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NEWTON POLYHEDRA (RESOLUTION OF SINGULARITIES)

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Some results are presented on the resolution of singularities and compactification of an algebraic manifold determined by a system of algebraic equations with fixed Newton polyhedra and rather general coefficients. Resolution and compactification are carried out by means of smooth toric manifolds which are described in the first half of the survey.

The Newton polyhedron of a polynomial depending on several variables is the convex hull of the exponents of the monomials contained in the polynomial with nonzero coefficients. The Newton polyhedron generalizes the concept of degree and plays an analogous role. Discrete characteristics of a joint level line of several polynomials in multidimensional complex space are the same for almost all values of the coefficients and are computed in terms of Newton polyhedra. Among the discrete characteristics computed are the number of solutions of a system of n equations in n unknowns, the Euler characteristic, the arithmetic and geometric genus of full intersections, and the Hodge numbers of a mixed Hodge structure on the cohomology of full intersections.

A Newton polyhedron is defined not only for polynomials but also for germs of analytic functions. For germs of analytic functions of general position with given Newton polyhedra the multiplicity of a joint solution of a system of analytic equations, the Milnor number and zeta function of the monodromy operator, the asymptotics of oscillating integrals, and the Hodge numbers of a mixed Hodge structure on vanishing cohomology are computed; in the two-dimensional case and the multidimensional quasihomogeneous case the modality of a germ of a function is computed.

In the answers quantities characterizing both the sizes of the polygons (the volume and the number of integer points contained inside the polygon) and their combinatorics (the number of faces of different dimensions and the numerical characteristics of their abutments) are encountered. These and other results connected with Newton polyhedra can be found in the works [1-9, 11-16, 18-24, 26-28].

A large part of the computations with Newton polyhedra is carried out by means of toric manifolds. "Elementary" computations in which it is possible to get by without their help are are most often exceptional. The basic step in applying toric manifolds consists in the explicit construction of a resolution of singularities and subsequent nonsingular compactification of the joint level line of several polynomials having sufficient general coefficients and fixed Newton polyhedra. The present paper is devoted to toric manifolds from the point of view of their applications to the resolution of singularities and compactification.

In the first half of the paper we present a detailed construction of smooth toric manifolds. Usually the description of these manifolds is presented in terms of spectra of rings which are common in algebraic geometry but are little suited for specialists in mathematical analysis. In our exposition the entire algebraic apparatus is reduced to linear algebra and to the simplest properties of integral lattices.

The second half of the paper is devoted to theorems on compactification and resolution of singularities. In the first of these (part 2.4) a nonsingular compactification of a joint

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level line of polynomials in the group T^n is set forth. This most simple (and, perhaps, most applicable) theorem was published in [18]. In the second theorem (part 2.6) a resolution of singularities with subsequent compactification of a joint level line of polynomials in $\mathbf{C}^n$ is set forth. In the third theorem (part 2.7) we present a resolution of singularities of the germ of a joint level line of several analytic functions (the case of a hypersurface was published in [28]).

1. Toric Manifolds

1.1. Integral Lattice. Suppose that in real n-dimensional space $\mathbf{R}^{n*}$ there is given an integral lattice (it is no accident that an asterisk appears in the notation: below stereometric constructions will be carried out in the space dual to the basic space $\mathbf{R}^n$). We shall need the simplest properties of the lattice. Vectors $a_1,\dots,a_n$ are called a basis of the integral lattice if integral linear combinations of them generate all integral vectors.

LEMMA 1. 1) Independent integral vectors form a basis in the lattice if and only if the parallelepiped $\Pi = \Sigma\lambda_i a_i$, $0 \leqslant \lambda_i < 1$ spanned by these vectors does not contain integral points distinct from the zero point. 2) Passage from one basis of the lattice to another is accomplished by an integral matrix with determinant equal to ±1.

Proof. 1) Indeed, the space $\mathbf{R}^{n*}$ is the disjoint union of parallelepiped Π_m, where $m = (m_1,\dots,m_n)$ are integral vectors, and Π_m consists of vectors $\Sigma\lambda_i a_i$ for $m_i \leqslant \lambda_i < m_i + 1$. All parallelepipeds Π_m differ by a shift by an integral vector and thus contain the same number of integral points. If this number is greater than 1, then $a_1,\dots,a_n$ do not form a basis of the lattice. 2) An integral matrix has an integral inverse if and only if its determinant is equal to ±1.

The second assertion of Lemma 1 makes it possible on the basis of a lattice to correctly define a volume element in $\mathbf{R}^{n*}$ so that the basis parallelepiped has unit volume.

A collection of integral vectors $a_1,\dots,a_k$ is called primitive if the parallelepiped $\Pi = \Sigma\lambda_i a_i$, $0 \leqslant \lambda_i < 1$ contains no integral points different from the point 0.

LEMMA 2. A collection of integral vectors $a_1,\dots,a_k$ is primitive if and only if it can be augmented to a basis of the lattice.

Proof. Let $a_{k+1},\dots,a_n$ be integral vectors such that the volume of the parallelepiped spanned by $a_1,\dots,a_k$, $a_{k+1},\dots,a_n$ has smallest possible (integral) nonzero volume. We shall show that this value is 1 (and hence that the collection of vectors is a basis of the lattice). Indeed, if this is not so, then the parallelepiped $\sum^n \lambda_i a_i$ for $0 \leqslant \lambda_i \leqslant 1$ contains some integral point b. By replacing one of the vectors $a_{k+1},\dots,a_n$ by the vector b, we reduce the volume of the parallelepiped.

LEMMA 3. In a k-dimensional plane in which there exists an integral basis there also exists a primitive basis.

The proof of Lemma 3 is similar to that of Lemma 2.

1.2. Conical Polyhedra and Their Subdecompositions. A rational cone in $\mathbf{R}^{n*}$ is a cone formed by linear combinations with nonnegative coefficients of a finite number of integral vectors.

It is known that the set of solutions of a finite number of linear equations $\langle x, m_i\rangle = 0$ and inequalities $\langle x, m_j\rangle \geqslant 0$ with integer coefficients is a rational cone. The cone is called pointed if it does not contain a linear subspace. A rational cone has a finite number of faces (the zero-dimensional face – the vertex of the cone – is included in the number of faces of a pointed cone). One-dimensional faces are called edges. The collection of irreducible integral vectors lying on the edges of a pointed rational cone is called the basis of the cone. A face of a pointed cone is determined by the collection of its edges.

A simplicial cone is a pointed rational cone whose number of edges is equal to its dimension. A simplicial cone is called primitive if its basis is primitive. The multiplicity of a k-dimensional simplicial cone is the volume of the parallelepiped spanned by its basis. (The volume is computed in a k-dimensional plane containing the cone whose volume element is determined by the integral lattice.) A simplicial cone is primitive if and only if it has multiplicity 1.

A conical polyhedron is a collection of a finite number of pointed rational cones in which any two cones can intersect only along faces and which together with each cone contains all its faces.

The Main Example. With an integral polygon Δ of full dimension $\dim\Delta = n$ lying in the space $\mathbf{R}^n$ there is connected a dual conical polyhedron Δ^* in the space $\mathbf{R}^{n*}$. Here is its definition. With a vector $a\in\mathbf{R}^{n*}$ there is connected the face Δ^a of the polygon Δ on which the scalar product of vectors lying in Δ with the vector a is minimal. Two vectors $a, b\in\mathbf{R}^{n*}$ are called equivalent if the faces connected with them coincide, i.e., $\Delta^a = \Delta^b$. The closure of the equivalence class of vectors connected with a face Δ_i forms a rational cone in $\mathbf{R}^{n*}$ which is called the cone dual to the face Δ_i. The collection of cones dual to all faces of a polygon Δ forms a conical polyhedron called the polyhedron dual to the polygon Δ and denoted by Δ^*. In the conical polyhedron Δ^* to a k-dimensional face of the polygon Δ there corresponds a dual $(n - k)$-dimensional cone. Thus, to the polygon Δ itself (which is considered an n-dimensional face) there corresponds the cone consisting of the single point 0 in $\mathbf{R}^{n*}$. To each $(n - 1)$-dimensional face there corresponds a ray in $\mathbf{R}^{n*}$ orthogonal to this face and "directed into the polyhedron Δ," etc.

With a conical polyhedron K there is connected a subset $|K|$ lying in $\mathbf{R}^{n*}$ which is the union of the cones defining K. The conical polyhedron K is determined not only by the set $|K|$ but also by the method of decomposing this set into rational cones. We say that a conical polyhedron M is a subdecomposition of the polyhedron K if $|M| = |K|$ and each cone of the conical polyhedron M lies inside some cone of the conical polyhedron K. A conical polyhedron is called simplicial if it is formed from a collection of simplicial cones and primitive if it is formed from a collection of primitive cones. A primitive subdecomposition of a polyhedron is called simple if no primitive cone of the original conical polyhedron is subdecomposed.

THEOREM 1. For any conical polyhedron there exists a simple subdecomposition.

Remark. In the book [25] it is proved that for any conical polyhedron there exists a primitive subdecomposition. The algorithm proposed in [25] actually reduces to a simple subdecomposition, but this refinement is not formulated in [25]. I learned the formulation of this very useful refinement from A. N. Varchenko.

The proof of Theorem 1 consists of two steps. At the first step we prove a version of the theorem in which primitive subdecompositions are replaced by simplicial subdecompositions. The second step consists in a primitive subdecomposition of simplicial cones.

Step 1. We fix an edge of one of the cones of the conical polyhedron. We perform the following operation: we span the cones by this edge and each face of all cones of the conical polyhedron containing this edge. We obtain a subdecomposition of the polyhedron for which the edges are the same as for the original conical polyhedron. If for each subdecomposition obtained there is an edge for which this operation is nontrivial, we perform this operation, etc. After a finite number of steps we must stop, since from a fixed number of edges it is possible to form only a finite number of conical polyhedra. We thus obtain the desired simplicial subdecomposition.

Step 2. Suppose that among the simplicial cones of the subdecomposition there are cones of multiplicity greater than 1. We choose one of the cones of highest multiplicity. Such a cone must contain an integral vector all of whose coordinates are less than one in its expansion in terms of the basis of the cone. By spanning the simplicial cones by this vector and all faces of all cones containing this vector (excepting, of course, that face strictly inside which the vector lies), we obtain a subdecomposition in which there are fewer cones of maximal multiplicity. Continuing this process, we annihilate all cones of highest multiplicity. We then all annihilate all cones of the next multiplicity, etc.

Remark. The proof of Theorem 1 contains an explicit algorithm for the simple decomposition of a conical polyhedron.

1.3. The Torus, Its Characters, and One-Parameter Subgroups. We denote by $\mathbf{C}_0^n$ n-dimensional complex space with coordinates $z_1,\ldots,z_n$ from which all coordinate planes have been removed, i.e., $z\in\mathbf{C}_0^n$, if $z_1 \neq 0,\ldots,z_n \neq 0$. The space $\mathbf{C}_0^n$ together with the operation of componentwise multiplication is a group. This commutative algebraic group is called an n-dimensional (complex) torus and is denoted by T^n. The group T^n with a fixed coordinate system $z_1,\ldots,z_n$ we call the standard torus.

A character χ of the torous (more precisely, an algebraic character of the torus) is an algebraic homomorphism of the torus T^n into the one-dimensional torus, $\chi:T^n \to T^1$. In coordinate notation each character is a monomial, i.e., a function of the form $z_1^{m_1}\cdot\ldots\cdot z_n^{m_n}$, where m_i are integers (not necessarily positive). We number the monomials by means of integral vectors $m = (m_1,\ldots,m_n)$ of the fixed real space R^n and use the abbreviated notation $z_1^{m_1}\cdot\ldots\cdot z_n^{m_n} = z^m$. We denote the corresponding character by χ^m. The characters form a group under multiplication. The enumeration gives an isomorphism of this group with the integral lattice of the space $\mathbf{R}^n$.

