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Newtonian Hydrodynamics in an Expanding Universe in Terms of the Scalar-Tensor Theory

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In view of a possible relevance of the scalar-tensor theory of gravity to various cosmological and astrophysical problems, Newtonian hydrodynamics suitable at the matter dominant stage of the Brans-Dicke universe is formulated and a formalism for dealing with various non-linear effects of cosmic fluid being important during the formation of galaxies is developed in comparison with the previous one based on the general relativistic cosmology. In contrast with the situation in the linear theory of gravitational instability, it is shown in the case of a density perturbation with spherical symmetry that the critical epoch and density ratio for the occurrence of gravitational binding at the center may considerably be different from those in the Friedmann universe. Dynamical equations for a rotating gaseous ellipsoid with uniform density are also derived and their general features are touched upon.

§ 1. Introduction

In previous two papers[1)] (these papers will be referred to as [GRI] and [GRII]), the author has developed a formalism for dealing with various non-linear effects of cosmic fluids which are important in the process of galaxy formation at some stage of an expanding universe, on the basis of Newtonian hydrodynamics[2)] in the general relativistic cosmology. In order to assess the importance of the coupling of vorticity with cosmic expansion before the establishment of gravitational binding of a density perturbed region, the author and Fujimoto[3)] (this paper will be referred to as [GRIII]) have recently applied the formalism to the case where a gaseous ellipsoid with uniform density is rotating relative to the cosmological background with flat 3-space. It has been revealed numerically that, if its angular momentum is lower than a certain limit, the presence of rotation does not so much modify Kihara's[4)] result or Tomita's[5)] refined result in the case of a density perturbation with spherical symmetry that its peak point becomes gravitationally bound at the instant when the density ratio ρ/ρ_B (ρ_B is the background density) reaches the value $(3\pi/4)^2=5.55$. This is due to the fact that the density promoting effects of gravity and shearing overwhelm the opposite effect of vorticity in such a way that their ratios at the critical epoch are about $1:10^{-1}:10^{-3}$.

On the other hand, in view of the possible relevance of the scalar-tensor theory of gravity[6)] to the resolution of various cosmological problems such as those concerning the abundance ratio of He to H at an early stage of cosmic

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expansion and the formation of galaxies at a more later stage, the gravitational instability in the Brans-Dicke universe with flat 3-space has been studied by the author.[7] As a result, it has been shown that, at the matter dominant stage, the density contrast can grow in time as $\delta\rho/\rho_B \propto t^{(2/3)\{1+2/(4+3\omega)\}}$ (ω is the coupling constant characteristic in the scalar-tensor theory), provided that the wave length λ of the density perturbation is much smaller than the Hubble length cH^{-1} but considerably larger than the Jeans wave length λ_J. The above growth rate is somewhat higher (if $5 \lesssim \omega < \infty$, as insisted upon by the advocators of the scalar-tensor theory) than its general relativistic counterpart,[8] i.e., $\delta\rho/\rho_B \propto t^{2/3}$, but needless to say, in the Brans-Dicke cosmology too, we must appeal to various non-linear effects of cosmic fluid in order to deal with the formation of galaxies in more detail.

The aim of this paper is to derive Newtonian hydrodynamical equations in the Brans-Dicke universe and to develop, using these equations, a formalism for dealing with the non-linear effects in question, which is to be compared with that in [GRI] ~ [GRIII]. In § 2 the background universe is dealt with, laying emphasis on its matter dominant stage relevant to the formation of a gravitationally bound system whose mass is of the order of $M = 10^{9\sim 11} M_\odot$. In § 3 the Newtonian hydrodynamical equations are derived from the field equations in the scalar-tensor theory for the perturbed universe by means of a suitable approximation. The description of cosmic fluid in a rotating frame of reference relative to the cosmological background is also dealt with. In § 4, by a procedure similar to that in [GRII], the coupled dynamical equations for the vorticity vector ω_i, the shearing tensor q_{ij} and the density ratio ρ/ρ_B are derived and it is shown that, if all non-linear effects are negligible, the last equation leads to the density contrast obtained in Ref. 7). In § 5 the method of local analysis around a peak point of any density perturbation is applied to the density perturbation with spherical symmetry and the dynamical equation for ρ/ρ_B at that point is solved in the cases of $\omega = 0$ and $\omega = -4/5$ in order to see how the critical value of ρ/ρ_B for the occurrence of gravitational binding depends upon the value of ω. Section 6 is devoted to the global analysis of a rotating gaseous ellipsoid with uniform density, whose relativistic cosmological counterpart has been numerically performed in [GRIII].

Summary of the scalar-tensor theory of gravity

For convenience sake, we shall summarize the field equations in the scalar-tensor theory of gravity, on the prescription that the metric is represented by

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3; i, j = 1, 2, 3) \tag{1·1}$$

where $g_{\mu\nu}$ is the metric tensor whose signature is $(-+++)$. Then we have

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \left(\frac{8\pi}{c^4\phi}\right)T_{\mu\nu} + \omega\left(\phi_\mu\phi_\nu - \frac{1}{2}g_{\mu\nu}\phi_\alpha\phi^\alpha\right)\Big/\phi^2 + (\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi)/\phi\,, \tag{1·2}$$

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$$\Box\phi \equiv g^{\mu\nu}\phi_{;\mu\nu} = \frac{8\pi}{c^4(3+2\omega)}T\,, \tag{1·3}$$

from which, by virtue of the well-known identity $G^{\mu\nu}{}_{;\nu} \equiv 0$, we obtain

$$T^{\mu\nu}{}_{;\nu} = 0\,, \tag{1·4}$$

where ϕ is a massless scalar field whose reciprocal is proportional to the gravitation constant G (in the general relativity theory), $\phi_\mu \equiv \phi_{,\mu} \equiv \partial_\mu \phi$ and $T_{\mu\nu}(T \equiv g^{\mu\nu}T_{\mu\nu})$ is the energy-momentum tensor of matter and radiation. It follows from Eqs. (1·2) and (1·3) that

$$R_{\mu\nu} = \left(\frac{8\pi}{c^4\phi}\right)\left\{T_{\mu\nu} - \left(\frac{1+\omega}{3+2\omega}\right)Tg_{\mu\nu}\right\} + \Phi_{\mu\nu}\,, \tag{1·5}$$

where

$$\Phi_{\mu\nu} \equiv \omega\phi_\mu\phi_\nu/\phi^2 + \phi_{;\mu\nu}/\phi\,.$$

If the assemblage of matter and radiation can be regarded as a perfect fluid with the 4-velocity $U^\mu (U_\mu U^\mu = -1)$, we have

$$T_{\mu\nu} = (c^2\rho + p)\,U_\mu U_\nu + pg_{\mu\nu}\,, \qquad T = 3p - c^2\rho\,, \tag{1·6}$$

where $\rho = \rho_m + \rho_r$ and $p = p_m + p_r (p_r = c^2\rho_r/3)$ are the total density and pressure, respectively. Equation (1·6) must of course be supplemented by a relativistic equation of thermodynamics and the equation of state for matter.

