

Nice triples and Grothendieck–Serre’s conjecture concerning principal \mathbf{G} -bundles over reductive group schemes

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Abstract

In a series of papers [Pan0], [Pan1], [Pan2], [Pan3] we give a detailed and better structured proof of the Grothendieck–Serre’s conjecture for semi-local regular rings containing a finite field. The outline of the proof is the same as in [P1],[P2],[P3]. If the semi-local regular ring contains an infinite field, then the conjecture is proved in [FP]. *Thus the conjecture is true for regular local rings containing a field.*

The present paper is the one [Pan1] in that new series. Theorem 1.1 is one of the main result of the paper. It is also one of the key steps in the proof of the Grothendieck–Serre’s conjecture for semi-local rings containing a field (see [Pan3]). The proof of the main theorem is completely geometric. It is based on an extension of theory of nice triples from [PSV] and [P]. In turn the theory of nice triples is inspired by the Voevodsky theory of standard triples [Voe]. Our refinement of that Voevodsky theory is based on the use of Artin’s elementary fibrations, on geometric lemma [OP2, Lemma 8.2] and construction [P, Constr. 4.2]. The latter construction is taken from [OP2, the proof of lemma 8.1]).

1 Main results

Let R be a commutative unital ring. Recall that an R -group scheme \mathbf{G} is called *reductive*, if it is affine and smooth as an R -scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $R \rightarrow \Omega$ the scalar extension \mathbf{G}_Ω is a connected reductive algebraic group over Ω . This definition of a reductive R -group scheme coincides with [SGA3, Exp. XIX, Definition 2.7]. A well-known conjecture due to J.-P. Serre and A. Grothendieck (see [Se, Remarque, p.31], [Gr1, Remarque 3, p.26-27], and [Gr2, Remarque 1.11.a]) asserts that given a regular local ring R and its field of fractions K and given a reductive group scheme \mathbf{G} over R , the map

$$H_{\text{ét}}^1(R, \mathbf{G}) \rightarrow H_{\text{ét}}^1(K, \mathbf{G}),$$

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induced by the inclusion of R into K , has a trivial kernel. If R contains an infinite field, then the conjecture is proved in [FP].

For a scheme U we denote by \mathbb{A}_U^1 the affine line over U and by \mathbb{P}_U^1 the projective line over U . Let T be a U -scheme. By a principal \mathbf{G} -bundle over T we understand a principal $\mathbf{G} \times_U T$ -bundle. We refer to [SGA3, Exp. XXIV, Sect. 5.3] for the definitions of a simple simply-connected group scheme over a scheme and a semi-simple simply-connected group scheme over a scheme.

Theorem 1.1. *Let k be a field. Let \mathcal{O} be the semi-local ring of finitely many closed points on a k -smooth irreducible affine k -variety X . Set $U = \text{Spec } \mathcal{O}$. Let \mathbf{G} be a reductive group scheme over U . Let \mathcal{G} be a principal \mathbf{G} -bundle over U trivial over the generic point of U . Then there exists a principal \mathbf{G} -bundle \mathcal{G}_t over the affine line $\mathbb{A}_U^1 = \text{Spec } \mathcal{O}[t]$ and a monic polynomial $h(t) \in \mathcal{O}[t]$ such that*

- (i) *the \mathbf{G} -bundle \mathcal{G}_t is trivial over the open subscheme $(\mathbb{A}_U^1)_h$ in \mathbb{A}_U^1 given by $h(t) \neq 0$;*
- (ii) *the restriction of \mathcal{G}_t to $\{0\} \times U$ coincides with the original \mathbf{G} -bundle \mathcal{G} .*
- (iii) *$h(1) \in \mathcal{O}$ is a unit.*

If the field k is infinite a weaker result is proved in [PSV, Thm.1.2]. The proof of Theorem 1.1 is given in Section 6. It is easily derived from Theorem 5.4 and even from the following result, which is weaker than Theorem 5.4.

Theorem 1.2 (Geometric). *Let X be an affine k -smooth irreducible k -variety, and let x_1, x_2, \dots, x_n be closed points in X . Let $U = \text{Spec}(\mathcal{O}_{X, \{x_1, x_2, \dots, x_n\}})$ and $f \in k[X]$ be a non-zero function vanishing at each point x_i . Let \mathbf{G} be a reductive group scheme over X , \mathbf{G}_U be its restriction to U . Then there is a monic polynomial $h \in \mathcal{O}_{X, \{x_1, x_2, \dots, x_n\}}[t]$, a commutative diagram of schemes with the irreducible affine U -smooth Y*

$$\begin{array}{ccccc}
 (\mathbf{A}^1 \times U)_h & \xleftarrow{\tau_h} & Y_h := Y_{\tau^*(h)} & \xrightarrow{(p_X)|_{Y_h}} & X_f \\
 \text{inc} \downarrow & & \downarrow \text{inc} & & \text{inc} \downarrow \\
 (\mathbf{A}^1 \times U) & \xleftarrow{\tau} & Y & \xrightarrow{p_X} & X
 \end{array} \tag{1}$$

and a morphism $\delta : U \rightarrow Y$ subjecting to the following conditions:

- (i) *the left hand side square is an elementary **distinguished** square in the category of affine U -smooth schemes in the sense of [MV, Defn.3.1.3];*
- (ii) *$p_X \circ \delta = \text{can} : U \rightarrow X$, where can is the canonical morphism;*
- (iii) *$\tau \circ \delta = i_0 : U \rightarrow \mathbf{A}^1 \times U$ is the zero section of the projection $pr_U : \mathbf{A}^1 \times U \rightarrow U$;*
- (iv) *$h(1) \in \mathcal{O}[t]$ is a unit;*
- (v) *for $p_U := pr_U \circ \tau$ there is a Y -group scheme isomorphism $\Phi : p_U^*(\mathbf{G}_U) \rightarrow p_X^*(\mathbf{G})$ with $\delta^*(\Phi) = id_{\mathbf{G}_U}$.*

A sketch of the proof of Theorem 1.1. In general, \mathbf{G} does not come from X . However we may assume, that as \mathbf{G} , so \mathcal{G} are defined over X and \mathbf{G} is reductive over X . Say, let \mathcal{G}' is a principal \mathbf{G} on X with $\mathcal{G}'|_U = \mathcal{G}$. In this case there two reductive group schemes on Y . Namelly, $p_U^*(\mathbf{G}|_U)$ and $p_X^*(\mathbf{G})$. Clearly, they coincides when restricted to $\delta(U)$. By the item (v) of Theorem 1.2 the scheme Y can be chosen such that two reductive group schemes $p_U^*(\mathbf{G}|_U)$ and $p_X^*(\mathbf{G})$ on Y are isomorphic via an isomorphism Φ and Φ is such that its restriction to $\delta(U)$ is the identity. Take again $p_X^*(\mathcal{G}')$ and regard it as a principal $p_U^*(\mathbf{G}|_U)$ -bundle using the isomorphism Φ . Denote that principal $p_U^*(\mathbf{G}|_U)$ -bundle ${}_U p_X^*(\mathcal{G})$. It is trivial on Y_h , since $p_X^*(\mathcal{G}')$ is trivial on Y_h . Take the trivial $pr_U^*(\mathbf{G}|_U)$ -bundle on $\mathbf{A}^1 \times U$ and glue it with ${}_U p_X^*(\mathcal{G})$ via an isomorphism over Y_h . This way we get a principal $\mathbf{G}|_U$ -bundle \mathcal{G}_t over $\mathbf{A}^1 \times U$. Clearly, it is the desired one. The polinomial h one should take as in Theorem 1.2. Details are given in Section 6.

