NLEVP: A Collection of Nonlinear Eigenvalue Problems

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We present a collection of 52 nonlinear eigenvalue problems in the form of a MATLAB toolbox. The collection contains problems from models of real-life applications as well as ones constructed specifically to have particular properties. A classification is given of polynomial eigenvalue problems according to their structural properties. Identifiers based on these and other properties can be used to extract particular types of problems from the collection. A brief description of each problem is given. NLEVP serves both to illustrate the tremendous variety of applications of nonlinear eigenvalue problems and to provide representative problems for testing, tuning, and benchmarking of algorithms and codes.

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1. INTRODUCTION

In many areas of scientific computing collections of problems are available that play an important role in developing algorithms and in testing and benchmarking software. Among the uses of such collections are

— tuning an algorithm to optimize its performance across a wide and representative range of problems;
— testing the correctness of a code against some measure of success, where the latter is typically an error or residual whose nature is suggested by the underlying problem;
— measuring the performance of a code—for example, speed, execution rate, or again an error or residual;
— measuring the robustness of a code, that is, the behavior in extreme situations, such as for very badly scaled and/or ill conditioned data;
— comparing two or more codes with respect these factors.

A collection ideally combines problems artificially constructed to reflect a wide range of possible properties with problems representative of real applications. Problems for which something is known about the solution are always particularly attractive.

The practice of reproducible research, whereby research is published in such a way that the underlying numerical (and other) experiments can be repeated by others, is of growing interest and visibility [Donoho et al. 2009; LeVeque 2009; Mesirov 2010]. Reproducible research is aided by the availability of well documented and maintained benchmark collections.

Two areas that have historically been well endowed with collections of problems implemented in software are linear algebra and optimization. In linear algebra an early collection is ACM Algorithm 694 [Higham 1991], which contains parametrized, mainly dense, test matrices, most of which were later incorporated into the MATLAB `gallery` function. The University of Florida Sparse Matrix Collection is a regularly updated collection of sparse matrices [Davis; Davis and Hu 2011], with over 2200 matrices from practical applications. It includes all the matrices from the earlier Matrix Market repository (although not the matrix generators) [Matrix Market], the Harwell-Boeing collection [Duff et al. 1989] of sparse matrices, and the NEP collection [Bai et al. 1997] of standard and generalized eigenvalue problems. The CONTEST toolbox [Taylor and Higham 2009] produces adjacency matrices describing random networks. In optimization we mention just the collections in the widely used Cute and Cuter testing environments [Bongartz et al. 1995; Gould et al. 2003], though various other, sometimes more specialized, collections are available.

The growing interest in nonlinear eigenvalue problems has created the need for a collection of problems in this area. The standard form of a nonlinear eigenvalue problem is $F(\lambda)x = 0$, where $F : \mathbb{C} \to \mathbb{C}^{m\times n}$ is a given matrix-valued function and $\lambda \in \mathbb{C}$ and the nonzero vector $x \in \mathbb{C}^n$ are the sought eigenvalue and eigenvector, respectively. Rational and polynomial functions are of particular interest, the most practically important case being the quadratic $Q(\lambda) = \lambda^2 A + \lambda B + C$, which corresponds to the quadratic eigenvalue problem. For recent surveys on nonlinear eigenproblems see Mehrmann and Voss [2004] and Tisseur and Meerbergen [2001]. Associated with an $n \times n$ matrix quadratic $Q(\lambda)$ are the matrix equations $X^2A + XB + C = 0$ and $AX^2 + BX + C = 0$, where the unknown $X \in \mathbb{C}^{n\times n}$ is called a solvent [Dennis et al. 1976; Gohberg et al. 2009; Higham and Kim 2000]. Thus a matrix polynomial $P(\lambda)$ defines both an eigenvalue problem and two matrix equations.

We have built a collection of nonlinear eigenvalue problems from a variety of sources. Some are from models of real-life applications, while others have been constructed specifically to have particular properties. Many of the matrices have been used in

Table I. Problems Available in the Collection and Their Identifiers

`qep`	quadratic eigenvalue problem
`pep`	polynomial eigenvalue problem
`rep`	rational eigenvalue problem
`nep`	other nonlinear eigenvalue problem

previous papers to test numerical algorithms. In order to provide focus and keep the collection to a manageable size we have chosen to exclude linear problems from the collection. The problems range from the old, such as the `wing` problem from the classic 1938 book of Frazer, Duncan, and Collar [Frazer et al. 1938], to the very recent, notably several problems from research in 3D vision that are not yet well known in the numerical analysis community.

Nonlinear eigenvalue problems are often highly structured, and it is important to take account of the structure both in developing the theory and in designing numerical methods. We therefore provide a thorough classification of our problems that records the most relevant structural properties.

We have chosen to implement the collection in MATLAB, as a toolbox, recognizing that it is straightforward to convert the matrices into a format that can be read by other languages by using either the built-in MATLAB I/O functions or those provided in Matrix Market. Care has been taken to make the toolbox compatible with GNU Octave [GNU Octave]. A criterion for inclusion of problems is that the underlying MATLAB code and data files are not too large, since we want to provide the toolbox as a single file that can be downloaded in a reasonable time.

The NLEVP toolbox is available, as both a zip file and a tar file.[1] For details of how to install and use the toolbox see Betcke et al. [2011].

In Section 2 we explain how we classify the problems through identifiers that can be used to extract specific types of problem from the collection. The main features of the problems are described in Section 3, while Section 4 describes the design of the toolbox. Conclusions are given in Section 5.

2. IDENTIFIERS

We give in Table I a list of identifiers for the types of problems available in the collection and in Table II a list of identifiers that specify the properties of problems in the collection. These properties can be used to extract specialized subsets of the collection for use in numerical experiments. All the identifiers are case insensitive. In the next two subsections we briefly recall some relevant definitions and properties of nonlinear eigenproblems.

2.1. Nonlinear Eigenproblems

The *polynomial eigenvalue problem* (PEP) is to find scalars λ and nonzero vectors x and y satisfying $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$, where

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i, \qquad A_i \in \mathbb{C}^{m\times n}, \quad A_k \neq 0 \tag{1}$$

[1] `http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html.`

Table II. List of Identifiers for the Problem Properties

`nonregular`	`symmetric`	`hyperbolic`
`real`	`hermitian`	`elliptic`
`nonsquare`	`T-even`	`overdamped`
`sparse`	`*-even`	`proportionally-damped`
`scalable`	`T-odd`	`gyroscopic`
`parameter-dependent`	`*-odd`	
`solution`	`T-palindromic`	
`random`	`*-palindromic`	
	`T-anti-palindromic`	
	`*-anti-palindromic`	

is an $m \times n$ matrix polynomial of degree k. Here, x and y are right and left eigenvectors corresponding to the eigenvalue λ. The *reversal* of the matrix polynomial (1) is defined by

$$\mathrm{rev}\big(P(\lambda)\big) = \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^{k-i} A_i.$$

A PEP is said to have an eigenvalue ∞ if zero is an eigenvalue of $\mathrm{rev}(P(\lambda))$.

A *quadratic eigenvalue problem* (QEP) is a PEP of degree $k = 2$. For a survey of QEPs see Tisseur and Meerbergen [2001]. Polynomial and quadratic eigenproblems are identified by `pep` and `qep`, respectively, in the collection (see Table I), and any problem of type `qep` is automatically also of type `pep`.

The matrix function $R(\lambda) \in \mathbb{C}^{m\times n}$ whose elements are rational functions

$$r_{ij}(\lambda) = \frac{p_{ij}(\lambda)}{q_{ij}(\lambda)}, \qquad 1 \le i \le m, \quad 1 \le j \le n,$$

where $p_{ij}(\lambda)$ and $q_{ij}(\lambda)$ are scalar polynomials of the same variable and $q_{ij}(\lambda) \not\equiv 0$, defines a *rational eigenvalue problem* (REP) $R(\lambda)x = 0$ [Kublanovskaya 1999]. Unlike for PEPs there is no standard format for specifying REPs. For the collection we use the form

$$R(\lambda) = P(\lambda)Q(\lambda)^{-1},$$

where $P(\lambda)$ and $Q(\lambda)$ are matrix polynomials, or the less general form (often encountered in practice)

$$R(\lambda) = A + \lambda B + \sum_{i=1}^{k-1} \frac{\lambda}{\sigma_i - \lambda} C_i, \tag{2}$$

where A, B, and the C_i are $m \times n$ matrices, and the σ_i are the poles. Which form is used is specified in the help for the M-file defining the problem. Rational eigenproblems are identified by `rep` in the collection.

As mentioned in the introduction, PEPs and REPs are special cases of *nonlinear eigenvalue problems* (NEPs) $F(\lambda)x = 0$, where $F : \mathbb{C} \to \mathbb{C}^{m\times n}$. A convenient general form for expressing an NEP is

$$F(\lambda) = \sum_{i=0}^{k} f_i(\lambda) A_i, \tag{3}$$

where the $f_i : \mathbb{C} \to \mathbb{C}$ are nonlinear functions and $A_i \in \mathbb{C}^{m\times n}$. Any problem that is not polynomial, quadratic, or rational is identified by `nep` in the collection (see Table I).

Table III. Some Identifiers and the Corresponding Spectral Properties

For parameter-dependent problems, the problem is classified as `real` or `hermitian` if it is so for real values of the parameter

Identifier	Property of $F(\lambda) \in \mathbb{C}^{m\times n}$	Spectral properties
`real`	$\overline{F(\lambda)} = F(\bar\lambda)$	eigenvalues real or come in pairs $(\lambda, \bar\lambda)$
`symmetric`	$m = n,\ (F(\lambda))^T = F(\lambda)$	none unless F is real
`hermitian`	$m = n,\ (F(\lambda))^* = F(\bar\lambda)$	eigenvalues real or come in pairs $(\lambda, \bar\lambda)$

Table IV. Some Identifiers and the Corresponding Spectral Symmetry Properties

Identifier	Property of $P(\lambda)$	Eigenvalue pairing
`T-even`	$P^T(-\lambda) = P(\lambda)$	$(\lambda, -\lambda)$
`*-even`	$P^*(-\lambda) = P(\lambda)$	$(\lambda, -\bar\lambda)$
`T-odd`	$P^T(-\lambda) = -P(\lambda)$	$(\lambda, -\lambda)$
`*-odd`	$P^*(-\lambda) = -P(\lambda)$	$(\lambda, -\bar\lambda)$
`T-palindromic`	$\mathrm{rev}P^T(\lambda) = P(\lambda)$	$(\lambda, 1/\lambda)$
`*-palindromic`	$\mathrm{rev}P^*(\lambda) = P(\lambda)$	$(\lambda, 1/\bar\lambda)$
`T-anti-palindromic`	$\mathrm{rev}P^T(\lambda) = -P(\lambda)$	$(\lambda, 1/\lambda)$
`*-anti-palindromic`	$\mathrm{rev}P^*(\lambda) = -P(\lambda)$	$(\lambda, 1/\bar\lambda)$

2.2. Some Definitions and Properties

Nonlinear eigenproblems are said to be *regular* if $m = n$ and $\det(F(\lambda)) \not\equiv 0$, and *nonregular* otherwise. Recall that a regular PEP possesses nk (not necessarily distinct) eigenvalues [Gohberg et al. 2009], including infinite eigenvalues. As the majority of problems in the collection are regular we identify only nonregular problems, for which the identifier is `nonregular`.