Moreover, Dicke[9] has asserted that Mach's principle finds its physical reality by the requirement that Eq. (1·3) must be solved in such a way as

$$\phi(x) = -\frac{8\pi}{c^4(3+2\omega)}\int D_{\rm ret}(x,x')\,T(x')\,\{-g(x')\}^{1/2}d^4x', \tag{1·7}$$

where $D_{\rm ret}(x,x')$ is the retarded bi-scalar Green's function for a massless scalar field in a curved space-time specified by the metric tensor $g_{\mu\nu}$ which in turn depends upon ϕ and its derivatives via Eq. (1·2).

§ 2. The background universe

For simplicity, let us consider an expanding homogeneous universe specified by the metric

$$ds_B{}^2 = -{}^Bg_{\mu\nu}dx^\mu dx^\nu = c^2dt^2 - a^2(t)\,(dx^2 + dy^2 + dz^2)\,, \tag{2·1}$$

where $x^\mu = (t, x_i)$ are the coordinates comoving with the fluid matter and radiation constituting the universe, so that we have

$${}^BT_0{}^0 = -c^2\rho_B(t), \quad {}^BT_0{}^i = 0\,, \quad {}^BT_i{}^j = p_B(t)\delta_i{}^j\,, \quad (U_B{}^\mu = c^{-1}\delta_0{}^\mu) \tag{2·2}$$

and

$$\phi = \phi_B(t) \text{ such as } \phi_B(t_0) = G^{-1}. \ (t_0 \text{ is the present epoch}) \tag{2·3}$$

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By making use of Eqs. (2·1)～(2·3), we may derive from Eqs. (1·2)～(1·4) the following fundamental equations in the Brans-Dicke cosmology:

$$8\pi\phi_B^{-1}\rho_B = 3(\dot{a}/a)^2 + 3(\dot{a}/a)(\dot{\phi}_B/\phi_B) - (\omega/2)(\dot{\phi}_B/\phi_B)^2, \tag{2·4}$$

$$\ddot{\phi}_B/\phi_B + 3(\dot{a}/a)(\dot{\phi}_B/\phi_B) = \frac{8\pi}{(3+2\omega)}(\rho_B - 3p_B/c^2), \tag{2·5}$$

$$\dot{\rho}_B + 3(\dot{a}/a)(\rho_B + p_B/c^2) = 0\,, \tag{2·6}$$

where a dot denotes differentiation with respect to the cosmic time t and $\rho_B = \rho_{Bm} + \rho_{Br}$, $p_B = p_{Bm} + p_{Br}$ ($\rho_{Br} = 3p_{Br}/c^2 = bT_{Br}^4/c^2$, T_{Br} is the radiation temperature whose present value is about 3°K[10] and b the Stefan-Boltzmann constant).

The matter dominant stage

Let us pay attention here to the matter dominant stage of the universe such as $\rho_{Bm} > \rho_{Br} = 3p_{Br}/c^2 \gg p_{Bm}/c^2$, because the formation of galaxies must occur at such a stage in the Brans-Dicke cosmology as well. At this stage, we can approximate the quantities $(\rho_B - 3p_B/c^2)$ and $(\rho_B + p_B/c^2)$ in Eqs. (2·5) and (2·6) by $\rho_B \simeq \rho_{Bm}$, so that Eq. (2·6) can be integrated as follows:

$$a^3\rho_B = a_0{}^3\rho_{B0} = \text{const}\,, \tag{2·7}$$

where ρ_{B0} is the value of ρ_B at the present epoch t_0 such as $\phi_{B0} = G^{-1}$ and $H_0 \equiv (\dot{a}/a)_0$ (the Hubble constant). On inserting Eq. (2·7) into Eqs. (2·4) and (2·5), we obtain

$$3(\dot{a}/a)^2 + 3(\dot{a}/a)(\dot{\phi}_B/\phi_B) - (\omega/2)(\dot{\phi}_B/\phi_B)^2 = 8\pi G\rho_{B0}\frac{(a^3\phi_B)_0}{a^3\phi_B}\,, \tag{2·8}$$

$$\ddot{\phi}_B/\phi_B + 3(\dot{a}/a)(\dot{\phi}_B/\phi_B) = \frac{8\pi G\rho_{B0}}{(3+2\omega)}\cdot\frac{(a^3\phi_B)_0}{a^3\phi_B}\,. \tag{2·9}$$

If we assume that $Ht = \text{const}$ like its counterpart $Ht = 2/3$ in the general relativistic cosmology, it follows from Eqs. (2·8) and (2·9) that

$$\begin{aligned} a = a_0(t/t_0)^m \quad (Ht = m), \qquad G\phi_B = (t/t_0)^{2-3m}, \\ 4\pi G\rho_{B0}t_0{}^2 = (1 - m/2), \end{aligned} \tag{2·10}$$

where

$$m \equiv (2+2\omega)/(4+3\omega). \tag{2·11}$$

While the above solution has been derived without recourse to Eq. (1·7) as a manifestation of Mach's principle, the consistency of Eqs. (2·10) and (1·7) has been verified by making use of our Green's function $D_{\text{ret}}(x, x')$ in the universe under consideration.[11]

It is easily seen that Eq. (2·10) provides us with an expanding model (if $m > 0$) such that the case of $m = 2/3$ or $\omega = \infty$ corresponds to the Einstein-de Sitter

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model in the general relativistic cosmology. Accordingly, we shall assume here that $0<m\leq 2/3$ or $-1<\omega\leq\infty$, while a more realistic range would be $5\lesssim\omega<\infty$.[9]

§ 3. Description of the perturbed universe in the Newtonian approximation

Let us consider the perturbed universe at its matter dominant stage. Such a universe may be represented by the following expressions for the metric and the scalar field:

$$ds^2=-g_{\mu\nu}dx^\mu dx^\nu=c^2\{1-2\mathfrak{V}/c^2+0(1/c^4)\}dt^2-a^2(t)\{1+0(1/c^2)\}\times(dx^2+dy^2+dz^2), \tag{3·1}$$

$$\phi=\phi_B(t)\{1+\psi/c^2+0(1/c^4)\}, \tag{3·2}$$

where $a(t)$ and $\phi_B(t)$ are given by Eq. (2·10), and $\mathfrak{V}=\mathfrak{V}(\boldsymbol{x},t)$ and $\psi=\psi(\boldsymbol{x},t)$ are the gravitational potential and the variation of ϕ divided by $c^2\phi_B$, respectively, due to the presence of local density perturbations.

