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Nijenhuis *G*-Manifolds and Lenard Bicomplexes: A New Approach to KP Systems*

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Abstract. We suggest a method to extend the theory of recursion operators to integrable Hamiltonian systems in two-space dimensions, like KP systems. The approach aims to stress the conceptual unity of the theories in one and two space dimensions. A sound explanation of the appearance of bilocal operators is also given.

1. Introduction

This paper deals with the theory of recursion operators for nonlinear Hamiltonian equations in two space dimensions. According to a common opinion [1, 2], the use of these operators cannot be extended beyond the theories in one space dimension. In fact, a new phenomenology occurs in two space dimensions, which seems to be incompatible with a classical recursion scheme, its more impressive feature being the appearance of bilocal operators [1]. Our aim is to correct this opinion. We believe that it is due to an incomplete understanding of the potentialities of the method of recursion operators. In our opinion, a basic element of the theory has been missed, that is the role of the symmetry algebra. Usually, in fact, the recursion operators are coupled with a peculiar algebra of vector fields leaving them invariant. This algebra is (in some sense) trivial in one space dimension, and this fact explains why its role has been so far underestimated. It becomes crucial in two space dimensions. In the present paper the symmetry algebra is taken, since the beginning, as a fundamental element of the theory, at the same level of the recursion operator. This leads us to develop, in Sect. 2, the theory of the *Nijenhuis G-manifolds*. (For conceptual reasons, we prefer to use the name of Nijenhuis tensor instead of the more common, but also less specific, term of recursion operator.) The main outcome is the quite natural notion of *Lenard bicomplex*. It is the basic tool allowing us to deal in a surprisingly unified way with the theories both in one and two space dimensions. The same differences which seemed before to cleanly mark the two cases, appear now of

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secondary importance. They accidentally arise in the concrete realization of the same abstract scheme. An indispensable tool in order to put this abstract scheme in action is the *reduction theory* of Nijenhuis G-manifolds and of Lenard bicomplexes. It is developed in Sects. 4 and 5. The application of the reduction techniques to the simple examples of algebraic Nijenhuis G-manifolds, constructed in Sect. 3, leads us directly to obtain, in Sect. 6, the recursion scheme well-fitted for both KdV and KP systems. Last (but not least) the theory of the Lenard bicomplex gives a sound explanation for the appearance of *bilocal operators*, as shown in Sect. 5. A concise but complete comparison with other approaches [3, 1] using these operators is finally given in the last section.

2. The Nijenhuis G-Manifolds and the Lenard Bicomplex

Let M be a differentiable manifold, N a tensor field on M of type (1,1). M is said to be a *Nijenhuis manifold* if N is torsion-free, i.e. if it fulfils the condition

$$[N\varphi, N\psi]-N[N\varphi, \psi]-N[\varphi, N\psi]+N^2[\varphi, \psi]=0 \qquad (2.1)$$

for any pair of vector fields φ, ψ on M (the bracket $[\varphi, \psi]$ denotes the commutator of the vector fields φ and ψ; the vector field $N\varphi$ is the image of φ under the linear mapping $N:\mathfrak{X}(M)\to\mathfrak{X}(M)$). Furthermore, let G be a Lie group acting on M, such that:

(i) the action $\Phi: G\times M\to M$ leaves N invariant,

(ii) G is itself a Nijenhuis manifold, with a left-invariant Nijenhuis tensor $\Delta:\mathfrak{X}(G)\to\mathfrak{X}(G)$.

Then M is said to be endowed with the structure of a *Nijenhuis G-manifold* (or *GN-manifold* for short): it will be denoted by (M, N, G, Φ, Δ).

Our interest for this kind of manifolds is due to the following property. Let l_a ($a\in\mathfrak{g}$, the Lie algebra of G) be any left-invariant vector field on G leaving Δ invariant. By means of the infinitesimal action of $\mathfrak{g}$ on M, $X_m=d\phi_m(e):\mathfrak{g}\to T_mM$, we construct the two-indices family of vector fields on M given by

$$\varphi_a^{jk}:=N^j\cdot X\cdot \Delta_e^k a \ , \qquad (2.2)$$

where Δ_e is the evaluation of Δ at the identity e of G. We claim that these vector fields leave N invariant and commute in pairs

$$[\varphi_a^{jk}, \varphi_a^{lm}]=0 \ . \qquad (2.3)$$

Moreover, if M is a Poisson-Nijenhuis manifold [4, 5] and the action of G is Poissonian, they are (locally) Hamiltonian.

To prove this statement, one has to recall that on any Poisson-Nijenhuis manifold (M, N, P) the vector fields

$$\varphi^j=N^j\varphi \qquad \psi^k=N^k\psi \ , \qquad (2.4)$$

recursively obtained from vector fields φ and ψ leaving N and P invariant, are locally Hamiltonian, leave N invariant and fulfil the commutation relations

$$[\varphi^j, \psi^k]=N^{j+k}[\varphi, \psi] \ . \qquad (2.5)$$

Then, one easily realizes that the vector fields $l_a^j = \Delta^j l_a$ are left-invariant and commute in pairs on G. By the action ϕ, this property is conveyed to the infinitesimal generators $\varphi_a^j = X \cdot \Delta_e^j a$ on M. Indeed, these vector fields leave N and P invariant (by definition), and commute in pairs since $[\varphi_a^j, \varphi_a^k] = -d\phi \cdot [l_a^j, l_a^k] = 0$. So, our statement follows from $[\varphi_a^{jk}, \varphi_a^{lm}] = N^{j+l}[\varphi_a^k, \varphi_a^m] = 0$.

We remark that neither the Lie group G nor the Nijenhuis tensor Δ are really needed to construct the vector fields φ^{jk}. It suffices to know the Lie algebra $\mathfrak{g}$, the infinitesimal action X and the evaluations of Δ and l_a at e, as it is shown by (2.2). These evaluations fulfil the conditions

$$[\Delta_e b_1, \Delta_e b_2] - \Delta_e[b_1, \Delta_e b_2] - \Delta_e[\Delta_e b_1, b_2] + \Delta_e^2[b_1, b_2] = 0 \ , \tag{2.6}$$

$$\mathrm{ad}_a \cdot \Delta_e = \Delta_e \cdot \mathrm{ad}_a \ , \tag{2.7}$$

which assure the existence of a left-invariant Nijenhuis tensor field Δ on G, and of a left-invariant vector field l_a leaving Δ invariant. For this reason, the two-indices family of vector fields φ^{jk} will be referred to as the *Lenard bicomplex* associated with the starting symmetry $a \in \mathfrak{g}$ of the $\mathfrak{g}N$ manifold $(M, N, \mathfrak{g}, X, \Delta_e)$.

3. Examples of $\mathfrak{g}N$ Manifolds

In this section we exhibit an explicit class of $\mathfrak{g}N$ manifolds modelled on associative algebras. The model is constructed so to encompass the theories of KdV and KP systems as particular cases. It aims to stress the unity of these two theories.

To construct the model we need an associative algebra A with unit, endowed with a derivation $D: A \to A$. This algebra is the *ambient space* where our theory takes place. In A we select two particular subalgebras. The first one is the algebra $K = \mathrm{Ker}\, D$ of the elements of the kernel of D. The second one is any subalgebra V fulfilling the following two conditions:

(V.1) V is stable with respect to D: $D(V) \subset V$,
(V.2) D, restricted to V, is kernel-free: $V \cap K = \emptyset$.