We consider the group of algebraic one-parameter subgroups, i.e., the algebraic homomorphisms $\lambda:T^1 \to T^n$. Each such homomorphism in coordinate notation has the form $z_1 = t^{\alpha_1},\ldots,z_n = t^{\alpha_n}$, where the α_i are integers or, more briefly, $z = t^\alpha$. We number the one-parameter subgroups λ by the points α of the integral lattice of the space $\mathbf{R}^{n*}$.

Between the one-parameter subgroups λ and the characters χ there is a scalar product equal to the degree of the composite homomorphism $\chi\cdot\lambda:T^1 \to T^1$. The scalar product of the character χ^m and the one-parameter group λ^α is equal to $\Sigma\alpha_i m_i$ where α_i and m_i are the coordinates of the integral vectors α and m. This scalar product extends to the spaces $\mathbf{R}^{n*}$ and $\mathbf{R}^n$ and gives a duality between them (for this scalar product the degree $\chi^m\cdot\lambda^\alpha:T^1 \to T^1$ is equal to $\langle\alpha, m\rangle$).

We consider the asymptotic behavior of a curve in the torus. Let $z:(C\setminus 0) \to T^n$ be the germ of a meromorphic curve in the torus, and suppose that the leading terms of the expansion of this curve have the form $z_i(t) = c_i t^{\alpha_i}(1 + O(t))$. Using the operation of multiplication in in T^n, the leading terms of the expansion can be written more simply, namely: $z(t) = Ct^\alpha(1 + O(t))$ where $C = c_1,\ldots,c_n$ is the vector of coefficients and $\alpha\in\mathbf{R}^{n^*}$ is the vector of degrees $\alpha = \alpha_1,\ldots,\alpha_n$. For us the following simple assertion plays an important role.

<u>Assertion.</u> Let χ^m be a character of the torus and let $z(t) = Ct^\alpha(1 + O(t))$ be a germ of a meromorphic curve in the torus. Then $\lim_{t\to 0}\chi^m(z(t))$ can be computed explicitly. Namely, it is equal to $\chi^m(C)$ if $\langle\alpha, m\rangle = 0$, it is zero if $\langle\alpha, m\rangle > 0$, and it is equal to infinity if $\langle\alpha, m\rangle < 0$.

Indeed, $\chi^m(Ct^\alpha(1+O(t))=\chi^m(C)\cdot\chi^m(t^\alpha)\cdot\chi^m(1+O(t))$. Further, $\lim_{t\to 0}\chi^m(1+O(t))=1$, and $\chi^m(t^\alpha)$ tends to one, zero, or infinity, respectively, depending on whether $\langle\alpha, m\rangle = 0$, $\langle\alpha, m\rangle > 0$, or $\langle\alpha, m\rangle \leqslant 0$.

If to an integral line in $\mathbf{R}^{n*}$ there corresponds a one-dimensional subgroup of the torus, then to a multidimensional plane in $\mathbf{R}^{n*}$ there corresponds a multidimensional subgroup.

A plane in $\mathbf{R}^{n*}$ is called integral if it is generated by the integral vectors lying on it. Let $\alpha_1,\ldots,\alpha_k$ be a primitive collection of vectors in a k-dimensional plane π. We define a k-dimensional subgroup $T(\pi)$ of the torus T^n as the set of points of the torus of the form $z = t_1^{\alpha_1}\cdot\ldots\cdot t_k^{\alpha_k}$ where $t_1,\ldots,t_k$ is an arbitrary collection of nonzero complex numbers. Not only the plane π but also the primitive collection of vectors participate in the definition of the group $T(\pi)$. It is easy to see, however, that the group $T(\pi)$ does not depend on the choice of the primitive collection. Fixing this collection gives an isomorphism of the group $T(\pi)$ with the standard k-dimensional torus T^k.

We denote by $\Phi(\pi)$ the factor group of the torus by the subgroup $T(\pi)$. The group $\Phi(\pi)$ is isomorphic to the $(n-k)$-dimensional torus. Indeed, suppose that the vectors $\alpha_{k+1},\ldots,\alpha_h$ augment the vectors $\alpha_1,\ldots,\alpha_k$ to a basis of the lattice. The factor group $\Phi[\pi]$ is isomorphic to the subgroup of points of the form $z = t_{k+1}^{\alpha_{k+1}}\cdot\ldots\cdot t_n^{\alpha_n}$.

<u>1.4. Toric Manifolds (as Sets of Points).</u> With each conical polyhedron in the space $\mathbf{R}^{n*}$ of one-parameter subgroups of the torus there is connected a certain algebraic manifold [25]. This manifold has no singularities if and only if the conical polyhedron is primitive. We shall need singular manifolds. We proceed to the description of nonsingular manifolds. We first describe the sets of points of such manifolds and then introduce on these sets the structure of an analytic manifold.

Let K be a primitive conical polyhedron in the space $\mathbf{R}^{n*}$ of one-parameter subgroups of the torus, and let $\{\sigma_i\}$ be the collection of its primitive cones, $K=\bigcup\{\sigma_i\}$. With each cone

σ_i we connect a group $\Phi[\sigma_i]$ as follows: we set $\Phi[\sigma_i]$ equal to the group $\Phi(\pi(\sigma_i))$ where $\pi(\sigma_i)$ is the integral plane generated by the cone σ_i.

Definition 1. The set of points of a toric manifold M_K corresponding to a primitive conical polyhedron K is the disjoint union of the factor groups $\Phi[\sigma_i]$ where $\{\sigma_i\}$ is the collection of simplicial cones of the polyhedron K.

The torus group acts in a natural manner on the set M_K. The orbits of this action are the factor groups $\Phi[\sigma_i]$.

Thus, to each simplicial cone σ of (real) dimension k in the set M_K there corresponds the torus $\Phi[\sigma]$ of (complex) dimension $n - k$. Each manifold M_K contains exactly one n-dimensional torus T^n; it corresponds to the point 0 — the vertex of all cones σ_i. The manifold M_K contains precisely as many $(n - 1)$-dimensional tori as there are one-dimensional cones in the conical polyhedron K, etc.

We note that the same group can be contained several times in the set M_K: if the planes generated by the cones σ_1 and σ_2 coincide, then the groups $\Phi[\sigma_1]$ and $\Phi[\sigma_2]$ also coincide. But in the set M_K in this case both $\Phi[\sigma_1]$ and $\Phi[\sigma_2]$ are present.

So far the set of points M_K is a conglomerate in no way connected which consists of the group T^n and a collection of its factor groups $\Phi[\sigma_i]$. The structure of an analytic manifold will later be introduced on this set. Looking ahead, we formulate an assertion (which we shall prove later and shall not use for the time being) necessary for an intuitive idea of the situation. In the torus T^n we consider the shifted one-parameter subgroup Ct^{a}. As $t \to 0$ this one-parameter subgroup will converge in M_K to a point $c \in \Phi[\sigma_i]$ if and only if the degree vector a lies strictly within the cone σ_i (i.e., it does not lie on a face of it), while the coefficient C goes over into c under the factorization $\rho: T^n \to \Phi[\sigma_i]$. The point c of the factor group $\Phi[\sigma_i]$ should be thought of as the limit as $t \to 0$ of the line Ct^{a}.

Definition 2. An orbit $\Phi[\sigma_1]$ is said to abut the orbit $\Phi[\sigma_2]$ if σ_1 is a face of the cone σ_2 (the orbit $\Phi[\sigma_1]$ has larger dimension than $\Phi[\sigma_2]$).

The topology on the set M_K (see part 1.6) is arranged such that one orbit abuts another if and only if the second orbit lies in the closure of the first.

The Basic Example. Let Δ be an n-dimensional integral polyhedron in $\mathbf{R}^n$ for which the dual polyhedron Δ^* in the space $\mathbf{R}^{n*}$ is primitive. Let M_{Δ^*} be the toric manifold constructed on the basis of this decomposition. The set of orbits of the manifold M_{Δ^*} is in one-to-one correspondence with the set of faces of the polyhedron Δ. Here to faces of real dimension k there correspond orbits of complex dimension k, and the orbits abut one another if and only if the faces corresponding to them abut one another (the polyhedron itself is considered an n-dimensional face; to it in M_{Δ^*} there corresponds the n-dimensional torus T^n).

Let K_1 and K_2 be two primitive conical polyhedra whereby the interior of each cone of the polyhedron K_1 belongs to the interior of some cone of the polyhedron K_2 (the interior of a cone spanned by vectors $a_1, \ldots, a_k$ is the set $\sum \lambda_i a_i$, where $\lambda_i > 0$). Under these conditions we define the mapping $g: M_{K_1} \to M_{K_2}$.

We assign to the cone σ_1 of the polyhedron K_1 the smallest cone σ_2 of the polyhedron K_2 containing the cone σ_1.

There exists a natural homomorphism of the factor group $\Phi[\sigma_1]$ into the group $\Phi[\sigma_2]$ (since the plane containing the cone σ_2 also contains the cone σ_1). We define the mapping $g: M_{K_1} \to M_{K_2}$ as the union of this conglomerate of homomorphisms.

Remark 1. The intuitive basis for this definition is the following. A point $c \in \Phi[\sigma_1]$ is to be thought of as the limit of a shifted one-parameter subgroup. The mapping g assigns to the point c the limit of the same one-parameter subgroup in M_{K_2}.

Let $g: M_{K_1} \to M_{K_2}$ be the mapping defined above, and let $\Phi[\sigma]$ be one of the orbits in M_{K_2}. The mapping injectively takes the subset $g^{-1}\Phi[\sigma]$ into the orbit $\Phi[\sigma]$ if and only if the cone σ is present among the cones of the polyhedron K_1.

1.5. Structure of the Analytic Manifold. Case of a Simplicial Cone. We begin with the definition of the structure of an analytic manifold for the simplest conical polyhedron $\bar{\sigma}$ consisting of a k-dimensional primitive cone σ and all its faces. A certain collection of functions will first be defined on the set of points $M_{\bar{\sigma}}$. The coordinate functions in different charts of the manifold $M_{\bar{\sigma}}$ will be chosen from this collection. We begin from stereometry.

The dual cone $\sigma^* \subset \mathbf{R}^n$ to the cone $\sigma \subset \mathbf{R}^{n*}$ is defined as the set of vectors $x \in \mathbf{R}^n$ for which $\langle a, x\rangle \geqslant 0$ for all $a \in \sigma$. The dual cone to a k-dimensional cone contains an $(n-k)$-dimensional subspace. The collection of integral vectors $m_1,\dots,m_n$ is called a basis of the cone σ^* if it is a basis of the lattice in $\mathbf{R}^n$ and any vector of the cone σ^* can be represented in the form $\Sigma\lambda_i m_i$ where the numbers $\lambda_1,\dots,\lambda_k$ are nonnegative and $\lambda_{k+1},\dots,\lambda_n$ are arbitrary real numbers.

Assertion 1. For a cone dual to a primitive cone there exists a basis (for $k < n$ the basis is not uniquely determined). If $a_1,\dots,a_k$ is a basis of the primitive cone σ, then the collection of vectors $\{m\}$ of the basis of the dual cone σ^* can by numbered so that $\langle a_i, m_j\rangle = \delta_{i,j}$ where i varies from 1 to k and j from 1 to n.

For the proof of the assertion it is necessary to augment the primitive collection of vectors $a_1,\dots,a_k$ to a basis and consider the cone σ^* in coordinates of the dual space equal to the scalar products with the basis vectors.