The identifiers `real`, `hermitian`, and `symmetric` are defined in Table III. For PEPs, the `real` identifier corresponds to P having real coefficient matrices, while `hermitian` corresponds to Hermitian (but not all real) coefficient matrices. Similarly, `symmetric` indicates (complex) symmetric coefficient matrices, and the `real` identifier is added if the coefficient matrices are real symmetric. For problems that are parameter-dependent the identifiers `real` and `hermitian` are used if the problem is real or Hermitian for real values of the parameter.

Definitions of identifiers for odd-even and palindromic-like square matrix polynomials, together with the special symmetry properties of their spectra [Mackey et al. 2006] are given in Table IV.

Gyroscopic systems of the form $Q(\lambda) = \lambda^2 M + \lambda G + K$ with M, K Hermitian, $M > 0$, and $G = -G^*$ skew-Hermitian are a subset of $*$-even (T-even when the coefficient matrices are real) QEPs and are identified with `gyroscopic`. Here, for a Hermitian matrix A, we write $A > 0$ to denote that A is positive definite and $A \geq 0$ to denote that A is positive semidefinite. When $K > 0$ the eigenvalues of Q are purely imaginary and semisimple [Duffin 1960; Lancaster 1966] and the quadratic $Q(i\lambda)$ is hyperbolic.

A Hermitian matrix polynomial $P(\lambda)$ is *hyperbolic* if there exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that $P(\mu)$ is positive definite and for every nonzero $x \in \mathbb{C}^n$ the scalar equation $x^*P(\lambda)x = 0$ has k distinct zeros in $\mathbb{R} \cup \{\infty\}$. All the eigenvalues of such a P are real, semisimple, and grouped in k intervals, each of them containing n eigenvalues [Al-Ammari and Tisseur 2012; Higham et al. 2009; Markus 1988]. These polynomials are identified in the collection by `hyperbolic`. *Overdamped* systems $Q(\lambda) = \lambda^2 M + \lambda C + K$ are particular hyperbolic QEPs for which $M > 0$, $C > 0$, and $K \geq 0$; they have the identifier `overdamped`. Finally, a QEP is said to be *proportionally damped* when M, C, and K are simultaneously diagonalizable by congruence or strict equivalence [Lancaster

Table V. Quadratic Eigenvalue Problems

acoustic_wave_1d	acoustic_wave_2d	bicycle	bilby
cd_player	closed_loop	concrete	damped_beam
dirac	foundation	gen_hyper2	gen_tantipal2
gen_tpal2	intersection	hospital	metal_strip
mobile_manipulator	omnicam1	omnicam2	pdde_stability
power_plant	qep1	qep2	qep3
qep4	qep5	railtrack	railtrack2
relative_pose_6pt	schrodinger	shaft	sign1
sign2	sleeper	speaker_box	spring
spring_dashpot	surveillance	wing	wiresaw1
wiresaw2			

Table VI. Other Eigenvalue Problems

Polynomial, degree ≥ 3	butterfly	mirror	orr_sommerfeld
	planar_waveguide	plasma_drift	relative_pose_5pt
Nonsquare polynomial	qep4	surveillance	
Nonregular polynomial	qep4	qep5	surveillance
Rational	loaded_string		
Nonlinear	fiber	gun	hadeler
	time_delay		

and Zaballa 2009] (a sufficient condition for which is that $C = \alpha M + \beta K$ with M and K simultaneously diagonalizable, hence the name), and such a QEP is identified by `proportionally-damped`.

Hermitian matrix polynomials $P(\lambda)$ with even degree k that are *elliptic*, that is, $P(\lambda) > 0$ for all $\lambda \in \mathbb{R}$ [Markus 1988, Section 34], are identified by `elliptic`. Elliptic matrix polynomials have nonreal eigenvalues.

The identifier `sparse` is used if the defining matrices are stored in the MATLAB sparse format. Problems that depend on one or more parameters are identified with `parameter-dependent`. Problems for which random numbers are used in the construction are identified with `random`. A separate identifier, `scalable`, is used to denote that the problem dimension (or an approximation of it) is a parameter. For parameter-dependent problems a default value of the parameter is provided, typically being a value used in previously published experiments.

For some problems a *supposed* solution is optionally returned, comprising eigenvalues and/or eigenvectors that are exactly known, approximate, or computed. These problems are identified with `solution`. The documentation for the problem provides information on the nature of the supposed solution.

Tables V and VI identify the QEPs, the PEPs that are of degree at least 3, the nonsquare PEPs, the REPs, and the nonlinear but nonpolynomial and nonrational problems in the collection.

3. COLLECTION OF PROBLEMS

This section contains a brief description of all the problems in the collection. The identifiers for the problem properties are listed inside curly brackets after the name of each problem. The problems are summarized in Table VII.

We use the following notation. $A \otimes B$ denotes the Kronecker product of A and B, namely the block matrix $(a_{ij}B)$ [Higham 2008, Sec. B.13]. The ith unit vector (that is, the ith column of the identity matrix) is denoted by e_i.

Table VII. Problems in NLEVP

acoustic_wave_1d	QEP from acoustic wave problem in 1 dimension.
acoustic_wave_2d	QEP from acoustic wave problem in 2 dimensions.
bicycle	2-by-2 QEP from the Whipple bicycle model.
bilby	5-by-5 QEP from bilby population model.
butterfly	Quartic matrix polynomial with T-even structure.
cd_player	QEP from model of CD player.
closed_loop	2-by-2 QEP associated with closed-loop control system.
concrete	Sparse QEP from model of a concrete structure.
damped_beam	QEP from simply supported beam damped in the middle.
dirac	QEP from Dirac operator.
fiber	NEP from fiber optic design.
foundation	Sparse QEP from model of machine foundations.
gen_hyper2	Hyperbolic QEP constructed from prescribed eigenpairs.
gen_tantipal2	T-anti-palindromic QEP with eigenvalues on the unit circle.
gen_tpal2	T-palindromic QEP with prescribed eigenvalues on the unit circle.
gun	NEP from model of a radio-frequency gun cavity.
hadeler	NEP due to Hadeler.
intersection	10-by-10 QEP from intersection of three surfaces.
hospital	QEP from model of Los Angeles Hospital building.
loaded_string	REP from finite element model of a loaded vibrating string.
metal_strip	QEP related to stability of electronic model of metal strip.
mirror	Quartic PEP from calibration of cadioptric vision system.
mobile_manipulator	QEP from model of 2-dimensional 3-link mobile manipulator.
omnicam1	9-by-9 QEP from model of omnidirectional camera.
omnicam2	15-by-15 QEP from model of omnidirectional camera.
orr_sommerfeld	Quartic PEP arising from Orr-Sommerfeld equation.
pdde_stability	QEP from stability analysis of discretized PDDE.
planar_waveguide	Quartic PEP from planar waveguide.
plasma_drift	Cubic PEP arising in Tokamak reactor design.
power_plant	8-by-8 QEP from simplified nuclear power plant problem.
qep1	3-by-3 QEP with known eigensystem.
qep2	3-by-3 QEP with known, nontrivial Jordan structure.
qep3	3-by-3 parametrized QEP with known eigensystem.
qep4	3-by-4 QEP with known, nontrivial Jordan structure.
qep5	3-by-3 nonregular QEP with known Smith form.
railtrack	QEP from study of vibration of rail tracks.
railtrack2	Palindromic QEP from model of rail tracks.
relative_pose_5pt	Cubic PEP from relative pose problem in computer vision.
relative_pose_6pt	QEP from relative pose problem in computer vision.
schrodinger	QEP from Schrodinger operator.
shaft	QEP from model of a shaft on bearing supports with a damper.
sign1	QEP from rank-1 perturbation of sign operator.
sign2	QEP from rank-1 perturbation of 2*sin(x) + sign(x) operator.
sleeper	QEP modelling a railtrack resting on sleepers.
speaker_box	QEP from model of a speaker box.
spring	QEP from finite element model of damped mass-spring system.
spring_dashpot	QEP from model of spring/dashpot configuration.
surveillance	21-by-16 QEP from surveillance camera callibration.
time_delay	3-by-3 NEP from time-delay system.
wing	3-by-3 QEP from analysis of oscillations of a wing in an airstream.
wiresaw1	Gyroscopic QEP from vibration analysis of a wiresaw.
wiresaw2	QEP from vibration analysis of wiresaw with viscous damping.

`acoustic_wave_1d` {pep, qep, symmetric, *-even, parameter-dependent, sparse, scalable}. This quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$ arises from the finite element discretization of the time-harmonic wave equation $-\Delta p - \left(2\pi f/c\right)^2 p = 0$ for the acoustic pressure p in a bounded domain, where the boundary conditions are partly Dirichlet ($p = 0$) and partly impedance $\left(\frac{\partial p}{\partial n} + \frac{2\pi i f}{\zeta} p = 0\right)$ [Chaitin-Chatelin and van Gijzen 2006]. Here, f is the frequency, c is the speed of sound in the medium, and ζ is the (possibly complex) impedance. We take $c = 1$ as in Chaitin-Chatelin and van Gijzen [2006]. The eigenvalues of Q are the resonant frequencies of the system, and for the given problem formulation they lie in the upper half of the complex plane. For more on the discretization of acoustics problems see, for example, Harari et al. [1996].

On the 1D domain $[0, 1]$ the $n \times n$ matrices are defined by

$$M = -4\pi^2 \frac{1}{n}\left(I_n - \frac{1}{2} e_n e_n^T\right), \quad C = 2\pi i \frac{1}{\zeta}\, e_n e_n^T, \quad K = n \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}.$$

`acoustic_wave_2d` {pep, qep, symmetric, *-even, parameter-dependent, sparse, scalable}. A 2D version of Acoustic wave 1D. On the unit square $[0, 1] \times [0, 1]$ with mesh size h the $n \times n$ coefficient matrices of $Q(\lambda)$ with $n = \frac{1}{h}\left(\frac{1}{h} - 1\right)$ are given by

$$M = -4\pi^2 h^2 I_{m-1} \otimes \left(I_m - \tfrac{1}{2} e_m e_m^T\right), \quad D = 2\pi i \tfrac{h}{\zeta} I_{m-1} \otimes \left(e_m e_m^T\right),$$
$$K = I_{m-1} \otimes D_m + T_{m-1} \otimes \left(-I_m + \tfrac{1}{2} e_m e_m^T\right),$$

where $\otimes$ denotes the Kronecker product, $m = 1/h$, ζ is the (possibly complex) impedance, and

$$D_m = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 4 & -1 \\ & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{m \times m}, \qquad T_{m-1} = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(m-1)\times(m-1)}.$$

The eigenvalues of Q are the resonant frequencies of the system, and for the given problem formulation they lie in the upper half of the complex plane.