The elements of K are denoted by $(a, b, c, \ldots)$ and are called the *constants*; those of V are denoted by $(v, \varphi, \psi, \ldots)$ and are called the *vectors*. The manifolds M we are looking for are the affine hyperplanes modelled on V, defined by

$$M = V + \{c\} \ , \tag{3.1}$$

where c is any constant such that

$$[c, V] \subset V \ . \tag{3.2}$$

These hyperplanes are endowed with the structure of Nijenhuis manifold by introducing the *Nijenhuis tensor* $N^{(1)}: M \times V \to V$ given by

$$N_u^{(1)}\varphi = \tfrac{1}{2}[u, D^{-1}\varphi] \ , \tag{3.3}$$

where $[,]$ is the commutator in A, $u \in M$, $\varphi \in V$. This tensor is well-defined on M by (3.2), and fulfils (2.1) since D is a derivation. To construct an *infinitesimal action* leaving $N^{(1)}$ invariant, we consider the Lie algebra $\mathfrak{g} \subset K$ of the constants a

commuting with c and leaving V invariant. Explicitly:

$$[a,c]=0 \ , \tag{3.4}$$

$$a(V)\subset V \ , \quad (V)a\subset V \ . \tag{3.5}$$

This algebra acts on M according to

$$X_u^{(1)}: a\in \mathfrak{g} \mapsto \varphi_a(u) := [a,u] \tag{3.6}$$

and this action leaves $N^{(1)}$ invariant (since a is a constant and D is a derivation). Finally, we observe that $\mathfrak{g}$ is an *associative* algebra with unit. Then, conditions (2.6) and (2.7) are simply fulfilled by setting

$$\Delta_e = R_{\bar{a}} \ , \tag{3.7}$$

$$a=1 \ , \tag{3.8}$$

where $R_{\bar{a}}$ denotes the right-multiplication by any fixed element $\bar{a}\in\mathfrak{g}$. So, we have explicitly constructed a whole class of $\mathfrak{g}N$ manifolds $(M, N^{(1)}, \mathfrak{g}, X^{(1)}, \Delta_e, a)$, depending on the choice of a derivation D and of two constants c and $\bar{a}$.

A particular case is of special interest. It corresponds to the following choice of D and c [6]:

$$D=[x, \ ] \ , \tag{3.9}$$

$$c=x^2+y \ , \tag{3.10}$$

where x and y are given elements of A fulfilling the following conditions:

$$(\mathfrak{g}N1) \quad [x,V]\subset V \ , \quad \{x,V\}\subset V \ ,$$

$$(\mathfrak{g}N2) \quad [x,v]=0 \Rightarrow v=0 \ ,$$

$$(\mathfrak{g}N3) \quad [y,V]\subset V \ ,$$

$$(\mathfrak{g}N4) \quad [x,y]=0 \ ,$$

and the bracket $\{,\}$ is the anticommutator in A. The conditions: ($\mathfrak{g}N$1–2) entail (V.1) and (V.2), ($\mathfrak{g}N$1–3) entail (3.2), and ($\mathfrak{g}N$4) means that c is a constant. Moreover, $x\in\mathfrak{g}$ by ($\mathfrak{g}N$1–4), so that we can set $\bar{a}=x$. By using the definition (2.2) we obtain the Lenard bicomplex of Fig. 1a, where $v_x := [x,v]$, $v_y := [y,v]$, ... As it will be shown later on, this scheme provides all the KdV and KP systems. It will be referred to as the KP *bicomplex*.

A different $\mathfrak{g}N$ structure (related with KP systems) is constructed on the same affine hyperplane (3.1) under the stronger assumptions

$$[c,V]\subset V \ , \quad \{c,V\}\subset V \ , \tag{3.2 bis}$$

on the constant c. The new structure is defined by the same symmetry algebra as in the previous example, and by the Nijenhuis tensor

$$4N_u^{(2)}\varphi = -D^2\varphi + \{Du, D^{-1}\varphi\} + 2\{u,\varphi\} + [u, D^{-1}[D^{-1}\varphi, u]] \ . \tag{3.3 bis}$$

The KP systems are obtained as follows. One observes that the action $X^{(1)}$, defined by (3.6), leaves $N^{(2)}$ invariant and that c belongs to the symmetry algebra by (3.2

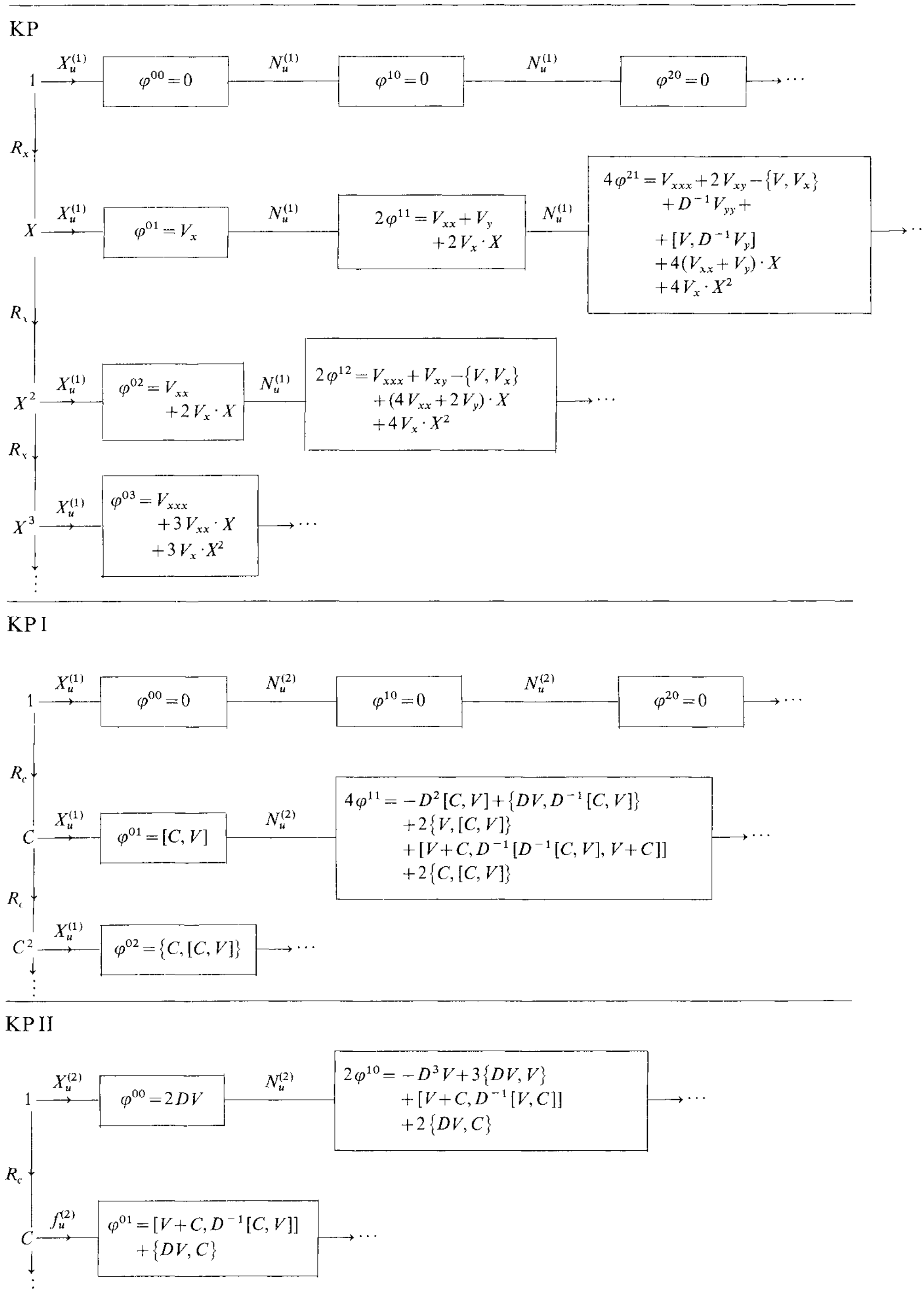


Fig. 1. The Lenard bicomplexes

bis). Then one constructs the Lenard bicomplex associated with the $\mathfrak{g}N$ structure $(M, N^{(2)}, \mathfrak{g}, X^{(1)}, \Delta_e = R_c, a=1)$. The final result is given in Fig. 1b. As it will be shown later on, this scheme provides *half* the KP systems: it will be referred to as the KPI *bicomplex*. To recover the whole KP systems, it is important to remark that a second infinitesimal action exists leaving the Nijenhuis tensor $N^{(2)}$ invariant [11]. It is defined by

$$X_u^{(1)} : a \in \mathfrak{g} \mapsto \{Du, a\} + [u, D^{-1}[a, u]] \ . \tag{3.11}$$