Let $m \in \sigma^*$, and suppose that χ^m is a character with index m. The function χ^m is defined on the torus T^n. We shall now define the function $\chi^m_{\bar\sigma}$ extending χ^m to $M_{\bar\sigma}$.

Definition. On the torus T^n the function $\chi^m_{\bar\sigma}$ is set equal to χ^m. On an orbit $\Phi[\sigma_i]$, where σ_i is a face of σ, $\chi^m_{\bar\sigma}$ is set equal to zero if for some (and hence any) vector a lying inside the face σ_i the inequality $\langle a, m\rangle > 0$ holds. Otherwise the function χ^m is constant on equivalent classes in T^n corresponding to points of the factor group $\Phi[\sigma_i]$. In this case the function is defined as the value of χ^m on the corresponding equivalence class.

LEMMA 1. Let c be a point of the factor group $\Phi[\sigma_i]$, let C be a point of the torus T^n going into c under the homomorphism of factorization, and let a be a vector lying strictly inside the cone σ_i. Then for any character χ^m for $m \in \sigma^*$ there is the equality

$$\lim_{t\to\infty} \chi^m(Ct^a) = \chi^m_{\bar\sigma}(c).$$

The proof of the lemma follows immediately from the computation of part 1.3. We note that lemma 1 agrees completely with the concept of points of the orbit $\Phi[\sigma_i]$ as limits of shifted one-parameter subgroups.

LEMMA 2. The extended functions satisfy the same relations as the original functions, more precisely, the equalities $\chi^{m+l}_{\bar\sigma} = \chi^m_{\bar\sigma}\cdot\chi^l_{\bar\sigma}$ hold where $m, l \in \sigma^*$.

Proof. The limit of the product is equal to the product of the limits.

We denote by $\mathbf{C}^n_k$ the domain in standard coordinate space $\mathbf{C}^n$ with coordinates $z_1,\dots,z_n$ defined by the inequalities $z_{k+1} \neq 0,\dots,z_n \neq 0$.

We shall construct a one-to-one mapping of the set M'_σ into $\mathbf{C}^n_k$ which plays the role of a coordinate system in $M_{\bar\sigma}$. This mapping will be constructed on the basis of a basis A in the cone σ^*. Let $A = \{m_1,\dots,m_n\}$ be a basis in σ^* with $\langle a_i, m_j\rangle = \delta_{i,j}$ where $a_1,\dots,a_k$ is a basis of the cone σ.

Definition. A coordinate mapping $f_A: M_{\bar\sigma} \to \mathbf{C}^n_k$ is a mapping taking a point $c \in M_{\bar\sigma}$ into the point with coordinates $\chi^m_{\bar\sigma}(c),\dots,\chi^{m_n}_{\bar\sigma}(c)$.

LEMMA 3. 1) Let $I \in (1,\dots,k)$ be a subset of indices, and let σ_I be the face spanned by the vectors a_i, $i \in I$. The mapping f_A establishes a one-to-one correspondence between the factor group $\Phi[\sigma_I]$ and the set defined in $\mathbf{C}^n_k$ by the equations $z_i = 0$ for $i \in I$ and the inequalities $z_j \neq 0$ for $j \notin I$. 2) The mapping f_A gives a one-to-one correspondence between $M_{\bar\sigma}$ and $\mathbf{C}^n_k$.

Proof. 1) The characters $\chi^{m_i}_{\bar\sigma}$ for $i \in I$ on the orbit $\Phi[\sigma_I]$ are equal to zero, since $\langle a_i, m_i\rangle = 1$ and hence $\langle x, m_i\rangle > 0$ if x lies inside the cone σ_I. The remaining characters χ^{m_j}, $j \notin I$, are constant on the subgroup $T(\pi(\sigma_I))$ and separate the equivalence classes relative to this subgroup, since the collection A forms a basis of the lattice. 2) Summing the images of all the factor groups, we obtain the required assertion.

In the domain $\mathbf{C}^n_k$ defined in $\mathbf{C}^n$ by the inequalities $x_{k+1} \neq 0,\dots,z_n \neq 0$ the monomials $z_1^{m_1}\cdot\ldots\cdot z_k^{m_k}\cdot z_{k+1}^{m_{k+1}}\cdot\ldots\cdot z_n^{m_n}$ for which the degree of the first k variables are nonnegative, i.e., $m_1 \geqslant 0,\dots,m_k \geqslant 0$, are regular.

LEMMA 4. The preimages of regular monomials in the domain $\mathbf{C}^n_k$ under the mapping $f_A: M_{\bar\sigma} \to \mathbf{C}^n_k$ are all characters $\chi^m_{\bar\sigma}$, where $m \in \sigma^*$, and only these.

Proof. An integral vector belongs to the cone σ^* if and only if the first k coordinates of its expansion in the basis A are nonnegative.

LEMMA 5. Let $f_B: M_{\bar{\sigma}} \to \mathbf{C}_k^n$ be the one-to-one mapping connected with the collection B of basis vectors in σ^*. Then the mapping $f_B f_A^{-1}: \mathbf{C}_k^n \to \mathbf{C}_k^n$ is bianalytic.

Proof. Let z_i be a coordinate function in $\mathbf{C}_k^n$. The function $f_B^* z_i$ is one of the characters $\chi_{\bar{\sigma}}^m$, where $m \in \sigma^*$. Therefore, $(f_A^{-1})^* f_B^* z_i$ is a regular monomial in the domain $\mathbf{C}_k^n$. Therefore, the mapping $f_B f_A^{-1}$ is analytic. The inverse mapping is analytic for the same reasons.

We introduce on $M_{\bar{\sigma}}$ the structure of an analytic manifold by means of the one-to-one mapping $f_A^{-1}: \mathbf{C}_k^n \to M_{\bar{\sigma}}$. The preceding assertion shows that the structure is well defined (does not depend on the basis in the cone σ^*).

Let $z(t) = Ct^{\alpha}(1 + O(t))$ be the germ of a meromorphic curve in the torus T^n. Now that an analytic structure has been introduced on $M_{\bar{\sigma}}$ it is necessary to prove the assertion regarding the behavior of the curve z(t) as $t \to 0$ mentioned in part 1.4.

LEMMA 6. 1) If a vector of degree α of the curve does not lie in the cone σ, then on the manifold $M_{\bar{\sigma}}$ there exists an analytic function whose restriction to the curve tends to ∞ as $t \to 0$; 2) if a vector of degree α lies in the cone σ, then as $t \to 0$ the curve has a limit in $M_{\bar{\sigma}}$, namely, if α lies inside the face σ_i of the cone σ, then the limit lies in $\Phi[\sigma_i]$ and is equal to the image of the point C in the group $\Phi[\sigma_i]$ under the homomorphism of factorization.

Proof. If α does not lie in the cone σ, then in the cone σ^* there is a vector m such that $\langle\alpha, m\rangle < 0$. The character χ^m extends analytically to $M_{\bar{\sigma}}$ (this extension is $\chi_{\bar{\sigma}}^m$). The limit as $t \to 0$ of $\chi^m(z(t))$ is equal to infinity, since $\langle\alpha, m\rangle < 0$. This proves the first assertion of the lemma. Suppose α lies in the face σ_I of the cone σ. The basis characters χ^{m_i} for $i \in I$ are constant on the subgroup $T(\pi(\sigma_I))$ and $\lim_{t\to 0} \chi^{m_i}(z(t)) = \chi^{m_i}(C)$. This proves the second assertion of the lemma.

LEMMA 7. Each character χ^m on the torus extends meromorphically to the manifold $M_{\bar{\sigma}}$. The corresponding meromorphic function $\chi_{\bar{\sigma}}^m$ is regular on $M_{\bar{\sigma}}$ if and only if $m \in \sigma^*$.

Proof. Each integral vector in $\mathbf{R}^m$ is the difference of two integral vectors lying in the cone σ^*. Therefore, each character on the torus coincides with the ratio of two holomorphic functions on $M_{\bar{\sigma}}$. Further, if $m \notin \sigma^*$, then there exists an integral vector $\alpha \in \sigma$ such that $\langle\alpha, m\rangle < 0$. The function χ^m along the one-parameter subgroup t^{α} as $t \to 0$ tends to infinity. The curve t^{α} as $t \to 0$ has a limit in $M_{\bar{\sigma}}$. Hence, the function χ^m is not regular.

LEMMA 8. Let σ_1 be a face of the cone σ. The analytic structures introduced on the sets $M_{\bar{\sigma}_1}$ and $M_{\bar{\sigma}}$ agree, i.e., the imbedding $g: M_{\bar{\sigma}_1} \to M_{\bar{\sigma}}$ is analytic.

Proof. A basis A of the cone σ^* is simultaneously a basis of the cone σ_1^* (but some of the basis vectors of the cone σ^* not invertible in this cone are invertible in the cone σ_1^*).

A chart of the manifold $M_{\bar{\sigma}_1}$ connected with the basis A is obtained from a chart of the manifold M_{σ} by dropping some coordinate hyperplanes (these hyperplanes correspond to vectors of the basis A not invertible in the cone σ^* but invertible in the cone σ_1^*).

LEMMA 9. Suppose that the cone σ_1 is contained inside another cone σ_2. Then the mapping $g: M_{\bar{\sigma}_1} \to M_{\bar{\sigma}_2}$ is analytic.

Proof. The dual cones are connected by the reverse inclusion $\sigma_1^* \supseteq \sigma_2^*$. We consider any character χ^m where $m \in \sigma_2^*$. We have the equality $g^* \chi_{\bar{\sigma}_2}^m = \chi_{\bar{\sigma}_1}^m$. Indeed, we choose an arbitrary shifted one-parameter subgroup passing into the point $c \in M_{\bar{\sigma}_1}$. Then the same one-parameter subgroup in $M_{\bar{\sigma}_2}$ passes into the point g(c). On the shifted one-parameter subgroup the functions $\chi_{\bar{\sigma}_1}^m$ and $\chi_{\bar{\sigma}_2}^m$ coincide with the character χ^m. Passing to the limit, we obtain the required equality. The assertion has been proved, since the collection of coordinate functions in $M_{\bar{\sigma}_2}$ is chosen among functions $\chi_{\bar{\sigma}_2}^m$ for $m \in \sigma_2^*$.

1.6. Structure of the Analytic Manifold (General Case). We proceed to the definition of the analytic structure on the set M_K. With each primitive cone σ of the conical polyhedron K there is connected a conical polyhedron $\bar{\sigma}$, the collection of cones of which is contained in the collection of cones of the conical polyhedron K. The set $M_{\bar{\sigma}}$ can therefore be identified with a subset in M_K. In each subset of $M_{\bar{\sigma}}$ an analytic structure has already been introduced. These analytic structures are consistent. Indeed, the common part of the sets $M_{\bar\sigma_1}$ and $M_{\bar\sigma_2}$ is $M_{\bar\sigma_3}$ where $\sigma_3=\sigma_1\cap\sigma_2$, and the imbeddings of the analytic manifold $M_{\bar\sigma_3}$ in $M_{\bar\sigma_1}$ and $M_{\bar\sigma_2}$ are analytic (see Lemma 8, part 1.5). We now define a topology on M_K as follows: we call a set $U\subset M_K$ open if and only if the intersections of U with all subsets of $M_{\bar\sigma}$ are open. There are sufficiently many open sets: the image of an open set in each chart of $M_{\bar\sigma}$ will be open in the entire manifold M_K (this follows from the consistency of the topologies of different charts). Thus, everything occurring in a small neighborhood of a point of the manifold M_K can be considered in any of the charts containing it.

Remark. In gluing together analytic manifolds it is possible to obtain sets with a bad topology. Here is the simplest example. For two copies of the complex line we identify all points with the exception of the origins (the set obtained can be thought of as the complex line with a double origin). On this set there are two charts with consistent analytic structures, but any two neighborhoods of the double origin intersect. We note that in this set there is an analytic curve having two distinct limits.