The article is organized as follows. In Section 2 we recall definition of nice triples from [PSV] and inspired by Voevodsky notion of perfect triples.

In Section 3 Theorem 3.1 on equating group schemes is proved. In Section 4 Theorem 4.3 is proved. In Section 5 a first application of the above machinery is given. Namely, Theorems 5.1 and 5.4 are proved. Theorem 5.1 is a one more geometric presentation theorem. Theorem 5.4 is a stronger version of Theorem 1.2. *Finally, in Section 6 Theorem 1.1 is proved.* In Section 7 the main result of [PSV] *is extended to the case of an arbitrary base field k .* Namelly, the following Theorem is proved there

Theorem 1.3. *Let k be a field. Let \mathcal{O} be the semi-local ring of finitely many closed points on a k -smooth irreducible affine k -variety X and let K be its field of fractions. Let \mathbf{G} be an isotropic simple simply connected group scheme over \mathcal{O} . Then for any Noetherian k -algebra A the map*

$$H_{et}^1(\mathcal{O} \otimes_k A, G) \rightarrow H_{et}^1(K \otimes_k A, G),$$

induced by the inclusion \mathcal{O} into K , has trivial kernel.

Stress the following. If the field k is finite, and $A = k$ and the group scheme G comes from the field k and it is simple simply connected, then the latter theorem is an unpublished result due to Gabber (see also [Pan0]).

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2 Nice triples

In the present section we recall and study certain collections of geometric data and their morphisms. The concept of a *nice triple* was introduced in [PSV, Defn. 3.1] and is very similar to that of a *standard triple* introduced by Voevodsky [Voe, Defn. 4.1], and was in fact inspired by the latter notion. Let k be a field, let X be a k -smooth irreducible

affine k -variety, and let $x_1, x_2, \dots, x_n \in X$ be a **family of closed points**. Further, let $\mathcal{O} = \mathcal{O}_{X, \{x_1, x_2, \dots, x_n\}}$ be the corresponding geometric semi-local ring.

After substituting k by its algebraic closure \tilde{k} in $k[X]$, we can assume that X is a \tilde{k} -smooth *geometrically irreducible* affine \tilde{k} -scheme. **The geometric irreducibility of X is required in the proposition ?? to construct an open neighborhood X^0 of the family $\{x_1, x_2, \dots, x_n\}$ and an elementary fibration $p : X^0 \rightarrow S$, where S is an open sub-scheme of the projective space $\mathbf{P}_k^{\dim X - 1}$. The proposition ?? is used in turn to prove the proposition 2.6. To simplify the notation, we will continue to denote this new \tilde{k} by k .**

Definition 2.1. Let $U := \text{Spec}(\mathcal{O}_{X, \{x_1, x_2, \dots, x_n\}})$. A nice triple over U consists of the following data:

- (i) a smooth morphism $q_U : \mathcal{X} \rightarrow U$, where \mathcal{X} is an irreducible scheme,
- (ii) an element $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$,
- (iii) a section Δ of the morphism q_U ,

subject to the following conditions:

- (a) each irreducible component of each fibre of the morphism q_U has dimension one,
- (b) the module $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/f \cdot \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is finite as a $\Gamma(U, \mathcal{O}_U) = \mathcal{O}$ -module,
- (c) there exists a finite surjective U -morphism $\Pi : \mathcal{X} \rightarrow \mathbf{A}^1 \times U$;
- (d) $\Delta^*(f) \neq 0 \in \Gamma(U, \mathcal{O}_U)$.

There are many choices of the morphism Π . Any of them is regarded as assigned to the nice triple.

Remark 2.2. Since Π is a finite morphism, the scheme \mathcal{X} is affine. We will write often below $k[\mathcal{X}]$ for $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. The only requirement on the morphism Δ is this: Δ is a section of q_U . Hence Δ is a closed embedding. We write $\Delta(U)$ for the image of this closed embedding. The composite map $\Delta^* \circ q_U^* : k[\mathcal{X}] \rightarrow \mathcal{O}$ is the identity. If $\text{Ker} = \text{Ker}(\Delta^*)$, then Ker is the ideal defining the closed subscheme $\Delta(U)$ in \mathcal{X} .

Definition 2.3. A morphism between two nice triples over U

$$(q' : \mathcal{X}' \rightarrow U, f', \Delta') \rightarrow (q : \mathcal{X} \rightarrow U, f, \Delta)$$

is an étale morphism of U -schemes $\theta : \mathcal{X}' \rightarrow \mathcal{X}$ such that

- (1) $q'_U = q_U \circ \theta$,
- (2) $f' = \theta^*(f) \cdot h'$ for an element $h' \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$,
- (3) $\Delta = \theta \circ \Delta'$.

Two observations are in order here.

- Item (2) implies in particular that $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\theta^*(f) \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a finite \mathcal{O} -module.

- It should be emphasized that no conditions are imposed on the interrelation of Π' and Π .

Let U be as in Definition 2.1 and $\text{can} : U \hookrightarrow X$ be the canonical inclusion of schemes.

Definition 2.4. A nice triple $(q_U : \mathcal{X} \rightarrow U, \Delta, f)$ over U is called special if the set of closed points of $\Delta(U)$ is contained in the set of closed points of $\{f = 0\}$.

Remark 2.5. Clearly the following holds: let (\mathcal{X}, f, Δ) be a special nice triple over U and let $\theta : (\mathcal{X}', f', \Delta') \rightarrow (\mathcal{X}, f, \Delta)$ be a morphism between nice triples over U . Then the triple $(\mathcal{X}', f', \Delta')$ is a special nice triple over U .

Proposition 2.6. One can shrink X such that x_1, x_2, \dots, x_n are still in X and X is affine, and then construct a special nice triple $(q_U : \mathcal{X} \rightarrow U, \Delta, f)$ over U and an essentially smooth morphism $q_X : \mathcal{X} \rightarrow X$ such that $q_X \circ \Delta = \text{can}$, $f = q_X^*(f)$.

Proof of Proposition 2.6. If the field k is infinite, then this proposition is proved in [PSV, Prop. 6.1]. So, we may and will assume that k is finite. To prove the proposition repeat literally the proof of [PSV, Prop. 6.1]. One has to replace the references to [PSV, Prop. 2.3] and [PSV, Prop.2.4] with the reference to [P, Prop. 2.3] and [P, Prop. 2.4] respectively.

□

Recall the definition [P, Defn. 3.7]. If U as in Definition 2.1 then for any U -scheme V and any closed point $u \in U$ set $V_u = u \times_U V$. For a finite set A denote $\sharp A$ the cardinality of A .

