in the space $\mathcal{R}(D)$ (see §4) in the case when the polygon Δ_D contains an interior integer-valued point.

Lemma. *Suppose that a point a lies on the one-dimensional orbit $M^{\vec{n}}_{\infty}$ of a surface M and is included in a divisor D with multiplicity $k(a)$. Let the divisor D be admissible and let the polygon Δ_D contain an interior integer-valued point. Then a general curve Γ in the space $\mathcal{R}(D)$ is smooth around the point a and Γ is tangent to the orbit $M^{\vec{n}}_{\infty}$ with multiplicity $k(a)$.*

Proof. Let c be the parameter of the point a on the orbit $M^{\vec{n}}$ and let $P = 0$ be the equation of the curve in $\mathbb{C}^{*2}$, where $\mathrm{Mul}_P = \mathrm{Mul}_D$.

We perform a unimodular power transformation of the torus $\mathbb{C}^{*2}$ that takes the vector $\vec{n}$ to the vector $\vec{e}_1$. After such a transformation and cancellation by an appropriate monomial, the equation $P = 0$ will have the following properties:

1) the Laurent polynomial P will be regular in the plane (x, y) outside the axis $y = 0$;
2) its restriction $P(0, y)$ to the axis $x = 0$ will coincide with the polynomial $P_{\vec{n}}(y)$.

In the coordinates (x, y) the axis $x = 0$ coincides with the orbit $M^{\vec{n}}_{\infty}$, and the point a coincides with the point $(0, c)$. If the polynomial $P_{\vec{n}}$ has a root of multiplicity 1 at the point c, then our lemma follows from the implicit function theorem. Now let c be a root of multiplicity $\geqslant 2$ of the polynomial $P_{\vec{n}}$. By the theorem in §5, there is an integer-valued point with coordinates $(1, k)$ inside the Newton polygon Δ, where k is some integer.

Therefore, for any values of the parameter λ, the polygon $P_\lambda = P + \lambda x y^k$ lies in the space considered, that is, $\mathrm{Mul}_{P_\lambda} = \mathrm{Mul}_D$. If $\lambda \neq -c^{-k}\dfrac{\partial P}{\partial y}(0, c)$, then the implicit function theorem can be applied to the equation $P_\lambda = 0$ around the point $(0, c)$. Our lemma is proved.

Theorem. *Let a divisor D be admissible, and let its Newton polygon Δ contain an interior integer-valued point. Then a general curve in the space $\mathcal{R}(D)$ is a smooth irreducible curve of genus $g = B(\Delta)$, where $B(\Delta)$ is the number of interior points of Δ. The intersection of the general curve in the space $\mathcal{R}(\Delta)$ with the torus $\mathbb{C}^{*2}$ is a sphere with g handles and with q points removed, where q is the number of geometrically distinct points in the support of the divisor D.*

Proof. The general equation $P = 0$, where $\mathrm{Mul}_P = \mathrm{Mul}_D$, defines a non-singular curve in the torus $\mathbb{C}^{*2}$. Indeed, let a point (k, m) be interior for the polygon Δ. We rewrite the equation $P(x, y) = 0$ in the form $\widetilde{P} x^{-k} y^{-m} = \lambda$, where $\widetilde{P} = P - \lambda x^k y^m$. By virtue of the Sard–Bertini theorem, the curve $\widetilde{P} x^{-k} y^{-m} = \lambda$ is non-singular for almost all values of the parameter λ. By our lemma, a general curve in the space $\mathcal{R}(D)$ is non-singular also at the points of intersection with one-dimensional orbits of the surface M. Therefore the general curve in the space $\mathcal{R}(D)$ has no singular points at all. By the implicit function theorem, a small change of coefficients of the equation of a smooth curve does not change the topology of this curve. Changing the coefficients of all monomials a little, we can make this equation Δ-non-degenerate. Applying well-known results on Δ-non-degenerate curves [5], we can make the general curve $\mathcal{R}(D)$ irreducible and have genus $B(\Delta)$. The number of geometrically distinct points of intersection of a smooth curve in the space $\mathcal{R}(D)$ with the union of one-dimensional orbits is equal to the number of geometrically distinct points in the divisor D.

§7. Polygon Δ_D without interior integer-valued points

Let a divisor D on the union of one-dimensional orbits of a torus surface M be admissible, and let us assume that its Newton polygon Δ_D does not contain interior integer-valued points. The complete description of this case is based on the complete enumeration of polygons without interior integer-valued points (see §5). If $\Delta(D)$ has no interior integer-valued points, then, up to a multiplier, there exists only one Laurent polynomial P such that $\mathrm{Mul}_P = \mathrm{Mul}_\Delta$ and only one curve in the space $\mathcal{R}(D)$. We denote this curve by $r(D)$.

Theorem 1. *By a power unimodular change of coordinates and multiplication by a monomial, the Laurent polynomial P can be reduced either to a polynomial of degree m with non-zero constant term independent of the first coordinate function, or to a quadratic polynomial with non-zero constant term and with non-zero coefficients of x^2 and y^2, or to the polynomial $yS_k(x) + Q_m(x)$, where S_k and Q_m are polynomials of degrees k and m, $m > 0$, $m \geqslant k \geqslant 0$, with non-zero constant terms.*

Proof. This follows immediately from the consideration of the list of polynomials without interior integer-valued points (see §5).

We additionally assume that the equation $P = 0$ is irreducible. Then after the power change of coordinates, this equation defines a horizontal line in the first case, a smooth quadric in the second case, and the graph of a rational function in the third case. We see that the irreducible curve $r(D)$ behaves exactly like a general curve in the space $\mathcal{R}(D)$ if the polygon Δ_D has interior integer-valued points.

Theorem 2. *Suppose that a divisor D is admissible and its Newton polygon Δ does not contain interior integer-valued points. Additionally, suppose that the unique curve $r(D)$ in the space $\mathcal{R}(D)$ is irreducible. Then it is a smooth rational curve on the surface M. If D contains a point a with multiplicity $k(a)$, then the curve $r(D)$ is tangent at the point a to the one-dimensional orbit passing through the point a with multiplicity $k(a)$.*

It remains to describe conditions for irreducibility of the curve in terms of the divisor D and to analyse this case.

The case of a segment. Let the function Mul_D of the divisor D have exactly two characteristic vectors $\vec{n}_1$ and $\vec{n}_2$ such that $\vec{n}_1 + \vec{n}_2 = 0$, and let it possess the reflexivity property $\mathrm{Mul}_D(\vec{n}_1, c) = \mathrm{Mul}_D(\vec{n}_2, c^{-1})$.