Since the quantity $\phi_B^{-1}(t)=G(t/t_0)^{3m-2}$ plays a role of the gravitation constant in the Brans-Dicke cosmology, it follows from a dimensional consideration for $\mathfrak{V}/c^2$ due to a spherical density perturbation with mass δM and radius L that $\mathfrak{V}/c^2\sim(G\delta M/c^2L)(t/t_0)^{3m-2}$ (which is nearly equal to $5\times10^{-(4+3m)}(2/3\geq m>0)$ for $\delta M=10^{11}M_\odot$, $L=10^3pc$ and $t=0.1t_0$). Accordingly, we may neglect $(\mathfrak{V}/c^2)^2$ compared with unity, just as in the case of general relativistic cosmology.[2] Similarly, it is reasonable to neglect $(\psi/c^2)^2$ compared with unity. Then it follows from Eq. (3·1) and the relation $U_\mu U^\mu=-1$ for the 4-velocity U^μ in Eq. (1·6) for the energy-momentum tensor of cosmic matter (assumed as a perfect fluid) in the perturbed universe that

$$U^0=c^{-1}\{1+\mathfrak{V}/c^2+\tfrac{1}{2}(v/c)^2+0(1/c^4)\},\qquad U^i=U^0u_i\,, \tag{3·3}$$

where $v_i\equiv au_i$ is the physical fluid velocity relative to the co-moving frame (t,x_i) such that $(v/c)^2$ can be regarded as being of the order of $\mathfrak{V}/c^2$. On inserting Eq. (3·3) into Eq. (1·6), we obtain

$$\begin{aligned}&T^{00}=\rho+0(1/c^2),\qquad T^{0i}=\rho u_i+0(1/c^2),\\&T^{ij}=\rho u_iu_j+a^{-2}p\delta_{ij}+0(1/c^2),\\&T=-c^2\rho\{1+0(1/c^2)\}.\end{aligned} \tag{3·4}$$

Moreover, by the use of Eqs. (3·1) and (3·2), we can represent R_{00} and Φ_{00} appearing in the (00)-component of Eq. (1·5) and $\Box\phi$ in Eq. (1·3) as follows:

$$\begin{aligned}&R_{00}=-3\ddot{a}/a-a^{-2}\nabla^2\mathfrak{V}+0(1/c^2),\\&\Phi_{00}=\ddot{\phi}_B/\phi_B+\omega(\dot{\phi}_B/\phi_B)^2+0(1/c^2),\\&\Box\phi=-(1/c^2)\{\ddot{\phi}_B+3(\dot{a}/a)\dot{\phi}_B-a^{-2}\phi_B\nabla^2\psi+0(1/c^2)\},\end{aligned} \tag{3·5}$$

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where $\nabla^2 \equiv \partial_i \partial_i$.

Now let us consider a Newtonian approximation in which all terms of the order of $O(1/c^2)$ have been neglected in the reduction of Eqs. (1·3), (1·4) and (1·5) by the use of Eqs. (2·10), (3·1) and (3·2). Then it follows from Eqs. (1·4) and (3·4) that

$$D_t \rho + (3H + \partial_i u_i)\rho = 0, \tag{3·6}$$

$$D_t u_i + 2H u_i = -a^{-2}(\rho^{-1}\partial_i p - \partial_i \mathfrak{B}), \tag{3·7}$$

which are the Newtonian hydrodynamical equations of the same form as their counterparts, e.g., Eqs. (2·2·1) and (2·2·2) in [GRI], in the general relativistic cosmology, where $D_t \equiv \partial_t + u_i \partial_i$. Moreover, we can verify that Eckart's[12] relativistic equation of thermodynamics is approximately reduced to the usual Newtonian form

$$c_v D_t T + p D_t(1/\rho) = 0 \quad \text{(for an adiabatic process)}, \tag{3·8}$$

where T is the matter temperature such as $p = (k/\mu m_H)\rho T$.

In contrast with the above three equations, the reduction of Eqs. (1·5) and (1·3) reveals a new situation characteristic in the Brans-Dicke cosmology. Namely, by the use of the Eqs. (2·10), (3·4) and (3·5), we obtain

$$a^{-2}\nabla^2 \mathfrak{B} = -4\pi\left(\frac{4+2\omega}{3+2\omega}\right)\phi_B{}^{-1}(\rho - \rho_B), \tag{3·9}$$

$$a^{-2}\nabla^2 \psi = -\frac{8\pi}{(3+2\omega)}\phi_B{}^{-1}(\rho - \rho_B), \tag{3·10}$$

which provide us with the following relation:

$$\delta\phi/\phi_B \equiv \psi/c^2 = \frac{1}{(2+\omega)}(\mathfrak{B}/c^2), \tag{3·11}$$

on the assumption that $\psi = 0$ in the background region such as $\rho = \rho_B$. Equation (3·9) is the scalar-tensorial version of Eq. (2·2·3) in [GRI] and its right-hand side includes the factor $\phi_B{}^{-1} = G(t/t_0)^{3m-2}$ as mentioned already, as well as an additional numerical factor $(4+2\omega)/(3+2\omega) = (4-4m)/(2-m)$ (which varies from 1 (for $m = 2/3$) to 12/7 (for $m = 1/4$)). Both the relation (3·11) and the appearance of that numerical factor in Eq. (3·9) for the gravitational potential are common with those in the case[13] where the background universe is the Minkowski space-time to be specified by $\rho_B = 0$, $a = 1$ and $\phi_B = \text{const}$. It is to be noticed, however, that the normalization $\phi_B{}^{-1} = \text{const} = G(3+2\omega)/(4+2\omega)$ in the latter case is incompatible with our normalization $\phi_B{}^{-1}(t_0) = G$ on which Eq. (2·10) relies.

Thus it may be said that Eqs. (3·6), (3·7), (3·8) and (3·9), supplemented by Eqs. (2·10) and (3·11), are the Newtonian hydrodynamical equations suitable

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at the matter dominant stage of the Brans-Dicke univese in the co-moving frame (t, x_i) of reference.