Definition 2.7. Let (\mathcal{X}, f, Δ) be a special nice triple over U . We say that the triple (\mathcal{X}, f, Δ) satisfies conditions 1* and 2* if either the field k is infinite or (if k is finite) the following holds

(1*) for $\mathcal{Z} = \{f = 0\} \subset \mathcal{X}$ and for any closed point $u \in U$, any integer $d \geq 1$ one has

$$\sharp\{z \in \mathcal{Z}_u \mid \deg[k(z) : k(u)] = d\} \leq \sharp\{x \in \mathbf{A}_u^1 \mid \deg[k(x) : k(u)] = d\}$$

(2*) for the vanishing locus \mathcal{Z} of f and for any closed point $u \in U$ the point $\Delta(u) \in \mathcal{Z}_u$ is the only $k(u)$ -rational point of $\mathcal{Z}_u = u \times_U \mathcal{Z}$.

3 Equating group schemes

The main result of the present section is Theorem 3.1. It is stated and proved at the very end the present section. We begin with the following result which extends [PSV, Prop.5.1].

Theorem 3.1. *Let S be a regular semi-local irreducible scheme. Assume that G_1 and G_2 are reductive S -group schemes which are forms of each other. Let $T \subset S$ be a connected non-empty closed sub-scheme of S , and $\varphi : G_1|_T \rightarrow G_2|_T$ be T -group scheme isomorphism. Then there exists a finite étale morphism $\tilde{S} \xrightarrow{\pi} S$ together with a section $\delta : T \rightarrow \tilde{S}$ of π over T and \tilde{S} -group scheme isomorphisms $\Phi : \pi^*G_1 \rightarrow \pi^*G_2$ such that*

(i) $\delta^*(\Phi) = \varphi$,

(ii) *the scheme \tilde{S} is irreducible.*

Proposition 3.2. *Theorem 3.1 holds in the case when the group schemes G_1 and G_2 are semi-simple.*

Proof of Proposition 3.2. **Let $s \in S$ be a closed point and let V be an S -scheme. In this section we will write $V(s)$ for the scheme $s \times_S V$.** The proof of the proposition literally repeats the proof of [PSV, Prop.5.1] except exactly one reference, which is the reference to [OP2, Lemma 7.2]. That reference one has to replace with the reference to the following

Lemma 3.3. *Let $S = \text{Spec}(R)$ be a regular semi-local scheme such that the residue field at any of its closed point is finite. Let T be a closed subscheme of S . Let \bar{W} be a closed subscheme of $\mathbb{P}_S^d = \text{Proj}(S[X_0, \dots, X_d])$ and $W = \bar{W} \cap \mathbf{A}_S^d$, where \mathbf{A}_S^d is the affine space defined by $X_0 \neq 0$. Let $W_\infty = \bar{W} \setminus W$ be the intersection of \bar{W} with the hyperplane at infinity $X_0 = 0$. Assume that over T there exists a section $\delta : T \rightarrow W$ of the canonical projection $W \rightarrow S$. Assume further that*

- (1) *W is smooth and equidimensional over S , of relative dimension r ;*
- (2) *For every closed point $s \in S$ the closed fibres of W_∞ and W satisfy*

$$\dim(W_\infty(s)) < \dim(W(s)) = r .$$

Then there exists a closed subscheme \tilde{S} of W which is finite étale over S and contains $\delta(T)$.

Proof of Lemma 3.3. To avoid technicalities we will give the proof in the case of $r = 1$ and left the general case to the reader. If $r = 1$, then for every closed point $s \in S$ the closed fibres of W_∞ and W satisfy

$$\dim(W_\infty(s)) < \dim(W(s)) = 1 .$$

Since S is semi-local, after a linear change of coordinates we may assume that δ maps T into the closed subscheme of \mathbf{P}_T^d defined by $X_1 = \dots = X_d = 0$. For each closed fibre $\mathbf{P}^d(s)$ of \mathbf{P}_S^d using [Poo, Thm.1.2], we can choose a **homogeneous** polynomial $H(s)$ such that the subscheme $Y(s)$ of $\mathbf{P}^d(s)$ defined by the equation

$$H(s) = 0$$

intersects $W(s)$ transversally and avoids $W_\infty(s)$.

Let $s \in S$ be a closed point of S . Let $s \in T$ be a closed point of T . Let $\mathbf{P}^d(s)$ be closed fibre of \mathbf{P}_S^d over the point s . Let $\mathbb{A}^d(s) \subset \mathbf{P}^d(s)$ be the affine subspace defined by the inequality $X_0 \neq 0$. Let $t_i := X_i/X_0$ ($i \in \{1, 2, \dots, d\}$) be the coordinate function on $\mathbb{A}^d(s)$. The origin $x_{0,s} = (0, 0, \dots, 0) \in \mathbb{A}^d(s)$ has the homogeneous coordinates $[1 : 0 : \dots : 0]$ in $\mathbf{P}^d(s)$. Let $W(s) \subset \mathbf{P}^d(s)$ be the fibre of W over the point s . If $x_{0,s}$ is in $W(s)$, then let $\tau(s) \subset \mathbb{A}^d(s)$ be the tangent space to $W(s)$ at the point $x_0 \in \mathbb{A}^d(s)$. In this case let $l_s = l_s(t_1, t_2, \dots, t_d)$ be a linear form in $k(s)[t_1, t_2, \dots, t_d]$ such that $l_s|_{\tau(s)} \neq 0$. If $x_{0,s}$ is not in $W(s)$, then set $l_s = t_1$. In all the cases

$$L_s := X_0 \cdot l_s \in k(s)[X_1, X_2, \dots, X_d]$$

is a homogeneous polynomial of degree 1.

Let $s \in S$ be a closed point. Suppose $x_{0,s} \in W(s)$. Then by [Poo, Thm. 1.2] there is an integer $N_1(s) \geq 1$ such that for any positive integer $N \geq N_1(s)$ there is a homogeneous polynomial

$$H_{1,N}(s) = X_0^N \cdot F_{0,N}(s) + X_0^{N-1} \cdot F_{1,N}(s) + \dots + X_0 \cdot F_{N-1,N}(s) + F_{N,N}(s) \in k(s)[X_0, X_1, \dots, X_d]$$

of degree N with homogeneous polynomials $F_{i,N}(s) \in k(s)[X_1, \dots, X_n]$ of degree i such that the following holds

- (i) $F_{0,N}(s) = 0$, $F_{1,N}(s) = L_s$;
- (ii) the subscheme $V(s) \subseteq \mathbf{P}^d(s)$ defined by the equation $H_{1,N}(s) = 0$ intersects $W(s)$ transversally;
- (iii) $V(s) \cap W_\infty(s) = \emptyset$.

Let $s \in S$ be a closed point. Suppose $x_{0,s}$ is not in $W(s)$. Then by [Poo, Thm. 1.2] there is an integer $N_1(s) \geq 1$ such that for any positive integer $N \geq N_1(s)$ there is a homogeneous polynomial

$$H_{1,N}(s) = X_0^N \cdot F_{0,N}(s) + X_0^{N-1} \cdot F_{1,N}(s) + \dots + X_0 \cdot F_{N-1,N}(s) + F_{N,N}(s) \in k(s)[X_0, X_1, \dots, X_d]$$

of degree N with homogeneous polynomials $F_{i,N}(s) \in k(s)[X_1, \dots, X_n]$ of degree i such that the following holds

- (i) $F_{0,N}(s) = 0$, $F_{1,N}(s) = L_s = X_1$;
- (ii) the subscheme $V(s) \subseteq \mathbf{P}^d(s)$ defined by the equation $H_{1,N}(s) = 0$ intersects $W(s)$ transversally;
- (iii) $V(s) \cap W_\infty(s) = \emptyset$.