The Newton polynomial Δ_D is a segment in this case. To describe the curve $r(D)$, we perform a unimodular automorphism of the torus $\mathbb{C}^{*2} \subset M$ taking the vector $\vec{n}_1$ from the Lie algebra of the torus $\mathbb{C}^{*2}$ to the vector $\vec{e}_1$, where $\vec{e}_1 = (1,0)$. After such a transformation, the curve $r(D)$ in $\mathbb{C}^{*2}$ with coordinates (x, y) will consist of the union of horizontal lines $y = c_i$; each line $y = c_i$ is included in the curve $r(D)$ with multiplicity equal to the multiplicity of the point with parameter c_i on the orbit $M^{\vec{n}_1}$ with which it is included in the divisor D.

In particular, the *curve* $r(D)$ *does not contain multiple components if and only if the function* Mul_D *is equal to* 0 *or* 1. *The curve* $r(D)$ *is not multiple and irreducible if and only if the function* Mul_D *is equal to* 1 *at exactly two points* $(\vec{n}_1, c)$ *and* $(-\vec{n}_1, c^{-1})$ *and vanishes at all the remaining points.*

Now we pass to the case of two-dimensional polygons Δ_D. In this case the admissible divisors D satisfy the Vieta condition. The combinatorial types of polygons Δ differ in the cases considered below, and the Pascal conditions are of different form.

Simplex with sides of integer-valued length 2. In this case the divisor D contains 2 points (taking account of multiplicity) on each of the three orbits corresponding to the vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$. The Pascal condition is of the form

$$\vec{n}_1 + \vec{n}_2 + \vec{n}_3 = 0.$$

Assertion 1. *The curve* $r(D)$ *is reducible if and only if one can choose, in the support of the divisor* D, *one point on each of the three orbits so that the product of their parameters is* -1. *In this case the curve consists of two components. These components merge into one multiple component if and only if, in addition, all points of the divisor are multiple. If these components differ, then they intersect at exactly one point of the surface* M. *This point is not a point of the torus* $\mathbb{C}^{*2}$ *if and only if one point of the divisor* D *is multiple.*

Proof. With a power transformation, the curve $r(D)$ is transformed to a quadric. The assertion is obvious for quadrics.

Remark. If an admissible divisor contains three double points, then by the Vieta condition $[(-\xi_1)(-\xi_2)(-\xi_3)]^2 = 1$, where ξ_i are the parameters of these points. Two cases are possible:

(1) the case $(-\xi_1)(-\xi_2)(-\xi_3) = 1$ corresponds to one component of multiplicity 2,

(2) the case $(-\xi_1)(-\xi_2)(-\xi_3) = -1$ corresponds to a non-singular curve; with a power transformation, such a curve can be transformed to a parabola tangent to the coordinate axes.

The case of a triangle of integer-valued height 1. Let the divisor D contain points on the three orbits corresponding to the vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$. Suppose that on the orbit corresponding to the vector $\vec{n}_1$ the divisor D contains at least as many points as on either of the remaining orbits. Let this number of points be equal to the number $k \geqslant 1$. Our case is defined by the following conditions:

(1) the divisor has exactly one point on each of the remaining orbits,

(2) the following relations hold: $k\vec{n}_1 + \vec{n}_2 + \vec{n}_3 = 0$, $|\det(\vec{n}_1, \vec{n}_2)| = |\det(\vec{n}_1, \vec{n}_3)| = 1$.

Under conditions (1) and (2) the admissibility condition for the divisor D is the Vieta condition.

Assertion 2. *Under these conditions, the curve $r(D)$ is always irreducible.*

Proof. A triangle with integer-valued height 1 cannot be decomposed into the Minkowski sum of integer-valued polygons. The Laurent polynomial with such a Newton polygon is irreducible.

Trapezium of integer-valued height 1. This case appears under the following circumstances: the divisor D contains points on the four orbits corresponding to the vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$, $\vec{n}_4$. On one of the orbits there are at least as many points as on the other orbits. Let this orbit correspond to the vector $\vec{n}_1$. We denote the number of points on it by m, $m \geqslant 1$. One of the vectors $\vec{n}_2$, $\vec{n}_3$, $\vec{n}_4$ is the vector $-\vec{n}_1$. Assume that it is the vector $\vec{n}_3$. The divisor must have exactly one point on each of the orbits corresponding to the vectors $\vec{n}_2$ and $\vec{n}_4$ and a number k, $k \geqslant 1$, of points on the orbit corresponding to the vector $-\vec{n}_1 = \vec{n}_3$. The Pascal relation $m\vec{n}_1 + \vec{n}_2 + k\vec{n}_3 + \vec{n}_4 = 0$ and the relation $|\det(\vec{n}_1, \vec{n}_2)| = |\det(\vec{n}_1, \vec{n}_3)| = 1$ must hold.

These discrete conditions are equivalent to the Pascal condition for the existence of a trapezium Δ_D and the requirement that Δ_D does not contain interior integer-valued points. These conditions imply that the divisor D is admissible if and only if the Vieta relation holds.

Now let us expand the divisor D in a linear combination with non-negative integer coefficients of some effective divisors D_i. Let $c_1, \dots, c_j$ be the parameters of geometrically distinct points in the support of the divisor D on the orbit corresponding to the vector $\vec{n}_1$. We associate with each number i, $1 \leqslant i \leqslant j$, the following two-point divisor D_i: the divisor contains the point A_i with parameter c_i on the orbit $M_\infty^{\vec{n}_1}$ and the point B_i with parameter c_i^{-1} on the orbit $M_\infty^{-\vec{n}_1}$. We put $k(i)$ equal to the minimum of the multiplicities of the points A_i and B_i in the divisor D. Let $D_\nu = \sum k(i)D_i$, and $D_0 = D - D_\nu$.

Assertion 3. *The above described expansion of the divisor D into the sum $D = D_0 + \sum k(i)D_i$ corresponds to the expansion of the curve $r(D)$ into irreducible components*

$$r(D) = r(D_0) + \sum_{i=1} k(i)r(D_i).$$

Each of the curves $r(D_i)$ is a smooth rational curve on a torus surface M. The curve $r(D)$ is irreducible if and only if all numbers $k(i)$ are equal to 0. The curve $r(D)$ contains no multiple components if and only if all numbers $k(i)$ are less than 2. For $i > 0$ the different curves $r(D_i)$ on the torus surface do not intersect each other. Each of the curves $r(D_i)$ for $i > 0$ intersects the curve $r(D_0)$ on the surface M at a single point. This intersection point does not lie in the torus $\mathbb{C}^{*2}$ if and only if the supports of the divisors D_0 and D_i intersect.