`bicycle` {pep,qep,real,parameter-dependent}. This is a 2×2 quadratic polynomial arising in the study of bicycle self-stability [Meijaard et al. 2007]. The linearized equations of motion for the Whipple bicycle model can be written as

$$M\ddot{q} + C\dot{q} + Kq = f,$$

where M is a symmetric mass matrix, the nonsymmetric damping matrix $C = vC_1$ is linear in the forward speed v, and the stiffness matrix $K = gK_0 + v^2 K_2$ is the sum of two parts: a velocity independent symmetric part gK_0 proportional to the gravitational acceleration g and a nonsymmetric part $v^2 K_2$ quadratic in the forward speed.

`bilby` {pep,qep,real,parameter-dependent}. This 5×5 quadratic matrix polynomial arises in a model from Bean et al. [1997] for the population of the greater bilby (*Macrotis lagotis*), an endangered Australian marsupial. Define the 5×5 matrix

$$M(g,x) = \begin{bmatrix} gx_1 & (1-g)x_1 0 & 0 & 0 & 0 \\ gx_2 & 0 & (1-g)x_2 & 0 & 0 \\ gx_3 & 0 & 0 & (1-g)x_3 & 0 \\ gx_4 & 0 & 0 & 0 & (1-g)x_4 \\ gx_5 & 0 & 0 & 0 & (1-g)x_5 \end{bmatrix}.$$

The model is a quasi-birth-death process some of whose key properties are captured by the elementwise minimal solution of the quadratic matrix equation

$$R = \beta\left(A_0 + RA_1 + R^2A_2\right), \qquad A_0 = M(g,b), \quad A_1 = M(g,e-b-d), \quad A_2 = M(g,d),$$

where b and d are vectors of probabilities and e is the vector of ones. The corresponding quadratic matrix polynomial is $Q(\lambda) = \lambda^2 A + \lambda B + C$, where

$$A = \beta A_2^T, \quad B = \beta A_1^T - I, \quad C = \beta A_0^T.$$

We take $g = 0.2$, $b = [1,\ 0.4,\ 0.25,\ 0.1,\ 0]^T$, and $d = [0,\ 0.5,\ 0.55,\ 0.8,\ 1]^T$, as in [Bean et al. 1997].

`butterfly` {pep,real,parameter-dependent,T-even,sparse,scalable}. This is a quartic matrix polynomial $P(\lambda) = \lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$ of dimension m^2 with T-even structure, depending on a 10×1 parameter vector c [Mehrmann and Watkins 2002]. Its spectrum has a butterfly shape. The coefficient matrices are Kronecker products, with A_4 and A_2 real and symmetric and A_3 and A_1 real and skew-symmetric, assuming c is real. The default is $m = 8$.

`cd_player` {pep,qep,real}. This is a 60×60 quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M = I_{60}$ arising in the study of a CD player control task [Chahlaoui and Van Dooren 2002, 2005; Draijer et al. 1992; Wortelboer et al. 1996]. The mechanism that is modeled consists of a swing arm on which a lens is mounted by means of two horizontal leaf springs. This is a small representation of a larger original rigid body model (which is also quadratic).

`closed_loop` {pep,qep,real,parameter-dependent}. This is a quadratic polynomial

$$Q(\lambda) = \lambda^2 I + \lambda \begin{bmatrix} 0 & 1+\alpha \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

associated with a closed-loop control system with feedback gains 1 and $1+\alpha$, $\alpha \geq 0$. The eigenvalues of $Q(\lambda)$ lie inside the unit disc if and only if $0 \leq \alpha < 0.875$ [Tisseur and Higham 2001].

`concrete` {pep,qep,symmetric,parameter-dependent,sparse}. This is a quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + (1 + i\mu)K$ arising in a model of a concrete structure supporting a machine assembly [Feriani et al. 2000]. The matrices have dimension 2472. M is real diagonal and low rank. C, the viscous damping matrix, is pure imaginary and diagonal. K is complex symmetric, and the factor $1 + i\mu$ adds uniform hysteretic damping. The default is $\mu = 0.04$.

`damped_beam` {pep,qep,real,symmetric,sparse,scalable}. This QEP arises in the vibration analysis of a beam simply supported at both ends and damped in the middle [Higham et al. 2008]. The quadratic $Q(\lambda) = \lambda^2 M + \lambda C + K$ has real symmetric coefficient matrices with $M > 0$, $K > 0$, and $C = ce_n e_n^T \geq 0$, where c is a damping parameter.

Half of the eigenvalues of the problem are purely imaginary and are eigenvalues of the undamped problem ($C = 0$).

`dirac` {`pep,qep,real,symmetric,parameter-dependent,scalable`}. The spectrum of this matrix polynomial is the second order spectrum of the radial Dirac operator with an electric Coulombic potential of strength α,

$$D = \begin{bmatrix} 1 + \frac{\alpha}{r} & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -1 + \frac{\alpha}{r} \end{bmatrix}.$$

For $-\sqrt{3}/2 < \alpha < 0$ and $\kappa \in \mathbb{Z}$, D acts on $L^2((0,\infty),\mathbb{C}^2)$ and it corresponds to a spherically symmetric decomposition of the space into partial wave subspaces [Thaller 1992]. The problem discretization is relative to subspaces generated by the Hermite functions of odd order. The size of the matrix coefficients of the QEP is $n+m$, corresponding to n Hermite functions in the first component of the L^2 space and m in the second component [Boulton and Boussaid 2010].

For $\kappa = -1$, $\alpha = -1/2$ and n large enough, there is a conjugate pair of isolated points of the second order spectrum near the ground eigenvalue $E_0 \approx 0.866025$. The essential spectrum, $(-\infty,-1] \cup [1,\infty)$, as well as other eigenvalues, also seem to be captured for large n.

`fiber` {`nep,sparse,solution`}. This nonlinear eigenvalue problem arises from a model in fiber optic design based on the Maxwell equations [Huang et al. 2010; Kaufman 2006]. The problem is of the form

$$F(\lambda)x = (A - \lambda I + s(\lambda)B)x = 0,$$

where $A \in \mathbb{R}^{2400\times 2400}$ is tridiagonal and $B = e_{2400}e_{2400}^T$. The scalar function $s(\lambda)$ is defined in terms of Bessel functions. The real, positive eigenvalues are the ones of interest.

`foundation` {`pep,qep,symmetric,sparse`}. This is a quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$ arising in a model of reinforced concrete machine foundations resting on the ground [Feriani et al. 2000]. The matrices have dimension 3627; M is real and diagonal, C is complex and diagonal, and K is complex symmetric.

`gen_hyper2` {`pep,qep,real,symmetric,hyperbolic,parameter-dependent,scalable, solution,random`}. This is a hyperbolic quadratic matrix polynomial generated from a given set of eigenvalues and eigenvectors (λ_k, v_k), $k = 1\colon 2n$, such that with

$$\begin{aligned} \Lambda &= \operatorname{diag}(\lambda_1,\ldots,\lambda_{2n}) =: \operatorname{diag}(\Lambda_1,\Lambda_2), \qquad \Lambda_1,\Lambda_2 \in \mathbb{R}^{n\times n}, \\ V &:= [\,v_1,\ldots,v_{2n}\,] =: [\,V_1 \quad V_2\,], \qquad V_1, V_2 \in \mathbb{R}^{n\times n}, \end{aligned}$$

$\lambda_{\min}(\Lambda_1) > \lambda_{\max}(\Lambda_2)$, V_1 is nonsingular, and $V_2 = V_1 U$ for some orthogonal matrix U. Then the $n \times n$ symmetric quadratic $Q(\lambda) = \lambda^2 A + \lambda B + C$ with

$$\begin{aligned} A &= \Gamma^{-1}, \qquad \Gamma = V_1\Lambda_1V_1^T - V_2\Lambda_2V_2^T, \\ B &= -A(V_1\Lambda_1^2V_1^T - V_2\Lambda_2^2V_2^T)A, \\ C &= -A(V_1\Lambda_1^3V_1^T - V_2\Lambda_2^3V_2^T)A + B\Gamma B, \end{aligned}$$

is hyperbolic and has eigenpairs (λ_k, v_k), $k = 1\colon 2n$ [Al-Ammari and Tisseur 2012; Guo et al. 2009a]. The quadratic $Q(\lambda)$ has the property that A is positive definite and $-Q(\mu)$ is positive definite for all $\mu \in (\lambda_{\max}(\Lambda_2), \lambda_{\min}(\Lambda_1))$. If $\lambda_{\max}(\Lambda) < 0$ then B and C are positive definite and $Q(\lambda)$ is overdamped.

`gen_tpal2` {pep,qep,real,T-palindromic,parameter-dependent,scalable,random}. This is a real T-palindromic quadratic matrix polynomial generated from a given set of eigenvalues

$$\lambda_{2j-1} = \cos t_j + i \sin t_j,\ \lambda_{2j} = \bar{\lambda}_{2j-1}, \quad t_j \in (0, \pi), \quad j = 1\colon n, \tag{4}$$

lying on the unit circle. Let

$$J = \operatorname{diag}\left(\begin{bmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \beta_n \\ -\beta_n & 0 \end{bmatrix} \right), \quad i\beta_j = \frac{\lambda_{2j-1} - 1}{\lambda_{2j-1} + 1} \in \mathbb{R}, \tag{5}$$

$$S = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \in \mathbb{R}^{2n \times 2n}, \quad X = [\,X_1 \quad X_1 H\,]\,\Pi \in \mathbb{R}^{n \times 2n},$$

where $X_1 \in \mathbb{R}^{n\times n}$ is nonsingular, $H \in \mathbb{R}^{n\times n}$ is a symmetric matrix, and Π is a permutation matrix such that $\Pi^T S \Pi = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. Then the $n \times n$ real quadratic $\widetilde{Q}(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ with

$$A_2 = (XJSX^T)^{-1},\ A_1 = -A_2 X J^2 S X^T A_2,\ A_0 = -A_2(XJ^2SX^TA_1 + XJ^3SX^TA_2) \tag{6}$$

is real T-even with eigenvalues $\pm i\beta_j, j = 1\colon n$. Finally,

$$Q(\lambda) = (\lambda+1)^2 \widetilde{Q}\Big(\frac{\lambda-1}{\lambda+1}\Big) = \lambda^2(A_2 + A_1 + A_0) + \lambda(-2A_2 + 2A_0) + (A_2 - A_1 + A_0) \tag{7}$$

is real T-palindromic with eigenvalues $\lambda_j, j = 1\colon 2n$ [Al-Ammari 2011].