Let $N_1 = \max_s \{N_1(s)\}$, where s runs over all the closed points of S . For any closed point $s \in S$ set $H_1(s) := H_{1,N_1}(s)$. Then for any closed point $s \in S$ the polynomial $H_1(s) \in k(s)[X_0, X_1, \dots, X_d]$ is homogeneous of the degree N_1 . By the chinese remainders' theorem for any $i = 0, 1, \dots, N$ there exists a common lift $F_{i,N_1} \in A[X_1, \dots, X_d]$ of all polynomials $F_{i,N_1}(s)$, s is a closed point of S , such that $F_{0,N_1} = 0$ and for any $i = 0, 1, \dots, N$ the polynomial F_{i,N_1} is homogeneous of degree i . By the chinese remainders' theorem there exists a common lift

$L \in A[X_1, \dots, X_d]$ of all polynomials L_s , s is a closed point of S , such that L is homogeneous of the degree one. Set

$$H_{1,N_1} := X_0^{N_1-1} \cdot L + X_0^{N_1-2} \cdot F_{2,N_1} \cdots + X_0 \cdot F_{N_1-1,N_1} + F_{N_1,N_1} \in A[X_0, X_1, \dots, X_d].$$

Note that for any closed point s in S the evaluation of H_{1,N_1} at the point s coincides with the polynomial $H_{1,N_1}(s)$ in $k(s)[X_0, X_1, \dots, X_d]$. Note also that $H_{1,N_1}[1 : 0 : \dots : 0] = 0$. Hence $H_{1,N_1}|_{\delta(T)} \equiv 0$.

Note that for any closed point s in S the evaluation of H_{1,N_1} at the point s coincides with the polynomial $H_{1,N_1}(s)$ in $k(s)[X_0, X_1, \dots, X_d]$. Since S is semi-local and W_∞ is projective quasi-finite over S it is finite over S . Let $V \subset \mathbb{P}_S^d$ be the subscheme defined by $\{H_{1,N_1} = 0\}$. Since for any closed point $s \in S$ one has $V(s) \cap W_\infty(s) = \emptyset$, hence $V \cap W_\infty = \emptyset$. Note also that $H_{1,N_1}[1 : 0 : \dots : 0] = 0$. Hence $H_{1,N_1}|_{\delta(T)} \equiv 0$.

We claim that the subscheme $\tilde{S} = V \cap W$ has the required properties. Note first that $V \cap W$ is finite over S . In fact, $V \cap W = V \cap \bar{W}$, which is projective over S and such that every closed fibre (hence every fibre) is finite. Since the closed fibres of $V \cap W$ are finite étale over the closed points of S , to show that $V \cap W$ is finite étale over S it only remains to show that it is flat over S . Noting that $V \cap W \subset W$ is defined in every closed fibre by a length one regular sequence of equations and localizing at each closed point of S , we see that flatness follows from [OP2, Lemma 7.3]. Whence the lemma. \square

Return to the proof of the proposition 3.2. Its proof literally repeats now the proof of [PSV, Prop.5.1]. Apriory the regular scheme \tilde{S} is not necessary irreducible. In that case replace \tilde{S} with its irreducible component containing the scheme $\delta(T)$. The proof of the proposition 3.2 is completed. \square

Proposition 3.4. *Theorem 3.1 holds in the case when the groups G_1 and G_2 are tori and, more generally, in the case when the groups G_1 and G_2 are of multiplicative type.*

We left a proof of this latter proposition to the reader. The following proposition follows easily from [SGA3, Exp. 9, Cor. 2.9].

Proposition 3.5. *Let T and S be the same as in Theorem 3.1. Let M_1 and M_2 be two S -group schemes of multiplicative type. Let $\alpha_1, \alpha_2 : M_1 \rightrightarrows M_2$ be two S -group scheme morphisms such that $\alpha_1|_T = \alpha_2|_T$. Then $\alpha_1 = \alpha_2$.*

Proof of Theorem 3.1. Let $Rad(G_r) \subset G_r$ be the radical of G_r and let $der(G_r) \subset G_r$ be the derived subgroup of G_r ($r = 1, 2$) (see [D-G, Exp.XXII, 4.3]). By the very definition the radical is a torus. The S -group scheme $der(G_r)$ is semi-simple ($r = 1, 2$). Set $Z_r := Rad(G_r) \cap der(G_r)$. The above embeddings induce natural S -group morphisms

$$\Pi_r : Rad(G_r) \times_S der(G_r) \rightarrow G_r$$

with Z_r as the kernel ($r = 1, 2$). By [D-G, Exp.XXII, Prop.6.2.4] Π_r is a central isogeny. Particularly, Π_r is a faithfully flat finite morphism by [D-G, Exp.XXII, Defn.4.2.9]. Let $i_r : Z_r \hookrightarrow Rad(G_r) \times_S der(G_r)$ be the closed embedding.

The T -group scheme isomorphism $\varphi : G_1|_T \rightarrow G_2|_T$ induces certain T -group scheme isomorphisms $\varphi_{der} : der(G_1|_T) \rightarrow der(G_2|_T)$, $\varphi_{rad} : rad(G_1|_T) \rightarrow rad(G_2|_T)$ and $\varphi_Z : Z_1|_T \rightarrow Z_2|_T$ **such that**

$$(\Pi_2)|_T \circ (\varphi_{der} \times \varphi_{rad}) = \varphi \circ (\Pi_1)|_T \text{ and } i_{2,T} \circ \varphi_Z = (\varphi_{rad} \times \varphi_{der}) \circ i_{1,T}.$$

By Propositions 3.2 and 3.4 there exist a finite étale morphism $\pi : \tilde{S} \rightarrow S$ (with an irreducible scheme \tilde{S}) and its section $\delta : T \rightarrow \tilde{S}$ over T and \tilde{S} -group scheme isomorphisms

$$\Phi_{der} : der(G_{1,\tilde{S}}) \rightarrow der(G_{2,\tilde{S}}) \quad , \quad \Phi_{Rad} : Rad(G_{1,\tilde{S}}) \rightarrow Rad(G_{2,\tilde{S}}) \quad \text{and} \quad \Phi_Z : Z_{1,\tilde{S}} \rightarrow Z_{2,\tilde{S}}$$

such that $\delta^*(\Phi_{der}) = \varphi_{der}$, $\delta^*(\Phi_{rad}) = \varphi_{rad}$ and $\delta^*(\Phi_Z) = \varphi_Z$.