The Lenard bicomplex associated with the $\mathfrak{g}N$ structure $(M, N^{(2)}, \mathfrak{g}, X^{(2)}, \Delta_e = R_c, a=1)$ is given in Fig. 1c: it will be referred to as the KP II bicomplex.

The origin of the $\mathfrak{g}N$ structure (3.3 bis) will be explained in Sect. 4, by means of the reduction theory of $\mathfrak{g}N$ manifolds. (However, the next section is not essential for the understanding of the rest of the paper.)

4. A Reduction Theorem for $\mathfrak{g}N$ Manifolds

The reduction technique is a powerful tool to obtain a large class of different $\mathfrak{g}N$ manifolds, starting from a few ones, simple enough to be dealt with easily. In this section it is applied to recover the $\mathfrak{g}N$ structure related with KP systems. We begin by summarizing the reduction technique [4, 5].

4.1. The Reduction Technique

Let $(M, N, \mathfrak{g}, X, \Delta_e)$ be a $\mathfrak{g}N$ manifold. In our search for submanifolds S of M carrying $\mathfrak{g}N$ structures, we consider first of all the characteristic distribution of N, defined by

$$\mathscr{D}_m := N_m(T_m M) \ , \quad m \in M \ . \tag{4.1}$$

If the rank of N is constant in M, $\mathscr{D}$ is integrable in the sense of Frobenius. Let S be an integral manifold of $\mathscr{D}$. At each point $m \in S$, the images and the kernels of the powers of N

$$\operatorname{Im} N_m^i := \{\psi(m) \in T_m S : \psi(m) = N_m^i \varphi(m) \ , \quad \text{for some} \quad \varphi(m) \in T_m S\} \ , \tag{4.2}$$

$$\operatorname{Ker} N_m^i := \{\chi(m) \in T_m S : N_m^i \chi(m) = 0\} \tag{4.3}$$

fulfil the obvious inclusion relations

$$\operatorname{Im} N^{i+1} \subset \operatorname{Im} N^i \qquad \operatorname{Ker} N^{i+1} \supset \operatorname{Ker} N^i \ . \tag{4.4}$$

We assume that there exists a finite index r (called the Riesz index of N on S) which is constant on S and such that for $i=r$ both sequences (4.4) become stationary

$$\operatorname{Im} N_m^{r+1} = \operatorname{Im} N_m^r \ , \qquad \operatorname{Ker} N_m^{r+1} = \operatorname{Ker} N_m^r \ . \tag{4.5}$$

Then one can show that $\operatorname{Im} N_m^r$ and $\operatorname{Ker} N_m^r$ are integrable and intersect transversally on $S: T_m S = \operatorname{Im} N_m^r \oplus \operatorname{Ker}_m^r$. If the leaves of the distribution $\operatorname{Ker} N^r$ are connected, the quotient space $M' := S|\operatorname{Ker} N^r$ is a manifold and the canonical projection $\pi : S \to M'$ is a surjective submersion, then M' inherits a kernel-free Nijenhuis structure defined as

$$N'_{m'} \cdot d\pi(m) = d\pi(m) \cdot N_m \ , \tag{4.6}$$

where m is any point of the fiber over $m' : m' = \pi(m)$. Thus M' is a new (reduced) Nijenhuis manifold.

As for the infinitesimal action X, it suffices to restrict it to the isotropy subalgebra $\mathfrak{g}_S \subset \mathfrak{g}$ of S, i.e. to consider only the subalgebra spanned by the vector fields $\varphi_a = X \cdot a$ which are tangent to S. They are automatically projectable on M', and their projections

$$\varphi'_a(m') := d\pi(m) \cdot \varphi_a(m) \quad (a \in \mathfrak{g}_S, m \in \pi^{-1}(m')) \tag{4.7}$$

define the symmetry algebra of N'. As for $\Delta_e : \mathfrak{g} \to \mathfrak{g}$, we require that it leaves $\mathfrak{g}_S$ invariant.

4.2. An Application

Let A, D, K, V be defined as in Sect. 3. We denote by A' the algebra $gl(2, A)$ of 2×2 matrices with entries in A, and we put $u'_{11} = u_1$, $u'_{12} = u_2$, $u'_{21} = u_3$, $u'_{22} = u_4$. In A' we consider the affine hyperplane $M' = V' + \{c'\}$, modelled on the subalgebra V' of matrices with entries in V, corresponding to any constant c' such that $c'(V') \subset V'$, $(V')c' \subset V'$. A class of $\mathfrak{g}N$ structures is constructed in M' by considering the two families of vector fields in A' given by

$$\varphi_{\xi'}(u') := D\xi' + [u', \xi'] \ , \tag{4.8}$$

$$\psi_{\xi'}(u') := [b', \xi'] \ , \tag{4.9}$$

where $u' \in M'$, $\xi' \in V'$ and b' is any given constant such that

$$[b', V'] \subset V' \ . \tag{4.10}$$

It is easy to check that these vector fields are tangent to M' on account of (4.10), and fulfil the commutation relations

$$[\varphi_{\xi'}, \varphi_{\eta'}] = \varphi_{[\eta', \xi']} \ , \tag{4.11}$$

$$[\varphi_{\xi'}, \psi_{\eta'}] + [\psi_{\xi'}, \varphi_{\eta'}] = \psi_{[\eta', \xi']} \ , \tag{4.12}$$

$$[\psi_{\xi'}, \psi_{\eta'}] = 0 \ , \tag{4.13}$$

since D is a derivation. If A is endowed with a trace form making D skew-symmetric,

$$\mathrm{tr}(D\xi \cdot \eta) + \mathrm{tr}(\xi \cdot D\eta) = 0 \tag{4.14}$$

they are also Hamiltonian with respect to a pair of compatible Poisson structures on A' [4, 5]. We make the assumption that the fields $\varphi_{\xi'}$ define a free and transitive action on M', i.e. that for every $\varphi' \in V'$ a unique $\xi' \in V'$ exists such that $\varphi' = D\xi' + [u', \xi']$ (this condition is easily verified in many applications). Under this assumption, one can define a family of tensor fields of type (1,1) on M', depending on the choice of b', and given by:

$$N' : \varphi_{\xi'} \mapsto \psi_{\xi'} \ . \tag{4.15}$$

They fulfil the Nijenhuis condition (2.1) on account of the commutation relations (4.11)–(4.13). Moreover, they are invariant with respect to the infinitesimal action (3.11), provided $X^{(1)}$ is restricted to the subalgebra $\mathfrak{g}_{b'} \subset \mathfrak{g}$ of the constants commuting with b':

$$\mathfrak{g}_{b'} = \{a' \in K' [a', c'] = [a', b'] = 0,\ a'(V') \subset V',\ (V')a' \subset V'\} \ . \qquad (4.16)$$

So, for any choice of b' fulfilling (4.10), one has a $\mathfrak{g}N$ manifold. It can be shown that different choices of b' correspond to different "Zakharov-Shabat spectral problems." We now proceed to the reduction of the tensor N' corresponding to $b' := \{b_1 = b_3 = b_4 = 0,\ b_2 = 1\}$ over the affine hyperplane associated with $c' := \{c_1 = c_4 = 0,\ c_3 = 1,\ c_2 = c\}$.