Proof. The polynomial $yS_k(x) + Q_m(x)$ is irreducible if and only if it is divisible by a polynomial $G(x)$. The components of the expansion correspond to the roots c_i of the polynomial $G(x)$, and the numbers k_i correspond to the multiplicities of these roots. The curves $r(D_i)$ correspond to the vertical lines $x = c_i$, and the curve $r(D_0)$ to the graph of the rational function $y = S_k(x)/Q_m(x)$ (the numerator and denominator of the fraction are to be divided by a common factor $G(x)$).

§8. Meromorphic vector functions on compact curves

We consider a triple Γ, f, g that consists of a compact (not necessarily connected) curve Γ and a meromorphic function (f, g) on it. We assume everywhere that on each connected component of the curve Γ

(1) neither of the functions f, g vanishes identically,
(2) the vector-function (f, g) is not a constant.

With each triple Γ, f, g we associate a function $\mathrm{Mul}_{\Gamma f g}$ on the product $\mathbb{Z}^2_{\mathrm{ir}} \times \mathbb{C}^*$ that maps a non-cancellable vector $\vec{n}$ and a non-zero complex number c to the sum of the multiplicities of germs at all points of the curve Γ at which the germ of the vector-function has type $\vec{n}$, and where the reduced Weil number of this germ is equal to c (see §2). In other words,

$$\mathrm{Mul}_{\Gamma f g}(\vec{n}, c) = \sum_a k(a),$$

where the summation is performed over all points a at which the type of the germ of (f, g) is equal to $\vec{n}$ and $[f, g]_a = c$.

In the present section we solve the following problems 1–3.

Problem 1. *Given a function* Mul *on* $\mathbb{Z}^2_{\mathrm{ir}} \times \mathbb{C}^*$ *having non-negative integer values and equal to zero everywhere except finitely many points, to find triples* Γ, f, g *for which* $\mathrm{Mul}_{\Gamma f g}$ *is equal to* Mul.

Solutions of Problem 1 are twofold: multiple and non-multiple. We say that a triple Γ, f, g is *non-multiple* if the vector-function (f, g) glues together at most finitely many points of the curve Γ.

Problem 1 has the following versions.

Problem 2. *To find non-multiple solutions of Problem* 1.

Problem 3. *To find non-multiple solutions* Γ, f, g *of Problem* 1 *for which the curve* Γ *is connected.*

Lemma 1. *Let* $\pi\colon (\Upsilon, b) \to (\Gamma, a)$ *be the germ of an* l*-sheeted mapping of the curve* Υ *to the curve* Γ *and let* (f, g) *be the germ of a meromorphic function at the point* $a \in \Gamma$. *Then the germ* $(\pi^* f, \pi^* g)$ *of a meromorphic vector-function at the point* $b \in \Upsilon$ *has the same type as the germ* (f, g), *and its multiplicity is* l *times larger than the multiplicity of the germ* (f, g). *The following equalities hold*:

(1) $[\pi^* f, \pi^* g]_b = [f, g]_a$,
(2) $\{\pi^* f, \pi^* g\}_b = \{f, g\}^l_a$.

Proof. One can choose local parameters u and v on the germs of the curves Γ and Υ in such a way that the mapping π is defined by the formula $\pi(v) = v^l = u$. If $f = c_1 u^{b_1} + \cdots$, $g = c_2 u^{b_2} + \cdots$, then $\pi^* f = c_1 v^{l b_1} + \cdots$, $\pi^* g = c_2 v^{l b_2} + \cdots$. From here the desired equations follow.

Let the curve Γ consist of the connected components $\Gamma_1, \dots, \Gamma_k$. Each triple Γ, f, g is related to k triples $\Gamma_1, f, g; \dots; \Gamma_k, f, g$. (The same symbol denotes a function on the curve Γ and the restriction of this function to the connected component Γ_i of this curve.) Let $\pi\colon \Upsilon \to \Gamma$ be a ramified covering of the curve Υ over the curve Γ such that the number of points in the inverse image $\pi^{-1}(a)$ of a common point a on the component Γ_i is equal to μ_i (we do not assume that the full inverse image $\pi^{-1}(\Gamma_i)$ of the component Γ_i is connected).

Lemma 2. *The following equation holds*:

$$\mathrm{Mul}_{\Upsilon FG} = \sum \mu_i \, \mathrm{Mul}_{\Gamma_i fg},$$

where $F = \pi^* f$ and $G = \pi^* g$.

Lemma 2 follows immediately from Lemma 1.

With a triple Υ, F, G we associate a plane curve Γ^{geom} that is the closure of the image of the curve Υ in the torus $\mathbb{C}^{*2}$ under a meromorphic mapping taking the point a to the point $(F(a), G(a))$.

For each irreducible component of this curve, its *multiplicity* μ_i is defined to be equal to the number of inverse images of the common point on this component under the mapping $(F, G)\colon \Upsilon \to \mathbb{C}^{*2}$.

A plane algebraic curve Γ^{alg} geometrically coinciding with the curve Γ^{geom} so that each component of it is considered as having the multiplicity μ_i is called the *characteristic plane curve* for the triple Υ, F, G.

A Laurent polynomial $P(x,y) = \prod P_i^{\mu_i}(x,y)$, where $P_i(x,y)$ is an irreducible Laurent polynomial vanishing at the ith component and μ_i is the multiplicity of this component, is called the *characteristic Laurent polynomial* for the triple Υ, F, G. The characteristic polynomial is defined up to multiplication by a non-zero constant (we identify two Laurent polynomials whose quotient is a monomial). The curve Γ^{alg} is defined by the equation $P = 0$.

The normalization $\overline{\Gamma}$ of the characteristic curve Γ^{geom} consists of components $\overline{\Gamma}_i$ that are the normalizations of the components Γ_i^{geom} of the characteristic curve. On the normalization $\overline{\Gamma}$ a meromorphic vector-function $(x,y)\colon \overline{\Gamma} \to \mathbb{C}^{*2}$ is defined that specifies its birational isomorphism with the curve Γ^{geom}.

Theorem 1. *Let $\pi\colon \Upsilon \to \overline{\Gamma}$ be a ramified covering over the normalization of the curve Γ^{geom} such that the number of inverse images of the common point on the component $\overline{\Gamma}_i$ is equal to the multiplicity μ_i of this component. Then the characteristic curve for the triple Υ, F, G, where $F = \pi^* x$ and $G = \pi^* y$, coincides with the curve Γ^{alg}. Conversely, if the characteristic curve for the triple Υ, F, G coincides with the curve Γ^{alg}, then there exists a ramified covering $\pi\colon \Upsilon \to \overline{\Gamma}$ such that $F = \pi^* x$, $G = \pi^* y$, and the number of inverse images of the common point on the component $\overline{\Gamma}_i$ is equal to the multiplicity μ_i of the component Γ_i^{geom} in Γ^{geom}.*

Theorem 1 immediately follows from the definitions.