`gen_tantipal2` {pep,qep,real,T-anti-palindromic,parameter-dependent,scalable, random}. This is a real T-anti-palindromic quadratic matrix polynomial generated from a given set of eigenvalues lying on the unit circle as in (4). Let J be as in (5) and

$$S = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \quad X = [\,X_1 \quad X_1 U\,],$$

where $X_1 \in \mathbb{R}^{n\times n}$ is nonsingular and $U \in \mathbb{R}^{n\times n}$ is orthogonal. Then the $n \times n$ real quadratic $\widetilde{Q}(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ with matrix coefficients as in (6) is real T-odd with eigenvalues $\pm i\beta_j$, $j = 1\colon n$. Finally, $Q(\lambda)$ in (7) is real T-anti-palindromic with eigenvalues $\lambda_j, j = 1\colon 2n$ [Al-Ammari 2011].

`gun` {nep,sparse}. This nonlinear eigenvalue problem models a radio-frequency gun cavity. The eigenvalue problem is of the form

$$F(\lambda)x = \big[K - \lambda M + i(\lambda - \sigma_1^2)^{\frac{1}{2}} W_1 + i(\lambda - \sigma_2^2)^{\frac{1}{2}} W_2\big]x = 0,$$

where M, K, W_1, W_2 are real symmetric matrices of size 9956×9956. K is positive semidefinite and M is positive definite. In this example $\sigma_1 = 0$ and $\sigma_2 = 108.8774$. The eigenvalues of interest are the λ for which $\lambda^{1/2}$ is close to 146.71 [Liao 2007, p. 59].

`hadeler` {nep,real,symmetric,scalable}. This nonlinear eigenvalue problem has the form

$$F(\lambda)x = \Big[(e^{\lambda} - 1)A_2 + \lambda^2 A_1 - \alpha A_0\Big]x = 0,$$

where $A_2, A_1, A_0 \in \mathbb{R}^{n\times n}$ are symmetric and α is a scalar parameter [Hadeler 1967]. This problem satisfies a generalized form of overdamping condition that ensures the existence of a complete set of eigenvectors [Ruhe 1973].

`hospital` {pep,qep,real}. This is a 24×24 quadratic polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M = I_{24}$, arising in the study of the Los Angeles University Hospital building

[Chahlaoui and Van Dooren 2002, 2005]. There are 8 floors, each with 3 degrees of freedom.

`intersection` {pep,qep,real}. This 10×10 quadratic polynomial arises in the problem of finding the intersection between a cylinder, a sphere, and a plane described by the equations

$$\begin{aligned} f_1(x,y,z) &= 1.6\text{e-}3x^2 + 1.6\text{e-}3y^2 - 1 = 0,\\ f_2(x,y,z) &= 5.3\text{e-}4x^2 + 5.3\text{e-}4y^2 + 5.3\text{e-}4z^2 + 2.7\text{e-}2x - 1 = 0,\\ f_3(x,y,z) &= -1.4\text{e-}4x + 1.0\text{e-}4y + z - 3.4\text{e-}3 = 0. \end{aligned} \tag{8}$$

Use of the Macaulay resultant leads to the QEP $Q(x)v = 0$, where

$$\begin{aligned} Q(x)v &= [yf_1 \quad zf_1 \quad f_1 \quad yf_2 \quad zf_2 \quad f_2 \quad yzf_3 \quad yf_3 \quad zf_3 \quad f_3]^T = (x^2A_2 + xA_1 + A_0)v,\\ v &= [y^3 \quad y^2z \quad y^2 \quad yz^2 \quad z^3 \quad z^2 \quad yz \quad y \quad z \quad 1]^T. \end{aligned}$$

The matrix A_2 is singular and the QEP has only four finite eigenvalues: two real and two complex. Let (λ_i, v_i), $i = 1, 2$ be the two real eigenpairs. With the normalization $v_i(10) = 1$, $i = 1, 2$, $(x_i, y_i, z_i) = (\lambda_i, v_i(8), v_i(9))$ are solutions of (8) [Manocha 1994].

`loaded_string` {rep,real,symmetric,parameter-dependent,sparse,scalable}. This rational eigenvalue problem arises in the finite element discretization of a boundary problem describing the eigenvibration of a string with a load of mass m attached by an elastic spring of stiffness k. It has the form

$$R(\lambda)x = \left(A - \lambda B + \frac{\lambda}{\lambda - \sigma}C\right)x = 0,$$

where the pole $\sigma = k/m$, and $A > 0$ and $B > 0$ are $n \times n$ tridiagonal matrices defined by

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}, \quad B = \frac{h}{6}\begin{bmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & 4 & 1 \\ & & 1 & 2 \end{bmatrix},$$

and $C = ke_ne_n^T$ with $h = 1/n$ [Solov'ëv 2006].

`metal_strip` {pep,qep,real}. Modeling the electronic behavior of a metal strip using partial element equivalent circuits (PEEC's) results in the delay differential equation [Bellen et al. 1999]

$$\begin{cases} D_1\dot{x}(t-h) + D_0\dot{x}(t) = A_0x(t) + A_1x(t-h), & t \geq 0, \\ x(t) = \varphi(t), & t \in [-h, 0), \end{cases}$$

where

$$A_0 = 100\begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}, \quad A_1 = 100\begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix},$$

$$D_1 = -\frac{1}{72}\begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}, \quad D_0 = I, \quad \varphi(t) = \left[\sin(t), \quad \sin(2t), \quad \sin(3t)\right]^T.$$

Assessing the stability of this delay differential equation by the method in Faßbender et al. [2008] and [Jarlebring 2008] leads to the quadratic eigenproblem $(\lambda^2 E + \lambda F + G)u = 0$ with

$$\begin{aligned} E &= (D_0 \otimes A_1) + (A_0 \otimes D_1), \quad G = (D_1 \otimes A_0) + (A_1 \otimes D_0), \\ F &= (D_0 \otimes A_0) + (A_0 \otimes D_0) + (D_1 \otimes A_1) + (A_1 \otimes D_1). \end{aligned}$$

This problem is PCP-palindromic [Faßbender et al. 2008], that is, there is an involutory matrix P such that $E = P\overline{G}P$ and $F = P\overline{F}P$.

`mirror` {pep,real,random}. The 9×9 quartic matrix polynomial $\lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$ is obtained from a homography-based method for calibrating a central cadioptric vision system, which can be built from a perspective camera with a hyperbolic mirror or an orthographic camera with a parabolic mirror [Zhang and Li 2008]. A_0 and A_4 have only two nonzero columns, so there are at least 7 infinite eigenvalues and 7 zero eigenvalues.

`mobile_manipulator` {pep,qep,real}. This is a 5×5 quadratic matrix polynomial arising from modelling a two-dimensional three-link mobile manipulator as a time-invariant descriptor control system [Bunse-Gerstner et al. 1999; Byers et al. 1998, Ex. 14]. The system in its second-order form is

$$\begin{aligned} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) &= Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

where the coefficient matrices are 5×5 and of the form

$$M = \begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_0 & -F_0^T \\ F_0 & 0 \end{bmatrix},$$

with

$$M_0 = \begin{bmatrix} 18.7532 & -7.94493 & 7.94494 \\ -7.94493 & 31.8182 & -26.8182 \\ 7.94494 & -26.8182 & 26.8182 \end{bmatrix}, D_0 = \begin{bmatrix} -1.52143 & -1.55168 & 1.55168 \\ 3.22064 & 3.28467 & -3.28467 \\ -3.22064 & -3.28467 & 3.28467 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 67.4894 & 69.2393 & -69.2393 \\ 69.8124 & 1.68624 & -1.68617 \\ -69.8123 & -1.68617 & -68.2707 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The quadratic $Q(\lambda) = \lambda^2 M + \lambda D + K$ is close to being nonregular [Byers et al. 1998; Higham and Tisseur 2002].

`omnicam1` {pep,qep,real}. This is a 9×9 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ arising from a model of an omnidirectional camera (one with angle of view greater than 180 degrees) [Mičušík and Pajdla 2003]. The matrix A_0 has one nonzero column, A_1 has 5 nonzero columns and rank 5, while A_2 has full rank. The eigenvalues of interest are the real eigenvalues of order 1.

`omnicam2` {pep,qep,real}. The description of `ominicam1` applies to this problem, too, except that the quadratic is 15×15.

`orr-sommerfeld` {pep,parameter-dependent,scalable}. This example is a quartic polynomial eigenvalue problem arising in the spatial stability analysis of the Orr_Sommerfeld equation [Tisseur and Higham 2001]. The Orr_Sommerfeld equation is a linearization of the incompressible Navier–Stokes equations in which the perturbations in velocity and pressure are assumed to take the form $\Phi(x,y,t) = \phi(y)e^{i(\lambda x - \omega t)}$, where λ is a wavenumber and ω is a radian frequency. For a given Reynolds number R, the Orr_Sommerfeld equation may be written

$$\left[\left(\frac{d^2}{dy^2}-\lambda^2\right)^2 - iR\left\{(\lambda U-\omega)\left(\frac{d^2}{dy^2}-\lambda^2\right)-\lambda U''\right\}\right]\phi = 0. \qquad (9)$$

In spatial stability analysis the parameter is λ, which appears to the fourth power in (9), so we obtain a quartic polynomial eigenvalue problem. The quartic is constructed using a Chebyshev spectral discretization. The eigenvalues λ of interest are those closest to the real axis and $\mathrm{Im}(\lambda) > 0$ is needed for stability. The default values $R = 5772$ and $\omega = 0.26943$ correspond to the critical neutral point corresponding to λ and ω both real for minimum R [Bridges and Morris 1984; Orszag 1971].