Among the integral leaves of N', we choose the submanifold S' defined by $u_3 = 1$, $u_1 + u_4 = 0$. On this submanifold, the Riesz index is $r = 1$ and the leaves of the null distribution $\mathrm{Ker} N'$ are given by:

$$u_3 = 1 \ , \qquad u_1 + u_4 = 0 \ , \qquad Du_1 + u_1^2 + u_2 = u \ . \qquad (4.17)$$

Consequently:

(i) the quotient manifold $S'/\mathrm{Ker} N'$ is identified with the affine hyperplane $M = V + \{c\}$ in A,

(ii) a suitable section of the projection $\pi : (u_1, u_2) \to u$ is given by

$$\gamma = \{u_1 = u_4 = 0,\ u_3 = 1,\ u_2 = u\} \ . \qquad (4.18)$$

A simple computation which consists in:

(i) evaluating the vector fields $\varphi_{\xi'}$ and $\psi_{\xi'}$ on the points of the section γ,

(ii) finding the unique $\xi' \in V'$ such that, for any $\varphi \in V$, $d\pi \cdot \psi_{\xi'} = \varphi$ and $\varphi_{\xi'}$ be tangent to γ,

(iii) computing $\psi = d\pi \cdot \varphi_{\xi'}$,

gives the reduced Nijenhuis tensor $N : \varphi \mapsto \psi$ on M. Explicitly, one finds

$$4 N_u^{(2)} \varphi = -D^2 \varphi + \{Du, D^{-1} \varphi\} + 2\{u, \varphi\} + [u, D^{-1}[D^{-1} \varphi, u]] \ . \qquad (4.19)$$

As for the symmetry algebra (4.16), the choice of b' and c' entails that

$$\mathfrak{g}_{b'} = \{a' : a_2 = a_3 = 0, a_1 = a_4 = a, [a, c] = 0, a(V) \subset V, (V)a \subset V\} \ . \qquad (4.20)$$

Under these assumptions the infinitesimal generator $\varphi_a'(u') := [a', u']$ is tangent to S' and then is automatically projectable on M. Its projection is given by

$$\varphi_a(u) = [a, u] \ , \qquad (4.21)$$

so that the first infinitesimal action $X_u^{(1)}$ is obtained.

4.3. *An Extension*

So far, the second symmetry generator (3.11) has been missed. The reason is due to the choice of the reduction technique. Recall that, at the beginning, only the vector fields $\varphi_{\xi'}$ and $\psi_{\xi'}$ were available but not the Nijenhuis tensor. To define this tensor, we were compelled to restrict our manifold to the affine hyperplane M', in order to make the action of $\varphi_{\xi'}$ free and transitive. Of course, this has reduced the symmetry algebra φ'_a and only the symmetry generator (4.21) survived. We can overcome this difficulty by changing the reduction technique. We shall now give a sketch of the new procedure by using the example at hand, without claiming for any general theory. The aim is mainly to justify formula (3.11).

Let us still consider the vector fields (4.8) and (4.9) without setting any limitation on u' and ξ'. This means that we regard these vector fields as defining two infinitesimal actions of A' (endowed with its natural Lie algebra structure) on itself, considered as a manifold. Each action defines its family of orbits. In particular, we consider the orbit

$$O=\{u'\in A': u_3=1,\, u_1+u_4=0\} \tag{4.22}$$

of the second action $\psi_{\xi'}$. We remark that this orbit properly includes the previous submanifold S', since now (u_1, u_4) are not restricted to V. Two relevant subalgebras are associated with any point u' of O. The first one is the isotropy algebra with respect to the action $\psi_{\xi'}$, i.e. the set of vectors ξ' such that $\psi_{\xi'}=0$ at u'. It is clearly independent of the point u', being given by

$$\mathfrak{g}_O^{(2)}=\{\xi'\in A': \xi_3=0,\, \xi_1=\xi_4\} \ . \tag{4.23}$$

The second algebra consists of all the vectors ξ' such that the infinitesimal generator $\varphi_{\xi'}(u')$ is tangent to O. It is given by

$$\begin{aligned}\mathfrak{g}_O^{(1)}(u')=\{\xi'\in A': \xi_1-\xi_4=&-D\xi_3+\{u_1,\xi_3\},\ \xi_1+\xi_4\\ &=D^{-1}([u_1, D\xi_3-\{u_1,\xi_3\}]-[u_2,\xi_3])+2a\} \ ,\end{aligned} \tag{4.24}$$

where $\xi_2, \xi_3\in A'$ and $a\in K$ are arbitrary parameters. These two algebras have the intersection

$$\mathfrak{g}^{(3)}(u')=\mathfrak{g}_O^{(1)}(u')\cap \mathfrak{g}_O^{(2)}=\{\xi'\in A': \xi_3=0,\, \xi_1=\xi_4=a\in K\} \tag{4.25}$$

parametrized by (ξ_2, a). The commutation relations in $\mathfrak{g}^{(3)}$ are easily found by considering the associated infinitesimal generators $\varphi_{(\xi_2,a)}$, and by computing their commutator as vector fields. One gets

$$[\varphi_{(\xi_2,a)}, \varphi_{(\eta_2,b)}]=\varphi_{([\eta_2,a]-[\xi_2,b],[a,b])} \ . \tag{4.26}$$

Analogous commutation relations (not of this simple form) can be also computed between $\varphi_{\xi_2}:=\varphi_{(\xi_2,0)}$ and $\psi_{\xi'}$ or $\varphi_{\xi'}$, with $\xi'\in\mathfrak{g}_O^{(1)}$. They all imply that the commutators $[\varphi_{\xi_2}, \psi_{\xi'}]$ and $[\varphi_{\xi_2}, \varphi_{\xi'}]$ are still fields of the type φ_{ξ_2} for a suitable choice of ξ_2, depending on the point $u'\in O$. Then, consider the orbits of φ_{ξ_2} on O. They are still defined by (4.17), without assuming $(u_1, u_4)\in V$. So, the quotient manifold $M'=O/\varphi_{\xi_2}$ is identified with A, and the canonical projection is $\pi: (u_1, u_2)\mapsto u=Du_1+u_1^2+u_2$. On account of the above-mentioned commutation relations, the vector fields $\psi_{\xi'}$ and $\varphi_{\xi'}$, $\xi'\in\mathfrak{g}^{(1)}$, project on M'. The projected vector fields are given by

$$\psi'_{\xi_3}=2D\xi_3 \ , \tag{4.27}$$

$$\varphi'_{(\xi_3,a)}=\tfrac{1}{2}(-D^3\xi_3+\{Du,\xi_3\}+2\{u,D\xi_3\}+[u,D^{-1}[\xi_3,u]])-[a,u] \ . \tag{4.28}$$

So, we started from A' with a pair of vector fields $(\psi_{\xi'}, \varphi_{\xi'})$, and we arrived to A with three vector fields $(\psi'_{\xi_3}, \varphi'_{\xi_3}, \varphi'_a)$. The last one is the first symmetry generator (4.21) already found.