Corollary 1. *If the characteristic curve has no multiple components, then the triple Υ, F, G is uniquely defined by the characteristic curve up to an isomorphism. It is the normalization of the characteristic curve together with the vector-function (x, y) on it.*

Corollary 1 follows from Theorem 1, since in this case the ramified covering is an isomorphism.

Corollary 2. *If the characteristic curve has multiple components, then the triple Υ, F, G can be uniquely recovered by fixing the following data: a collection of any finite sets A_i on irreducible components $\overline{\Gamma}_i$ of the normalization of the characteristic curve having multiplicities $\mu_i \geqslant 2$, and a collection of homomorphisms of the fundamental groups $\pi_1(\overline{\Gamma}_i \setminus A_i)$ to groups of permutations of μ_i elements.*

Corollary 2 follows from Theorem 1 and the classification of ramified coverings. Corollary 2 shows that there exist multiparameter families of different triples Υ, F, G with a fixed characteristic curve having multiple components. The continuous parameters are sets of branch points A_i on the multiple components that can be chosen arbitrarily (and that can include any number of points).

Theorem 2. *For any triple Υ, F, G, the following equality holds:*

$$\mathrm{Mul}_{\Upsilon FG} = \mathrm{Mul}_P,$$

where P is the characteristic Laurent polynomial of the triple Υ, F, G.

Proof. It suffices to use Theorem 1$'$ on Newton polygons for the equation $P = 0$ of the characteristic curve of the triple Υ, F, G and Theorem 1 from this section.

Now we have the complete solution of Problem 1.

Indeed, we know the conditions necessary and sufficient for the function Mul to be the function Mul_P of a Laurent polynomial P. Each solution P of the problem $\mathrm{Mul}_P = \mathrm{Mul}$ is related to the solutions of Problem 1 for which the curve $P = 0$ in $\mathbb{C}^{*2}$ is characteristic. If the curve $P = 0$ has no multiple components, then up to isomorphism exactly one triple solving Problem 1 is related to it. Namely, such a triple is the normalization of the characteristic curve together with the vector-function (x, y) on it (see Corollary 1). But if the curve $P = 0$ has multiple components, then there are many solutions of Problem 1 related to it, but all of them are multiple. Each triple with the characteristic curve $P = 0$ is obtained from the covering over the characteristic curve. The set of such coverings contains infinitely many components of increasing dimension (see Corollary 2).

Now we proceed to Problems 2 and 3. In view of Theorems 1 and 2, the solution of Problem 2 (Problem 3) uses Laurent polynomials P without multiple factors (irreducible Laurent polynomials P) such that $\mathrm{Mul}_P = \mathrm{Mul}$. Solutions of Problem 2 (Problem 3) provide the normalizations of the curves $P = 0$ along with the vector-functions (x, y) on them.

We consider the cases that are possible here.

1. *The case when $\Delta(\mathrm{Mul})$ contains an interior integer-valued point.*

Let the function Mul satisfy the conditions for the existence of a Laurent polynomial P such that $\mathrm{Mul}_P = \mathrm{Mul}$, and let the polygon $\Delta(\mathrm{Mul})$ contain an interior integer-valued point. Then, for a general Laurent polynomial P such that $\mathrm{Mul}_P = \mathrm{Mul}$, the curve $P = 0$ in $\mathbb{C}^{*2}$ is connected and non-singular (see §6). Therefore, in this case the solubility of Problem 1 implies the solubility of both Problem 2 and Problem 3. Solutions of these problems form complex manifolds of dimension $B(\Delta)$, where $B(\Delta)$ is the number of interior points of the polygon $\Delta(\mathrm{Mul})$.

2. *The case when* $\Delta(\mathrm{Mul})$ *does not contain an interior integer-valued point.*

If the problem of finding the Laurent polynomial such that $\mathrm{Mul}_P = \mathrm{Mul}$ is soluble, then it has a unique solution up to an invertible multiplier. If the corresponding Laurent polynomial P is irreducible, then the curve $P = 0$ together with the vector-function (x, y) presents a unique solution of Problems 1–3. If the corresponding Laurent polynomial is reducible but has no multiple roots, then the curve $P = 0$ together with the vector-function (x, y) presents a unique solution of Problems 1 and 2. Problem 3 is insoluble in this case.

If the curve $P = 0$ has multiple components, then Problems 2 and 3 are insoluble (whereas Problem 1 has infinitely many solutions).

Using the function Mul, we can easily recognize the case we are dealing with (see §7). Thus, we have obtained the solubility conditions for Problems 1, 2 and 3, and completely described the solutions of these problems.

§9. Refinement of Weil's theorem

How can necessary and sufficient conditions for Mul to be the function $\mathrm{Mul}_{\Gamma fg}$ be interpreted in the theory of functions on curves?

1. The Pascal condition $\sum \mathrm{Mul}(\vec{n}, c)\vec{n} = 0$. The Pascal condition for the function $\mathrm{Mul} = \mathrm{Mul}_{\Gamma fg}$ means that the degree of the divisor as a meromorphic function f or g on a compact curve Γ is equal to zero. In the geometry of curves, the validity of this condition is obvious. If it holds, then one can construct the polygon $\Delta = \Delta(\mathrm{Mul})$ from the function Mul.

2. The Vieta condition $\sum (-c)^{\mathrm{Mul}(\vec{n},c)} = 1$. The Vieta condition for the function $\mathrm{Mul} = \mathrm{Mul}_{\Gamma fg}$ coincides with the following theorem of Weil.

The Weil theorem. *For any compact analytic curve* Γ *and two meromorphic functions* f *and* g *on it, the following relation holds*:

$$\prod_{a\in\Gamma} \{f, g\}_a = 1.$$

By these arguments, firstly we have proved the Weil theorem. Secondly, we have shown that if the discrete Pascal condition $\sum \mathrm{Mul}(\vec{n}, c) = 0$ holds and the polygon $\Delta(\mathrm{Mul})$ is not a segment, then Weil's theorem provides a unique necessary condition for the existence of a curve Γ and a vector-function (f, g) such that $\mathrm{Mul}_{\Gamma fg} = \mathrm{Mul}$.