For later usefulness, we shall further rewrite the above hydrodynamical equations in other frames of reference. If we go over from the co-moving frame (t, x_i) to the inertial frame (t, X_i) with $X_i \equiv a(t)x_i$, Eqs. (3·6), (3·7), (3·8) and (3·9) are reduced to

$$d\rho/dt + (\partial V_i/\partial X_i)\rho = 0, \tag{3·12}$$

$$dV_i/dt = -\rho^{-1}\partial p/\partial X_i + \partial\mathfrak{B}/\partial X_i - \frac{m(1-m)}{t^2}X_i, \tag{3·13}$$

$$c_v dT/dt + p(\partial V_i/\partial X_i) = 0, \tag{3·14}$$

$$\Delta\mathfrak{B} = -4\pi\left(\frac{4-4m}{2-m}\right)\phi_B^{-1}(\rho-\rho_B), \tag{3·15}$$

where $V_i \equiv HX_i + v_i$, and $d/dt \equiv \partial_t + V_i\partial/\partial X_i$, $\Delta \equiv \partial^2/\partial X_i\partial X_i$. Equations (3·12)$\sim$(3·15) are reduced to Eqs. (2·6)$\sim$(2·9) in [GRIII] if we put $m=2/3$, as should be the case. Needless to say, Eqs. (2·10) and (3·11) are valid in the inertial frame as well. If we go over from the inertial frame (t, X_i) to a non-inertial frame (t, X_i') which is rotating with the angular velocity Ω_i relative to the former, there arise again Eqs. (3·12), (3·14) and (3·15), but Eq. (3·13) must be rewritten as

$$dV_i/dt + 2\varepsilon_{ijk}\Omega_j V_k = -\rho^{-1}\partial p/\partial X_i + \partial\mathfrak{B}/\partial X_i - \left[\left\{\frac{m(1-m)}{t^2} - \Omega^2\right\}\delta_{ij} + \Omega_i\Omega_j - \varepsilon_{ijk}d\Omega_k/dt\right]X_j, \tag{3·13'}$$

with a proviso that X_i and V_i in Eqs. (3·12), (3·13'), (3·14) and (3·15) are an abbreviation of X_i' and V_i', respectively, where ε_{ijk} is Kronecker's anti-symmetric tensor.

§ 4. Dynamical equations for ω_i, q_{ij} and ρ/ρ_B

As in [GRII], let us define the density ratio, the vorticity and shearing of the velocity field u_i in the following way:

$$\Phi \equiv \ln(\rho/\rho_B) \equiv \ln(1+K),$$
$$\omega_i \equiv \tfrac{1}{2}\varepsilon_{ijk}\partial_k u_j, \tag{4·1}$$
$$q_{ij} = q_{ji} \equiv \tfrac{1}{2}(\partial_j u_i + \partial_i u_j) - \tfrac{1}{3}(\partial_k u_k)\delta_{ij}. \quad (q_{ii}=0)$$

*)

Similarly, let us consider a case such that Eq. (3·8) can be replaced by

$$\partial_i p = v_s^2\partial_i\rho = \rho v_s^2\partial_i\Phi, \tag{4·2}$$

*) Just as $v_i = au_i$, both ω_i and q_{ij} are physically measurable quantities.

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where $v_s=v_s(t)$ (if $\partial_i T=0$) is the velocity of sound in the perturbed region. By making use of Eqs. (4·1) and (4·2), we obtain from Eqs. (3·6), (3·7) and (3·9) the following set of dynamical equations:

$$(D_t+2H)u_i=\partial_i\{a^{-2}(\mathfrak{B}-v_s^2\Phi)\}, \qquad (4\cdot3)$$

$$\{D_t+2H-(2/3)(D_t\Phi)\}\omega_i=q_{ij}\omega_j, \qquad (4\cdot4)$$

$$\{D_t+2H-(2/3)(D_t\Phi)\}q_{ij}+q_{ik}q_{jk}+\omega_i\omega_j-\tfrac{1}{3}(q_{mn}^2+\omega^2)\delta_{ij}$$
$$=(\partial_i\partial_j-\tfrac{1}{3}\delta_{ij}\nabla^2)\{a^{-2}(\mathfrak{B}-v_s^2\Phi)\}, \qquad (4\cdot5)$$

$$\{D_t^2+2HD_t-(v_s/a)^2\nabla^2\}\Phi-\frac{1}{3}(D_t\Phi)^2-4\pi\left(\frac{4-4m}{2-m}\right)(\rho_B/\Phi_B)(e^\Phi-1)$$
$$-q_{mn}^2+2\omega^2=0, \qquad (4\cdot6)$$

where

$$\mathfrak{B}=\left(\frac{4-4m}{2-m}\right)a^2(\rho_B/\phi_B)\int\frac{K(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}d\boldsymbol{x}'. \qquad (4\cdot7)$$

Equations (4·3)~(4·5) are equivalent to Eqs. (2·6)~(2·8) in [GRII], but Eqs. (4·6) and (4·7) are different from Eqs. (2·9) and (2·10) in [GRII] on account of the variability of gravitation constant.

Let us assume in particular that all non-linear terms in the above dynamical equations are negligible. Then it follows from Eqs. (4·4)~(4·6) that

$$\partial_t(a^2\omega_i)=0, \qquad (4\cdot8)$$

$$\partial_t(a^2q_{ij})=(\partial_i\partial_j-\tfrac{1}{3}\delta_{ij}\nabla^2)(\mathfrak{B}-c_s^2K), \qquad (4\cdot9)$$

$$\left\{\partial_t^2+2H\partial_t-(c_s/a)^2\nabla^2-4\pi\left(\frac{4-4m}{2-m}\right)(\rho_B/\phi_B)\right\}K=0, \qquad (4\cdot10)$$

where c_s stands for the velocity of sound in the background universe. Equation (4·8) shows the conservation of vorticity and Eq. (4·10) is the dynamical equation for $K=\delta\rho/\rho_B(\ll 1)$ whose relativistic cosmological counterpart has been given by Nariai et al.[14] If K is known, Eqs. (4·7) and (4·9) permit us to determine q_{ij} as a set of known functions of t and $\boldsymbol{x}$. In a special case such as

$$K(\boldsymbol{x},t)=\hat{K}(t)\frac{\sin(nr)}{nr}, \quad (r\equiv(x_ix_i)^{1/2},\ n=\text{const}) \qquad (4\cdot11)$$

Eq. (4·10) reduces to

$$d^2\hat{K}/dt^2+(2m/t)d\hat{K}/dt-\{1-(\lambda_J/\lambda)^2\}\{(2-2m)/t^2\}\hat{K}=0, \qquad (4\cdot12)$$

by the use of Eq. (2·10), where $\lambda\equiv 2\pi a/n$ and $\lambda_J\equiv\{2/(1-m)\}^{1/2}(\pi c_s t)$ stand for the wave length of the density perturbation and the Jeans wave length, respectively. If $\lambda\gg\lambda_J$, the general solution of Eq. (4·12) is given by

$$\hat{K}=c_1t^{2-2m}+c_2t^{-1}, \quad (c_1, c_2=\text{const}) \qquad (4\cdot13)$$

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which is equivalent to Eq. (5·12) in Ref. 7), as should be the case. On inserting Eq. (4·7) with $K(\boldsymbol{x}, t)$ given by Eqs. (4·11) and (4·13) into Eq. (4·9) and performing the required integrations by the use of Eq. (2·10), we obtain

$$q_{ij}(\boldsymbol{x}, t) = \{(2-2m)(c_1 t + c_3) t^{-2m} - c_2 t^{-2}\} Q_n(r) \left(\frac{1}{3}\delta_{ij} - \frac{x_i x_j}{r^2}\right), \qquad (4\cdot14)$$

with

$$Q_n(r) \equiv \frac{\sin(nr)}{nr} - \frac{3}{(nr)^2}\left\{\frac{\sin(nr)}{nr} - \cos(nr)\right\}, \qquad (4\cdot15)$$

which is nearly equal to $-(nr)^2/15$ for $nr \ll 1$ or $r \simeq 0$, where c_3 is an integration constant. The above example shows that, even in the case where a spherically symmetric density perturbation coexists with an irrotational flow, the fluid has generally a non-vanishing shearing which, however, may vanish at a peak point of the density perturbation.