Now, we can repeat the whole process in A, by considering the vector fields $(\psi'_{\xi_3}, \varphi'_{\xi_3})$. Let us restrict our attention to the constants b such that $b(V)\subset V$ and $(V)b\subset V$. Without loss of generality, we can then assume $A=V\oplus K$, so that we can characterize the orbits of ψ'_{ξ_3} as the affine hyperplanes modelled on V. The isotropy algebra $\mathfrak{g}^{(2)}$, for each of these orbits, is the algebra K of the constants. Fix an orbit $M=V+\{c\}$. The algebra $\mathfrak{g}_c^{(1)}$ consists of all the vectors $\xi_3=\xi+b$, such that $\xi\in V$ and b is a constant commuting with c: $[b,c]=0$. So, the algebra $\mathfrak{g}_c^{(3)}=\mathfrak{g}_c^{(1)}\cap\mathfrak{g}^{(2)}$ consists of all the constant b such that $[b,c]=0$. In particular, consider the vector field (4.28) for $\xi_3=b\in\mathfrak{g}_c^{(3)}$. It is tangent to the manifold M, and gives the second symmetry generator (3.11), exactly as the vector field $\varphi_{\xi'}$ for $\xi'=\{\xi_1=\xi_4=a, \xi_2=\xi_3=0\}\in\mathfrak{g}_O^{(3)}$ gives the first symmetry generator. Finally, consider the vector fields (4.27) and (4.28) for $\xi_3=\xi\in V$. The action of ψ'_ξ is free and transitive on M. Consequently, a unique tensor field N is defined on M such that $N:\psi'_\xi\mapsto\varphi'_\xi$. It is the Nijenhuis tensor (4.19) previously found. This completes our deduction.

5. A Reduction Theorem for Lenard Bicomplexes

The reduction techniques for $\mathfrak{g}N$ manifolds are not sufficient to recover all the interesting examples of integrable systems known from the literature. This is due to the rather restrictive nature of the conditions defining $\mathfrak{g}N$ manifolds modelled on associative algebras. Take, for example, the condition ($\mathfrak{g}N1$) of Sect. 3. One has to find an element $x\in A$ and a subalgebra $V\subset A$ such that $x(V)\subset V$ and $(V)x\subset V$. Clearly, these conditions entail $(V)x^2\subset V$, $(V)x^3\subset V,\ldots$ and so on, so that V must fulfil the condition

$$\sum_{k\geqq 0} v_k x^k\in V \qquad (v_k\in V)\ . \tag{5.1}$$

Consequently, if the order of x is infinite the dimension of V cannot be finite. This compels us to work with infinite dimensional $\mathfrak{g}N$ manifolds, and so to deal with evolution equations in an infinite number of fields. We can overcome this difficulty by means of the *reduction technique* for the bicomplexes. They admit reductions onto finite-dimensional manifolds, even if this is not possible for the entire $\mathfrak{g}N$ structure. This is the new feature we aim to display in this section.

Let S be a submanifold of M, $\mathfrak{g}_S\subset\mathfrak{g}$ its isotropy algebra, i.e. the set of the elements $a\in\mathfrak{g}$ such that $X\cdot a$ is tangent to S. We consider the linear mappings $X_j:\mathfrak{g}\to\mathfrak{X}(M)$ recursively defined by

$$X_{j+1}:=N\cdot X_j-X_j\cdot\Delta_e\ ,\qquad X_0=X\ . \tag{5.2}$$

Clearly, if TS is invariant with respect to N and $\mathfrak{g}_S$ with respect to Δ_e, then

$$X_j(\mathfrak{g}_S)\subset\mathfrak{X}(S) \tag{5.3}$$

and S is itself a $\mathfrak{g}N$-manifold. Let us assume (5.3) to hold true even if $\mathfrak{g}_S$ and TS are not invariant, and let us consider the subsequence of vector fields

$$\varphi_a^j := \sum_0^j {}_k (-1)^k \binom{j}{k} \varphi_a^{j-k,k} = \sum_0^j {}_k \binom{j}{k} N^{j-k} \cdot X \cdot (-\Delta_e)^j a \tag{5.4}$$

extracted from the bicomplex. It is straightforward to check that they are tangent to S. Indeed, (5.2) and (5.4) can be given the form

$$\hat{X}(\lambda) = \exp \lambda N \cdot X \cdot \exp - \lambda \Delta_e \ , \tag{5.5}$$

$$\varphi_a(\lambda) = (\exp \lambda N \cdot X \cdot \exp - \lambda \Delta_e) a \ , \tag{5.6}$$

by introducing the formal series

$$\varphi_a(\lambda) = \sum_{k \geqq 0} \varphi_a^k \frac{\lambda^k}{k!} \ , \tag{5.7}$$

$$\hat{X}(\lambda) = \sum_{k \geqq 0} X_k \frac{\lambda^k}{k!} \ . \tag{5.8}$$

Clearly, $\hat{X}(\lambda)$ verifies (5.3) and this entails the required property for $\varphi_a(\lambda)$. Thus we can conclude that a commuting hierarchy of vector fields is naturally defined on each submanifold S fulfilling (5.3). This hierarchy will be referred to as the *standard reduction* of the Lenard bicomplex.

We now specialize our study to the algebraic $\mathfrak{g}N$ manifolds of Sect. 3. The right hint is given by the remark that both the Nijenhuis tensors (3.3) [with the choices (3.9) and (3.10) for D and c] and (4.19), and the infinitesimal action (4.21) and (3.11) have a peculiar structure. Indeed, by making explicit their dependence on the constants $(x, y, c, \ldots)$, they can be written at any point $u = v + c$ of M in the form

$$2N_v^{(1)} = (D + (L_v - R_v) \cdot D^{-1} + \mathrm{ad}_y \cdot D^{-1}) + 2R_x \ , \tag{5.9}$$

$$\begin{aligned} 4N_v^{(2)} = (&-D^2 + (L_{v_x} + R_{v_x}) \cdot D^{-1} + 2(L_v + R_v) + 2\,\mathrm{ad}_c + \\ &-(L_v - R_v + \mathrm{ad}_c) \cdot D^{-1} \cdot (L_v - R_v + \mathrm{ad}_c) \cdot D^{-1}) + 4R_c \ , \end{aligned} \tag{5.10}$$

$$X_v^{(1)} = L_v - R_v + \mathrm{ad}_c \ , \tag{5.11}$$

$$X_v^{(2)} = L_{Dv} + R_{Dv} - (L_v - R_v + \mathrm{ad}_c) \cdot D^{-1} \cdot (L_v - R_v + \mathrm{ad}_c) \ . \tag{5.12}$$

This suggests to consider a more general class of tensor fields of the following form:

$$N_v = p(L_v, R_v, D_1, \ldots, D_l, \mathrm{ad}_{x_1}, \ldots, \mathrm{ad}_{x_n}) + R_x \ , \tag{5.13}$$

$$X_v = q(L_v, R_v, D_1, \ldots, D_l, \mathrm{ad}_{x_1}, \ldots, \mathrm{ad}_{x_n}) \ , \tag{5.14}$$

where p and q are polynomial functions of L_v, R_v and of a given set $(D_1, \ldots, D_l, \mathrm{ad}_{x_1}, \ldots, \mathrm{ad}_{x_n})$ of commuting derivations leaving V invariant:

$$[D_j, D_k] = 0 \ , \qquad \mathrm{ad}_{x_j} D_k = D_k \mathrm{ad}_{x_j} \ , \qquad [x_j, x_k] = 0 \ , \tag{5.15}$$

$$D_j(V) \subset V \ , \qquad \mathrm{ad}_{x_j}(V) \subset V \ . \tag{5.16}$$

The additional term R_x in (5.13) (where x may be anyone of the constants $x_1, \ldots, x_n$) is the troubling term. Due to its presence we must require condition (5.1) on V, preventing the reduction of N over finite-dimensional submanifolds. Instead we shall now show that it does not forbid standard reductions of the Lenard bicomplex.