3. The reflexivity condition $\mathrm{Mul}(\vec{n}, c) = \mathrm{Mul}(-\vec{n}, c^{-1})$. This condition must necessarily hold only in the special case when the function Mul has at most two characteristic vectors $\vec{n}_1$ and $\vec{n}_2$. We already have a complete description of this case. However, since firstly we are dealing with a refinement of Weil's theorem in this special case, and secondly the complete analysis of this case is also very simple using the theory of functions on the curve Γ, we present it here.

We say that the triple Γ, f, g is *exceptional* if there are no more than two nonzero vectors that are of the type of the germ of $(f, g)_a$ for some point $a \in \Gamma$. We say that a vector $\vec{n} \in \mathbb{Z}^2_{\mathrm{ir}}$ is characteristic for the function $\mathrm{Mul}_{\Gamma fg}$ if there exists a number $c \in \mathbb{C}^*$ such that $\mathrm{Mul}_{\Gamma fg}(\vec{n}, c) \neq 0$.

Lemma 1. *For any exceptional triple Γ, f, g there exist exactly two characteristic vectors, and their sum vanishes.*

This lemma follows from the Pascal condition.

We show that the case of any pair of characteristic vectors $\vec{n}_1$, $\vec{n}_2$ such that $\vec{n}_1 + \vec{n}_2 = 0$ is reduced to the easily described case when one of them (say the vector $\vec{n}_1$) is equal to the vector $\vec{e}_1 = (1, 0)$. Let us denote by A a unimodular matrix such that $A\vec{e}_1 = \vec{n}_1$.

Lemma 2. (1) *A triple Γ, f, g is exceptional with characteristic vectors $\vec{n}_1$ and $-\vec{n}_1$ if and only if the triple Γ, F, G, in which the vector-function (F, G) is obtained by the power transformation with matrix A^{-1} from the vector-function (f, g), has characteristic vectors $\vec{e}_1$ and $-\vec{e}_1$.*

(2) *An exceptional triple Γ, F, G has characteristic vectors $\vec{e}_1$ and $-\vec{e}_1$ if and only if the function G is constant on each connected component of the curve Γ.*

Proof. The first statement of Lemma 2 follows from Lemma 2 of §2. The vector-function (F, G) has characteristic vectors $\vec{e}_1$ and $-\vec{e}_1$ if and only if the function G does not vanish at any point of the curve and has no poles. Such a function is constant on each connected component of the curve Γ. Lemma 2 is proved.

Theorem 1. *For each exceptional triple Γ, f, g the function* $\mathrm{Mul} = \mathrm{Mul}_{\Gamma fg}$ *has the reflexivity property*

$$\mathrm{Mul}_{\Gamma fg}(\vec{n}, c) = \mathrm{Mul}_{\Gamma fg}(-\vec{n}, c^{-1}). \qquad (*)$$

Proof. (1) Suppose that the curve Γ is connected and the function g is constant: $g \equiv c$. Then at zero points a of the function f we have $[f, c]_a = c$; therefore the function $\mathrm{Mul}_{\Gamma fg}$ on the pair $(\vec{e}_1, c)$ is equal to $\deg f$. At poles of the function f we have $[f, c]_c = c^{-1}$; therefore the function $\mathrm{Mul}_{\Gamma fg}$ on the pair $(-\vec{e}_1, c^{-1})$ is equal to $\deg f$. At all the remaining points the function $\mathrm{Mul}_{\Gamma fg}$ vanishes.

(2) Lemma 2 reduces the general case to the case considered in item (1).

Theorem 2. *Suppose that a function* Mul *with characteristic vectors $\vec{n}_1$ and $-\vec{n}_1$ has the reflexivity property, the function $\varphi(c) = \mathrm{Mul}(\vec{n}_1, c)$ does not vanish at the points $c_1, \dots, c_m$, and $\varphi(c_i) = k_i$. The triples Γ, f, g for which $\mathrm{Mul}_{\Gamma fg} = \mathrm{Mul}$ are in one-to-one correspondence with collections of m ramified coverings over the Riemann sphere $\pi_i\colon \Gamma_i \to \overline{\mathbb{C}}$ of degrees k_i (the curves Γ_i are not necessarily connected). The following triple Γ, f, g corresponds to a collection of coverings. Suppose that Γ, F, G is a triple in which the curve $\Gamma = \cup\Gamma_i$ and (F, G) is a vector-function such that its restriction to the curve Γ_i is the vector-function (c_i, π_i). The triple Γ, f, g is a power transformation of the triple Γ, F, G with unimodular matrix A such that $A\vec{e}_1 = \vec{n}_1$.*

The proof of Theorem 2 follows from Lemma 2 and the calculation of the function $\mathrm{Mul}_{\Gamma fg}$ for the triple in which the curve Γ is connected and the function g is constant (see item 1 of the proof of Theorem 1).

Corollary. 1. *For any non-zero exceptional function* Mul *possessing the reflexivity property there exist triples* Γ, f, g *for which* $\mathrm{Mul}_{\Gamma f g} = \mathrm{Mul}$.

2. *Non-multiple triples* Γ, f, g *exist if and only if all non-zero values of the function* Mul *are equal to* 1. *For each such function* Mul *there exists exactly one triple* Γ, f, g.

3. *A non-multiple triple* Γ, f, g *in which the curve* Γ *is connected exists if and only if the function* Mul *is non-zero at exactly two points and is equal to* 1 *at these points.*

We have repeated the analysis of the exceptional case without resorting to Newton polygons explicitly, but using only the simplest properties of meromorphic functions on compact curves.

§10. Abel's theorem and the Vieta relation

Let D be a divisor lying on the union of one-dimensional orbits M_∞ of the torus surface M. Among conditions for the existence of a curve intersecting M_∞ along the divisor D there is the continuous Vieta condition (see §4). Usually, a continuous condition of this kind arises from theorems of the type of Abel's theorem (see [6]). In this section we present the simplest version of Abel's theorem, which is closely connected with the Vieta relation.

Let a finite collection $\{A_i\}$ of non-intersecting finite sets $A_1, \dots, A_m$, $A_i \cap A_j = \varnothing$, be defined on a compact (but not necessarily connected) curve Λ.

We say that a meromorphic form ω on a curve Λ is *$\{A_i\}$-regular* if

(1) the form ω is regular on the complement of the curve Λ to the set $\bigcup_{i=1}^m A_i$;

(2) the form ω has poles of order not higher than 1 at the points of the set $\bigcup_{i=1}^m A_i$;

(3) for each set A_i the sum of residues of the form ω at the points of the set A_i is equal to 0.