Since Z_r is contained in the center of $der(G_r)$ and is of multiplicative type Proposition 3.5 yields the equality

$$i_{2,\tilde{S}} \circ \Phi_Z = (\Phi_{Rad} \times \Phi_{der}) \circ i_{1,\tilde{S}} : Z_{1,\tilde{S}} \rightarrow Rad(G_{2,\tilde{S}}) \times_{\tilde{S}} der(G_{2,\tilde{S}}).$$

Thus $(\Phi_{Rad} \times \Phi_{der})$ induces an \tilde{S} -group scheme isomorphism

$$\Phi : G_{1,\tilde{S}} \rightarrow G_{2,\tilde{S}}$$

such that $\Pi_{2,\tilde{S}} \circ (\Phi_{Rad} \times \Phi_{der}) = \Phi \circ \Pi_{1,\tilde{S}}$. The latter equality yields the following one $(\Pi_2)|_T \circ \delta^*(\Phi_{Rad} \times \Phi_{der}) = \delta^*(\Phi) \circ (\Pi_1)|_T$, which in turn yields the equality

$$(\Pi_2)|_T \circ (\varphi_{rad} \times \varphi_{der}) = \delta^*(\Phi) \circ (\Pi_1)|_T.$$

Comparing it with the equality $(\Pi_2)|_T \circ (\varphi_{rad} \times \varphi_{der}) = \varphi \circ (\Pi_1)|_T$ and using the fact that $(\Pi_1)|_T$ is faithfully flat we conclude the equality $\delta^*(\Phi) = \varphi$. □

4 Nice triples and group schemes

We need in an extension of the [P, Thm. 3.9]. For that it is convenient to give a definition under the following set up. Let U be as in Definition 2.1. Let (\mathcal{X}, f, Δ) be a special nice triple over U and let $G_{\mathcal{X}}$ be a reductive \mathcal{X} -group scheme and $G_U := \Delta^*(G_{\mathcal{X}})$ and $G_{\text{const}} := q_U^*(G_U)$. Let $\theta : (q' : \mathcal{X}' \rightarrow U, f', \Delta') \rightarrow (q : \mathcal{X} \rightarrow U, f, \Delta)$ be a morphism between nice triples over U .

Definition 4.1 (Equating group schemes). We say that the morphism θ *equates* the reductive \mathcal{X} -group schemes $G_{\mathcal{X}}$ and G_{const} , if there is an \mathcal{X}' -group scheme isomorphism $\Phi : \theta^*(G_{\text{const}}) \rightarrow \theta^*(G_{\mathcal{X}})$ with $(\Delta')^*(\Phi) = id_{G_U}$.

Remark 4.2. Let $\rho : (\mathcal{X}'', f'', \Delta'') \rightarrow (\mathcal{X}', f', \Delta')$ and $\theta : (\mathcal{X}', f', \Delta') \rightarrow (\mathcal{X}, f, \Delta)$ be morphisms of nice triples over U . If θ equates $G_{\mathcal{X}}$ and G_{const} , then $\theta \circ \rho$ also equates $G_{\mathcal{X}}$ and G_{const} .

Theorem 4.3. *Let U be as in Definition 2.1. Let (\mathcal{X}, f, Δ) be a special nice triple over U . Let $G_{\mathcal{X}}$ be a reductive \mathcal{X} -group scheme and $G_U := \Delta^*(G_{\mathcal{X}})$ and $G_{\text{const}} := q_U^*(G_U)$. Then there exist a morphism $\theta'' : (q'' : \mathcal{X}'' \rightarrow U, f'', \Delta'') \rightarrow (q : \mathcal{X} \rightarrow U, f, \Delta)$ between nice triples over U such that*

- (i) *the morphism θ'' equates the reductive \mathcal{X} -group schemes G_{const} and $G_{\mathcal{X}}$*
- (ii) *the triple $(\mathcal{X}'', f'', \Delta'')$ is a special nice triple over U subjecting to the conditions (1*) and (2*) from Definition 2.7.*

Proof of Theorem 4.3. Let U be as in the theorem. Let (\mathcal{X}, f, Δ) be a special nice triple over U as in the theorem. By the definition of a nice triple there exists a finite surjective morphism $\Pi : \mathcal{X} \rightarrow \mathbf{A}^1 \times U$ of U -schemes. The first part of the construction [P, Constr. 4.2] gives us now the data $(\mathcal{Z}, \mathcal{Y}, S, T)$, where $(\mathcal{Z}, \mathcal{Y}, T)$ are closed subsets of \mathcal{X} finite over U . If $\{y_1, \dots, y_n\}$ are all the closed points of \mathcal{Y} , then $S = \text{Spec}(\mathcal{O}_{\mathcal{X}, y_1, \dots, y_n})$.

Further, let $G_U = \Delta^*(G_{\mathcal{X}})$ be as in the hypotheses of Theorem 4.3 and let G_{const} be the pull-back of G_U to \mathcal{X} . Finally, let $\varphi : G_{\text{const}}|_T \rightarrow G_{\mathcal{X}}|_T$ be the canonical isomorphism. Recall that by assumption \mathcal{X} is U -smooth and irreducible, and thus S is regular and irreducible. By Theorems 3.1 there exists a finite étale morphism $\theta_0 : S' \rightarrow S$, a section $\delta : T \rightarrow S'$ of θ_0 over T and an isomorphism $\Phi_0 : \theta_0^*(G_{\text{const}, S}) \rightarrow \theta_0^*(G_{\mathcal{X}}|_S)$ such that $\delta^*(\Phi_0) = \varphi$, and *where the scheme S' is irreducible.*

Consider now the diagram (4) from the construction [P, Constr.4.2].

$$\begin{array}{ccccc}
 & & S' & \xrightarrow{\quad} & \mathcal{V}' \\
 & \nearrow \delta & \downarrow \theta_0 & & \downarrow \theta \\
 T & \xrightarrow{\quad} & S & \xrightarrow{\quad} & \mathcal{V} \xrightarrow{\quad} \mathcal{X}
 \end{array} \tag{2}$$

Recall that here $\theta : \mathcal{V}' \rightarrow \mathcal{V}$ is finite étale (and the square is cartesian). We may and will now suppose that the neighborhood \mathcal{V} of the points $\{y_1, \dots, y_n\}$ from that diagram is chosen such that there is \mathcal{V}' -group schemes isomorphism $\Phi : \theta^*(G_{\text{const}, \mathcal{V}}) \rightarrow \theta^*(G_{\mathcal{X}}|_{\mathcal{V}})$ with $\Phi|_{S'} = \Phi_0$. Clearly, $\delta^*(\Phi) = \varphi$.

Applying the second part of the construction [P, Constr.4.2] and also the proposition [P, Prop. 4.3] to the finite étale morphism $\theta : \mathcal{V}' \rightarrow \mathcal{V}$ and to the section $\delta : T \rightarrow \mathcal{V}'$ of θ over T we get

- 0) firstly, an open subset $\mathcal{W} \subseteq \mathcal{V}$ containing \mathcal{Y} (and hence containing S) and endowed with a finite surjective U -morphism $\Pi^* : \mathcal{W} \rightarrow \mathbf{A}^1 \times U$;
- 1) secondly, a triple $(\mathcal{X}', f', \Delta')$;
- 2) thirdly, the étale morphism of U -schemes $\theta : \mathcal{X}' \rightarrow \mathcal{X}$;
- 3) fourthly, inclusions of U -schemes $S \subset \mathcal{W}$ and $S' \subset \mathcal{X}'$.