Proposition 5.1. *(Standard reduction of algebraic Lenard bicomplexes). Let M be an algebraic $\mathfrak{g}N$ manifold modelled on an associative algebra A with unit. Assume that:*

(i) *M is an affine hyperplane of equation $M = V + \{c\}$, where V is any subalgebra of A fulfilling both conditions* (5.1), *with respect to x, and* (5.16),

(ii) N and X have the form (5.13) *and* (5.14),

(iii) $\mathfrak{g}$ is an associative algebra with unit,

(iv) x belongs to $\mathfrak{g}$.

Moreover, let Q be any subalgebra of V fulfilling (5.16) *but not* (5.1). *Then the vector fields*

$$\varphi^k := \sum_{j=0}^{k} (-1)^j \binom{k}{j} (N^{k-j} \cdot X \cdot R_x^j) \cdot 1 \tag{5.17}$$

commute in pairs and are tangent to the affine hyperplane $S = Q + \{c\}$ modelled on Q.

To prove this statement, we use the fact that both M and $\mathfrak{g}$ are submanifolds of the ambient space A to write the bicomplex (5.6) in the equivalent form

$$\varphi(\lambda) = (\exp \lambda N \cdot \exp - \lambda R_x) \cdot (\exp \lambda R_x \cdot X \cdot \exp - \lambda R_x) \cdot 1 \ . \tag{5.18}$$

By setting

$$N(\lambda) := \exp \lambda R_x \cdot N \cdot \exp - \lambda R_x = \mathrm{Ad}_{\exp \lambda R_x} N \ , \tag{5.19}$$

$$X(\lambda) := \exp \lambda R_x \cdot X \cdot \exp - \lambda R_x = \mathrm{Ad}_{\exp \lambda R_x} X \ , \tag{5.20}$$

and deriving (5.17) k times with respect to λ one easily obtains:

$$\frac{d^k \varphi(\lambda)}{d\lambda^k} = (\exp \lambda N \cdot \exp - \lambda R_x) \cdot \left(N(\lambda) - R_x + \frac{d}{d\lambda} \right)^k \cdot X(\lambda) \cdot 1 \ . \tag{5.21}$$

So, one has the basic relation

$$\varphi^k = \left(N(\lambda) - R_x + \frac{d}{d\lambda} \right)^k \cdot X(\lambda) \cdot 1 \Big|_{\lambda = 0} \tag{5.22}$$

entailing that the fields of the hierarchy are completely determined by the linear mappings $N(\lambda)$ and $X(\lambda)$. These mappings are computed by observing that

$$[R_x, L_v] = 0 \ , \quad [R_x, R_v] = -R_{[x,v]} \ ,$$
$$[R_x, D_k] = 0 \ , \quad [R_x, \mathrm{ad}_{x_j}] = 0 \ , \tag{5.23}$$

and consequently that

$$\mathrm{Ad}_{\exp \lambda R_x} L_v = L_v \ , \quad \mathrm{Ad}_{\exp \lambda R_x} R_v = R_{v(\lambda)} \ ,$$
$$\mathrm{Ad}_{\exp \lambda R_x} D_i = D_i \ , \quad \mathrm{Ad}_{\exp \lambda R_x} \mathrm{ad}_{x_j} = \mathrm{ad}_{x_j} \ , \tag{5.24}$$

where

$$v(\lambda) := \exp(-\lambda \mathrm{ad}_x) \cdot v \tag{5.25}$$

is the orbit of v for the flow generated by ad_x. This orbit clearly belongs to Q on account of the assumption (5.16). Thus

$$\begin{aligned} N(\lambda) &= \mathrm{Ad}_{\exp \lambda R_x}(p(L_v, R_v, D_k, \mathrm{ad}_{x_j}) + R_x) \\ &= p(\mathrm{Ad}_{\exp \lambda R_x} L_v, \ldots, \mathrm{Ad}_{\exp \lambda R_x} \mathrm{ad}_{x_j}) + R_x \\ &= p(L_v, R_{v(\lambda)}, D_k, \mathrm{ad}_{x_j}) + R_x \ , \end{aligned} \tag{5.26}$$

and similarly

$$X(\lambda) = q(L_v, R_{v(\lambda)}, D_k, \mathrm{ad}_{x_j}) \ , \tag{5.27}$$

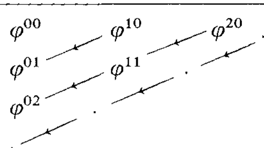
so that $N(\lambda)$ and $X(\lambda)$ are obtained from N and X simply by replacing R_v by $R_{v(\lambda)}$. They are *bilocal* operators, because of the simultaneous dependence on v and $v(\lambda)$. Finally, the Proposition (5.1) is proved by observing that $\left(N(\lambda) - R_x + \frac{d}{d\lambda}\right)$ and $X(\lambda)$ leave Q invariant on account of condition (5.16).

$\varphi^0 = \varphi^{00}$

$\varphi^1 = \varphi^{10} - \varphi^{01}$

$\varphi^2 = \varphi^{20} - 2\varphi^{11} + \varphi^{02}$

. . . .



KP

$$\begin{aligned} \varphi^0 &= 0 \\ \varphi^1 &= -q_x \\ \varphi^2 &= -q_y \\ 4\varphi^3 &= -(q_{xxx} + 3\{q, q_x\} + 3D^{-1}q_{yy} + 3[q, D^{-1}q_y]) \\ 2\varphi^4 &= -(q_{xxy} + \{q_x, D^{-1}q_y\} + 2\{q, q_y\} + D^{-2}q_{yyy} \\ &\quad + [q, D^{-2}q_{yy} + D^{-1}[q, D^{-1}q_y]] + D^{-1}[q, D^{-1}q_y]_y) \end{aligned}$$

KP I

$$\begin{aligned} \varphi^0 &= 0 \\ \varphi^1 &= -q_y \\ 2\varphi^2 &= q_{xxy} - \{q_x, D^{-1}q_y\} - 2\{q, q_y\} + D^{-2}q_{yyy} + \\ &\quad + [q, D^{-2}q_{yy} - D^{-1}[D^{-1}q_y, q]] + D^{-1}[q, D^{-1}q_y]_y \end{aligned}$$

KP II

$$\begin{aligned} \varphi^0 &= 2q_x \\ 2\varphi^1 &= -(q_{xxx} - 3\{q, q_x\} + 3D^{-1}q_{yy} + 3[q, D^{-1}q_y]) \end{aligned}$$

Fig. 2. The standard reductions

Possible variants and generalizations of this reduction scheme will not be considered in this paper. A suitable geometrical interpretation of bilocal operators $N(\lambda)$ and $X(\lambda)$ is suggested in [6].

We end this section by applying the previous result to the Lenard bicomplexes defined in Sect. 3. One easily gets the schemes of Fig. 2, showing that KP is equivalent to KPI + KPII if $D=[x,\cdot]$ and $c=y$ in the last two bicomplexes. By the first scheme, the whole KP hierarchy is obtained with a fixed step, starting from a unique initial symmetry.

6. KP Systems

We consider a few *realizations* of the previous abstract schemes, in order to obtain from a unified point of view the explicit form of KdV and KP hierarchies known from the literature. As it has been already remarked, the present approach clearly shows the unity of these two theories, corresponding to two different realizations of the *same* abstract scheme over different base algebras.