We say that a meromorphic function φ on a curve Λ is *$\{A_i\}$-meromorphic* if the function φ has one and the same value (possibly ∞) at all points of the set A_i, that is, for $i = 1, \dots, m$, if $a, b \in A_i$ then $\varphi(a) = \varphi(b)$.

Theorem 1 (a version of Abel's theorem). *The trace of the $\{A_i\}$-regular form ω of the $\{A_i\}$-meromorphic mapping $\varphi\colon \Lambda \to \mathbb{C}$ vanishes identically.*

Proof. We start with a local evaluation. Let $\varphi(u) = u^k$ and $\omega = a\dfrac{du}{u} + \omega_1$, where ω_1 is a holomorphic form. Then the trace of the form ω has a pole of multiplicity 1 with the same residue as the form ω. Indeed, let $v = u^k$, let $v^{1/k}$ be a branch of a root, and let ε_i, $i = 1, \dots, k$, be kth roots of 1. Then

$$\operatorname{Tr}\omega = \operatorname{Tr}\omega_1 + a \sum_{1 \leqslant i \leqslant k} \left(\frac{\frac{1}{k}\varepsilon_i v^{1/k-1}}{\varepsilon_i v^{1/k}} \right) dv = \operatorname{Tr}\omega_1 + a\frac{dv}{v}.$$

By Abel's theorem, the trace of the holomorphic form ω_1 is holomorphic.

From the local evaluation and the usual Abel theorem, it follows firstly that the form $\operatorname{Tr}\omega$ could only have poles of order not greater than 1 and only at points of the image of the set $\bigcup A_i$. Secondly, it follows from the local evaluation and the definition of a $\{A_i\}$-regular form and of a $\{A_i\}$-meromorphic mapping that the form $\operatorname{Tr}\omega$ has no poles at all. Consequently, $\operatorname{Tr}\omega \equiv 0$.

As in the usual Abel theorem, the vanishing of the trace of the form implies the following corollary.

Corollary. *Let φ be a $\{A_i\}$-meromorphic function on the curve Λ, whose divisor has a support not intersecting the union of the sets A_i. Let γ be a 1-chain on the curve Λ such that $\partial\gamma$ coincides with the divisor of the function φ. Then for each $\{A_i\}$-regular form ω the relation*

$$\int_\gamma \omega = 0 \mod(R\omega)$$

holds, where $R\omega$ is the lattice of periods for the form ω on the manifold $\Lambda \setminus \left(\bigcup A_i\right)$.

Let us consider a special case. Let the curve Λ consist of n copies of the Riemann sphere Λ_i, numbered cyclically. On each sphere Λ_i, two different points (P_i, Q_i) are selected. We consider the curve Λ with a collection of sets of points $A_1, \dots, A_n$, where the set A_i for $1 \leqslant i < n$ contains the pair of points (Q_i, P_{i+1}) and the set A_n contains the pair of points (Q_n, P_1). The curve Λ with the collection $\{A_i\}$ can be interpreted as an 'algebraic polygon' with 'sides' Λ_i and 'vertices' A_i, at which the points Q_i and P_{i+1} are glued together.

An example of such an algebraic polygon is the normalization of a set of one-dimensional orbits on a torus surface. The inverse image of each null-dimensional orbit arises on the normalization of two one-dimensional orbits. These normalizations must be glued together along inverse images of the common null-dimensional orbits.

On an algebraic polygon Λ there exists only one $\{A_i\}$-regular form ω up to a coefficient, namely, a form ω such that its restriction to Λ_i has poles of order 1 only at the points Q_i and P_i, its residue at the point Q_i being equal to $+1$, and its residue at the point P_i being equal to -1.

The existence of one $\{A_i\}$-meromorphic form implies the existence of one restriction to divisors of $\{A_i\}$-meromorphic functions. This restriction is as follows. Let t_i be a parameter on a sphere Λ_i having a zero of order 1 at the point Q_i and a pole of order 1 at the point P_i. Such a parameter is unique up to a non-zero constant multiplier.

Suppose that φ is a $\{A_i\}$-meromorphic function, N_i and L_i are the sets of zeros and poles of its restriction to the component Λ_i, μ is the multiplicity of a zero or a pole, and γ_i is a 1-chain whose boundary $\partial\gamma_i$ coincides with the divisor of the restriction of the function φ to the component Λ_i.

Assertion 1. *For any algebraic polygon Λ and any $\{A_i\}$-meromorphic function on it, the following equality holds*:

$$\prod_{1\leqslant i\leqslant m} \prod_{a_j\in N_i} \left(t_i(a_j)\right)^{\mu(a_j)} = \prod_{1\leqslant i\leqslant m} \prod_{b_j\in L_i} \left(t_i(b_j)\right)^{\mu(b_j)}. \tag{1}$$

Proof. Indeed, by Abel's theorem we have

$$\sum_{1\leqslant i\leqslant m} \int_{\gamma_i} \frac{dt_i}{t_i} = 2k\pi\sqrt{-1}.$$

Integrating and exponentiating, we obtain the desired relation.

Assertion 1 is invertible (as expected, since Assertion 1 is a special case of the version of Abel's theorem considered).

Assertion 2. *Any divisor on an algebraic polygon Λ for which the Abel relation* (1) *holds is the divisor of an $\{A_i\}$-meromorphic function.*

Proof. The restriction of a function φ to a sphere Λ_i is uniquely recovered from its zeros and poles up to a multiplier. The question is whether it is possible to choose these multipliers consistently on all spheres Λ_i. Let us recover the ratio $\varphi(P_i)/\varphi(Q_i)$. Writing the rational function φ on the sphere Λ_i explicitly, we obtain

$$\varphi(P_i) \prod_{a_j \in N_i} t_i(a_j)^{k(a_j)} = \varphi(Q_i) \prod_{b_j \in L_i} t_i(b_j)^{k(b_j)}. \tag{2}$$

The numbers $z_i = \varphi(P_i)/\varphi(Q_i)$ found from (2) are connected with the Abel relation $\prod z_i = 1$. Using this relation we can consistently recover the multipliers on the spheres Λ_i (compare with the proof of the theorem from §1).

Remark. The Abel relation (1) for algebraic polygons is reduced to the Vieta formula for the product of roots of a polynomial. Indeed, relation (2) is none other than the Vieta formula for a rational function φ on the Riemann sphere Λ_i. Multiplying these relations on all spheres, we obtain the Abel relation (1).

We need the following rather unexpected statement.

