# NILPOTENT MATRICES WITH INVERTIBLE TRANSPOSE[1]

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**1. Introduction.** Let $R$ be an associative ring with identity. Denote by $R_0$, the anti-isomorphic ring to $R$, obtained by defining a new composition $a \circ b = b \cdot a$, in $R$. For any matrix $A$ over $R$, we denote by $T(A)$, the transpose of $A$ regarded as a matrix over $R_0$. One checks easily that $T(AB) = T(B)T(A)$ for any two matrices $A$, $B$ over $R$ suitably sized for multiplication. From here it is easy to conclude that a matrix $A$ over $R$ is invertible iff $T(A)$ is invertible over $R_0$. (An $m \times n$ matrix $A$ is said to be invertible if $\exists$ a matrix $B$ such that $AB = I_m$, $BA = I_n$.)

For a square matrix $A$ over $R$, $T(A^s) = (T(A))^s$, where $s$ is any positive integer. From this last relation we conclude that $A$ is nilpotent of index $k$ iff $T(A)$ is nilpotent of index $k$. If $R$ is commutative then $R_0 = R$. $A$ is invertible over $R$ iff $T(A)$ is invertible over $R$. A matrix $A$ over $R$ is nilpotent of index $k$ iff $T(A)$ is nilpotent of index $k$, over $R$.

We return to the case when $R$ is arbitrary. Since we shall have no further occasion to refer to $R_0$, we shall denote the transpose of $A$ by $A^t$. The mapping $t$ which takes a matrix over $R$ to its transpose over $R$ does not even possess the property $(A^2)^t = (A^t)^2$. Therefore one can easily construct matrices $A$ such that $A^2 = 0$ and $(A^t)^2 \neq 0$ (see Example 2.3). In fact $(A^2)^t = (A^t)^2$ implies the commutativity of $R$. This fact is clear by putting

$$A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Also it is well known that the transpose of an invertible matrix over a division ring is not necessarily invertible [**2**, p. 24, Exercise 3].

We show here that over a division ring $D$, which is not commutative, there exists a $2 \times 2$ matrix $A$ which is nilpotent and whose transpose is invertible. This is equivalent to the existence of a $2 \times 2$ invertible matrix over $D$ whose transpose is nilpotent. The following two results, which have some independent interest are observed as a part of the proof. A division ring in the multiplicative group of which any two conjugate elements commute is commutative. A division ring satisfying the polynomial identity $xy^2x = yx^2y$ is commutative.

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2.1 LEMMA. *$A$ is a $2\times 2$ nilpotent matrix over a division ring $D$ if and only if*

$$A=\begin{bmatrix}0 & 0\\ a & 0\end{bmatrix} \quad \textit{or} \quad A=\begin{bmatrix}-yz & y\\ -zyz & zy\end{bmatrix}$$

*for some $a$, $y$, $z$ in $D$.*

PROOF. Assume $A$ is a $2\times 2$ nilpotent matrix. Let

$$V=\{(d_1,d_2)\,|\,d_1,d_2\in D\}.$$

Either $A=0$ or the properly descending chain $V\supset VA\supset VA^2$ of subspaces of ${}_DV$, gives $VA^2=0$, therefore $A^2=0$. Now if $(1,0)A=(0,0)$. Suppose $(0,1)A=(a,b)$, then $(0,0)=(0,1)A^2=(0,1)A\cdot A=(a,b)A=(a(1,0)+b(0,1))A=b(a,b)=(ba,b^2)\Rightarrow b^2=0\Rightarrow b=0$. So that

$$A=\begin{bmatrix}(1,0)A\\ (0,1)A\end{bmatrix}=\begin{bmatrix}0 & 0\\ a & 0\end{bmatrix}.$$

If $(1,0)A=(x,y)\neq(0,0)$, then $(0,1)A=(zx,zy)$ as ${}_D(VA)$ is of dimension one. Now

$$\begin{aligned}(0,0) &= (1,0)A^2=(1,0)A\cdot A=(x,y)A=(x(1,0)+y(0,1))A\\ &= x(x,y)+y(zx,zy)=(x+yz)(x,y),\end{aligned}$$

which implies $x+yz=0$, therefore $x=-yz$. Therefore

$$A=\begin{bmatrix}(1,0)A\\ (0,1)A\end{bmatrix}=\begin{bmatrix}x, y\\ zx, zy\end{bmatrix}=\begin{bmatrix}-yz, y\\ -zyz, zy\end{bmatrix}.$$

Converse is trivial.

2.2 LEMMA. *If*

$$A=\begin{bmatrix}-yz & y\\ -zyz & zy\end{bmatrix} y\neq 0, \qquad z\neq 0, \quad y,z\in D$$

*then $A^t$ is singular (noninvertible) if and only if $(y^{-1}zy)z=z(y^{-1}zy)$.*

PROOF.

$$A^t=\begin{bmatrix}-yz & -zyz\\ y & zy\end{bmatrix}$$

is singular iff its rows are left linearly dependent over $D$. Now $(-yz, -zyz)$, $(y, zy)$ are left linearly dependent iff $(-yz, -zyz)$, $(1, y^{-1}zy)$ are linearly dependent iff $-zyz=(-yz)(y^{-1}zy)$ iff $(y^{-1}zy)z=z(y^{-1}zy)$.