§ 5. Local analysis at the center of a density perturbation with spherical symmetry

Let us pay attention to a peak point (denoted by $\boldsymbol{x}=0$ without loss of generality) of $\Phi \equiv \ln(\rho/\rho_B)$ such that $(u_i)_{x=0}=0$ at any time. It has been shown in [GRII], however, that the local analysis at that point of the dynamical equations for ω_i, q_{ij} and Φ (such as those given by Eqs. (4·4)$\sim$(4·7)) is useless, except in a special class such as $(q_{ij})_{x=0}=0$ of spherically symmetric density perturbations, i.e., $\Phi=\Phi(r, t)$ and $\omega_i=0$. Accordingly we shall consider only this class in what follows.

If the effective scale $L \equiv 2\pi a|\rho\Phi/\nabla^2\rho|^{1/2}_{x=0}$ is considerably larger than the Jeans scale $L_J \equiv \{2/(1-m)\}^{1/2}(\pi v_s t)$, it follows from Eq. (4·6) withe $\omega_i=(q_{ij})_{x=0}=0$ that

$$d^2\Psi/dt^2 + (2m/t)d\Psi/dt - \frac{1}{3}(d\Psi/dt)^2 - \frac{(2-2m)}{t^2}(e^{\Psi}-1) = 0, \qquad (5\cdot1)$$

where $\Psi \equiv \Phi(0, t)$. The above equation is a non-linear version of Eq. (4·12) with $\lambda \gg \lambda_J$. In order to deal with the occurrence of gravitational binding, let us put

$$\begin{cases} \rho_B = \rho_{B0}\tau^{-3m}, & (\tau \equiv t/t_0) \\ \rho \equiv \rho_{B0} y^{-3}, \end{cases} \qquad (5\cdot2)$$

where ρ is an abbreviation of $\rho(0, t)$. Then Eq. (5·1) is reduced to

$$d^2y/d\tau^2 = \frac{(1-m)}{3}\{(2-3m)\tau^{-2}y - 2\tau^{3m-2}y^{-2}\}, \quad (2/3 \geq m > 0) \qquad (5\cdot3)$$

which is a generalized version of Eq. (4·10) in [GRI], i.e., the case of $m=2/3$. The dynamical system governed by Eq. (5·3) with $m \neq 2/3$ is analogous to that

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for the free fall of a test particle in the gravitational field due to a central body with variable mass ($\propto\tau^{3m-2}$) and in an additional repulsive field ($\propto\tau^{-2}y$). Such a complicated situation disappears and the dynamical system has the energy integral $\frac{1}{2}(dy/d\tau)^2-(2/9)y^{-1}=E=\text{const}$, provided that $m=2/3$. If $E<0$, we arrive at the following result[4),5)] for the occurrence (at $\tau=\tau_c$) of gravitational binding ($d\rho/d\tau=0, d^2\rho/d\tau^2>0$):

$$\tau_c/\tau_\infty=1/2\,,\qquad (\rho/\rho_B)_c=(3\pi/4)^2=5.55\,,\quad (\text{for } m=2/3) \tag{5·4}$$

where τ_∞ stands for the epoch at which ρ becomes infinitely large.

In order to simplify Eq. (5·3), let us further put

$$Y=\tau^{3m-2}y\,,\qquad dT=\pm\tau^{2(3m-2)}d\tau\,,\quad (2/3>m>0) \tag{5·5}$$

where $\pm$ is an abbreviation of $+$ (if $2/3>m\geq1/2$) or $-$ (if $1/2>m>0$). Then we obtain

$$d^2Y/dT^2=\tfrac{2}{3}(2-3m)(4m-1)\{\tau(T)\}^{6(1-2m)}Y-\tfrac{2}{3}(1-m)Y^{-2}, \tag{5·6}$$

which has the "energy" integral in the special two cases $m=1/4$ and $m=1/2$, as well as in the case $m=2/3$ mentioned already.

The case of $m=1/4$ *or* $\omega=-4/5$

In this case, Eq. (5·6) can be integrated as

$$dY/dT=(Y^{-1}+2E)^{1/2},\qquad (Y=\tau(\rho/\rho_B)^{-1/3},\qquad T=\tfrac{2}{3}\tau^{-3/2}) \tag{5·7}$$

where $E=\text{const}$ stands for the total "energy" of the dynamical system specified by (Y,T) and we have chosen the $+$ sign in front of $(Y^{-1}+2E)^{1/2}$ so as to make possible $d\rho/d\tau=0$ at some epoch. In order that the gravitational binding ($d\rho/d\tau=0, d^2\rho/d\tau^2>0$) may occur at $\tau=\tau_c$ or $T=T_c$ corresponding to $Y=Y_c$, we must have

$$1+2EY_c=(25/36)\,Y_c^3/T_c^2,\qquad 3-4EY_c>0\,, \tag{5·8}$$

which are permissible not only in the case $E<0$, but also in other cases $E=0$ and $(3/4)\,Y_c^{-1}>E>0$. Taking account of this point, we can integrate Eq. (5·7) as follows:

(i) $E=0$: $\rho=\rho_{B0}\tau^{-3/4}\{1-(\tau/\tau_\infty)^{3/2}\}^{-2},$ (5·9)

(ii) $E<0$:
$$\begin{cases}\rho=\dfrac{9}{2}\rho_B(\tau_\infty)(n\tau_\infty)^{-3}(1-\cos\theta)^{-3}\{1+(n\tau_\infty)^{3/2}(\theta-\sin\theta)\}^{5/2},\\[2mm] \tau=\tau_\infty\{1+(n\tau_\infty)^{3/2}(\theta-\sin\theta)\}^{-2/3},\end{cases} \tag{5·10}$$