Theorem 2. *Let Γ be a curve on a torus surface M not passing through its fixed points. Then there is a linearly equivalent curve $\widetilde{\Gamma}$ not passing though the fixed points that intersects one-dimensional orbits only at points with parameter ξ equal to -1.*

Proof. Let a curve Γ be defined in $\mathbb{C}^{*2}$ by the equation $P = 0$, and let Δ be the Newton polygon for the Laurent polynomial P. We consider the Laurent polynomial Q described in the corollary from §1 that was constructed from the polygon Δ. For this Laurent polynomial Q, all polynomials $Q_{\vec{n}}$, $\vec{n} \in \mathbb{Z}^2_{\text{ir}}$, have no roots different from -1. The closure $\widetilde{\Gamma}$ of the curve defined in $\mathbb{C}^{*2}$ by the equation $Q = 0$ has the desired properties. Indeed, the divisor of the rational function $\Phi = \dfrac{P}{Q}$ on the surface M is equal to $\Gamma - \widetilde{\Gamma}$. (The divisor does not contain one-dimensional orbits, since the Newton polygons for the Laurent polynomials P and Q are equal.) That is, the curves Γ and $\widetilde{\Gamma}$ are linearly equivalent. Further, the curve $\widetilde{\Gamma}$ intersects one-dimensional orbits only at points with parameter -1, since all the polynomials $Q_{\vec{n}}$ have no roots different from -1.

Now we explain the Vieta relation (see §4) using Abel's theorem and Theorem 2. For a curve Γ not passing through the fixed points of the torus surface, we consider an equivalent curve $\widetilde{\Gamma}$ intersecting one-dimensional orbits only at points with parameters equal to -1. Such a curve exists by Theorem 2.

Let Φ be a meromorphic function on the torus surface M whose zeros coincide with the curve Γ, and whose poles coincide with the curve $\widetilde{\Gamma}$. Applying Assertion 1 to the restriction φ of this curve to the union of one-dimensional orbits, we obtain

$$\prod_{1\leqslant i\leqslant m}\prod_{a_j\in N_i}(\xi_i(a_j))^{\mu(a_j)}=\prod_{1\leqslant i\leqslant m}(-1)^{\sum_{a_j\in N_i}\mu(a_j)}$$

or

$$\prod(-\xi(a_j))^{\mu(a_j)}=1.$$

We have obtained the Vieta relation once again. So, from the viewpoint of Abel's theorem, the sign -1 in this relation can be explained by the existence in each equivalence class of a curve $\widetilde{\Gamma}$ intersecting one-dimensional orbits only at points whose parameters are equal to -1.

§11. Singularities of the characteristic curve and Newton polygons

A Newton polygon Δ corresponds to each triple Γ, f, g; it is a polygon constructed from the function $\mathrm{Mul} = \mathrm{Mul}_{\Gamma fg}$ (see §1). For each polygon Δ there exist non-multiple triples Γ, f, g for which Δ is a Newton polygon. The situation is changed if we impose a restriction on the curve Γ. Below we will present a necessary condition for the polygon Δ to be the Newton polygon of a non-multiple triple Γ, f, g, in which Γ is a connected curve of genus g (see Corollary 1 in this section). The restriction to the polygon Δ is a consequence of an evaluation of the sum of the genera of singular points of the characteristic curve of a non-multiple triple Γ, f, g. We present this evaluation below.

We consider a triple Γ, f, g in which Γ is a compact but not necessarily connected curve and (f,g) is a meromorphic vector-function. Let $V\subset\mathbb{Z}^2_{\mathrm{ir}}$ denote a set of characteristic vectors of the function $\mathrm{Mul}_{\Gamma fg}$. In other words, the vector $\vec{n}\in\mathbb{Z}^2_{\mathrm{ir}}$ is contained in the set V if and only if it is the type of the germ of (f,g) at some point of the curve. We say that a torus surface $M\supset\mathbb{C}^{*2}$ is *sufficiently complete* for a triple Γ, f, g if the fan of this surface contains rays generated by all vectors $\vec{n}$ from the set V. Let $\pi\colon\Gamma\to\mathbb{C}^{*2}$ denote a characteristic meromorphic mapping of the triple Γ, f, g to the torus $\mathbb{C}^{*2}$ taking the point a to a point $\pi(a)=(f(a),g(a))$.

Lemma 1. *Let $M\supset\mathbb{C}^{*2}$ be a torus surface that is sufficiently complete for the triple Γ, f, g. Then the characteristic meromorphic mapping $\pi\colon\Gamma\to\mathbb{C}^{*2}$ is continued to an analytic mapping $\widetilde{\pi}\colon\Gamma\to M$. Under this holomorphic mapping, the image of Γ does not pass through the null-dimensional orbits of the surface M.*

The proof follows from the Assertion of §3.

We need some invariants of isolated singular points (see [1]). Let P be the germ of a holomorphic function of two variables, having an isolated singularity at the point a. We fix a small ball B with the centre at the point a. For small $\varepsilon\neq0$ the equation $P=\varepsilon$ defines in the ball B a non-singular connected curve that is a sphere with q handles and k holes. The number of holes k is equal to the number of locally irreducible branches of the analytic curve $P=0$ passing through the point a. The *genus* of a singular point a of the curve $P=0$ is the number $(k-1)+q$.

Assertion 1. *The genus of the curve $P=0$ is equal to $\frac{1}{2}(\chi_0-\chi_\varepsilon)$, where χ_0 and χ_ε are Euler characteristics of the normalizations of curves defined in the ball B by the equations $P=0$ and $P=\varepsilon$, where $\varepsilon\neq0$ is a sufficiently small complex number.*

Proof. The number χ_0 is equal to the number k of locally irreducible branches of the curve $P = 0$ passing through the point a. The number χ_ε is equal to $2 - 2q - k$. From here the assertion follows.

Theorem 1. *If a triple Γ, f, g is non-multiple and the corresponding Newton polygon Δ is two-dimensional (that is, it does not degenerate to a segment), then the sum of the genera of singular points of the closure of its characteristic curve $\pi(\Gamma) \subset \mathbb{C}^{*2}$ in a sufficiently complete torus surface $M \supset \mathbb{C}^{*2}$ is equal to $(K-1)+(B(\Delta)-g)$, where K is the number of connected components of the curve Γ, g is its genus, and $B(\Delta)$ is the number of interior integer-valued points in the polygon Δ.*

Proof. We consider the equation $P = 0$ of a characteristic curve $\pi(\Gamma) \subset \mathbb{C}^{*2}$. The Newton polygon of a Laurent polynomial P coincides with the Newton polygon of the triple Γ, f, g. By a small change of the coefficients of the Laurent polynomial P, we obtain a Laurent polynomial $\widetilde{P}$ that is Δ-non-degenerate for the polynomial Δ (see [4]). By the theory of Newton polygons (see [5]), if the Newton polygon Δ is two-dimensional, then the closure of the curve $\widetilde{P} = 0$ in the sufficiently complete torus surface is a sphere with $B(\Delta)$ handles.