We shall prove that in the set M_K any two points have nonintersecting neighborhoods (or, in other words, that the topology on the set M_K is Hausdorff). The proof is based on the fact that a meromorphic curve in M_K has no more than one limit.

LEMMA 1. Let $z(t) = Ct^{\alpha}(1 + O(t))$ be the germ of a meromorphic curve in the torus T^n, and let M_K be the toric manifold constructed on the basis of a conical polyhedron K. If a vector α of the degree of the curve lies in the set $|K|$, then the curve $z(t)$ as $t \to 0$ has a unique limit in M_K. This limit lies in the orbit $\Phi[\sigma]$, where σ is a cone of the polyhedron containing the vector α in its interior, and is equal to the image of C in the group $\Phi[\sigma]$ under the factorization homomorphism. If $\alpha \notin |K|$, then as $t \to 0$ the curve $z(t)$ has no limit points in M_K.

The proof follows immediately from the consideration of the curve $z(t)$ in the charts of $M_{\bar\sigma}$ carried out in Lemma 6 of part 1.5. For us the following version of the theorem on selection of curves familiar in algebraic geometry will play an important role.

THEOREM (Selection of One-Parameter Subgroups). Let $\{\chi^m\}$ be a finite set of characters of the torus. Suppose that in the torus T^n there is given a sequence of points along which all characters of the finite set tend to limits (finite and infinite). Then there is a shifted one-parameter subgroup Ct^{α} along which all characters tend to the same limits as $t \to 0$.

LEMMA 2. Suppose there is given a finite set of real linear functions on $\mathbf{R}^{n*}$. Suppose in $\mathbf{R}^{n*}$ there is given a sequence of points along which all the linear functions of the set tend to limits (finite and infinite). Then there exists a shifted ray $p\tau + q$, where $p, q\in\mathbf{R}^{n*}$ and $\tau\in\mathbf{R}$, along which the linear functions tend to the same limits as $\tau \to -\infty$. If all the linear functions have integer coefficients, then the vector p can be chosen to be an integer.

Proof. Let $\{l_i\}$, $\{f_i\}$, $\{g_i\}$ be subsets of linear functions which along the sequence tend, respectively, to $+\infty$, to $-\infty$, and to finite limits. We denote by σ the cone defined by the inequalities $l_i \geqslant 0$, $f_i \leqslant 0$. We denote by $L + q$ the shifted linear subspace defined by the equations $g_i = c_i$ where c_i is the limit of the function g_i along the given sequence.

Inside the cone σ arbitrarily far from its faces there exist points arbitrarily close to the shifted subspace $L + q$ (in particular, the cone σ has full dimension $\dim\sigma = n$). Indeed, for such points it is possible to take points of the sequence with sufficiently large indices.

The space L cannot intersect the cone σ along a face: otherwise all points lying a small distance from the plane $L + q$ would be a finite distance from the face of the cone. Therefore, in the space L there is a vector r pointing strictly into the cone σ. If the space L is defined by integer equations, then by a slight perturbation of r it is possible to make it rational and then, by multiplying by a natural number, to make it an integral vector. The ray $p\tau + q$, where $p = -r$, possesses all the required properties.

LEMMA 3. Suppose in the hypotheses of the theorem $\{\chi^{g_i}\}$ is the subset of characters tending to finite limits $c_i \neq 0$; then there exists a point $z\in T^n$ at which $\chi^{g_i}(z) = c_i$.

Proof. Suppose that in the subset of vectors $\{g_i\}$ corresponding to characters tending to finite limits the vectors $g_1,\dots,g_k$ are linearly independent. The equations $\chi^{g_i} = c_i$ for $i = 1,\dots,k$ are compatible and define a finite number of surfaces in the torus on which the remaining characters of the subset are constant. The sequence tends to one of the surfaces. Any point of this surface satisfies the conditions of the lemma.

We proceed to the proof of the theorem. We consider the homomorphism ρ of the group T^n into the linear space $\mathbf{R}^{n*}$ defined by the formula $\rho(z)=\ln|z|$ where $z = (z_1,\ldots,z_n)$ and $\ln|z|=(\ln|z_1|,\ldots,\ln|z_n|)$. Under this mapping for any point $x\in\mathbf{R}^{n*}$ and character χ^m, $m\in\mathbf{R}^n$, the following relation is satisfied: the logarithm of the modulus of the character χ^m is constant on the preimage $\rho^{-1}(x)$ of the point x and is equal to $\langle x, m\rangle$. Suppose the characters $\{\chi^{l_i}\}$, $\{\chi^{f_i}\}$, and $\{\chi^{g_i}\}$ along the given sequence of points tend, respectively, to ∞, to 0, and to the complex number $c_i \neq 0$. Then the linear functions on $\mathbf{R}^{n*}$ defined by the vectors $\{l_i\}$, $\{f_i\}$, $\{g_i\}$ in $\mathbf{R}^n$ will tend along the image of the sequence to ∞, to $-\infty$, and to $\ln|c_i|$, respectively. By Lemma 2 there exists a ray $p\tau + q$ along which the linear functions tend as $\tau \to -\infty$ to the same limits. The shifted one-parameter subgroup zt^p, where z is the point found in Lemma 3, satisfies all the conditions of the theorem.

LEMMA 4. The topology of M_K is Hausdorff.

Lemma 4 together with the lemma on bianalyticity of the functions of passing from one chart to another show that the topology introduced makes the set M_K a complex analytic manifold.

Proof. All open sets in M_K intersect the torus T^n (in each chart except the torus are found only certain points on coordinate hyperplanes). Suppose that all neighborhoods of points a and b intersect. Choosing according to a point of intersection with the torus small neighborhoods of the points a and b and then reducing the neighborhoods, we obtain a sequence of points of the torus having as its limit both the point a and the point b. We fix charts containing the points a and b. Let $\{\chi^m\}$ be a finite collection of characters corresponding to the coordinate functions of these charts. The functions $\{\chi^m\}$ tend to finite limits (equal to the coordinates of the point a and the coordinates of the point b in these coordinate systems) along the sequence of points constructed. By the theorem on selection of one-parameter subgroups there is a shifted one-parameter subgroup along which the functions $\{\chi^m\}$ tend to the same limits. Considering this curve in a coordinate chart containing the point a, we see that this curve tends to the point a. Considering this curve in a chart containing the point b, we see that it tends to the point b. However, the shifted one-parameter subgroup as $t \to 0$ has no more than one limit point in M_K, and hence the points a and b coincide.

COROLLARY. The system of charts in the set M_K converts it into a complex-analytic manifold. The mapping of the manifold M_K into M_{K_2} defined in part 1.4 is analytic.

We have verified the first part of the assertion. The second part reduces to a local consideration of Lemma 9 of part 1.5.

1.7. Criteria of Compactness and Properness. THEOREM (Criterion of Compactness of the Manifold M_K). The manifold M_K is compact if and only if the primitive conical polyhedron K covers the entire space of one-parameter lines of $\mathbf{R}^{n*}$, i.e., $|K| = \mathbf{R}^{n*}$.

Proof. If $|K| \neq \mathbf{R}^{n*}$, then there exists a vector $a\in\mathbf{R}^{n*}$ not lying in $|K|$. The one-parameter line of degree a as $t \to 0$ has no limit points in M_K. Hence, the manifold M_K is noncompact.

Suppose now that $|K| = \mathbf{R}^{n*}$. We shall show that from any sequence of points of M_K it is possible to select a convergent subsequence. The manifold M_K consists of a finite number of orbits. Any sequence has an infinite subsequence lying entirely in one of the orbits. We first consider the case where such a subsequence lies in the torus T^n itself.

In each of a finite M_σ^- in M_K we fix a coordinate system. Let $\{\chi^m\}$ be the finite collection of characters in the torus corresponding to these coordinate functions. We choose a subsequence along which each of the functions has a finite or infinite limit (such a subsequence exists: the values of each function lie in the space $\mathbf{C}^1$ compactified by the infinitely distant point, and the product of compact spaces is compact). We shall prove that this subsequence converges in M_K. Indeed, we choose a shifted one-parameter subgroup along which the functions $\{\chi^m\}$ tend to the same limits. This curve, just as any meromorphic curve in the torus, has a limit in M_K (see Lemma 1 of part 1.6). The subsequence also converges to this limit. Indeed, in the collection of functions $\{\chi^m\}$ there are coordinate functions of a chart in which the limit of the curve lies. The subsequence converges to the point with the same coordinates, since the limits of the coordinate functions coincide. We now consider a sequence of points in the orbit $\Phi[\sigma]$. Let $a_1,\ldots,a_k$ be a basis of the cone σ, and let $m_1,\ldots,m_k$ be vectors of the cone σ^* such that $\langle a_i, m_j\rangle = \delta_{i,j}$. We consider all cones of the conical polyhedron K for which σ is a face. For each such cone σ_I we fix a basis in σ_I^* containing the vectors $m_1,\ldots,m_k$ (see part 1.5). We now consider the collection of characters $\{\chi\}$ corresponding to all vectors of the fixed bases. This collection contains, in particular, the characters $\chi^{m_1},\ldots,\chi^{m_k}$. All the characters $\{\chi\}$ are regular on $M_{\bar{\sigma}}$ (since all the cones σ_I^* lie in the cone σ^*). We now shift the q-th point of the sequence from the orbit $\Phi[\sigma]$ into the torus T^n so that each function of the collection $\{\chi\}$ changes by no more than 2^{-q} in modulus. From the shifted points we choose a subsequence such that each character of $\{\chi\}$ has a finite or infinite limit along it. We shall show that the corresponding subsequence of the original sequence converges. Let Ct^{a} be a shifted one-parameter subgroup tending to the same limit as the shifted subsequence. We consider the new shifted one-parameter subgroup Ct^{a+b} where the vector b lies strictly within the cone σ. The limits of all the characters $\{\chi^m\}$ do not depend on b.

Indeed, for the characters $\{\chi^m\}$ the limit as $t \to 0$, $\chi^m(t^b) = 0$ is equal to zero (just as for points of the predeformed sequence), while for the remaining characters of $\{\chi\}$ there is the identity $\chi\{t^b\} = 1$. For a sufficiently large vector b the vector $a + b$ lies inside one of the cones σ_I a face of which is the cone σ (because the union of such cones σ_I covers a neighborhood of the cone σ in $\mathbf{R}^{n*}$). The limit of the shifted one-parameter subgroup Ct^{a+b} is found in the chart M_{σ_I}. All coordinates of the chart M_{σ_I} are present in the collection of characters. The subsequence of points selected has the same limits of all coordinate functions as the curve Ct^{a+b} and therefore has the same limit.

The coordinate functions of the corresponding subsequence of points in the orbit $\Phi[\sigma]$ differ from the coordinates of the shifted points by no more than 2^{-q}. Therefore, they also converge to the same limit.

Remark. An orbit of a toric manifold of dimension k together with all orbits abutting it, in turn, forms a toric manifold of dimension k. We shall describe the conical polyhedron corresponding to this manifold. Let $\rho: T^n \to \Phi[\sigma]$ be the factorization homomorphism. Under this homomorphism the space of one-parameter subgroups of the torus T^n goes over into the space of one-parameter subgroups of the torus $\Phi[\sigma]$, whereby a plane spanned by an $(n-k)$-dimensional simplicial cone σ goes over into 0. The images of the simplicial cones σ_I containing σ as a face are simplicial cones in the space of one-parameter subgroups of the torus $\Phi[\sigma]$. The collection of these cones forms the conical polyhedron of the k-dimensional toric manifold corresponding to the closure of the orbit. The last part of the proof of the compactness criterion is actually based on this construction.