(iii) $E>0$:
$$\begin{cases}\rho=\dfrac{9}{2}\rho_B(\tau_\infty)(n\tau_\infty)^{-3}(\text{ch}\,\theta-1)^{-3}\{1+(n\tau_\infty)^{3/2}(\text{sh}\,\theta-\theta)\}^{5/2},\\[2mm] \tau=\tau_\infty\{1+(n\tau_\infty)^{3/2}(\text{sh}\,\theta-\theta)\}^{-2/3},\end{cases} \tag{5·11}$$

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where $\rho_B=\rho_{B0}\tau^{-3/4}$, τ_∞ is the catastrophic epoch at which ρ becomes infinitely large and $n\equiv(3/16)^{2/3}|E|^{-1}$ (if $E\neq0$). From the above three equations, we obtain

$$\tau_c/\tau_\infty=5^{-2/3}=0.342\,,\qquad (\rho/\rho_B)_c=(5/4)^2=1.56\,,\quad (\text{if } E=0) \tag{5·12}$$

and

$$\tau_c/\tau_\infty=\begin{cases}\{1+(n\tau_\infty)^{3/2}(\theta_c-\sin\theta_c)\}^{-2/3}, & (\text{if } E<0)\\ \{1+(n\tau_\infty)^{3/2}(\text{sh}\,\theta_c-\theta_c))^{-2/3}, & (\text{if } E>0)\end{cases} \tag{5·13}$$

$$(\rho/\rho_B)_c=(75/24)\begin{cases}(1+\cos\theta_c)^{-1}, & (\text{if } E<0)\\ (\text{ch}\,\theta_c+1)^{-1}, & (\text{if } E>0)\end{cases} \tag{5·14}$$

where

$$n\tau_\infty=\begin{cases}\left\{\dfrac{(1-\cos\theta_c)\,(11+\cos\theta_c)}{6\sin\theta_c}-\theta_c\right\}^{-2/3}, & (\text{if } E<0)\\ \left\{\theta_c-\dfrac{(\text{ch}\,\theta_c-1)\,(\text{ch}\,\theta_c+11)}{6\,\text{sh}\,\theta_c}\right\}^{-2/3}, & (\text{if } E>0)\end{cases} \tag{5·15}$$

which are permissible only when $0<\theta_c\lesssim\pi/3$ (if $E<0$) and $0<\theta_c\lesssim2$ (if $E>0$). Examples in the latter cases are as follows:

$$\begin{cases}\tau_c/\tau_\infty=0.370\,, & (\rho/\rho_B)_c=1.83\,, & n\tau_\infty=12.4\,, & (\text{if } E<0 \text{ and } \theta_c=\pi/4)\\ \tau_c/\tau_\infty=0.245\,, & (\rho/\rho_B)_c=1.33\,, & n\tau_\infty=5.10\,. & (\text{if } E>0 \text{ and } \theta_c=3/2)\end{cases} \tag{5·16}$$

The case of $m=1/2$ or $\omega=0$

In this case, Eq. (5·6) can be integrated as

$$dY/dT=-\left(\frac{Y^3+2}{3Y}+2E\right)^{1/2},\qquad (Y=(\rho/\rho_B)^{-1/3},\ T=\ln\tau) \tag{5·17}$$

where E is an integration constant specifying the total "energy" of the rewritten dynamical system and the $-$ sign on the right-hand side is necessary for the security of $d\rho/d\tau=0$ at some epoch. Taking account of the inequalities $1\geq Y\geq0$ due to the definition of Y such that $Y=0$ corresponds to $\tau=\tau_\infty$ at which ρ becomes infinitely large, we can integrate Eq. (5·17) as follows:

$$\begin{cases}\rho=\rho_{B0}\tau^{-3/2}Y^{-3}, & (\rho_B=\rho_{B0}\tau^{-3/2})\\ \tau=\tau_\infty\exp\left\{-\displaystyle\int_0^Y\left(\frac{3Y}{Y^3+6EY+2}\right)^{1/2}dY\right\}.\end{cases} \tag{5·18}$$

Now the gravitational binding occurs at $\tau=\tau_c$ which corresponds to $Y=Y_c$ such as

$$Y_c^3+24EY_c+8=0\,,\qquad 1+2EY_c>0\,, \tag{5·19}$$

which has the required solution ($1>Y_c>0$), provided that $E<-3/8$. Then we have

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$$\begin{cases} \tau_c/\tau_\infty = \exp\left\{-\int_0^{Y_c}\left(\frac{3Y}{Y^3+6EY+2}\right)^{1/2} dY\right\}, \quad (E<-3/8) \\ (\rho/\rho_B)_c = Y_c^{-3}. \end{cases} \tag{5·20}$$

We shall present typical examples of the above solution in what follows.
(i) $E=-1/2$: In this case, the integration in Eq. (5·18) can be performed by quadrature in such a way as

$$\tau/\tau_\infty = \left\{\frac{(1-Y)}{1+2Y+\sqrt{3Y(Y+2)}}\right\}\{1+Y+\sqrt{Y(Y+2)}\}^{\sqrt{3}}, \quad (Y=(\rho/\rho_B)^{-1/3}) \tag{5·21}$$

and it follows from Eqs. (5·19)~(5·21) that

$$\tau_c/\tau_\infty = 0.365\,, \quad (\rho/\rho_B)_c = 2.94\,, \quad (\text{for } E=-1/2) \tag{5·22}$$

since $Y_c = 4\ \sin(\pi/18) = 0.698$.
(ii) $E=-2/5$: In this case, the relation between ρ and τ must be represented by such an elliptic integral as

$$\tau/\tau_\infty = \exp\left\{-\int_0^{Y}\left(\frac{15Y}{5Y^3-12Y+10}\right)^{1/2} dY\right\}, \quad (Y=(\rho/\rho_B)^{-1/3}) \tag{5·23}$$

but it follows from Eqs. (5·19) and (5·20) that

$$\tau_c/\tau_\infty \sim 0.60\,, \quad (\rho/\rho_B)_c = 4.52\,, \quad (\text{for } E=-2/5) \tag{5·24}$$

since $Y_c = (8/\sqrt{5})\sin\{\pi/6-(1/3)\arcsin(131/256)\} = 0.605$.

Moreover, adopting the value $(\rho/\rho_B)_i = 1.01$ as an initial (at $\tau=\tau_i$) density ratio, we obtain

$$\tau_i/\tau_\infty = \begin{cases} 2.97\times10^{-4}, & (\text{if } m=2/3 \text{ and } E<0) \\ 2.91\times10^{-2}, & (\text{if } m=1/4 \text{ and } E=0) \\ 4.62\times10^{-3}. & (\text{if } m=1/2 \text{ and } E=-1/2) \end{cases} \tag{5·25}$$

Summarizing the above analyses for the occurrence of gravitational binding at the center of a density perturbation with spherical symmetry in the Brans-Dicke universe specified by Eq. (2·10), we obtain Table I which shows how the numerical values of t_i/t_∞ (corresponding to $(\rho/\rho_B)_i=1.01$), t_c/t_∞ and $(\rho/\rho_B)_c$ depend upon the value of ω, where $t_c \equiv t_0\tau_c$, etc.