`pdde_stability` {`qep,pep,scalable,parameter-dependent,sparse,symmetric`}. This problem arises from the stability analysis of a partial delay-differential equation (PDDE) [Faßbender et al. 2008; Jarlebring 2008, Ex. 3.22]. Discretization gives rise to a time-delay system

$$\dot{x}(t) = A_0x(t) + A_1x(t-h_1) + A_2x(t-h_2),$$

where $A_0 \in \mathbb{R}^{n\times n}$ is tridiagonal and $A_1,\ A_2 \in \mathbb{R}^{n\times n}$ are diagonal with

$$(A_0)_{kj} = \begin{cases} -2(n+1)^2/\pi^2 + a_0 + b_0 \sin(j\pi/(n+1)) & \text{if } \ k=j, \\ (n+1)^2/\pi^2 & \text{if } \ |k-j| = 1, \end{cases}$$

$$(A_1)_{jj} = a_1 + b_1\frac{j\pi}{n+1}\left(1-e^{-\pi(1-j/(n+1))}\right),$$

$$(A_2)_{jj} = a_2 + b_2\frac{j\pi^2}{n+1}\left(1-j/(n+1)\right).$$

Here, the a_k and b_k are real scalar parameters and $n \in \mathbb{N}$ is the number of uniformly spaced interior grid points in the discretization of the PDDE. Asking for the delays h_1, h_2 such that the delay system is stable leads to the quadratic eigenvalue problem $(\lambda^2E + \lambda F + G)\,v = 0$ of dimension $n^2 \times n^2$ with

$$E = I \otimes A_2, \quad F = \left(I \otimes (A_0 + e^{-i\varphi_1}A_1)\right) + \left((A_0 + e^{i\varphi_1}A_1) \otimes I\right), \quad G = A_2 \otimes I,$$

where i is the imaginary unit and $\varphi_1 \in [-\pi, \pi]$ is a parameter. (To answer the stability question, the QEP has to be solved for many values of φ_1.)

Following Jarlebring [2008] and Faßbender et al. [2008] the default values are

$$n = 20,\ a_0 = 2,\ b_0 = 0.3,\ a_1 = -2,\ b_1 = 0.2,\ a_2 = -2,\ b_2 = -0.3,\ \varphi_1 = -\pi/2.$$

This problem is PCP-palindromic [Faßbender et al. 2008], that is, there is an involutory matrix P such that $E = P\overline{G}P$ and $F = P\overline{F}P$. Moreover, only the four eigenvalues on the unit circle are of interest in the application. The exact corresponding eigenvectors can be written as $x_j = u_j \otimes v_j$ for $j = 1\colon 4$.

`planar_waveguide` {`pep,real,symmetric,scalable`}. This 129×129 quartic matrix polynomial $P(\lambda) = \lambda^4A_4 + \lambda^3A_3 + \lambda^2A_2 + \lambda A_1 + A_0$ arises from a finite element solution of the equation for the modes of a planar waveguide using piecewise linear basis functions ϕ_i, $i = 0\colon 128$. The coefficient matrices are defined by

$$A_1 = \frac{\delta^2}{4}\,\mathrm{diag}(-1, 0, 0, \dots, 0, 0, 1), \quad A_3 = \mathrm{diag}(1, 0, 0, \dots, 0, 0, 1),$$
$$A_0(i,j) = \frac{\delta^4}{16}(\phi_i, \phi_j), \quad A_2(i,j) = (\phi_i', \phi_j') - (q\phi_i, \phi_j), \quad A_4(i,j) = (\phi_i, \phi_j).$$

Thus A_1 and A_3 are diagonal, while A_0, A_2, and A_4 are tridiagonal. The parameter δ describes the difference in refractive index between the cover and the substrate of the waveguide; q is a function used in the derivation of the variational formulation and is constant in each layer [Stowell 2010; Stowell and Tausch 2010]. This particular waveguide has been studied in the literature in connection with other solution methods [Chilwell and Hodgkinson 1984; Petráček and Singh 2002].

`plasma_drift` {pep}. This cubic matrix polynomial of dimension 128 or 512 results from the modeling of drift instabilities in the plasma edge inside a Tokamak reactor [Tokar et al. 2005]. It is of the form $P(\lambda) = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$, where A_0 and A_1 are complex, A_2 is complex symmetric, and A_3 is real symmetric. The desired eigenpair is the one whose eigenvalue has the largest imaginary part.

`power_plant` {pep,qep,symmetric,parameter-dependent}. This is a QEP $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$ describing the dynamic behavior of a nuclear power plant simplified into an eight-degrees-of-freedom system [Itoh 1973; Tisseur and Meerbergen 2001]. The mass matrix M and damping matrix D are real symmetric and the stiffness matrix has the form $K = (1+i\mu)K_0$, where K_0 is real symmetric (hence $K = K^T$ is complex symmetric). The parameter μ describes the hysteretic damping of the problem. The matrices are badly scaled.

`qep1` {pep,qep,real,solution}. This is a 3×3 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ from Tisseur and Meerbergen [2001, p. 250] with

$$A_2 = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & -6 & 0 \\ 2 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_0 = I.$$

The six eigenpairs (λ_k, x_k), $k = 1\colon 6$, are given by

k	1	2	3	4	5	6
λ_k	$1/3$	$1/2$	1	i	$-i$	∞
x_k	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$

Note that x_1 is an eigenvector for both of the distinct eigenvalues λ_1 and λ_2.

`qep2` {pep,qep,real,solution}. This is the 3×3 quadratic matrix polynomial [Tisseur and Meerbergen 2001, p. 256]

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = \lambda_4 = 1$, and $\lambda_5 = \lambda_6 = \infty$. The Jordan structure is given by

$$X_F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_F = \operatorname{diag}\left(-1, 1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

for the finite eigenvalues and and

$$X_\infty = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad J_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

for the infinite eigenvalues (see Gohberg et al. [2009] or Tisseur and Meerbergen [2001, Sec. 3.6] for definitions of Jordan structure).

qep3 {pep,qep,real,parameter-dependent,solution}. This is a 3×3 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ from Dedieu and Tisseur [2003, p. 89] with

$$A_2 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -1-\epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 2 & 0 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

The eigenpairs (λ_k, x_k), $k = 1{:}\ 6$, are given by

k	1	2	3	4	5	6
λ_k	0	1	$1+\epsilon$	2	3	∞
x_k	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ \frac{\epsilon-1}{\epsilon+1} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

For the default value of the parameter, $\epsilon = -1 + 2^{-53/2}$, the first and third eigenvalues are ill conditioned.

qep4 {pep,qep,nonregular,nonsquare,real,solution}. This is the 3×4 quadratic matrix polynomial [Byers et al. 2008, Ex. 2.5]

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The eigensystem includes an eigenvalue $\lambda_1 = 0$ with right eigenvectors $[\,2\ 1\ 0\ -1]^T$ and e_1 and an eigenvalue $\lambda = \infty$ with right eigenvector $[\,0\ 0\ 1\ 0]^T$. The Jordan and Kronecker structure is fully described in Byers et al. [2008, Ex. 2.5].

qep5 {pep,qep,nonregular,real}. This is the 3×3 quadratic matrix polynomial [Van Dooren and Dewilde 1983, Ex. 1]

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 1 & 4 & 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 3 & 0 \\ 1 & 4 & 2 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Its Smith form [Gantmacher 1959] is given by

$$D(\lambda) = \begin{bmatrix} 1 & -1 & -1 \\ -\lambda & 1+\lambda & \lambda \\ 0 & -\lambda & 1 \end{bmatrix} Q(\lambda) \begin{bmatrix} 1 & -3 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and since $\det(Q(\lambda)) = \det(D(\lambda)) \equiv 0$ this problem is nonregular.

`railtrack` {pep,qep,t-palindromic,sparse}. This is a T-palindromic quadratic matrix polynomial of size 1005: $Q(\lambda) = \lambda^2 A^T + \lambda B + A$ with $B = B^T$. It stems from a model of the vibration of rail tracks under the excitation of high speed trains, discretized by classical mechanical finite elements [Hilliges 2004; Hilliges et al. 2004; Ipsen 2004; Mackey et al. 2006]. This problem has the property that the matrix A is of the form

$$A = \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} \in \mathbb{C}^{1005\times 1005},$$

where $A_{21} \in \mathbb{C}^{201\times 67}$, that is, A has low rank ($\mathrm{rank}(A) = 67$). Hence this eigenvalue problem has many eigenvalues at zero and infinity.

`railtrack2` {pep,qep,t-palindromic,sparse,scalable,parameter-dependent}. This is a T-palindromic quadratic matrix polynomial of size $705m \times 705m$: $Q(\lambda) = \lambda^2 A^T + \lambda B + A$ with

$$A = \begin{bmatrix} 0 & \cdots & 0 & H_1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} H_0 & H_1^T & & 0 \\ H_1 & H_0 & \ddots & \\ & \ddots & \ddots & H_1^T \\ 0 & & H_1 & H_0 \end{bmatrix} = B^T,$$

where $H_0, H_1 \in \mathbb{C}^{705\times 705}$ depend quadratically on a parameter ω, whose default value is $\omega = 1000$. The default for the number of block rows and columns of A and B is $m = 51$. The structure of A implies that there are many eigenvalues at zero and infinity.

Like the problem `railtrack` this problem is from a model of the vibration of rail tracks, but here triangular finite elements are used for the discretization [Chu et al. 2008; Guo and Lin 2010; Huang et al. 2008]. The parameter ω denotes the frequency of the external excitation force.

`relative_pose_5pt` {pep,real}. The cubic matrix polynomial $P(\lambda) = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$, $A_i \in \mathbb{R}^{10\times 10}$, comes from the five point relative pose problem in computer vision [Kukelova et al. 2008, 2011]. In this problem the images of five unknown scene points taken with a camera with a known focal length from two distinct unknown viewpoints are given and it is required to determine the possible solutions for the relative configuration of the points and cameras. The matrix A_3 has one nonzero column, A_2 has 3 nonzero columns and rank 3, A_1 has 6 nonzero columns and rank 6, while A_0 is of full rank. The solutions to the problem are obtained from the last three components of the finite eigenvectors of P.

`relative_pose_6pt` {pep,qep,real}. The quadratic matrix polynomial $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$, where $A_i \in \mathbb{R}^{10\times 10}$, comes from the six point relative pose problem in computer vision [Kukelova et al. 2008, 2011]. In this problem the images of six unknown scene points taken with a camera of unknown focal length from two distinct unknown camera viewpoints are given and it is required to determine the possible solutions for the relative configuration of the points and cameras. The solutions to the problem are obtained from the last three components of the finite eigenvectors of P.

`schrodinger` {pep,qep,real,symmetric,sparse}. The spectrum of this matrix polynomial is the second order spectrum, relative to a subspace $\mathcal{L} \subset H^2(\mathbb{R})$, of the Schrödinger operator $Hf(x) = f''(x) + (\cos(x) - e^{-x^2})f(x)$ acting on $L^2(\mathbb{R})$ [Boulton and Levitin 2007]. The subspace $\mathcal{L}$ has been generated using fourth order Hermite elements on a uniform mesh on the interval $[-49, 49]$, subject to clamped boundary conditions. The corresponding quadratic matrix polynomial is given by $K - 2\lambda C + \lambda^2 B$ where

$$K_{jk} = \langle Hb_j, Hb_k \rangle, \qquad C_{jk} = \langle Hb_j, b_k \rangle \qquad \text{and} \qquad B_{jk} = \langle b_j, b_k \rangle.$$

Here $\{b_k\}$ is a basis of $\mathcal{L}$. The matrices are of size 1998.