Further we get

- (i) the special nice triple $(q_U \circ \theta : \mathcal{X}' \rightarrow U, f', \Delta')$ over U ;
- (ii) the morphism θ is a morphism $(\mathcal{X}', f', \Delta') \rightarrow (\mathcal{X}, f, \Delta)$ between the nice triples, which equates the \mathcal{X} -group schemes G_{const} and $G_{\mathcal{X}}$;
- (iii) the equality $f' = \theta^*(f)$.

To complete the proof of the theorem just apply the theorem [P, Thm. 3.9] to the the special nice triple $(\mathcal{X}', f', \Delta')$ and use the remark 4.2. \square

5 First application of the theory of nice triples

Theorem 5.1. *Let X be an affine k -smooth irreducible k -variety, and let x_1, x_2, \dots, x_n be closed points in X . Let $U = \text{Spec}(\mathcal{O}_{X, \{x_1, x_2, \dots, x_n\}})$. Let \mathbf{G} be a reductive X -group scheme and let $\mathbf{G}_U = \text{can}^*(\mathbf{G})$ be the pull-back of \mathbf{G} to U . Given a non-zero function $f \in k[X]$ vanishing at each point x_i , there is a diagram of the form*

$$\begin{array}{ccccc} \mathbf{A}^1 \times U & \xleftarrow{\sigma} & \mathcal{X} & \xrightarrow{q_X} & X \\ & \searrow \text{pr}_U & \downarrow q_U & \searrow \Delta & \nearrow \text{can} \\ & & U & & \end{array} \quad (3)$$

with an irreducible **affine** scheme \mathcal{X} , a smooth morphism q_U , a finite surjective U -morphism σ and an essentially smooth morphism q_X , and a function $f' \in q_X^*(f)k[\mathcal{X}]$, which enjoys the following properties:

- (a) if \mathcal{Z}' is the closed subscheme of \mathcal{X} defined by the principal ideal (f') , the morphism $\sigma|_{\mathcal{Z}'} : \mathcal{Z}' \rightarrow \mathbf{A}^1 \times U$ is a closed embedding and the morphism $q_U|_{\mathcal{Z}'} : \mathcal{Z}' \rightarrow U$ is finite;
- (a') $q_U \circ \Delta = \text{id}_U$ and $q_X \circ \Delta = \text{can}$ and $\sigma \circ \Delta = i_0$
(the first equality shows that $\Delta(U)$ is a closed subscheme in \mathcal{X});
- (b) σ is étale in a neighborhood of $\mathcal{Z}' \cup \Delta(U)$;
- (c) $\sigma^{-1}(\sigma(\mathcal{Z}')) = \mathcal{Z}' \amalg \mathcal{Z}''$ scheme theoretically for some closed subscheme \mathcal{Z}''
and $\mathcal{Z}'' \cap \Delta(U) = \emptyset$;
- (d) $\mathcal{D}_0 := \sigma^{-1}(\{0\} \times U) = \Delta(U) \amalg \mathcal{D}'_0$ scheme theoretically for some closed subscheme \mathcal{D}'_0
and $\mathcal{D}'_0 \cap \mathcal{Z}' = \emptyset$;
- (e) for $\mathcal{D}_1 := \sigma^{-1}(\{1\} \times U)$ one has $\mathcal{D}_1 \cap \mathcal{Z}' = \emptyset$.
- (f) there is a monic polynomial $h \in \mathcal{O}[t]$ such that $(h) = \text{Ker}[\mathcal{O}[t] \xrightarrow{\sigma^*} k[\mathcal{X}] \xrightarrow{\bar{\cdot}} k[\mathcal{X}]/(f')]$,
where $\mathcal{O} := k[U]$ and the map bar takes any $g \in k[\mathcal{X}]$ to $\bar{g} \in k[\mathcal{X}]/(f')$;
- (g) there is an \mathcal{X} -group scheme isomorphism $\Phi : p_U^*(\mathbf{G}_U) \rightarrow p_X^*(\mathbf{G})$ with $\Delta^*(\Phi) = \text{id}_{\mathbf{G}_U}$.

Proof of Theorem 5.1. By Proposition 2.6 one can shrink X such that x_1, x_2, \dots, x_n are still in X and X is affine, and then to construct a special nice triple $(q_U : \mathcal{X} \rightarrow U, \Delta, f)$ over U and an essentially smooth morphism $q_X : \mathcal{X} \rightarrow X$ such that $q_X \circ \Delta = \text{can}$, $f = q_X^*(f)$ and the set of closed points of $\Delta(U)$ is contained in the set of closed points of $\{f = 0\}$.

Set $\mathbf{G}_X = q_X^*(\mathbf{G})$, then $\Delta^*(\mathbf{G}_X) = \text{can}^*(\mathbf{G})$. Thus the U -group scheme \mathbf{G}_U from Theorem 4.3 and the U -group scheme \mathbf{G}_U from Theorem 5.1 are the same. By Theorem 4.3 there exists a morphism $\theta : (\mathcal{X}_{\text{new}}, f_{\text{new}}, \Delta_{\text{new}}) \rightarrow (\mathcal{X}, f, \Delta)$ such that the triple

$(\mathcal{X}_{new}, f_{new}, \Delta_{new})$ is a special nice triple over U subject to the conditions (1*) and (2*) from Definition 2.7. And, additionally, there is an isomorphism

$$\Phi : (q_U \circ \theta)^*(\mathbf{G}_U) = \theta^*(G_{\text{const}}) \rightarrow \theta^*(G_X) = (q_X \circ \theta)^*(\mathbf{G}) \text{ with } (\Delta_{new})^*(\Phi) = id_{G_U}$$

The triple $(\mathcal{X}_{new}, f_{new}, \Delta_{new})$ is a special nice triple **over** U subject to the conditions (1*) and (2*) from Definition 2.7. Thus by [P, Thm. 3.8] there is a finite surjective morphism $\mathbf{A}^1 \times U \xleftarrow{\sigma_{new}} \mathcal{X}_{new}$ of the U -schemes satisfying the conditions (a) to (f) from Theorem 5.1. Hence one has a diagram of the form

$$\begin{array}{ccccc} \mathbf{A}^1 \times U & \xleftarrow{\sigma_{new}} & \mathcal{X}_{new} & \xrightarrow{q_X \circ \theta} & X \\ & \searrow \text{pr}_U & \downarrow q_U \circ \theta & \curvearrowright \Delta' & \nearrow \text{can} \\ & & U & & \end{array} \quad (4)$$

with the irreducible scheme \mathcal{X}_{new} , the smooth morphism $q_{U,new} := q_U \circ \theta$, the finite surjective morphism σ_{new} and the essentially smooth morphism $q_{X,new} := q_X \circ \theta$ and with the function $f_{new} \in (q_{X,new})^*(f)k[\mathcal{X}_{new}]$, which after identifying notation enjoy the properties (a) to (f) from Theorem 5.1. The isomorphism Φ is the desired ones. Whence the Theorem 5.1. \square

We keep notation of the theorems 5.1. To formulate a consequence of the theorem 5.1 (see Corollary 5.2 below), note that using the items (b) and (c) of Theorem 5.1 one can find an element $g \in I(\mathcal{Z}'')$ such that

- (1) $(f') + (g) = \Gamma(\mathcal{X}, \mathcal{O}_X)$,
- (2) $\text{Ker}(\Delta^*) + (g) = \Gamma(\mathcal{X}_{new}, \mathcal{O}_{\mathcal{X}_{new}})$,
- (3) $\sigma_g = \sigma|_{\mathcal{X}_g} : \mathcal{X}_g \rightarrow \mathbf{A}_U^1$ is étale.