As Table I shows, the smaller the value of ω, the easier the occurrence of gravitational binding. This is due to the fact that, the smaller the value of ω, the stronger was gravitational interaction ($\propto \phi_B^{-1}$) at $t=t_i$ compared with the present epoch $t=t_0$. If $\omega \gtrsim 5$ as insisted upon by Dicke,[9] we probably arrive at the values of t_i/t_∞, t_c/t_∞ and $(\rho/\rho_B)_c$ which are between those in the cases of $\omega=\infty$ and $\omega=0$.

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Table I. Numerical values of t_i/t_∞, t_c/t_∞ and $(\rho/\rho_B)_c$ in the general relativistic universe and in the Brans-Dicke universe.

	$GR(\omega=\infty)$	$BD(\omega=0)$		$BD(\omega=-4/5)$		
$\frac{8\pi}{3}(\rho_B/H^2)_0$	1	2		28/3		
Ht	2/3	1/2		1/4		
$G\phi_B$	1	$(2H_0t)^{1/2}$		$(4H_0t)^{5/4}$		
E [a)]	0	$-2/5$	$-1/2$	<0 $(n\tau_\infty=12.4)$	0	>0 $(n\tau_\infty=5.10)$
t_i/t_∞	2.97×10^{-4}		4.62×10^{-3}		2.92×10^{-2}	
t_c/t_∞	0.5	~0.60	0.365	0.370	0.342	0.245
$(\rho/\rho_B)_c$	5.55	4.52	2.94	1.83	1.56	1.33

a) The constant E is the total energy of the dynamical system specified by the auxiliary valiables Y and T, except when $\omega=\infty$.

§ 6. Global analysis of a rotating gaseous ellipsoid with uniform density

Let us consider a gaseous ellipsoid with uniform density which is rotating with the angular velocity Ω_i relative to the cosmological background specified by Eq. (2·10). Then, as in [GRIII], we may put

$$\rho=\rho(t),\qquad (\mu m_H/k)\,(p/\rho)=T=T_c(t)\,S(\boldsymbol{X},t),\qquad V_i=\alpha_{ij}(t)\,X_j\,, \tag{6·1}$$

where the ellipsoidal configuration is specified by

$$S(\boldsymbol{X},t)=1-(X/\alpha_1)^2-(Y/\alpha_2)^2-(Z/\alpha_3)^2=0\,, \tag{6·2}$$

in which $\alpha_i=\alpha_i(t)$ stands for the i-th principal semi-axis and $X_i=(X, Y, Z)$. Since the coefficient α_{ij} of the velocity field V_i can be decomposed into three parts, i.e., α_{kk}, ω_i (vorticity) and q_{ij} (shearing), in such a way as

$$\alpha_{ij}=\tfrac{1}{3}\alpha_{kk}\delta_{ij}+q_{ij}+\varepsilon_{ijk}\omega_k,\qquad q_{ij}\equiv\tfrac{1}{2}(\alpha_{ij}+\alpha_{ji})-\tfrac{1}{3}\alpha_{kk}\delta_{ij}\,,\qquad \omega_i\equiv\tfrac{1}{2}\varepsilon_{ijk}\alpha_{jk}\,, \tag{6·3}$$

the required condition $dS/dt\equiv\partial_t S+V_i\partial S/\partial X_i=0$ leads to

$$q_i\equiv q_{ii}=\tfrac{1}{3}(2\dot{\alpha}_i/\alpha_i-\dot{\alpha}_j/\alpha_j-\dot{\alpha}_k/\alpha_k),\quad (\alpha_{ii}=\dot{\alpha}_i/\alpha_i)$$
$$r_i\equiv q_{jk}=-\left(\frac{\alpha_j^2-\alpha_k^2}{\alpha_j^2+\alpha_k^2}\right)\omega_i\,, \tag{6·4}$$
$$\lambda_i=\frac{2\omega_i}{(\alpha_j^2+\alpha_k^2)},\quad (\alpha_{jk}=-\lambda_i\alpha_j^2,\ \alpha_{kj}=\lambda_i\alpha_k^2),$$

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where (i, j, k) are cyclic permutations of $(1, 2, 3)$ and a dot denotes differentiation with respect to t. Moreover, an integration of Eq. (3·15) over the ellipsoidal region under consideration leads to

$$\mathfrak{B} = -\pi\left(\frac{4-4m}{2-m}\right)\left(\frac{\rho-\rho_B}{\phi_B}\right)(\mathcal{A}_{ij}X_iX_j - I), \tag{6·5}$$

where $\mathcal{A}_{ij}$ is a diagonal matrix whose diagonal components $\mathcal{A}_i$, together with I, are given by

$$\begin{aligned} I &\equiv \int_0^\infty \frac{ds}{\sqrt{\varphi(s)}}, \quad \{\varphi(s) \equiv (\alpha_1^2+s)(\alpha_2^2+s)(\alpha_3^2+s)\} \\ \mathcal{A}_i &\equiv \int_0^\infty \frac{ds}{(\alpha_i^2+s)\sqrt{\varphi(s)}}. \quad (\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3 = 2/\alpha_1\alpha_2\alpha_3) \end{aligned} \tag{6·6}$$

On inserting Eqs. (6·1) and (6·4) into Eqs. (3·12) and (3·14), we obtain

$$(4\pi/3)\rho\alpha_1\alpha_2\alpha_3 = M = \text{const}, \tag{6·7}$$

and

$$T_c \propto (\alpha_1\alpha_2\alpha_3)^{-2/3} \propto \rho^{2/3}, \tag{6·8}$$

which specify the conservation of mass and the T_c versus ρ relation (for an adiabatic process), respectively. Similarly, Eq. (3·13′) with $\mathfrak{B}$ given by Eq. (6·5) is reduced to the following dynamical equations for α_i and the following two integrals specifying the conservation of angular momentum and velocity circulation:

$$\begin{aligned} &\ddot{\alpha}_i/\alpha_i - (\lambda_k^2\alpha_j^2 + \lambda_j^2\alpha_k^2)\alpha_i^2 - 2(\lambda_k\Omega_k\alpha_j^2 + \lambda_j\Omega_j\alpha_k^2) - (\Omega_j^2 + \Omega_k^2) \\ &= -6\left(\frac{1-m}{2-m}\right)M\phi_B^{-1}\mathcal{A}_i - \frac{(1-m)}{t^2}(m - \alpha_1\alpha_2\alpha_3\mathcal{A}_i) + \frac{2kT_c}{\mu m_H}\alpha_i^{-2}, \end{aligned} \tag{6·9}$$

where (i, j, k) are cyclic permutations of $(1, 2, 3)$, and

$$\left(\frac{4\pi}{15}\rho\alpha_1\alpha_2\alpha_3\right)^2\Big[\{(\alpha_1^2+\alpha_2^2)\Omega_3 + 2(\alpha_1\alpha_2)^2\lambda_3\}^2 + (\text{cyclic permutations of } (1,2,3))\Big] = \left(\frac{2}{5}ML\right)^2 = \text{const}, \tag{6·10}$$

$$\{2\Omega_3 + (\alpha_1^2+\alpha_2^2)\lambda_3\}^2(\alpha_1\alpha_2)^2 + (\text{cyclic permutations of } (1,2,3)) = (2C)^2 = \text{const}. \tag{6·11}$$