THEOREM (Criterion That a Mapping Be Proper). A mapping $g: M_{K_1} \to M_{K_2}$ of toric manifolds (defined in the case where each cone of the conical polyhedron K_1 is contained in some cone of the conical polyhedron K_2) is proper if and only if $|K_1| = |K_2|$. A proper mapping is a mapping "onto."

Proof. Both manifolds M_{K_1} and M_{K_2} contain as an open, dense set the torus T^n on which the mapping g is an isomorphism. Therefore, a proper mapping is a mapping "onto." Further, we suppose that there exists a vector a lying in $|K_2|$ but not belonging to $|K_1|$. The one-parameter line t^a of degree a in the torus T^n as $t \to 0$ has a limit in M_{K_2} but has no limit in M_{K_1} which contradicts the property that the mapping be proper.

For $|K_1| = |K_2|$ the preimage of any curve having a limit in M_{K_2} has a limit in M_{K_1}. To complete the proof that the mapping is proper it is necessary to use the theorem on selection of one-parameter subgroups (in the same way as this was done in detail in the proof of the compactness criterion).

2. Compactification and Resolution of Singularities

2.1. Laurent Polynomials and Their Newton Polyhedra. The simplest toric manifold is the torus group T^n itself. The conical polyhedron corresponding to this manifold consists of the single point 0. On the other hand, the collection of regular functions on T^n is the richest collection.

A Laurent polynomial on T^n is a finite linear combination of characters $P=\sum c_m\chi^m$. The support supp (P) of a Laurent polynomial P is the finite set of points $\{m\}\subset\mathbf{R}^n$ for which the coefficient c_m is nonzero. The Newton polyhedron $\Delta(P)$ of a Laurent polynomial P is the convex hull of its support. The support function of a Laurent polynomial P is the function

$H_{\Delta}(P)$ on the dual space $\mathbf{R}^{n*}$ defined by the formula $H_{\Delta(P)}(a)=\min_{x\in\Delta(P)}\langle a, x\rangle$. The contraction P^{α} of a Laurent polynomial P by a vector $a\in\mathbf{R}^{n*}$ is $\sum_{m\in\Delta^a} c_m\chi^m$ — the linear combination of characters whose degrees lie on the support face $\Delta^{\alpha}(P)$ of the polygon $\Delta(P)$ in the direction α [i.e., the face of the polygon $\Delta(P)$ on which the scalar product with α achieves its minimum], and each character χ^m of degree $m\in\Delta^a(P)$ is contained in the Laurent polynomial P^{α} with the same coefficient as in the Laurent polynomial P.

LEMMA 1. Let Ct^{α} be a shifted one-parameter subgroup, and let P be a Laurent polynomial. Suppose that the contraction P^{α} of the Laurent polynomial P of order α at the point $C\in T^n$ is not equal to zero, i.e., $P^a(C)\neq 0$. Then the lowest-order term as $t\to 0$ of the restriction of P to the line Ct^{α} has the form $P^a(C)\,t^{H_{\Delta(P)}(a)}$. If $P^{\alpha}(C) = 0$, then the lowest-order term of the restriction has higher degree.

Proof. Restricting the Laurent polynomial $P = \sum c_m\chi^m$ to the shifted one-parameter group Ct^{α}, we obtain $\sum c_m\chi^m(C)\cdot\chi^m(t^a)$. We separate out the terms of lowest degree in t:

$$\sum c_m\chi^m(C)\,\chi^m(t^a)=t^{H_{\Delta(P)}(a)}\sum_{m\in\Delta^a(P)} c_m\chi^m(C)+\ldots=t^{H_{\Delta(P)}(a)}P^a(C)+\ldots$$

[here the dots denote terms of degree higher than $H_{\Delta(P)}(\alpha)$].

The proof of Lemma 1 is complete.

Let P and Q be two Laurent polynomials, and let $R = P\cdot Q$.

LEMMA 2. In multiplication of Laurent polynomials 1) the contractions by any integral vector α are multiplied: $R^{\alpha} = P^{\alpha}Q^{\alpha}$; 2) the support functions of the Newton polyhedra add: $H_{\Delta(R)} = H_{\Delta(P)} + H_{\Delta(Q)}$; 3) the Newton polyhedra add: $\Delta(R) = \Delta(P) + \Delta(Q)$.

Proof. Let C be any point of the torus for which $P^{\alpha}(C) \neq 0$ and $Q^{\alpha}(C) \neq 0$ (almost all points of the torus T^n satisfy this condition, since no Laurent polynomial vanishes identically).

By restricting the Laurent polynomials P and Q to the line Ct^{α}, we find the lowest-order term of the expansion of the restriction of the polynomial R to this line. This lowest-order term is $P^a(C)\,Q^a(C)\,t^{H_{\Delta(P)}(a)+H_{\Delta(Q)}(a)}$.

This computation (together with Lemma 1) proves part 1) and also part 2) for integral vectors. The support functions are homogeneous and continuous, and hence the assertion of part 2) extends to rational and then to arbitrary (vectorial) arguments of the support functions. The additivity of the Newton polyhedra follows from the additivity of the support functions.

2.2. Conditions of Nondegeneracy. A system of equations $P_1 = \ldots = P_k = 0$ with Laurent polynomials $P_1,\ldots,P_k$ is called regular in the torus T^n if at each root of this system the differentials of the functions P_i are linearly independent.

LEMMA 1. For almost all collections of coefficients c_m^i of Laurent polynomials $P_i = \Sigma c_m^i\chi^m$ with fixed supports the system of equations $P_1 = \ldots = P_k = 0$ is regular in T^n. More precisely, in the space of coefficients there exists an algebraic subset of complex codimension $\geqslant 1$ (real codimension $\geqslant 2$) in whose complement the corresponding system is regular. Moreover, if for $m < k$ the system $P_1 = \ldots = P_m = 0$ is regular, then for almost all (in the same sense) coefficients of the remaining Laurent polynomials $P_{m+1},\ldots,P_k$ the system $P_1 = \ldots = P_{m+1} = \ldots = P_k = 0$ is regular.

Proof. Suppose the system $P_1 = \ldots = P_m = 0$ is regular. Then it defines a nonsingular manifold $X\subset T^n$. In each of the equations $P_j = 0$ for $j > m$ we separate out a character χ^{m_j} and represent the equation $P_j = 0$ in the form $\tilde{P}_j=-c^j_{m_j}\chi^{m_j}$ where $\tilde{P}_j=P_j-c^j_{m_j}\chi^{m_j}$. The complete system of equations is equivalent to the system $\tilde{P}_j\chi^{-m_j}=-c^j_{m_j}$ on the manifold X. By the Sard–Bertini theorem for almost all coefficients $c^j_{m_j}$ this system is nondegenerate on X. For these collections of coefficients the system $P_1 = \ldots = P_k = 0$ is regular in the torus.

The system of equations $P_1 = \ldots = P_k = 0$ with Laurent polynomials $P_1, \ldots, P_k$ is called a-regular for an integral vector $a \in \mathbf{R}^{n*}$ if the system $P_1^a = \ldots = P_k^a = 0$ is regular (thus, regular systems are a-regular for $a = 0$).

LEMMA 2. In Lemma 1 regularity can be replaced by a-regularity.

For the proof it suffices to use Lemma 1 for the system $P_1^a = \ldots = P_k^a = 0$.

The system $P_1 = \ldots = P_k = 0$ is called Δ-nondegenerate if it is a-regular for any integral vector $a \in \mathbf{R}^{n*}$.

LEMMA 3. For a given system $P_1 = \ldots = P_k = 0$ there exists only a finite number of different conditions for a-regularity. Namely, all distinct conditions are obtained by choosing one vector a in the partition Δ^* of the space R^{n*} dual to the polyhedron $\Delta = \Delta(P_1) + \ldots + \Delta(P_k)$.

Proof. In order that for two vectors a and b the support faces of the polyhedron of the sum coincide, i.e., in order that $\Delta^a = \Delta^b$, it is necessary and sufficient that their support faces of the polyhedral terms coincide, i.e., that $\Delta^a(P_i) = \Delta^b(P_i)$ for $i = 1, \ldots, k$. If $\Delta^a(P_i) = \Delta^b(P_i)$ for $i = 1, \ldots, k$, then the condition of a-regularity does not differ from the condition of b-regularity.

LEMMA 4. In Lemma 1 regularity can be replaced by Δ-nondegeneracy.

According to Lemma 3, for the proof it suffices to use Lemma 2 a finite number of times.

2.3. Laurent Polynomials on Toric Manifolds. A Laurent polynomial is a finite linear combination of characters of the torus. Together with the characters the Laurent polynomials extend as meromorphic functions to toric manifold.

Let $P = \sum c_m \chi^m$ be a Laurent polynomial with Newton polyhedron $\Delta(P)$. Let σ be a primitive cone in the space R^{n*}. We are interested in how the function $P_{\bar\sigma}$ obtained by extending P to the manifold $M_{\bar\sigma}$ behaves. The situation is especially simple in the case where the support function $H_{\Delta(P)}$ of the polyhedron $\Delta(P)$ is linear on the cone σ. Thus, suppose that on the cone σ the function $H_{\Delta(P)}$ coincides with the scalar product with a vector m (the vector m is not uniquely determined if $\dim \sigma < n$; the scalar products with a vector m on the cone σ are uniquely determined).

LEMMA 1. Suppose that for vectors $b \in \sigma$ the equality $H_{\Delta(P)}(b) = \langle b, m\rangle$ holds; then 1) the function $P_{\bar\sigma}\chi^{-m}$ is regular on the manifold $M_{\bar\sigma}$; 2) the restriction of the function $P_{\bar\sigma}\chi^{-m}$ to the orbit $\Phi[\sigma]$ is $P^a\chi^{-m}$ where a is any vector lying strictly within σ (more precisely, the value of the function $P_{\bar\sigma}\chi^{-m}$ at a point c of the orbit $\Phi[\sigma]$ coincides with the value of the Laurent polynomial $P^a\chi^{-m}$ at any point C of the torus T_n that goes over into c under the factorization homomorphism $\rho: T^n \to \Phi[\sigma]$).

Proof. For each character χ^l contained in the Laurent polynomial the scalar product $\langle b, l\rangle$ with the vector b on the cone σ, $b \in \sigma$, is not less than the support function of the polyhedron $H_{\Delta(P)}$. Therefore, the character χ^{l-m} extends in a regular way to the manifold $M_{\bar\sigma}$. Further, on the orbit $\Phi[\sigma]$ those and only those characters χ^{l-m} vanish for which $\langle a, l - m\rangle > 0$. Characters for which $\langle a, l - m\rangle = 0$ correspond to points l on the face $\Delta^a(P)$ of the Newton polyhedron, and they extend so that $\chi_{\bar\sigma}^{l-m}(c) = \chi^{l-m}(C)$ where the point C is any point going over into c under the factorization homomorphism $\rho: T^n \to \Phi[\sigma]$.

On a one-dimensional cone the function $H_{\Delta(P)}$ is always linear.

COROLLARY. The order of a zero of the Laurent polynomial P with Newton polyhedron $\Delta(P)$ on the unique $(n - 1)$-dimensional orbit of the manifold $M_{\bar\sigma}$, where $\bar\sigma$ is the one-dimensional cone in which the vector a is an integral generator, is equal to the value of the support function of the polyhedron $\Delta(P)$ on the vector a [a zero of negative order on an $(n - 1)$-dimensional orbit means, as usual, a pole of the corresponding order].

Suppose now that $P_1, \ldots, P_k$ is a collection of Laurent polynomials with Newton polyhedra $\Delta_1, \ldots, \Delta_k$ whose support functions are linear on the cone σ. We shall be interested in how the manifold X defined in T^n by the system $P_1 = \ldots = P_k = 0$ behaves near the orbit $\Phi[\sigma]$ of the manifold $M_{\bar\sigma}$.