2.3 EXAMPLE. $D$ be a division ring of quaternions over any field $F$, in which $a^2+b^2+c^2+d^2=0$ implies $a=0$, $b=0$, $c=0$, $d=0$. Let $y=i$, $z=(1+i+j+k)$ then one checks that

$$(y^{-1}zy)z = 2(1 + i - j + k)$$

and

$$z(y^{-1}zy) = 2(1 + i + j - k).$$

Therefore $(y^{-1}zy)z\neq z(y^{-1}zy)$. Consequently

$$A = \begin{bmatrix} -yz & y \\ -zyz & zy \end{bmatrix} = \begin{bmatrix} 1 - i + j - k & i \\ 2(1 - i + j + k) & -1 + i + j - k \end{bmatrix}$$

is such that $A^2=0$ and $A^t$ is invertible.

2.4 LEMMA. *In a group $G$ $(y^{-1}zy)z=z(y^{-1}zy)$ $\forall y, z$ in $G$ if and only if $xy^2x=yx^2y$ $\forall x, y$ in $G$.*

PROOF. Assume $(y^{-1}zy)z=z(y^{-1}zy)$. Let $a, b\in G$. Putting $y=a$, $z=ab$, we get $a^{-1}abaab=aba^{-1}aba$. Therefore $ba^2b=ab^2a$. Assume $ab^2a=ba^2b$ $\forall a, b$ in $G$. Putting $a=y$, $b=y^{-1}z$, we get $yy^{-1}zy^{-1}zy=y^{-1}zy^2y^{-1}z$. Therefore $z(y^{-1}zy)=(y^{-1}zy)z$.

2.5 THEOREM. *If $D$ is a division ring such that the transpose of every $2\times 2$ nilpotent matrix over $D$ is singular (noninvertible) then $D$ is commutative.*

PROOF. In view of Lemmas 2.1 and 2.2 it follows that $D$ satisfies the hypothesis if and only if $(y^{-1}zy)z=z(y^{-1}zy)$ $\forall y, z$, $y\neq 0$, $z\neq 0$, in $D$. In view of Lemma 2.4 this condition is satisfied iff $ab^2a=ba^2b$ $\forall a, b$ in $D$. Therefore $D$ satisfies a polynomial identity of degree 4 over its center. Let $C$ be the center of $D$. $(D:C)\leqq 4$ [1, Theorem 1 p. 226]. If $D\neq C$, then $(D:C)=4$ [1, Proposition 1, p. 180]. Let $a$ be an element of $D$ outside the center $C$. Then $C(a)$, the subfield generated by $C$ and $a$ does not coincide with $D$ because $C(a)$ is commutative. Also $(D:C)=(D:C(a))_L\cdot(C(a):C)$ [1, Proposition 1, p. 157]. Consequently $(C(a):C)=2$. Therefore $C(a)=C\oplus Ca$. By Cartan-Brauer-Hua Theorem [1, p. 186, Corollary], there exists $x\in D$ such that $x^{-1}ax\notin C(a)$. Let $b=x^{-1}ax$. Clearly $ab=ba$. Let $D_1=C\oplus Ca\oplus Cb$. Now $ab\notin D_1$ because if $ab\in D_1$, then the additive subgroup $D_1$ is closed under multiplication and therefore $D_1$ is a division subring [**1**, p. 158, Proposition 2] of $D$ such that $(D_1:C)=3$. This is impossible because $(D:C)=(D:D_1)_L\cdot(D_1:C)$ [**1**, Proposition 1, p. 157]. Therefore $ab\notin D_1$ and hence $D=C\oplus Ca\oplus Cb\oplus Cab$. But now in view of $ab=ba$, we note any two element of the basis $\{1, a, b, ab\}$ of $D$ over $C$ commute. Therefore $D$ is commutative.

2.6 COROLLARY. *If $D$ is a division ring such that the transpose of every* $2\times 2$ *nilpotent matrix over $D$ is nilpotent, then $D$ is commutative.*

2.7 COROLLARY. *If the transpose of every* $2\times 2$ *invertible matrix over $D$ is nonnilpotent, then $D$ is commutative.*

I am grateful to Professor E. G. Straus for pointing out to me the following result which in conjunction with Lemmas 2.1 and 2.2 provides an explicit construction for the matrices mentioned in the title, and therefore an alternative proof of 2.5.

THEOREM (E. G. STRAUS). *If $x$, $y$ be two noncommuting elements of a division ring $D$, then $\exists$ at most one element $z$ in the coset $y+C_x$ such that $z^{-1}xz\in C_x$, where $C_x=\{c\in D: cx=xc\}$.*

PROOF. Let if possible $z$, $z'$ be distinct elements of $y+C_x$ such that $z^{-1}xz$ and $z'^{-1}xz'\in C_x$, $z'=z+c$, $0\neq c\in C_x$. Set $z^{-1}xz=x_2$. Clearly $x_1\neq x$, $xz=zx_1$. Now

$$\begin{aligned} xz' &= x(c+z) = xc+xz = cx+zx_1 \\ &= c(x-x_1)+(c+z)x_1 = c(x-x_1)+z'x_1. \end{aligned}$$

Consequently $z'^{-1}xz'=z'^{-1}c(x-x_1)+x_1$. As $z'^{-1}xz'$, $c(x-x_1)$ and $x_1\in C_x$, and $c(x-x_1)\neq 0$, it follows that $z'^{-1}\in C_x$. Therefore $z'\in C_x$, which is impossible.

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