While Eqs. (6·10) and (6·11) are the same as Eqs. (3·17) and (3·18) in [GRIII], Eq. (6·9) is different from Eq. (3·16) in [GRIII] owing to the variability of the gravitational interaction ($\propto\phi_B^{-1}$) and the relation $Ht = m$.

Contracting Eq. (6·9) with respect to the suffix i and making use of Eqs. (6·4) and (6·7), we obtain

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$$(\dot{\rho}/\rho)^{\cdot}-\frac{1}{3}(\dot{\rho}/\rho)^2-q_{ij}{}^2+2(\omega_i+\Omega_i)^2+\frac{2kT_c}{\mu m_H}(\alpha_1{}^{-2}+\alpha_2{}^{-2}+\alpha_3{}^{-2})$$
$$-\left(\frac{1-m}{2-m}\right)\left\{16\pi\rho/\phi_B-\frac{(2-m)(2-3m)}{t^2}\right\}=0\,. \tag{6·12}$$

If $\Omega_i=\omega_i=0$ and $\alpha_1=\alpha_2=\alpha_3=\alpha$ in particular, it follows from Eqs. (6·1), (6·4), (6·7) and (6·8) that $q_{ij}=0$, $\rho\alpha^3=3M/4\pi$ and $p_c=(k/\mu m_H)\rho T_c=B\rho^{5/3}$ ($B=$const). Accordingly, by the use of Eq. (2·10), we can reduce Eq. (6·12) to

$$d^2y/d\tau^2=\frac{(1-m)}{3}\{(2-3m)\tau^{-2}y-2\tau^{3m-2}y^{-2}+2y_{J0}y^{-3}\}, \tag{6·13}$$

where

$$\begin{cases} y\equiv(4\pi\rho_{B0}/3M)^{1/3}\alpha=(\rho/\rho_{B0})^{-1/3}, \qquad \tau\equiv t/t_0\,, \\ y_J\equiv y_{J0}\tau^{2-3m}=B\left\{\left(\frac{3}{1-m}\right)^{3/2}\left(\frac{2-m}{6}\right)\left(\frac{\rho_{B0}t_0}{GM}\right)\right\}^{2/3}\tau^{2-3m}. \end{cases} \tag{6·14}$$

A comparison of Eq. (6·13) with Eq. (5·3) shows that the latter is a special case of the former such as $y\gg y_J$ (which has been specified by $L\gg L_J$ in the local analysis). Thus it has been shown that the results obtained in § 5 are applicable to the dynamical development of a gaseous sphere with uniform density and low temperature ($y\gg y_J$).

Now let us consider a special case such as

$$\Omega_a=\lambda_a=\omega_a=0, \quad (a=1,2)$$
$$\Omega_3=\alpha_1\alpha_2\lambda_3=\frac{2\alpha_1\alpha_2}{(\alpha_1{}^2+\alpha_2{}^2)}\omega_3=\frac{2L}{(\alpha_1+\alpha_2)^2}, \quad (C=L) \tag{6·15}$$

which are consistent with Eqs. (6·10) and (6·11). Then Eq. (6·9) is reduced to

$$\ddot{\alpha}_1=-\frac{(1-m)}{t^2}(m-\alpha_1\alpha_2\alpha_3\mathcal{A}_1)\alpha_1-6\left(\frac{1-m}{2-m}\right)GM\tau^{3m-2}\mathcal{A}_1\alpha_1$$
$$+\frac{2kT_c}{\mu m_H}\alpha_1{}^{-1}+\frac{8L^2}{(\alpha_1+\alpha_2)^3}, \tag{6·16·1}$$

$$\ddot{\alpha}_2=-\frac{(1-m)}{t^2}(m-\alpha_1\alpha_2\alpha_3\mathcal{A}_2)\alpha_2-6\left(\frac{1-m}{2-m}\right)GM\tau^{3m-2}\mathcal{A}_2\alpha_2$$
$$+\frac{2kT_c}{\mu m_H}\alpha_2{}^{-1}+\frac{8L^2}{(\alpha_1+\alpha_2)^3}, \tag{6·16·2}$$

$$\ddot{\alpha}_3=-\frac{(1-m)}{t^2}(m-\alpha_1\alpha_2\alpha_3\mathcal{A}_3)\alpha_3-6\left(\frac{1-m}{2-m}\right)GM\tau^{3m-2}\mathcal{A}_3\alpha_3$$
$$+\frac{2kT_c}{\mu m_H}\alpha_3{}^{-1}, \tag{6·16·3}$$

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which corresponds to Eq. (3·21) in [GRIII]. Similarly, Eq. (6·12) is reduced to

$$\ddot{\rho}/\rho-\frac{4}{3}(\dot{\rho}/\rho)^2-\left(\frac{1-m}{2-m}\right)\left\{16\pi G\rho\tau^{3m-2}-\frac{(2-m)(2-3m)}{t^2}\right\}$$

$$-(q_1^2+q_2^2+q_3^2)+\frac{8L^2}{\alpha_1\alpha_2(\alpha_1+\alpha_2)^2}+\frac{2kT_c}{\mu m_H}(\alpha_1^{-2}+\alpha_2^{-2}+\alpha_3^{-2})=0, \quad (6\cdot17)$$

where q_i is the i-th diagonal component of q_{ij} such as $q_1+q_2+q_3=0$. While Eq. (6·17) is derivable from Eqs. (6·16) and (6·7), it will be useful to see the instant at which the gravitational binding ($\dot{\rho}=0, \ddot{\rho}>0$) occurs.

It is interesting to study how the results obtained in [GRIII] for the dynamics of a rotating gaseous ellipsoid must be modified in its scalar-tensorial version. Accordingly, numerical analyses of Eq. (6·16), supplemented by Eqs. (6·6), (6·7) and (6·8), will be performed in the near future.

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