LEMMA 2. Suppose that the Laurent polynomials $P_1, \ldots, P_k$ are a-regular for some (and hence for any) vector a lying strictly within σ. Then the closure $\bar X$ of the set X in the manifold $M_{\bar{\sigma}}$ in a neighborhood of the orbit $\Phi[\sigma]$ is a nonsingular analytic manifold transversally intersecting the orbit $\Phi[\sigma]$. The intersection of the closure $\bar{X}$ with the orbit $\Phi[\sigma]$ is hereby given in $\Phi[\sigma]$ by the system of equations

$$P_1^a\chi^{-m_1}=\ldots=P_k{}^a\chi^{-m_k}=0.$$

Proof. In a neighborhood of each point of the orbit $\Phi[\sigma]$ the lemma follows from the implicit function theorem and Lemma 1. To complete the proof it remains to take the union of these neighborhoods of its points as a neighborhood of the orbit.

Remark. Lemma 2 remains valid not only for Laurent polynomials but also for Laurent series $f_1 = \ldots = f_k = 0$ where the summation goes over an infinite set of indices. For the validity of such a generalization it suffices that the series of $f_1,\ldots,f_k$ converge in a neighborhood of the orbit. The generalization of Lemma 2 is especially convenient if all the contractions f_i^a are Laurent polynomials. Old considerations suffice to prove the generalization, since the implicit function theorem is valid for analytic functions as well as for algebraic functions.

Suppose now that K is an arbitrary primitive conical polyhedron in $\mathbf{R}^{n*}$, $P_1,\ldots,P_k$ are Laurent polynomials, and $P_{1,K},\ldots,P_{k,K}$ are meromorphic extensions of the Laurent polynomials to the manifold M_K. We summarize the results obtained.

THEOREM. 1) The order of a meromorphic function $P_{i,K}$ on the manifold M_K on the (n — 1)-dimensional orbit corresponding to an edge σ_1 in K with generator a is equal to $H_{\Delta(P_i)}(a)$. 2) If the support functions $H_{\Delta(P_i)}$ are linear on a cone σ of the conical polyhedron K and the system $P_1 = \ldots = P_k$ is a-regular for a lying strictly within σ, then the closure $\bar{X}$ of the set X of solutions of the system in T^n in a neighborhood of $\Phi(\sigma)$ is a nonsingular manifold transversally intersecting $\Phi[\sigma]$. The equations of the intersection $\bar{X}$ with $\Phi[\sigma]$ in the factor torus $\Phi[\sigma]$ are $P_1{}^a\chi^{-m_1}=\ldots=P_k{}^a\chi^{-m_k}=0$.

The proof of the theorem follows automatically from the local computations of Lemmas 1 and 2.

2.4. Compactification (Case of T^n). Let $P_1 = \ldots = P_k = 0$ be a Δ-nondegenerate system of equations in T^n with polyhedra $\Delta_1,\ldots,\Delta_k$, and let X be the manifold in T^n defined by this system.

Let K be an arbitrary primitive conical polyhedron giving a subdecomposition of the polyhedron Δ^* in $\mathbf{R}^{n*}$ dual to the polyhedron $\Delta = \Delta_1 + \ldots + \Delta_k$. The imbedding g of the torus T^n in the toric manifold M_K is called a compactification resolving the collection of polyhedra $\Delta_1,\ldots,\Delta_k$. We shall identify the image of the torus under the imbedding g with the original torus and the image of the manifold X with the original manifold X.

THEOREM. The closure $\bar{X}$ of the manifold X in the toric manifold is compact, nonsingular, and transversally intersects all orbits of the manifold M_K.

Proof. By the implicit function theorem the manifold $X\subset T^n$ is nonsingular in T^n. Near each orbit $\Phi[\sigma]$ of the manifold M_K the set $\bar{X}$ is an analytic manifold transversally intersecting the orbit $\Phi[\sigma]$. This was proved in the theorem of part 2.3.

Remark 1. The Laurent polynomials $P_1,\ldots,P_k$ are meromorphic functions on the manifold M_K. The orders of these functions on (n — 1)-orbits of the manifold M_K are determined in the theorem of part 2.3. The equations of the intersection of the manifold X with an orbit of any dimension of the manifold M_K are also determined in that theorem.

Remark 2. It is possible to explicitly construct a compactification resolving a given collection of polyhedra $\Delta_1,\ldots,\Delta_k$. For this it suffices to use the explicit algorithm for a simple subdecomposition of the polyhedron Δ^* presented in the theorem of part 1.2.

Remark 3. It follows from the theorem that the discrete characteristics of the manifold X depend only on the Newton polyhedra (and do not depend on the prescription of concrete coefficients of a Δ-nondegenerate system). Indeed, the Δ-nondegenerate systems form a set of real codimension no less than two in the space of systems with given Newton polyhedra. Therefore, it is possible to pass continuously from one Δ-nondegenerate system to another without passing through degenerate systems. The corresponding manifolds X will be deformed while remaining compact, analytic, and transversal orbits of the manifold M_K. A considerable part of the discrete invariants of a manifold X does not change under such deformations. Computations show that the main discrete invariants can be explicitly and rather simply expressed in terms of Newton polyhedra. In the expressions for the invariants such geometric characteristics of polyhedra as the number of integer points lying on their faces, the volumes of the polyhedra, etc. are encountered.

2.5. Compatibility Conditions. A collection of $l + 1$ polyhedra in $\mathbf{R}^n$ is called degenerate if there exists an l-dimensional subspace in which it is possible to parallel transport all the polyhedra. For example, the collection of one polyhedron consisting of a single point, the collection of two polyhedra consisting of two parallel segments, etc. are degenerate.

Definition. A collection of polyhedra in $\mathbf{R}^n$ is called dependent if it contains a degenerate subcollection.

THEOREM 1. A Δ-nondegenerate system of equations $P_1 = \ldots = P_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$ is incompatible in T^n if the Newton polyhedra $\Delta_1, \ldots, \Delta_k$ are dependent. Otherwise it defines in T^n an analytic $(n - k)$-dimensional manifold.

Proof. Let $\Delta_1, \ldots, \Delta_{l+1}$ be a degenerate collection of polyhedra. Then the system of equations $P_1 = \ldots = P_{l+1} = 0$ actually depends on l variables. Therefore, by an arbitrarily small change of its coefficients (in order to make this subsystem regular) it is possible to arrange that this subsystem, and hence also the entire system, becomes incompatible. The original (unperturbed) system cannot be compatible. Indeed, if it were compatible, then by the implicit function theorem under a small perturbation of the coefficients the manifold of solutions of the original system would not vanish but only be slightly deformed.

To prove the second part of the theorem we need Bernshtein's theorem on the number of roots [4]. We add to the system $P_1 = \ldots = P_k = 0$ the equations $P_{k+1} = \ldots = P_n = 0$ with Newton polyhedra $\Delta_{k+1}, \ldots, \Delta_n$ of full dimension so that the total system remains Δ-nondegenerate (this can be done by Lemma 4 of part 2.2). The system of equations $P_1 = \ldots = P_k = P_{k+1} = \ldots = P_n = 0$ is compatible in T^n, since the number of its solutions by Bernshtein's theorem is equal to the mixed volume of the Newton polyhedra $\Delta_1, \ldots, \Delta_n$ multiplied by $n!$, and the mixed volume of independent polyhedra is not equal to zero.

Let $P_1 = \ldots = P_k = 0$ be a Δ-nondegenerate system in the torus T^n which is invariant under the action of a q-dimensional subgroup of the torus. Such a situation arises if all the newton polyhedra $\Delta(P_1), \ldots, \Delta(P_k)$ lie in an $(n - q)$-dimensional plane L_{n-q} of the space R^n orthogonal to a q-dimensional plane $\pi \subset \mathbf{R}^{n*}$ corresponding to a q-dimensional subgroup of the torus. By multiplying the equations of the system by characters, the system $P_1 = \ldots = P_k = 0$ whose Newton polyhedra lie in planes parallel to the plane L_{n-q} can be reduced to this situation. We consider the new Δ-nondegenerate system $P_1 = \ldots = P_k = P_{k+1} = \ldots = P_m = 0$ containing the old equations and some new equations.

When is there at least one root of the extended system $P_1 = \ldots = P_m = 0$ in each orbit of the action of the q-dimensional group $T(\pi)$ on the manifold of solutions of the original system $P_1 = \ldots = P_k = 0$? The next theorem provides an answer to this question.

THEOREM 2. On each orbit there is a root of the extended system if and only if the polyhedra $\Delta_1, \ldots, \Delta_m$ are independent (the case in which the polyhedra $\Delta_1, \ldots, \Delta_k$ are dependent is an exception; in this case the set.*

Proof. The restriction of the equations $P_{k+1} = \ldots = P_m = 0$ to an orbit of the group $T(\pi)$ lying in the manifold of solutions of the system $P_1 = \ldots = P_k = 0$ is a Δ-nondegenerate system in the torus T^n whose Newton polyhedra are the images of the polyhedra $\Delta_{k+1}, \ldots, \Delta_n$ under the factorization of the space $\mathbf{R}^n$ by the subspace L_{n-q}. Theorem 2 now follows from Theorem 1.

2.6. Resolution of Singularities and Compactification (Case $\mathbf{C}^n$). In this subsection a resolution of singularities and a compactification of the resolved manifold are constructed for manifolds defined in $\mathbf{C}^n$ by a Δ-nondegenerate system of equations. The construction is entirely determined by the collection of Newton polyhedra of the system of equations. The cross of the singularity — the union of several coordinate planes along which it is necessary to blow up $\mathbf{C}^n$ in resolving singularities — is determined by the collection of Newton polyhedra. The cross of a singularity decomposes into orbits of three types: unnecessary

*Omission in Russian original — Publisher.

orbits, attainable orbits, and unattainable orbits. This decomposition is determined by the collection of Newton polyhedra and possesses the following properties. First of all, the manifold defined by a Δ-nondegenerate system does not intersect unnecessary orbits. Secondly, each point of the intersection of this manifold with an attainable orbit is a limit point for the toric part of the manifold (i.e., for the subset of points of the manifold lying on the torus). Third, all points of the manifold lying on unattainable orbits do not belong to the closure of the toric part of the manifold.

The space $\mathbf{C}^n$ is a toric manifold whose conical polyhedron is the positive octant in $\mathbf{R}^{n*}$. The characters whose degrees lie in the positive octant of $\mathbf{R}^n$ extend holomorphically to the space $\mathbf{C}^n$ and are ordinary monomials. Their linear combinations are the polynomials on $\mathbf{C}^n$.

With each polynomial on $\mathbf{C}^n$ there is connected a Laurent polynomial: the restriction of this polynomial to the torus T^n. Admitting some liberty of speech, we shall call the Newton polyhedron of this Laurent polynomial, the contraction of this Laurent polynomial etc. the Newton polyhedron of the polynomial, the contraction of the polynomial, etc.

Let $\Delta_1,\ldots,\Delta_k$ be polyhedra lying in the positive octant of the space $\mathbf{R}^n$. We restrict the support functions of the polyhedra $\Delta_1,\ldots,\Delta_k$ to the positive octant in $\mathbf{R}^{n*}$. A face σ of the positive octant in $\mathbf{R}^{n*}$ is called nonsingular for the collection of polyhedra $\Delta_1,\ldots,\Delta_k$ if the support functions of all the polyhedra of the collection are identically equal to zero on σ. All the remaining faces of the octant are called singular.

It follows from the definition that all faces of the octant (including the octant itself) are nonsingular if and only if all polyhedra contain the point $0\in\mathbf{R}^n$.