The essential spectrum of H consists of a set of bands separated by gaps. The end points of these bands are the Mathieu characteristic values. The presence of the short-range potential gives rise to isolated eigenvalues of finite multiplicity. The portion of the second order spectrum that lies in the box $[-1/2, 2] \times [-10^{-1}, 10^{-1}]$ is very close to the spectrum of H.

`shaft` {pep,qep,real,symmetric,sparse}. The quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M, C, K \in \mathbb{R}^{400\times 400}$, comes from a finite element model of a shaft on bearing supports with a damper [Kowalski 2000, Ex. 5.6]. The matrix M has rank 199 and so contributes a large number of infinite eigenvalues. C has a single nonzero element, in the $(20, 20)$ position. The coefficients M, C, and K are very sparse.

`sign1` {pep,qep,hermitian,parameter-dependent,scalable}. The spectrum of this quadratic matrix polynomial is the second order spectrum of the linear operator $Mf(x) = \operatorname{sign}(x)f(x) + a\widehat{f}(0)$ acting on $L^2(-\pi, \pi)$ with respect to the Fourier basis $\mathcal{B}_n = \{e^{-inx}, \dots, 1, \dots, e^{inx}\}$, where $\widehat{f}(0) = (1/2\pi)\int_{-\pi}^{\pi} f(x)\,dx$ [Boulton 2007]. The corresponding QEP is given by $K_n - 2\lambda C_n + \lambda^2 I_n$ where

$$K_n = \Pi_n M^2 \Pi_n, \qquad C_n = \Pi_n M \Pi_n$$

and I_n is the identity matrix of size $2n + 1$. Here Π_n is the orthogonal projector onto $\operatorname{span}(\mathcal{B}_n)$.

As n increases, the limit set of the second order spectrum is the unit circle, together with two real points: $\lambda_\pm$. The intersection of this limit set with the real line is the spectrum of M. The points $\lambda_\pm$ comprise the discrete spectrum of M.

`sign2` {pep,qep,hermitian,parameter-dependent,scalable}. This problem is analogous to problem `sign1`, the only difference being that the operator is $Mf(x) = (2\sin(x) + \operatorname{sign}(x))f(x) + a\widehat{f}(0)$.

Near the real line, the second order spectrum accumulates at $[-3, -1] \cup [1, 3] \cup \{\lambda_\pm\}$ as n increases. The two accumulation points $\lambda_\pm \approx \{-0.7674, 3.5796\}$ are the discrete spectrum of M.

`sleeper` {pep, qep, real, symmetric, sparse, scalable,proportionally-damped, solution}. This QEP describes the oscillations of a rail track resting on sleepers [Lancaster and Rózsa 1996]. The QEP has the form

$$Q(\lambda) = \lambda^2 I + \lambda(I + A^2) + A^2 + A + I,$$

where A is the circulant matrix with first row $[-2,\ 1,\ 0,\ \dots,\ 0,\ 1]$. The eigenvalues of A and corresponding eigenvectors are explicitly given as

$$\mu_k = -4\sin^2\left(\frac{(k-1)\pi}{n}\right), \quad x_k(j) = \frac{1}{\sqrt{n}}\exp\left(\frac{-2i\pi(j-1)(k-1)}{n}\right), \qquad k = 1\colon n.$$

The eigenvalues of Q can be determined from the scalar equations

$$\lambda^2 + \lambda(1 + \mu_k^2) + (1 + \mu_k + \mu_k^2) = 0.$$

Due to the symmetry, manifested in $\sin(\pi - \theta) = \sin(\theta)$, there are several multiple eigenvalues.

`speaker_box` {pep,qep,real,symmetric}. The quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M, C, K \in \mathbb{R}^{107\times 107}$, is from a finite element model of a speaker box that includes both structural finite elements, representing the box, and fluid elements, representing the air contained in the box [Kowalski 2000, Ex. 5.5]. The matrix coefficients are highly structured and sparse. There is a large variation in the norms: $\|M\|_2 = 1$, $\|C\|_2 = 5.7 \times 10^{-2}$, $\|K\|_2 = 1.0 \times 10^7$.

`spring` {pep,qep,real,symmetric,proportionally-damped, parameter-dependent, sparse,scalable}. This is a QEP $Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0$ arising from a linearly damped mass-spring system [Tisseur 2000]. The damping constants for the dampers and springs connecting the masses to the ground, and those for the dampers and springs connecting adjacent masses, are parameters. For the default choice of the parameters, the $n \times n$ matrices K, C, and M are

$$M = I, \quad C = 10\,T, \quad K = 5\,T, \quad T = \begin{bmatrix} 3 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 3 \end{bmatrix}.$$

`spring_dashpot` {pep, qep, real, parameter-dependent,sparse,scalable,random}. Gotts [2005] describes a QEP arising from a finite element model of a linear spring in parallel with Maxwell elements (a Maxwell element is a spring in series with a dashpot). The quadratic matrix polynomial is $Q(\lambda) = \lambda^2 M + \lambda D + K$, where the mass matrix M is rank deficient and symmetric, the damping matrix D is rank deficient and block diagonal, and the stiffness matrix K is symmetric and has arrowhead structure. This example reflects the structure only, since the matrices themselves are not from a finite element model but randomly generated to have the desired properties of symmetry etc. The matrices have the form

$$M = \operatorname{diag}(\rho \widetilde{M}_{11}, 0), \qquad D = \operatorname{diag}(0, \eta_1 \widetilde{K}_{11}, \ldots, \eta_m \widetilde{K}_{m+1,m+1}),$$

$$K = \begin{bmatrix} \alpha_\rho \widetilde{K}_{11} & -\xi_1 \widetilde{K}_{12}, & \ldots & -\xi_m \widetilde{K}_{1,m+1} \\ -\xi_1 \widetilde{K}_{12} & e_1 \widetilde{K}_{22} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -\xi_m \widetilde{K}_{1,m+1} & 0 & 0 & e_m \widetilde{K}_{m+1,m+1} \end{bmatrix},$$

where $\widetilde{M}_{ij}$ and $\widetilde{K}_{ij}$ are element mass and stiffness matrices, ξ_i and e_i measure the spring stiffnesses, and ρ is the material density.

`surveillance` {pep,qep,real,nonsquare,nonregular}. This is a 21×16 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ arising from calibration of a surveillance camera using a human body as a calibration target [Mičušík and Pajdla 2010]. The eigenvalue represents the focal length of the camera. This particular data set is synthetic and corresponds to a 600×400 pixel camera.

`time_delay` {nep,real}. This 3×3 nonlinear matrix function has the form $R(\lambda) = -\lambda I_3 + A_0 + A_1 \exp(-\lambda)$ with

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -b_3 & -b_2 & -b_1 \end{bmatrix},$$

and is the characteristic equation of a time-delay system with a single delay and constant coefficents [Jarlebring 2012; Jarlebring and Michiels 2010, 2011]. The problem has a double non-semisimple eigenvalue $\lambda = 3\pi i$.

`wing` {pep,qep,real}. This example is a 3×3 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ from Frazer et al. [1938, Sec. 10.11], with numerical values modified as in Lancaster [1966, Sec. 5.3]. The eigenproblem for $Q(\lambda)$ arose from the analysis of the oscillations of a wing in an airstream. The matrices are

$$A_2 = \begin{bmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix}.$$

`wiresaw1` {pep, qep, real, t-even, gyroscopic, parameter-dependent,scalable}. This gyroscopic QEP arises in the vibration analysis of a wiresaw [Wei and Kao 2000]. It takes the form $Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0$, where the $n \times n$ coefficient matrices are defined by

$$M = I_n/2, \quad K = \operatorname*{diag}_{1\le j\le n}\left(j^2\pi^2(1-v^2)/2\right),$$

and

$$C = -C^T = (c_{jk}), \quad \text{with} \quad c_{jk} = \begin{cases} \dfrac{4jk}{j^2-k^2}v, & \text{if } j+k \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Here, v is a real nonnegative parameter corresponding to the speed of the wire. Note that for $0 < v < 1$, K is positive definite and the quadratic

$$G(\lambda) := -Q(-\imath\lambda) = \lambda^2 M + \lambda(\imath C) - K$$

is hyperbolic (but not overdamped).

`wiresaw2` {pep,qep,real,parameter-dependent,scalable}. When the effect of viscous damping is added to the problem in `wiresaw1`, the corresponding quadratic has the form [Wei and Kao 2000]

$$\widetilde{Q}(\lambda) = \lambda^2 M + \lambda(C + \eta I) + K + \eta C,$$

where M, C, and K are the same as in `wiresaw1` and the damping parameter η is real and nonnegative.

4. DESIGN OF THE TOOLBOX

The problems in the NLEVP collection are accessed via a single MATLAB function `nlevp`, which is modelled on the MATLAB `gallery` function. This function calls those that actually generate the problems, which reside in a `private` directory located within the `nlevp` directory. This approach avoids the problem of name clashes with existing MATLAB functions and also provides an elegant interface to the collection.

All problems have input parameters comprising the problem name followed by (where applicable) the dimension and other parameters, and the coefficient matrices defining the problem (as specified in Section 2.1) are returned in a cell array. To illustrate, the following example sets up the `omnicam2` problem, finds its eigenvalues and eigenvectors with `polyeig`, and then prints the largest modulus of the eigenvalues:

```
>> coeffs = nlevp('omnicam2')
coeffs =
    [15x15 double]    [15x15 double]    [15x15 double]
>> [X,e] = polyeig(coeffs{:}); max(abs(e))
ans =
  3.6351e-001
```

The nonlinear function $F(\lambda)$ in (3) can be evaluated by calling `nlevp` with `eval` as its first argument. This is useful for evaluating the residual of an approximate eigenpair, for example:

```
>> lam = e(end); x = X(:,end); Fx = nlevp('eval','omnicam2',lam)*x; norm(Fx)
ans =
  5.8137e-032
```

The second output argument from `nlevp` is a function handle that enables the nonlinear scalar functions $f_i(\lambda)$ in (3) and their derivatives to be evaluated. This facilitates the use of numerical methods that require derivatives, especially for the nonpolynomial problems, for which obtaining the derivatives can be nontrivial. For example, the following code evaluates $f_i(0.5)$, $i = 1\colon 3$, and the first two derivatives (denoted `fp`, `fpp`), for the `fiber` problem:

```
>> [coeffs,fun] = nlevp('fiber');
>> [f,fp,fpp] = fun(0.5)
f =
  1.0000e+000 -5.0000e-001 -7.0746e-001
fp =
            0 -1.0000e+000 -7.0725e-001
fpp =
            0            0  7.0696e-001
```

Problems and their properties are stored in a simple database made from cell arrays. The database is accessed with the `nlevp_query` function in the `private` directory, which is invoked using the `query` argument to `nlevp`. For example, the properties for the `butterfly` problem are returned in a cell array by the following call (whose syntax illustrates the command/function duality of MATLAB [Higham and Higham 2005, Sec. 7.5]):

```
>> nlevp query butterfly
ans =
    'pep'
    'real'
    'parameter-dependent'
    'T-even'
    'scalable'
    'sparse'
```

A more sophisticated example finds the names of all PEPs of degree 3 or higher:

```
>> pep = nlevp('query','pep'); qep = nlevp('query','qep');
>> pep_cubic_plus = setdiff(pep,qep)
pep_cubic_plus =
    'butterfly'
    'mirror'
    'orr_sommerfeld'
    'planar_waveguide'
    'plasma_drift'
    'relative_pose_5pt'
```

The cell array pep_cubic_plus can then easily be used to extract these problems. For example, the first problem in pep_cubic_plus can be solved using

```
coeffs = nlevp(pep_cubic_plus{1}); [X,e] = polyeig(coeffs{:});
```

Tables V and VI were generated automatically in MATLAB using appropriate nlevp('query',...) calls.