Example 1 (KdV hierarchy). Let $\mathscr{A}[z]$ be the algebra of polynomials in $z\in\mathbb{C}$, with coefficients in an associative algebra $\mathscr{A}$ with unit (e.g., $\mathscr{A}=\mathrm{Mat}_n(\mathbb{C})$ gives rise to the non-Abelian KdV hierarchy [7], $\mathscr{A}=\mathbb{C}$ to the usual KdV theory). In this example, the following choices are made:

(i) A is the current algebra [8] of C^∞ functions on $\mathbb{R}$ taking their values in $\mathscr{A}[z]$.

(ii) $D=\partial/\partial x$ is the usual partial derivative with respect to x, so that K is given by the polynomials with constant coefficients; in particular, we set $c=z$.

(iii) V is the subalgebra of polynomial functions rapidly vanishing for $|x|\to\infty$; it fulfils the conditions (5.1) and (5.16) with respect to D and $x_1=c$.

(iv) M is the affine hyperplane with equation

$$u(x,z)=\sum_{k\geq 0} v_k(x)z^k+z\ , \tag{6.1}$$

where v_k are rapidly vanishing functions for $|x|\to\infty$. Since $c=z$ commute with any polynomial, KPI is clearly trivial, whereas KPII takes the form shown in Fig. 3a. The subalgebra Q of the polynomials with degree zero is the only subalgebra of V with a finite number of field functions fulfilling (5.16). The standard reduction on the affine hyperplane $S=\{u\in M: u=q+z\}$ gives the vector fields of the non-Abelian KdV chain, e.g.

$$\varphi^0=\varphi^{00}=2q_x \qquad \varphi^1=\varphi^{10}-\varphi^{01}=\tfrac{1}{2}(-q_{xxx}+3\{q,q_x\})\ , \tag{6.2}$$

and so on [7].

*Example 2 (*KP *Hierarchy).* The algebra of $\mathscr{A}$-valued polynomial functions on $\mathbb{R}$ is now replaced by the algebra A of differential operators

$$u=\sum_{k\geq 0} u_k(x,y)\,\frac{\partial^k}{\partial y^k} \tag{6.3}$$

whose coefficients are $\mathscr{A}$-valued C^∞ functions on $\mathbb{R}^2$. D is still the partial derivative with respect to x and $c=\partial_y$. V is the subalgebra of differential operators with rapidly

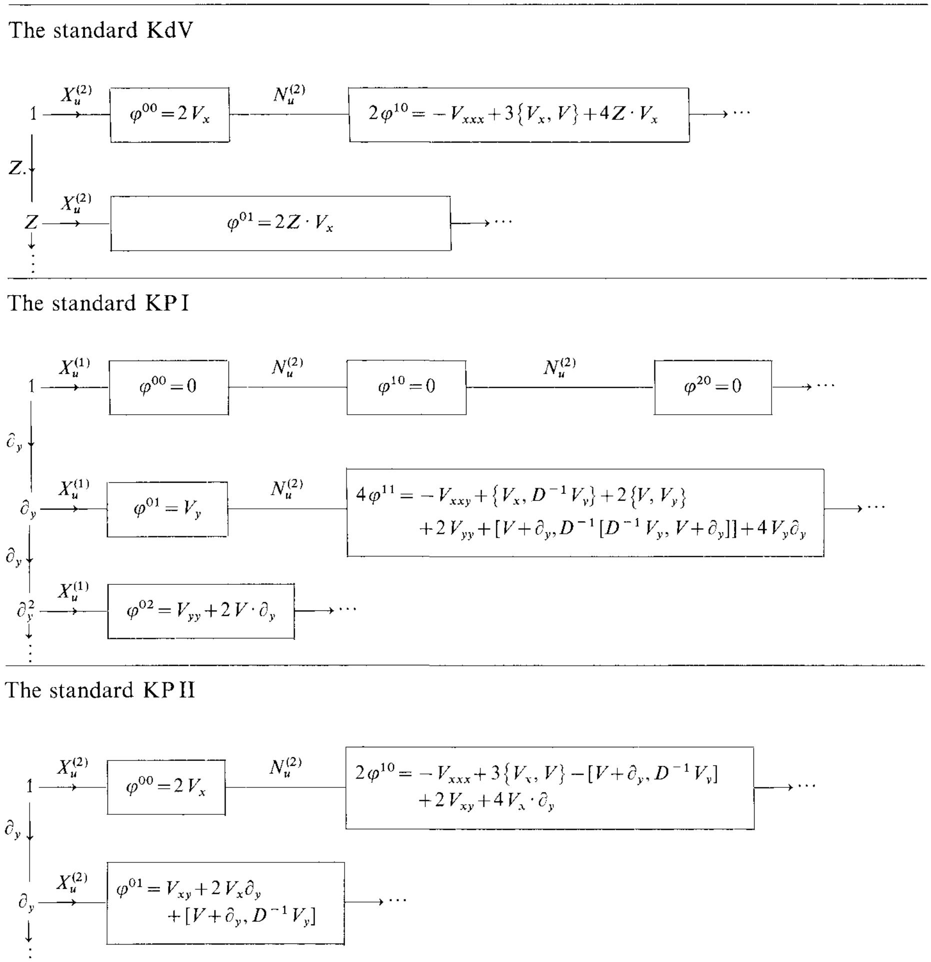


Fig. 3. The standard realization

vanishing coefficients for $x^2+y^2\to\infty$, and Q is the subalgebra of zero order operators (the choice $c=\partial_y$ is essential to make Q fulfil condition (5.16)). Since c does not commute with u, both bicomplexes KPI and KPII must be considered. By using the notations $u_y:=[c,u]$, $u_x:=Du$, one obtains for KPI and KPII the schemes of Fig. 3b and c. The standard reduction on Q gives the usual *(non-Abelian)* KP chains. The first equations are obtained from the schemes of Fig. 2b–c by setting $D=\frac{\partial}{\partial x}$ and by restricting q to be a function $q(x,y,t)$.

Example 3 (Unified KP *Hierarchy)*. Let A be the algebra of $\mathscr{A}$-valued differential operators in $\mathbb{R}^2$,

$$u=\sum_{k,j\geq 0} u_{jk}(x,y)\,\frac{\partial^j}{\partial x^j}\,\frac{\partial^k}{\partial y^k}\,, \tag{6.4}$$

V the algebra of rapidly decreasing differential operators, Q the subalgebra of zero-order operators, $x=\partial/\partial x$ and $y=\partial/\partial y$. Clearly, all the conditions (gN1) – (gN4) of Sect. 3 are fulfilled, so that the abstract KP bicomplex of Fig. 1a can be realized in this algebra. The first equations of the standard reduction to Q are obtained from the scheme of Fig. 2a by setting $D=\frac{\partial}{\partial x}$ and $q=q(x,y,t)$. One still recovers the whole KP hierarchy (KPI being the even component and KPII the odd one). This is a first example of two different bicomplexes, defined on different algebras, whose standard reduction over different submanifolds coincide. (As far as we know, the idea of using the algebra (6.4) is due to Bruschi [13], see also [9, 10]).

Example 4 (Integral Representation of KP *Hierarchies).* In this last example, we consider the algebra of integral operators

$$(uf)(x,y_1):=\int_R u(x,y_1,y_2)f(x,y_2)dy_2 \tag{6.5}$$

with the product

$$(u^*v)(x,y_1,y_2):=\int_R u(x,y_1,y_3)v(x,y_3,y_2)dy_3 \tag{6.6}$$

and the usual derivation $D=\partial_x$.