In the same torus surface we consider the closure of the original characteristic curve $P = 0$. Let us encircle singular points a_i of the closure of this curve with small balls B_i. Outside the union of the balls B_i, the closures of the curves $\widetilde{P} = 0$ and $P = 0$ are topologically equivalent and have the same Euler characteristics. The curve Γ is the normalization of its own characteristic curve. For the ball B_i, by virtue of Assertion 1 we have the relation

$$\chi(\pi^{-1}(B_i)) - \chi(\widetilde{\Gamma} \cap B_i) = 2g(a_i), \qquad (*)$$

where χ is the Euler characteristic and $g(a_i)$ the genus at the singular point a_i of the characteristic curve. The Euler characteristic is additive. Therefore the sum of the differences $(*)$ of the Euler characteristics over all balls B_i is equal to the difference of the Euler characteristics of the curve Γ and of the closure of the curve $\widetilde{P} = 0$, that is, it is equal to $(2K - 2g) - (2 - 2B(\Delta))$. Theorem 1 is proved.

Corollary 1. *On a connected curve Γ of genus g there exists no vector-function (f, g) such that the triple Γ, f, g is non-multiple and the Newton polygon of this triple is two-dimensional and includes fewer than g interior integer-valued points.*

The curve Γ defined in the torus $\mathbb{C}^{*2}$ by the general equation $P = 0$, with Newton polygon Δ, together with the vector-function (x, y) presents an example of a non-multiple triple for which Γ is connected and its genus is equal to the number of integer-valued points inside the Newton polygon of this triple.

Corollary 2. *Up to an isomorphism, there exist no other triples Γ, f, g having this property.*

Corollary 3. *If the genus of a connected curve Γ is less than the number of integer-valued points in the Newton polygon of the non-multiple triple Γ, f, g, then the closure of the characteristic curve of this triple in the sufficiently complete torus surface has singular points.*

For sufficiently general triples Γ, f, g, all singular points of the closure of the characteristic curve lie in the torus $\mathbb{C}^{*2}$ and are not the points of one-dimensional orbits of a sufficiently complete torus surface. We explicitly describe the corresponding conditions in the general case. To do this, it is not sufficient to know only the first terms of the expansions of the functions f and g; information on the second terms is required. Let us consider the germ of the analytic curve Γ at a point a with local parameter u, $u(a) = 0$, and the germ of the vector-function (f, g)

$$f = c_1 u^{b_1}(1 + \rho_1 u + \cdots), \qquad g = c_2 u^{b_2}(1 + \rho_2 u + \cdots),$$

for $c_1 \neq 0$, $c_2 \neq 0$ and a non-zero vector $\vec{b} = (b_1, b_2)$. Let $\vec{n}$ be the type of this germ, and let k be its multiplicity, that is, $\vec{n} \in \mathbb{Z}^2_{\mathrm{ir}}$, $\vec{b} = k\vec{n}$, where k is a positive integer. Let $\pi\colon \Gamma \to \mathbb{C}^{*2}$ denote the germ of the characteristic mapping, and let $\widetilde{\pi}\colon \Gamma \to M$ be its analytic continuation to the torus surface containing the one-dimensional orbit $M^{\vec{n}}_{\infty}$ corresponding to the vector $\vec{n}$.

Assertion 2. *The image of the germ $\widetilde{\pi}(\Gamma)$ on the surface M is the germ of a smooth curve that transversely intersects the orbit $M^{\vec{n}}_{\infty}$ if and only if the exponent vector $\vec{b}$ is irreducible (that is, the multiplicity k is equal to 1). The image of the germ $\widetilde{\pi}(\Gamma)$ on the surface M is the germ of a smooth curve tangent to the orbit $M^{\vec{n}}_{\infty}$ if and only if the exponent vector $\vec{b}$ is reducible (that is, the multiplicity k is greater than 1) and the coefficients ρ_1 and ρ_2 are not connected by the relation $b_1\rho_2 = b_2\rho_1$.*

This statement is verified by straightforward calculation.

Remark. The coefficients c_1, ρ_1, c_2, ρ_2 depend on the choice of the local parameter u. But the relations $c_1 \neq 0$, $c_2 \neq 0$, $b_1\rho_2 \neq b_2\rho_1$ are invariant with respect to a change of the local parameter.

We say that a germ (f, g) is *smooth at infinity* if it satisfies the conditions of Assertion 2.

Theorem 2. *If a triple Γ, f, g is non-multiple and the corresponding Newton polygon is two-dimensional, then the sum of the genera of the singular points of its characteristic curve $\pi(\Gamma) \subset \mathbb{C}^{*2}$ does not exceed the number $(K - 1) + (B(\Delta) - g)$, where K, $B(\Delta)$ and g are the same as in Theorem 1. Equality is attained if and only if all germs of the vector-function (f, g) of non-zero type $\vec{n}$ are smooth at infinity, and there is no pair of different points at which the germs of the vector-function have the same non-zero type and identical reduced Weil numbers.*

Proof. Indeed, the genus of a singular point is non-negative and vanishes only for a non-singular point. Therefore the inequality in Theorem 2 follows from Theorem 1. The inequality proves to be an equality if and only if the closure of the characteristic curve is non-singular at the points of its intersection with one-dimensional orbits on the surface M. This is possible if two smooth branches do not intersect at one point. To complete the proof it suffices to use Assertion 2 from this section and Assertion 2 from §3.

The following Theorem 3 (along with Theorems 1 and 2) shows that the Newton polygons of the triples Γ, f, g play the same role as the usual Newton polygons.

We say that for two triples Γ_1, f_1, g_1 and Γ_2, f_2, g_2 the non-degeneracy condition holds if there is no pair of points $a \in \Gamma_1$ and $b \in \Gamma_2$ at which the germs of the vector-functions (f_1, g_1) and (f_2, g_2) have the same type and identical reduced Weil numbers.

Theorem 3. *For any two triples Γ_1, f_1, g_1 and Γ_2, f_2, g_2 the number of isolated points of intersection of their images under characteristic mappings (calculated with regard to their multiplicity) does not exceed twice the mixed volume of their Newton polygons. Equality is attained if and only if the triples possess the non-degeneracy property.*

Proof. This follows from Bernstein's theorem [2] applied to characteristic curves of the triples Γ_1, f_1, g_1 and Γ_2, f_2, g_2.

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