The toolbox function nlevp_example provides a test that the toolbox is correctly installed. It solves all the PEPs in the collection of dimension less than 500 using MATLAB's polyeig and then plots the eigenvalues. It produces Figure 1 and output to the command window that begins as follows:

```
NLEVP contains 52 problems in total,
of which 47 are polynomial eigenvalue problems (PEPs).
Run POLYEIG on the PEP problems of dimension at most 500:

             Problem   Dim  Max and min magnitude of eigenvalues
             -------   ---  ------------------------------------
    acoustic_wave_1d    10  3.14e+000, 4.59e-001
    acoustic_wave_2d    30  2.61e+000, 6.83e-001
             bicycle     2  1.41e+001, 3.23e-001
               bilby     5        Inf, 4.84e-016
           butterfly    64  2.01e+000, 3.59e-001
           cd_player    60  1.87e+006, 2.23e-004
         closed_loop     2  1.07e+000, 3.31e-001
            concrete   2472 is a PEP but is too large for this test.
                 ...
```

The nlevp_example.m function can be used as a template by the user wishing to test a given solver on subsets of the NLEVP problems.

The toolbox function nlevp_test.m automatically tests that the problems in the collection have the claimed properties. It is primarily intended for use by the developers as new problems are added, but it can also be used as a test for correctness of the installation. While many of the tests are straightforward, some are less so. For example, we test for hyperbolicity of a Hermitian matrix polynomial by computing the eigensystem and checking the types of the eigenvalues, using a characterization in Al-Ammari and Tisseur [2012, Thm. 3.4, P1]. To test for proportional damping we use necessary

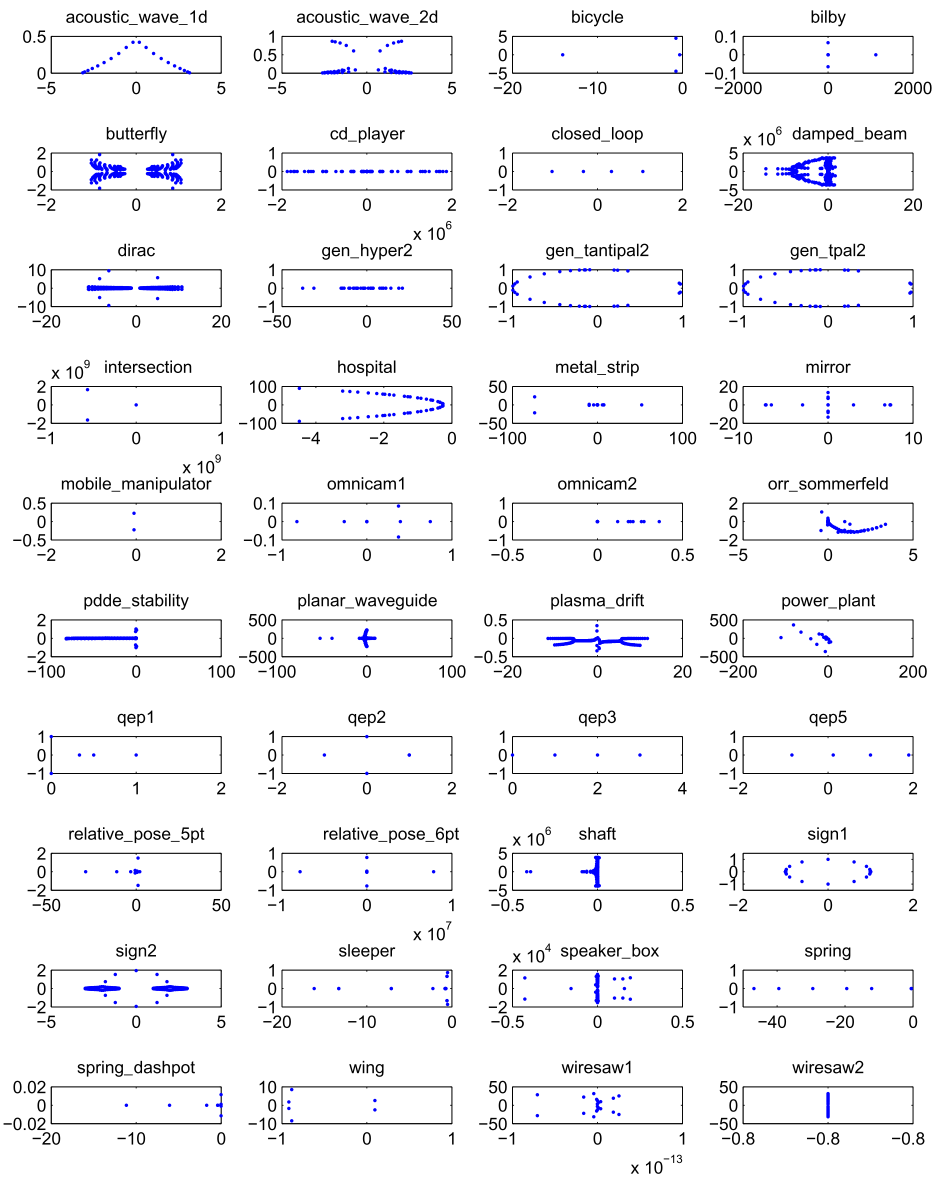

Fig. 1. Eigenvalue plots for PEP problems produced by `nlevp_example.m`.

and sufficient conditions from Lancaster and Zaballa [2009, Thms. 2, 4]. We reproduce part of the output:

```
>> nlevp_test
Testing the NLEVP collection
Testing generation of all problems
Testing T-palindromicity
Testing *-palindromicity
...
Testing proportionally damping
```

```
Testing given solutions
NLEVP collection tests completed.
***  Errors: 0
```

5. CONCLUSIONS

The NLEVP collection demonstrates the tremendous variety of applications of nonlinear eigenvalue problems and provides representative problems for testing in the form of a MATLAB toolbox. Version 1.0 of the toolbox was released in 2008, and at the time of writing the current version is 3.0, dated 22-Dec-2011. The toolbox has already proved useful in our own work and that of others [Asakura et al. 2010; Betcke 2008; Betcke and Kressner 2011; Grammont et al. 2011; Guo et al. 2009b; Hammarling et al. 2011; Jarlebring et al. 2010; Su and Bai 2011; Tisseur et al. 2011] and we hope it will find broad use in developing, testing, and comparing new algorithms. By classifying important structural properties of nonlinear eigenvalue problems, and providing examples of these structures, this work should also be useful in guiding theoretical developments.

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REFERENCES

Al-Ammari, M. 2011. Analysis of structured polynomial eigenvalue problems. Ph.D. thesis, MIMS EPrint 2011.89, Manchester Institute for Mathematical Sciences, University of Manchester, Manchester, UK.

Al-Ammari, M. and Tisseur, F. 2012. Hermitian matrix polynomials with real eigenvalues of definite type. Part I: Classification. *Linear Algebra Appl. 436,* 10, 3954–3973.

Asakura, J., Sakurai, T., Tadano, H., Ikegami, T., and Kimura, K. 2010. A numerical method for polynomial eigenvalue problems using contour integral. *Japan J. Ind. Appl. Math. 27,* 1, 73–90.

Bai, Z., Day, D., Demmel, J., and Dongarra, J. 1997. A test matrix collection for non-Hermitian eigenvalue problems (release 1.0). Tech. rep. CS-97-355, Department of Computer Science, University of Tennessee. LAPACK Working Note 123.

Bean, N. G., Bright, L., Latouche, G., Pearce, C. E. M., Pollett, P. K., and Taylor, P. G. 1997. The quasi-stationary behavior of quasi-birth-and-death processes. *Ann. Appl. Prob. 7,* 1, 134–155.

Bellen, A., Guglielmi, N., and Ruehli, A. E. 1999. Methods for linear systems of circuit delay-differential equations of neutral type. *IEEE Trans. Circuits Syst. Fundam. Theory Appl. 46,* 1, 212–216.

Betcke, T. 2008. Optimal scaling of generalized and polynomial eigenvalue problems. *SIAM J. Matrix Anal. Appl. 30,* 4, 1320–1338.

Betcke, T. and Kressner, D. 2011. Perturbation, extraction and refinement of invariant pairs for matrix polynomials. *Linear Algebra Appl. 435*, 514–536.

Betcke, T., Higham, N. J., Mehrmann, V., Schröder, C., and Tisseur, F. 2011. NLEVP: A collection of nonlinear eigenvalue problems. Users' guide. MIMS EPrint 2011.117, Manchester Institute for Mathematical Sciences.

Bongartz, I., Conn, A. R., Gould, N., and Toint, P. L. 1995. CUTE: Constrained and unconstrained testing environment. *ACM Trans. Math. Softw. 21,* 1, 123–160.

Boulton, L. 2007. Non-variational approximation of discrete eigenvalues of self-adjoint operators. *IMA J. Numer. Anal. 27,* 1, 102–121.

Boulton, L. and Boussaid, N. 2010. Non-variational computation of the eigenstates of Dirac operators with radially symmetric potentials. *LMS J. Comput. Math. 13*, 10–32.

Boulton, L. and Levitin, M. 2007. On approximation of the eigenvalues of perturbed periodic Schrödinger operators. *J. Phys. A: Math. Theor. 40*, 9319–9329.

Bridges, T. J. and Morris, P. J. 1984. Differential eigenvalue problems in which the parameter appears nonlinearly. *J. Comput. Phys. 55*, 437–460.

Bunse-Gerstner, A., Byers, R., Mehrmann, V., and Nichols, N. K. 1999. Feedback design for regularizing descriptor systems. *Linear Algebra Appl. 299*, 119–151.

Byers, R., He, C., and Mehrmann, V. 1998. Where is the nearest non-regular pencil? *Linear Algebra Appl. 285*, 81–105.