A singular face σ of the octant in $\mathbf{R}^{n*}$ is called unnecessary in the following case. We consider all polyhedra of the collection $\Delta_1,\ldots,\Delta_k$ for which the support function on the face σ is identically equal to zero, and we consider their support faces for which the support function is equal to zero. If the collection of support faces obtained is dependent, then the face is called unnecessary.

A singular face σ is called attainable if there exists a vector a lying strictly within the face σ such that the support faces $\Delta_1^a,\ldots,\Delta_k^a$ are independent.

A singular face is called unattainable if it is not unnecessary and is not attainable.

An orbit of the toric manifold $\mathbf{C}^n$ (i.e., a coordinate plane from which all smaller coordinate planes have been eliminated) is called singular (nonsingular, unnecessary, attainable, unattainable) for the Newton polyhedra $\Delta_1,\ldots,\Delta_k$ if the cone corresponding to this orbit (which is a face in $\mathbf{R}^{n*}$) is singular (nonsingular, unnecessary, attainable, unattainable) for these polyhedra. The union of all singular orbits fills the union of several coordinate planes and is called the cross of the singularity for the polyhedra $\Delta_1,\ldots,\Delta_k$.

We connect with a collection of polyhedra $\Delta_1,\ldots,\Delta_k$ a decomposition Δ^* of the space $\mathbf{R}^{n*}$ dual to the polyhedron $\Delta = \Delta_1 + \ldots + \Delta_k$ and the decomposition Δ_+^* of the positive octant in $\mathbf{R}^{n*}$ induced by the decomposition Δ^*.

We call a simple subdecomposition K^+ of the polyhedron Δ_+^* a resolving subdecomposition for the collection of polyhedra $\Delta_1,\ldots,\Delta_k$. We call a pair of primitive polyhedra $K^+\subset K$, where K^+ is a subpolyhedron of K, a compactifying pair for the collection of polyhedra $\{\Delta\}$ if K^+ is a resolving polyhedron for these polyhedra and K is a primitive subdecomposition of the polyhedron Δ^*.

LEMMA 1. On the basis of polyhedra $\Delta_1,\ldots,\Delta_k$ it is possible to explicitly construct some resolving polyhedra and compactifying pairs of the polyhedra.

Proof. To construct resolving polyhedra it is necessary to apply the theorem of part 1.2 for the polyhedron Δ_+^*. To construct compactifying pairs of polyhedra it suffices to apply this same theorem to the polyhedron Δ_∞^* dual to the polyhedron $\Delta_\infty = \Delta_0 + \Delta_1 + \ldots + \Delta_k$, where Δ_0 is a standard simplex (defined by the inequalities $0 \leqslant x_i$; $\Sigma x_i \leqslant 1$).

Definition. A toric manifold M_{K^+} together with the projection $g: M_{K^+} \to \mathbf{C}^n$ is called a resolution for the polyhedra $\Delta_1,\ldots,\Delta_k$ if K^+ is a resolving polyhedron for these polyhedra; toric manifolds M_{K^+}, M_K with projection $g: M_{K^+} \to \mathbf{C}^n$ and imbedding $g_0: M_{K^+} \to M_K$ are called a resolution with compactification for the polyhedra $\Delta_1,\ldots,\Delta_k$ if the polyhedra K^+, K form a compactifying pair.

LEMMA 2. 1) A resolution $g:M_{K^+} \to \mathbf{C}^n$ is a proper mapping which is single-valued away from the cross of a singularity of the collection of polyhedra $\Delta_1,\ldots,\Delta_k$. In particular, the mapping is one-to-one if all the polyhedra contain the point $0 \in \mathbf{R}^n$. 2) The imbedding g_0: $M_{K^+} \to M_K$ is a compactification of the space M_{K^+}. The complement of the image of M_{K^+} consists of transversally intersecting divisors in M_K.

Proof. By definition, support functions of polyhedra are equal to zero on nonsingular faces of the positive octant in $\mathbf{R}^{n*}$. These faces are primitive and are contained in the polyhedron Δ_+^*. Since K_+ is a simple decomposition, these faces do not subdecompose further, and the mapping $g:M_{K^+} \to \mathbf{C}^n$ is one-to-one on nonsingular orbits. The second part of the assertion follows from general properties of toric manifolds.

We proceed to the connection of a resolution for polyhedra $\Delta_1,\ldots,\Delta_k$ with Δ-nondegenerate systems with the same Newton polyhedra. Let $P_1 = \ldots = P_k = 0$ be a system of polynomial equations in $\mathbf{C}^n$ with Newton polyhedra $\Delta_1,\ldots,\Delta_k$; let $X \subset \mathbf{C}^n$ be the manifold defined by this system; let X_c be the intersection of this manifold with the cross of a singularity for the polyhedra $\Delta_1,\ldots,\Delta_k$. Let $g:M_{K^+} \to \mathbf{C}^n$ be a resolution, and let $g:M_{K^+} \to \mathbf{C}^n$, $g_0:M_{K^+} \to M_K$ be a resolution with compactification for the polyhedra $\Delta_1,\ldots,\Delta_k$. We denote by X_0 the preimage of the set $X \setminus X_c$ in M_{K^+}, by X^+ its closure in M_{K^+}, and by $\overline{X}^+$ the closure of its image in M_K.

THEOREM. Under the conditions formulated below on the coefficients of the equations $P_1,\ldots,P_k$, which are almost always satisfied, the following assertions hold: 1) X^+ is an analytic manifold in M_{K^+} intersecting its orbits transversally; 2) X_0 is obtained from X^+ by eliminating its intersections with the collection of transversally intersecting hypersurfaces which represent the preimage of the cross of the singularity under the projection g; 3) the projection g establishes a mutually analytic correspondence between the manifolds X_0 and the set $X \setminus X_c$; 4) the projection g maps X^+ onto the manifold X from which intersections with all orbits of unattainability for the polyhedra $\{\Delta_i\}$ have been eliminated; 5) if the polyhedra K^+, K form a compactifying pair, then the closure $\overline{X}^+$ of the manifold X^+ is a nonsingular compact manifold transversally intersecting the "infinitely distant" hypersurfaces of the manifold M_K, and X^+ is obtained from $\overline{X}^+$ by eliminating these intersections.

We shall formulate the conditions on the coefficients under which the theorem is valid. Let $\mathbf{R}^I$ be any coordinate plane in $\mathbf{R}^n$ (the space $\mathbf{R}^n$ is also considered a coordinate plane). We denote by $\{\Delta_i^I\}$ the collection of polyhedra $\Delta_i^I = \Delta_i \cap \mathbf{R}^I$ (in which empty polyhedra are deleted). We connect with the plane $\mathbf{R}^I$ the system $\{P_i\}^I$ of equations $P_1^I = \ldots = P_k^I = 0$ where P_i^I is the contraction of the polynomial P_i on the face Δ_i^I. Parts 1-4) of the theorem are satisfied for systems of equations for which the systems $\{P_i\}^I$ for all planes of $\mathbf{R}^I$ are a-regular for all vectors of the positive octant $a \in \mathbf{R}_+^{n*}$. For the validity of part 5) it is necessary to require that this condition be satisfied for all vectors $a \in \mathbf{R}^{n*}$.

Remark. The orders of the function g^*P_i on $(n-1)$-dimensional orbits of the manifolds M_{K^+} and M_K are determined by means of the theorem of part 2.3. By means of this same theorem we determine the equations of the intersection of the manifolds X^+ and $\overline{X}^+$, respectively, with the orbits of the manifolds M_{K^+} and M_K.

We note three special cases of Theorem 1.

1. All Newton polyhedra contain the origin. In this case the cross of the singularity is empty, the manifold M_K coincides with $\mathbf{C}^n$, and the manifold X^+ coincides with the original manifold X. In this case Theorem 1 sets forth a toric compactification of the space $\mathbf{C}^n$ in which the closure $\overline{X}$ of the manifold X is nonsingular and intersects the infinitely distant orbits transversally. In this case Theorem 1 is altogether analogous to the theorem on compactification for T^n (see part 2.4).

2. The cross of a singularity for Newton polyhedra contains only unnecessary orbits. For this case Theorem 1 sets forth a blow-up M_K of the space $\mathbf{C}^n$ containing the manifold X^+ which is bianalytically equivalent to the manifold X and a compactification of the space M_{K^+} in which the closure $\overline{X}$ of the manifold X is nonsingular and transversally intersects infinitely distant orbits.

3. The cross of a singularity for the Newton polyhedra contains only unnecessary and attainable orbits. This case differs from the preceding case in that the nonsingular manifold X^+ is not equivalent to the (singular) manifold X but maps onto the manifold X in one-to-one fashion on an open, dense set.

We proceed to the proof of the theorem. Parts 1)-3) and 5) are proved in the same way as in the theorem of part 2.4. The only part needing verification is part 4. It is necessary to prove that no point of the set X lying on an unattainable orbit is a limit point for the manifold $X \setminus X_c$ and that any point of the set X lying on an attainable orbit is a limit point for this set.

If a point on an unattainable orbit lies in the closure of $X \setminus X_c$, then there exists a meromorphic curve in $X \setminus X_c$ tending to this point (by the theorem on the selection of curves [17]). Let $z(t) = Ct^{\alpha} \times (1 + O(t))$ be this curve. Then the vector α lies inside the dual cone to an unattainable face, while the point C is a root of the system of equations $P_1^{\alpha} = \ldots = P_k^{\alpha} = 0$ [the latter condition is obtained by separating out the leading terms in the identities $P_i(z(t)) \equiv 0$]. However, by the condition of unattainability the polyhedra $\Delta_1^{\alpha}, \ldots, \Delta_k^{\alpha}$ are dependent, and the nondegenerate system $P_1^{\alpha} = \ldots = P_k^{\alpha} = 0$ is incompatible (Theorem 1 of part 2.5).

Suppose now that x is an arbitrary point of the set X lying on an attainable orbit. Since x lies on X, it follows that $P_i^I(x) = 0$ (here P_i^I are the restrictions of the polynomials P_i to the coordinate plane which is the closure of an attainable orbit). By hypothesis there exists a vector α lying within the dual cone to this orbit for which the polyhedra $\Delta_1^{\alpha}, \ldots, \Delta_k^{\alpha}$ are independent. We consider the toric manifold $M_{\bar{\sigma}_1}$ where σ_1 is the one-dimensional cone generated by the vector α. We consider the closure $\bar{X}_{\sigma_1}$ in $M_{\bar{\sigma}_1}$ of the manifold defined in T^n by the system $P_1 = \ldots = P_k = 0$. According to the theorem of part 2.3, this closure is an analytic manifold, and its intersection with the orbit $\Phi[\sigma_1]$ on $\Phi[\sigma_1]$ is given by the system $P_i^I = 0$. According to Theorem 2 of part 2.5, for each solution x of the system $P_i^I = 0$ there is a solution of the system $P_i^{\alpha} = 0$ equivalent to it relative to the action of the group $T(\pi)$ (here π is the plane containing the cone σ corresponding to the attainable orbit). Let $\tilde{x}$ be this solution. The manifold X_{σ_1} transversally intersects the orbit $\Phi[\sigma_1]$ of the manifold M_{σ_1}; therefore, there exists a curve issuing from the point $\tilde{x}$ in T^n and lying in X_{σ_1}. The projection of this curve under the mapping $g: M_{\sigma_1} \to \mathbf{C}^n$ issues from the point x and lies in $X \setminus X_c$. The proof of the theorem is complete.

2.7. Resolution of Singularities (Local Version). In this subsection we discuss a local version of the global constructions presented above. A conical Newton polyhedron is defined for a germ of an analytic function in $(\mathbf{C}^n, 0)$. For a system of analytic equations given in a neighborhood of zero the condition of Δ-nondegeneracy is defined which is satisfied for almost all systems of equations with fixed conical Newton polyhedra. The main result of the subsection is the resolution of singularities of a Δ-nondegenerate system of equations by means of a suitable toric manifold.