Corollary 5.2 (Corollary of Theorem 5.1). *The function f' from Theorem 5.1, the polynomial h from the item (f) of that Theorem, the morphism $\sigma : \mathcal{X} \rightarrow \mathbf{A}_U^1$ and the function $g \in \Gamma(\mathcal{X}, \mathcal{O}_X)$ defined just above enjoy the following properties:*

- (i) *the morphism $\sigma_g = \sigma|_{\mathcal{X}_g} : \mathcal{X}_g \rightarrow \mathbf{A}^1 \times U$ is étale,*
- (ii) *data $(\mathcal{O}[t], \sigma_g^* : \mathcal{O}[t] \rightarrow \Gamma(\mathcal{X}, \mathcal{O}_X)_g, h)$ satisfies the hypotheses of [C-T/O, Prop.2.6], i.e. $\Gamma(\mathcal{X}, \mathcal{O}_X)_g$ is a finitely generated $\mathcal{O}[t]$ -algebra, the element $(\sigma_g)^*(h)$ is not a zero-divisor in $\Gamma(\mathcal{X}, \mathcal{O}_X)_g$ and $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_X)_g/h\Gamma(\mathcal{X}, \mathcal{O}_X)_g$,*
- (iii) $(\Delta(U) \cup \mathcal{Z}') \subset \mathcal{X}_g$ and $\sigma_g \circ \Delta = i_0 : U \rightarrow \mathbf{A}^1 \times U$,
- (iv) $\mathcal{X}_{gh} \subseteq \mathcal{X}_{gf'} \subseteq \mathcal{X}_{f'} \subseteq \mathcal{X}_{q_X^*(f)}$,
- (v) $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_X)/(f')$ and $h\Gamma(\mathcal{X}, \mathcal{O}_X) = (f') \cap I(\mathcal{Z}'')$ and $(f') + I(\mathcal{Z}'') = \Gamma(\mathcal{X}, \mathcal{O}_X)$.

Proof. Just repeat literally the proof of [P, Cor. 7.2]. \square

Remark 5.3. The item (ii) of this corollary shows that the cartesian square

$$\begin{array}{ccc} \mathcal{X}_{gh} & \xrightarrow{\text{inc}} & \mathcal{X}_g \\ \sigma_{gh} \downarrow & & \downarrow \sigma_g \\ (\mathbf{A}^1 \times U)_h & \xrightarrow{\text{inc}} & \mathbf{A}^1 \times U \end{array} \quad (5)$$

can be used to glue principal \mathbf{G} -bundles for a reductive U -group scheme \mathbf{G} .

Set $Y := \mathcal{X}_g$, $p_X = q_X : Y \rightarrow X$, $p_U = q_U : Y \rightarrow U$, $\tau = \sigma_g$, $\tau_h = \sigma_{gh}$, $\delta = \Delta : U \rightarrow Y$ and note that $pr_U \circ \tau = p_U$. Take the monic polynomial $h \in \mathcal{O}[t]$ from the item (f) of Theorem 5.1. With this replacement of notation and with the element h we arrive to the following

Theorem 5.4. *Let the field k , the variety X , its closed points x_1, x_2, \dots, x_n , the semi-local ring $\mathcal{O} = \mathcal{O}_{X, \{x_1, x_2, \dots, x_n\}}$, the semi-local scheme $U = \text{Spec}(\mathcal{O})$, the function $f \in k[X]$ be the same as in Theorem 5.1. Let \mathbf{G} be a reductive X -group scheme and let $\mathbf{G}_U = \text{can}^*(\mathbf{G})$ be the pull-back of \mathbf{G} to U . Then one has a well-defined commutative diagram of affine schemes with the irreducible affine U -smooth Y , a section $\delta : U \rightarrow Y$ of the structure morphism $p_U : Y \rightarrow U$, and the monic polynomial $h \in \mathcal{O}[t]$*

$$\begin{array}{ccccc} (\mathbf{A}^1 \times U)_h & \xleftarrow{\tau_h} & Y_h := Y_{\tau^*(h)} & \xrightarrow{(p_X)|_{Y_h}} & X_f \\ \text{inc} \downarrow & & \downarrow \text{inc} & & \downarrow \text{inc} \\ (\mathbf{A}^1 \times U) & \xleftarrow{\tau} & Y & \xrightarrow{p_X} & X \end{array} \quad (6)$$

subject to the following conditions:

- (i) the left hand side square is an elementary **distinguished** square in the category of affine U -smooth schemes in the sense of [MV, Defn.3.1.3];
- (ii) $p_X \circ \delta = \text{can} : U \rightarrow X$, where can is the canonical morphism,
- (iii) $\tau \circ \delta = i_0 : U \rightarrow \mathbf{A}^1 \times U$ is the zero section of the projection $pr_U : \mathbf{A}^1 \times U \rightarrow U$;
- (iv) $h(1) \in \mathcal{O}[t]$ is a unit;
- (v) there is a Y -group scheme isomorphism $\Phi : p_U^*(\mathbf{G}_U) \rightarrow p_X^*(\mathbf{G})$ with $\delta^*(\Phi) = \text{id}_{\mathbf{G}_U}$.

Proof. The items (i) and (iv) of the Corollary 5.2 show that the morphisms $\delta(U) : U \rightarrow Y$ and $(p_X)|_{Y_h} : Y_h \rightarrow X_f$ are well defined. The items (i), (ii) of that Corollary show that the left hand side square in the diagram (6) is an elementary **distinguished** square in the category of smooth U -schemes in the sense of [MV, Defn.3.1.3]. The equalities $p_X \circ \delta = \text{can}$ and $\tau \circ \delta = i_0$ are obvious. The property (iv) of the polynomial h follows from the items (e),(f) and (a) of Theorem 5.1. The property (v) of the isomorphism Φ follows from the item (g) of Theorem 5.1. \square

6 Second application of the theory of nice triples

Proof of Theorem 1.1. The k -algebra \mathcal{O} is of the form $\mathcal{O}_{X, \{x_1, x_2, \dots, x_n\}}$, where X is a k -smooth irreducible affine variety. We may and will assume in this proof that the reductive group scheme \mathbf{G} and the principal \mathbf{G} -bundle \mathcal{G} are both defined over the variety X . Furthermore we may and will assume that there is given a non-zero function $f \in k[X]$ such that the \mathbf{G} -bundle \mathcal{G} is trivial on X_f and the function f vanishes at each point x_i in $\{x_1, x_2, \dots, x_n\}$. By Theorem 5.4 there is a diagram of the form (1) enjoying the properties (i) to (iv) from Theorem 5.4. Moreover there is a Y -group scheme isomorphism $\Phi : p_U^*(\mathbf{G}_U) \rightarrow p_X^*(\mathbf{G})$ such that $\delta^*(\Phi) = id_{\mathbf{G}_U}$.