In this case:

i) V is the subalgebra of the kernels

$$v(x,y_1,y_2)=\sum_{k\geq 0} v_k(x,y_1,y_2)\delta^{(k)}(y_1-y_2) \ , \tag{6.7}$$

where $\delta^{(0)}(y_1-y_2)$ is the Dirac distribution, $\delta^{(k)}(y_1-y_2):=\frac{\partial^k}{\partial y_1^k}\delta^{(0)}(y_1-y_2)$ and v_k are rapidly vanishing functions.

ii) c is given by $c=\delta^{(1)}(y_1-y_2)$ and M is the affine hyperplane $u_{12}=\sum_{k\geq 0} v_{12k}\delta_{12}^{(k)}+\delta_{12}^{(1)}$ (the two indices notation is used for shortness)

iii) Q is the subalgebra of the kernels $v_{12}=q_1\delta_{12}^{(0)}$.

Consequently, on M the Nijenhuis tensor N (4.19) takes the form

$$\begin{aligned}4(N_u\varphi)_{12}= &-D^2\varphi_{12}+\int_R (Du_{13}\cdot D^{-1}\varphi_{32}+D^{-1}\varphi_{13}\cdot Du_{32})dy_3\\ &+2\int_R (u_{13}\cdot\varphi_{32}+\varphi_{13}\cdot u_{32})dy_3\\ &+\int_R u_{13}\left(D^{-1}\int_R (D^{-1}\varphi_{34}\cdot u_{42}-u_{34}\cdot D^{-1}\varphi_{42})dy_4\right)dy_3\\ &-\int_R D^{-1}\left(\int_R (D^{-1}\varphi_{14}\cdot u_{43}-u_{14}\cdot D^{-1}\varphi_{43})dy_4\right)\cdot u_{32}dy_3 \ .\end{aligned} \tag{6.8}$$

Its evaluation at the points of the submanifolds $S=Q+\{c\}$ is given by

$$4N_{12}=-D^2+Dq_{12}^{+}D^{-1}+2q_{12}^{+}-q_{12}^{-}D^{-1}q_{12}^{-}D^{-1} \ , \tag{6.9}$$

where

$$q_{12}^{\pm}=q_1 \pm q_2+(D_1 \mp D_2) \ , \qquad D_i=\frac{\partial}{\partial y_i} \ (i=1,2) \ . \tag{6.10}$$

This is exactly the recursion operator discovered by Fokas and Santini [3].

The infinitesimal actions (4.21) and (3.11), evaluated at S, are given by

$$X_a^{(1)}=q_{12}^{-} \ . \qquad X_a^{(2)}=(Dq_{12}^{+}-q_{12}^{-}D^{-1}q_{12}^{-}) \ . \tag{6.11}$$

So, we could compute the bicomplexes KPI and KPII by taking the right translation by $\delta_{12}^{(1)}$ as Nijenhuis tensor Δ_e on the symmetry algebra and $a=\delta_{12}^{(0)}$ (the unit of A) as the starting symmetry. We would obtain the KP hierarchy in the bilocal formalism of Fokas and Santini.

7. The Relation with the Bilocal Approach

In this section we explicitly compute the bilocal operators $N(\lambda)$ and $X(\lambda)$ entering the standard reduction, for every realization of the algebra A considered in Sect. 6. This will enable us to point out the strict link between the bicomplex approach and that based on bilocal operators introduced by Konopelchenko [1].

Example 1 (KdV *Theory).* Since $c=z$ commutes with every polynomial, the adjoint action of c is trivial, and the operators N and X are invariant, so that

$$\varphi^n:=\left(N(\lambda)-R_c+\frac{d}{d\lambda}\right)^n \cdot X(\lambda)\Big|_{\lambda=0} \cdot 1=\frac{1}{4^n}\,(-D^2\cdot+\{q_x,D^{-1}\cdot\}+2\{q,\cdot\}$$
$$+[q,D^{-1}[D^{-1}\cdot,q]])^n\cdot X\cdot 1 \ . \tag{7.1}$$

In this case, the standard reduction coincides with the usual Lenard chain [7].

Example 2 (KP *Chain).* In this case the adjoint action of $c=\partial_y$ is given by

$$\mathrm{ad}_c^k v=\frac{\partial^k v}{\partial y^k} \ , \tag{7.2}$$

so that $v(\lambda)=v(y-\lambda)$. By using the notations

$$[v,\varphi]_\lambda=v\cdot\varphi-\varphi\cdot v(y-\lambda) \quad \{v,\varphi\}_\lambda=v\cdot\varphi+\varphi\cdot v(y-\lambda) \ , \tag{7.3}$$

one easily obtains, at any $v\in V$

$$4\left(N_\lambda-R_c+\frac{d}{d\lambda}\right)\varphi=-D^2\varphi+2\{u,\varphi\}_\lambda+2\varphi_y+\{u_x,D^{-1}\varphi\}_\lambda+$$
$$-[v+\partial_y,D^{-1}[v+\partial_y,D^{-1}\varphi]_\lambda]_\lambda+4\frac{d}{d\lambda}\varphi \ , \tag{7.4}$$

$$X_a^{(1)}(\lambda)=[v+\partial_y,a]_\lambda \quad X_a^{(2)}(\lambda)=\{Dv,a\}_\lambda+[v+\partial_y,D^{-1}[a,v+\partial_y]_\lambda]_\lambda \ . \tag{7.5}$$

This is exactly the formulation of the KP theory in bilocal form. Indeed, let us put $y_1=y$, $y_2=y-\lambda$, $v=q$, $\varphi(y,\lambda)=\tilde{\varphi}(y_1,y_2)$. Then

$$\left(N(\lambda)-R_{\partial_y}+\frac{d}{d\lambda}\right)\cdot\varphi(y,\lambda)=N_{12}\tilde{\varphi}(y_1,y_2) \tag{7.6}$$

and

$$X(\lambda)\cdot\varphi(y,\lambda)=X_{12}\cdot\tilde{\varphi}(y_1,y_2)\ , \tag{7.7}$$

where N_{12} and X_{12} are the operators (6.9)–(6.11) giving the realizations of N and X on the algebra of the integral operators. Consequently, the basic formula (5.22) defining the standard reduction

$$\varphi^k=\left(N(\lambda)-R_{\partial_y}+\frac{d}{d\lambda}\right)^k\cdot X(\lambda)\cdot 1\Big|_{\lambda=0} \tag{7.8}$$

becomes

$$\varphi^k=(N_{12})^k\cdot X_{12}\cdot 1|_{y_1=y_2}\ . \tag{7.9}$$

This is the "fixed-step" recursion formula for KP hierarchy first discovered by Fokas and Santini [3].

8. Concluding Remarks

With this paper we hope to have clarified the role of the method of recursion operators (here called Nijenhuis tensors) in the theory of Hamiltonian equations solvable by the inverse scattering technique. The apparent drawback of this method in 2+1 dimensions [1, 2] is due rather to an incomplete understanding of the symmetry algebra than to its intrinsic limitations. A systematic use of recursion relations also in the symmetry algebra (leading to what we have called a $\mathfrak{g}N$ manifold) allows us to overcome these difficulties, and to deal with systems like KP, Davey-Stewartson and so on with an unexpected ease. Moreover, we believe that the technique of algebraic $\mathfrak{g}N$ manifolds leads to a gain in clarity and unity, throwing some light, for example, in the appearance of bilocal operators. Naturally related to this subject is the problem of considering more general deformations than the adjoint action by a constant element. On the basis of some interesting results of Fokas and Santini [12] (whose deep meaning must yet be fully understood) it seems reasonable to arrive in this way to a better understanding of Bäcklund transformations. We hope to be able to do that in a forthcoming paper.

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