A we call an infinite, closed, convex polyhedron Δ conical if, first, Δ lies in the positive octant $\mathbf{R}_+^n$, second, Δ with each point m contains the translated positive octant $m + \mathbf{R}_+^n$ with vertex at the point m, and, third, if all the vertices of Δ are integral points.

An analytic function f in a neighborhood of the point $0 \in \mathbf{C}^n$ can be expanded in the Taylor series $f(z) = \sum c_m \chi^m(z)$ where the summation goes over the set of all vectors m lying in the positive octant $\mathbf{R}_+^n$, and $\chi^m(z) = z_1^{m_1} \cdot \ldots \cdot z_n^{m_n}$. The support supp (f) of the germ of an analytic function is the set of points $m \in \mathbf{R}_+^n$ for which the coefficient $c_m \neq 0$. The conical Newton polyhedron $\Delta(f)$ of the germ of an analytic function f is the convex hull of the set of translated positive octants with vertices at all points of the support of the function f, i.e., the convex hull of the set $\bigcup_{m \in \mathrm{supp}(f)} \{m + \mathbf{R}_+^n\}$

LEMMA 1. The conical polyhedron has only a finite number of vertices.

Proof. A conical polyhedron with an infinite number of vertices has an infinite increasing sequence of imbedded conical polyhedra. We assign to each conical polyhedron the ideal in the ring of convergent power series in a neighborhood of the point 0 of the space $\mathbf{C}^n$ consisting of all analytic functions with supports contained in the given conical polyhedron. Under this correspondence to a large polyhedron there corresponds a large ideal. Therefore, the existence of a conical polyhedron with an infinite number of vertices contradicts Hilbert's theorem on the breaking off of an increasing chain of ideals [10]. Of course, Lemma 1 can also be proved geometrically without appealing to Hilbert's theorem.

COROLLARY. The conical polyhedron has a finite number of compact faces.

The union of the compact faces of the conical polyhedron is called its diagram.

The conical polyhedron is called favorable if it has a vertex on each edge of the positive octant.

The support function of the conical polyhedron is defined for vectors in the positive octant $\mathbf{R}_+^{n*}$: the minimum over $x \in \Delta$ of the scalar product $\langle a, x\rangle$ of a vector $a \in \mathbf{R}^{n*}$ with nonnegative components is achieved at one of a finite number of vertices (if at least one coordinate of the vector a is negative, then this minimum is equal to $-\infty$). We note that on a face of the positive octant of $\mathbf{R}^{n*}$ the support function is equal to zero if and only if the conical polyhedron intersects the dual face of the positive octant in $\mathbf{R}^n$.

Let $f_1 = \ldots = f_k = 0$ be a system of analytic equations in a neighborhood of 0 with the conical Newton polyhedra $\Delta_1,\ldots,\Delta_k$. For each vector $a \in \mathbf{R}^{n*}$ with positive coordinates there are defined compact faces $\Delta_1^a,\ldots,\Delta_k^a$ of the conical Newton polyhedra $\Delta_1,\ldots,\Delta_k$. The definition of the contraction f_i^a of the analytic function f_i to order a literally repeats the definition of the contraction of a Laurent polynomial.

The system of equations $f_1 = \ldots = f_k = 0$ is called a-regular if the system of equations $f_1^a = \ldots = f_k^a = 0$ (in which f_i^a are already Laurent polynomials) is regular in T^n.

The system of equations $f_1 = \ldots = f_k = 0$ is called Δ-nondegenerate if it is a-regular for any vector a with positive coordinates.

The conditions of a-regularity and Δ-nondegeneracy are satisfied for almost all systems of analytic functions with given conical polyhedra.

Conical Newton polyhedra $\Delta_1,\ldots,\Delta_k$ are called dependent if for any vector a with positive coefficients the polyhedra $\Delta_1^a,\ldots,\Delta_k^a$ are dependent.

THEOREM 1. The germ X of an analytic set which is defined near the point $0 \in \mathbf{C}^n$ by a Δ-nondegenerate system of analytic equations $f_1 = \ldots = f_k = 0$ lies entirely in the union of coordinate planes if the conical Newton polyhedra of these equations are dependent. If the conical Newton polyhedra are independent, then the germ X, after eliminating its intersections with the coordinate planes, becomes an analytic $(n - k)$-dimensional manifold.

Proof. We shall show that for dependent conical polyhedra the germ X lies entirely in coordinate planes.

Indeed, we suppose that the point 0 is a limit point of the set obtained from the germ X after eliminating its intersections with coordinate planes. Then by the theorem on the selection of curves [17] there exists an analytic curve $z(t) = Ct^a(1 + O(t))$ lying in the set constructed and tending to zero as $t \to 0$. The identities $f_i(z(t)) \equiv 0$ are hereby satisfied. Separating out the lowest-order terms in these identities, we find that the point C is a joint root of the system $f_1{}^a = \ldots = f_k{}^a = 0$.

However, by Theorem 1 of part 2.5 the system $f_1^a = \ldots = f_k^a = 0$ is incompatible in the torus T^n, since the polyhedra $\Delta_1^a,\ldots,\Delta_k^a$ are dependent. The second part of the theorem can be deduced from the theorem on resolution of singularities.

We proceed to the resolution of singularities of a manifold defined in a neighborhood of the point zero in $\mathbf{C}^n$ by a Δ-nondegenerate system of analytic equations. The construction is entirely determined by the collection of conical Newton polyhedra and is a local modification of the construction of part 2.6.

The support functions of the conical manifolds are defined in the positive octant of space $\mathbf{R}^{n*}$. In part 2.6, in particular, the singular and nonsingular faces of the positive octant in $\mathbf{R}^{n*}$, the resolving polyhedrons K^+, the cross of the singularity, and the resolution consisting of a toric manifold M_{K^+} together with the projection $g: M_{K^+} \to \mathbf{C}^n$ were determined on the basis of the collection of Newton polyhedra. The definition of all these objects carries over literally to the case of conical polyhedra. The definition of unnecessary, attainable, and unattainable orbits of the cross of a singularity needs modification.

We recall that conical polyhedra are called dependent if for any covector with positive coordinates the support faces of these polyhedra corresponding to the covector are dependent.

A coordinate plane $\mathbf{R}^I$ in the space $\mathbf{R}^n$ is called unnecessary for the conical polyhedra $\Delta_1,\ldots,\Delta_k$ if the nonempty conical polyhedra $\Delta_1^I,\ldots,\Delta_k^I$, where $\Delta_i^I = \Delta_i \cap \mathbf{R}^I$, are dependent.

A coordinate plane $\mathbf{R}^I$ is called counnecessary for the conical polyhedra $\Delta_1,\ldots,\Delta_k$ if the conical polyhedra in the complementary coordinate plane of $\mathbf{R}^I$, which are the projections of those polyhedra of $\Delta_1,\ldots,\Delta_k$ for which the intersection with the plane $\mathbf{R}^I$ are empty, are dependent.

A singular orbit of the space $\mathbf{C}^n$ for a collection of conical polyhedra is called unnecessary if the coordinate plane ascribed to the orbit in $\mathbf{R}^n$ is unnecessary for these polyhedra. A singular orbit is called unattainable if it is unnecessary and the coordinate plane ascribed to it is counnecessary. The remaining singular orbits (for which the corresponding planes are independent and are not codependent) are called attainable. The coordinate plane $\mathbf{R}^I$ ascribed to an orbit of the space $\mathbf{C}^n$ is determined as follows. To the orbit there corresponds a simplicial face of the positive octant in $\mathbf{R}^{n*}$. A cone in $\mathbf{R}^n$ is dual to this space. We call the coordinate plane ascribed to the orbit the maximal linear subspace contained in this cone.

We proceed to the connection of the resolution for conical polyhedra $\Delta_1,\ldots,\Delta_k$ with Δ-nondegenerate systems of analytic equations with the same Newton polyhedra. Let $f_1 = \ldots = f_k = 0$ be a system of germs of analytic equations in a neighborhood of the point 0 in $\mathbf{C}^n$ with conical Newton polyhedra $\Delta_1,\ldots,\Delta_k$, let X be the germ of the manifold defined by this system, and let X_c be the intersection of this germ with the cross of the singularity for the conical manifolds $\Delta_1,\ldots,\Delta_k$. Let $g:M_{K^+} \to \mathbf{C}^n$ be the resolution for this collection of polyhedra. We denote by X_0 the preimage of the germ $X \setminus X_c$ in M_{K^+} and by $\overline{X}_0$ its closure in M_{K^+}.

THEOREM. The following assertions hold under the conditions on the coefficients of the initial segments of the Taylor series of the functions f_i which are almost always satisfied:

1) $\overline{X}_0$ is the germ of an analytic manifold in M_{K^+} transversally intersecting its orbits;

2) X_0 is obtained from $\overline{X}_0$ by eliminating intersections with the collection of transversally intersecting hypersurfaces — the preimages of the cross of the singularity for the projection g;

3) the projection g establishes a mutual analytic correspondence between X_0 and $X \setminus X_c$;

4) the projection g gives a proper mapping of $\overline{X}_0$ onto the germ of the manifold X from which intersections with all orbits in $\mathbf{C}^n$ which are unattainable for the polyhedra $\{\Delta_i\}$ have been eliminated;

5) assertions 1)-4) remain in force if by X and X_c we understand the parts of the corresponding analytic manifolds lying in a sufficiently small ball about the point zero, by X_0 we understand the preimage of $X \setminus X_c$, and by $\overline{X}_0$ we understand its closure.

We shall formulate conditions on the coefficients under which the theorem holds. We connect with the coordinate plane $\mathbf{R}^I$ the system of equations $f_1^I = \ldots = f_k^I = 0$, where f_i^I is the restriction of f_i to the coordinate plane $\mathbf{C}^I$ in $\mathbf{C}^n$ (restrictions identically equal to zero are not considered). The theorem holds for systems of equations which, first of all, are themselves Δ-nondegenerate and, secondly, all systems $\{f_i^I\}$ are Δ-nondegenerate.

The proof of the theorem is similar to the proofs of the theorems of parts 2.4 and 2.6, and we shall therefore not consider it. We note interesting special cases.

1. All conical Newton polyhedra are favorable, and hence the cross of the singularity consists of the single point 0. In this case the theorem shows that the manifold X has the isolated singular point 0 and gives a resolution of singularities $g:M_{K^+} \to \mathbf{C}^n$ blowing up only this point 0.

2. The cross of the singularity for the conical Newton polyhedra, with the exception of the point zero, contains only unnecessary orbits. In this case the theorem shows that the manifold X has the isolated singular point 0. The resolution of singularities $g:M_{K^+} \to \mathbf{C}^n$ blows up in $\mathbf{C}^n$ more than the point 0. However, the restriction of g to the manifold $\overline{X}_0$ blows up only the singular point 0 on the manifold X.

3. The cross of the singularity contains only unnecessary and attainable orbits. In this case the theorem gives a nonsingular manifold $\overline{X}_0$ together with the proper projection $g:\overline{X}_0 \to X$ which is an isomorphism on an open, dense set. This case differs from the preceding case in that the singular point 0 of the manifold X, generally speaking, is not isolated. The mapping $g:\overline{X}_0 \to X$ is a mapping "onto" and is an isomorphism away from a proper analytic set.

Remark. Under the conditions of the theorem the orders of the functions g^*f_i on hyperplanes lying in the preimage of zero are determined by means of the theorems of part 2.3. The equations of the intersection of the manifold $\bar{X}_0$ with orbits of the manifold M_{K+} lying in the preimage of zero are determined in a similar way.

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