Consider the commutative diagram (6). Given a \mathbf{G} -bundle \mathcal{G} over X , which is trivial on X_f take its pull-back $p_X^*(\mathcal{G})$ to Y . Using the isomorphism Φ we may and will regard the $p_X^*(\mathbf{G})$ -bundle $p_X^*(\mathcal{G})$ as a $p_U^*(\mathbf{G}_U)$ -bundle, t.e. as a \mathbf{G}_U -bundle. We will denote that \mathbf{G}_U -bundle by ${}_U p_X^*(\mathcal{G})$.

The \mathbf{G} -bundle is trivial on X_f . Hence the $p_X^*(\mathbf{G})$ -bundle $p_X^*(\mathcal{G})$ is trivial on $Y_{p_X^*(f)}$. Thus the \mathbf{G}_U -bundle ${}_U p_X^*(\mathcal{G})$ is trivial on $Y_{p_X^*(f)}$. Hence it is trivial also on Y_h .

Take a trivial \mathbf{G}_U -bundle over $(\mathbf{A}_U^1)_h$ and glue it with the \mathbf{G}_U -bundle ${}_U q_X^*(\mathcal{G})|_Y$ patching over Y_h (it can be done due to Theorem 5.4(i)). We get a \mathbf{G}_U -bundle \mathcal{G}_t over \mathbf{A}_U^1 which has particularly the following properties:

(a) the restriction of \mathcal{G}_t to $(\mathbf{A}_U^1)_h$ is trivial (by the construction);

(b) there is an isomorphism $\psi : {}_U q_X^*(\mathcal{G})|_Y \rightarrow \sigma_g^*(\mathcal{G}_t)$ of the \mathbf{G}_U -bundles;

It remains to check that the restriction of the \mathbf{G}_U -bundle \mathcal{G}_t to $0 \times U$ is isomorphic to the \mathbf{G}_U -bundle $can^*(\mathcal{G})$. To do that note that Theorem 5.4(ii) and Theorem 5.4(iii) yield the equalities

$$\mathbf{G}_U = \delta^*(q_X^*(\mathbf{G})) \quad \text{and} \quad can^*(\mathcal{G}) = \delta^*(q_X^*(\mathcal{G})).$$

There are two interesting \mathbf{G}_U -bundles over U . Namely, the \mathbf{G}_U -bundle $can^*(\mathcal{G}) = \delta^*(q_X^*(\mathcal{G}))$ and the \mathbf{G}_U -bundle $\delta^*({}_U q_X^*(\mathcal{G}))$. They coincide since $\delta^*(\Phi) = id_{\mathbf{G}_U}$. Thus

$$can^*(\mathcal{G}) = \delta^*({}_U q_X^*(\mathcal{G})) \cong \delta^*(\sigma_g^*(\mathcal{G}_t)) = \mathcal{G}_t|_{0 \times U},$$

where the middle \mathbf{G}_U -bundle isomorphism is the isomorphism $\delta^*(\psi)$. The latter equality holds by Theorem 5.4(iii). Whence the Theorem 1.1. □

Remark 6.1. Here is *the motivic view point* on the above arguments (in the constant case). The distinguished elementary square (5) defines a motivic space isomorphism $\mathcal{X}_g/\mathcal{X}_{gh} \xleftarrow{\sigma} \mathbf{A}_U^1/(\mathbf{A}_U^1)_h$ (just a Nisnevich sheaf isomorphism), hence there is a composite morphism of motivic spaces of the form

$$\varphi : \mathbf{A}_U^1/(\mathbf{A}_U^1)_h \xrightarrow{\sigma^{-1}} \mathcal{X}_g/\mathcal{X}_{gh} \rightarrow \mathcal{X}_g/\mathcal{X}_{q_X^*(f)} \xrightarrow{q} X/X_f.$$

Let $i_0 : 0 \times U \rightarrow \mathbf{A}_U^1/(\mathbf{A}_U^1)_h$ be the natural morphism. By the properties (a') and (d) from Theorem 5.1 the morphism $\varphi \circ i_0$ equals to the one

$$U \xrightarrow{can} X \xrightarrow{p} X/X_f,$$

where $p : X \rightarrow X/X_f$ is the canonical morphisms.

Now assume that \mathbf{G}_0 is a reductive group scheme over the field k . A \mathbf{G}_0 -bundle over X , trivialized on X_f , is "classified" by a morphism $\rho : X/X_f \rightarrow (B\mathbf{G}_0)_{et}$ in an appropriate category. Thus the morphism $\rho \circ \varphi$ "classifies" a \mathbf{G}_0 -bundle \mathcal{G}_t over \mathbf{A}_U^1 trivialized on $(\mathbf{A}_U^1)_h$. The equality $\varphi \circ i_0 = p \circ can$ shows that the \mathbf{G}_0 -bundles $\mathcal{G}_t|_{0 \times U}$ and $can^*(\mathcal{G})$ are isomorphic. This "proves" Theorem 1.1 in the constant case.

7 An extension of Theorem [PSV, Thm 1.1]

Theorem 7.1. *Let k be a field. Let \mathcal{O} be the semi-local ring of finitely many closed points on a k -smooth irreducible affine k -variety X and let K be its field of fractions. Let \mathbf{G} be an isotropic simple simply connected group scheme over \mathcal{O} . Then for any Noetherian k -algebra A the map*

$$H_{et}^1(\mathcal{O} \otimes_k A, G) \rightarrow H_{et}^1(K \otimes_k A, G),$$

induced by the inclusion \mathcal{O} into K , has trivial kernel.

Proof. If the field k is infinite, this theorem is exactly Theorem [PSV, Thm 1.1]. So, there is nothing to prove in this case. If the field k is finite, then repeat literally the proof of [PSV, Thm 1.1] and replace the reference to [PSV, Thm 1.2] with the reference to Theorem 7.2. \square

Theorem 7.2. *Let k, \mathcal{O}, K, A be the same as in Theorem 7.1. Let G be a not necessarily isotropic simple simply connected group scheme over \mathcal{O} . Let \mathcal{G} be a principal G -bundle over $\mathcal{O} \otimes_k A$ which is trivial over $K \otimes_k A$. Then there exists a principal G -bundle \mathcal{G}_t over $\mathcal{O}[t] \otimes_k A$ and a monic polynomial $f(t) \in \mathcal{O}[t]$ such that*

- (i) the G -bundle \mathcal{G}_t is trivial over $(\mathcal{O}[t]_f) \otimes_k A$,*
- (ii) the evaluation of \mathcal{G}_t at $t = 0$ coincides with the original G -bundle \mathcal{G} ,*
- (iii) $f(1) \in \mathcal{O}$ is invertible in \mathcal{O} .*

Proof of Theorem 7.2. If $A = k$, then Theorem 7.2 coincides with Theorem 1.1 and there is nothing to prove. The general case we left to the reader (follow literally the arguments from the proof of Theorem 1.1 given in Section 6). \square

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