Byers, R., Mehrmann, V., and Xu, H. 2008. Trimmed linearizations for structured matrix polynomials. *Linear Algebra Appl. 429*, 2373–2400.

Chahlaoui, Y. and Van Dooren, P. M. 2002. A collection of benchmark examples for model reduction of linear time invariant dynamical systems. MIMS EPrint 2008.22, Manchester Institute for Mathematical Sciences.

Chahlaoui, Y. and Van Dooren, P. M. 2005. Benchmark examples for model reduction of linear time-invariant dynamical systems. In *Dimension Reduction of Large-Scale Systems*, P. Benner, V. Mehrmann, and D. C. Sorensen Eds., Lecture Notes in Computational Science and Engineering Series, vol. 45, Springer, 380–392.

Chaitin-Chatelin, F. and van Gijzen, M. B. 2006. Analysis of parameterized quadratic eigenvalue problems in computational acoustics with homotopic deviation theory. *Numer. Linear Algebra Appl. 13*, 487–512.

Chilwell, J. and Hodgkinson, I. 1984. Thin-films field-transfer matrix theory of planar multilayer waveguides and reflection from prism-loaded waveguides. *J. Opt. Soc. Amer. A 1,* 7, 742–753.

Chu, E. K.-W., Hwang, T.-M., Lin, W.-W., and Wu, C.-T. 2008. Vibration of fast trains, palindromic eigenvalue problems and structure-preserving doubling algorithms. *J. Comput. Appl. Math. 219,* 1, 237–252.

Davis, T. A. University of Florida sparse matrix collection. http://www.cise.ufl.edu/research/sparse/matrices/.

Davis, T. A. and Hu, Y. 2011. The University of Florida sparse matrix collection. *ACM Trans. Math. Software 38,* 1, 1:1–1:25.

Dedieu, J.-P. and Tisseur, F. 2003. Perturbation theory for homogeneous polynomial eigenvalue problems. *Linear Algebra Appl. 358*, 71–94.

Dennis, Jr., J. E., Traub, J. F., and Weber, R. P. 1976. The algebraic theory of matrix polynomials. *SIAM J. Numer. Anal. 13,* 6, 831–845.

Donoho, D. L., Maleki, A., Rahman, M., Shahram, I. U., and Stodden, V. 2009. Reproducible research in computational harmonic analysis. *Comput. Sci. Eng. 11,* 1, 8–18.

Draijer, W., Steinbuch, M., and Bosgra, O. H. 1992. Adaptive control of the radial servo system of a compact disc player. *Automatica 28,* 3, 455–462.

Duff, I. S., Grimes, R. G., and Lewis, J. G. 1989. Sparse matrix test problems. *ACM Trans. Math. Softw. 15,* 1, 1–14.

Duffin, R. J. 1960. The Rayleigh-Ritz method for dissipative or gyroscopic systems. *Q. Appl. Math. 18*, 215–221.

Faßbender, H., Mackey, N., Mackey, D. S., and Schröder, C. 2008. Structured polynomial eigenproblems related to time-delay systems. *Electron. Trans. Numer. Anal. 31*, 306–330.

Feriani, A., Perotti, F., and Simoncini, V. 2000. Iterative system solvers for the frequency analysis of linear mechanical systems. *Comput. Meth. Appl. Mech. Eng. 190*, 1719–1739.

Frazer, R. A., Duncan, W. J., and Collar, A. R. 1938. *Elementary Matrices and Some Applications to Dynamics and Differential Equations*. Cambridge University Press. 1963 printing.

Gantmacher, F. R. 1959. *The Theory of Matrices*. Vol. 1, Chelsea, New York.

GNU Octave. http://www.octave.org.

Gohberg, I., Lancaster, P., and Rodman, L. 2009. *Matrix Polynomials*. SIAM, Philadelphia, PA.

Gotts, A. 2005. Report regarding model reduction, model compaction research project. Manuscript, University of Nottingham.

Gould, N. I. M., Orban, D., and Toint, P. L. 2003. CUTEr and SifDec: A constrained and unconstrained testing environment, revisited. *ACM Trans. Math. Softw. 29,* 4, 373–394.

Grammont, L., Higham, N. J., and Tisseur, F. 2011. A framework for analyzing nonlinear eigenproblems and parametrized linear systems. *Linear Algebra Appl. 435,* 3, 623–640.

Guo, C.-H. and Lin, W.-W. 2010. Solving a structured quadratic eigenvalue problem by a structure-preserving doubling algorithm. *SIAM J. Matrix Anal. Appl. 31,* 5, 2784–2801.

Guo, C.-H., Higham, N. J., and Tisseur, F. 2009a. Detecting and solving hyperbolic quadratic eigenvalue problems. *SIAM J. Matrix Anal. Appl. 30,* 4, 1593–1613.

Guo, C.-H., Higham, N. J., and Tisseur, F. 2009b. An improved arc algorithm for detecting definite Hermitian pairs. *SIAM J. Matrix Anal. Appl. 31,* 3, 1131–1151.

Hadeler, K. P. 1967. Mehrparametrige und nichtlineare Eigenwertaufgaben. *Arch. Rational Mech. Anal. 27,* 4, 306–328.

Hammarling, S., Munro, C. J., and Tisseur, F. 2011. An algorithm for the complete solution of quadratic eigenvalue problems. MIMS EPrint 2011.86, Manchester Institute for Mathematical Sciences. *ACM Trans. Math. Softw.* (To appear).

Harari, I., Grosh, K., Hughes, T. J. R., Malhotra, M., Pinsky, P. M., Steward, J. R., and Thompson, L. L. 1996. Recent developments in finite element methods for structural acoustics. *Arch. Comput. Meth. Eng. 3,* 2-3, 131–309.

Higham, D. J. and Higham, N. J. 2005. *MATLAB Guide* 2nd Ed. SIAM, Philadelphia, PA.

Higham, N. J. 1991. Algorithm 694: A collection of test matrices in MATLAB. *ACM Trans. Math. Softw. 17,* 3, 289–305.

Higham, N. J. 2008. *Functions of Matrices: Theory and Computation*. SIAM, Philadelphia, PA.

Higham, N. J. and Kim, H.-M. 2000. Numerical analysis of a quadratic matrix equation. *IMA J. Numer. Anal. 20,* 4, 499–519.

Higham, N. J. and Tisseur, F. 2002. More on pseudospectra for polynomial eigenvalue problems and applications in control theory. *Linear Algebra Appl.* 351–352, 435–453.

Higham, N. J., Mackey, D. S., Tisseur, F., and Garvey, S. D. 2008. Scaling, sensitivity and stability in the numerical solution of quadratic eigenvalue problems. *Int. J. Numer. Methods Eng. 73,* 3, 344–360.

Higham, N. J., Mackey, D. S., and Tisseur, F. 2009. Definite matrix polynomials and their linearization by definite pencils. *SIAM J. Matrix Anal. Appl. 31,* 2, 478–502.

Hilliges, A. 2004. Numerische Lösung von quadratischen Eigenwertproblemen mit Anwendung in der Schienendynamik. Diplomarbeit, TU Berlin.

Hilliges, A., Mehl, C., and Mehrmann, V. 2004. On the solution of palindromic eigenvalue problems. In *Proceedings of the European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS'04)*, P. Neittaanmäki, T. Rossi, S. Korotov, E. Oñate, J. Périaux, and D. Knörzer Eds., http://www.mit.jyu.fi/eccomas2004/proceedings/proceed.html.

Huang, T.-M., Lin, W.-W., and Qian, J. 2008. Structure-preserving algorithms for palindromic quadratic eigenvalue problems arising from vibration of fast trains. *SIAM J. Matrix Anal. Appl. 30,* 4, 1566–1592.

Huang, X., Bai, Z., and Su, Y. 2010. Nonlinear rank-one modification of the symmetric eigenvalue problem. *J. Comput. Math. 28,* 2, 218–234.

Ipsen, I. C. F. 2004. Accurate eigenvalues for fast trains. *SIAM News 37,* 9, 1–2.

Itoh, T. 1973. Damped vibration mode superposition method for dynamic response analysis. *Earthquake Eng. Struct. Dyn. 2*, 47–57.

Jarlebring, E. 2008. The spectrum of delay-differential equations: Numerical methods, stability and perturbation. Ph.D. thesis, TU Braunschweig, Institut Computational Mathematics, Carl-Friedrich-Gauß-Fakultät, 38023 Braunschweig, Germany.

Jarlebring, E. 2012. Convergence factors of Newton methods for nonlinear eigenvalue problems. *Linear Algebra Appl. 436*, 10, 3943–3953.

Jarlebring, E. and Michiels, W. 2010. Invariance properties in the root sensitivity of time-delay systems with double imaginary roots. *Automatica 46*, 1112–1115.

Jarlebring, E. and Michiels, W. 2011. Analyzing the convergence factor of residual inverse iteration. *BIT 51,* 4, 937–957.

Jarlebring, E., Michiels, W., and Meerbergen, K. 2010. A linear eigenvalue algorithm for the nonlinear eigenvalue problem. Rep. TW580, Katholieke Universiteit Leuven, Heverlee, Belgium.

Kaufman, L. 2006. Eigenvalue problems in fiber optic design. *SIAM J. Matrix Anal. Appl. 28,* 1, 105–117.

Kowalski, T. R. 2000. Extracting a few eigenpairs of symmetric indefinite matrix pencils. Ph.D. thesis, Department of Mathematics, University of Kentucky.

Kublanovskaya, V. N. 1999. Methods and algorithms of solving spectral problems for polynomial and rational matrices. *J. Math. Sci. 96,* 3, 3085–3287.

Kukelova, Z., Bujnak, M., and Pajdla, T. 2008. Polynomial eigenvalue solutions to the 5-pt and 6-pt relative pose problems. In *Proceedings of the 19th British Machine Vision Conference (BMVC'08)*, M. Everingham, C. Needham, and R. Fraile Eds., Vol. 1, 565–574.

Kukelova, Z., Bujnak, M., and Pajdla, T. 2011. Polynomial eigenvalue solutions to minimal problems in computer vision. *IEEE Trans. Pattern Anal. Mach. Intell. 33,* 12.

Lancaster, P. 1966. *Lambda-Matrices and Vibrating Systems*. Pergamon Press, Oxford, UK.

Lancaster, P. and Rózsa. 1996. The spectrum and stability of a vibrating rail supported by sleepers. *Comput. Math. Appl. 31,* 4/5, 201–213.

Lancaster, P. and Zaballa, I. 2009. Diagonalizable quadratic eigenvalue problems. *Mech. Syst. Sig. Process. 23,* 4, 1134–1144.

LeVeque, R. J. 2009. Python tools for reproducible research on hyperbolic problems. *Comput. Sci. Eng. 11,* 1, 19–27.

Liao, B.-S. 2007. Subspace projection methods for model order reduction and nonlinear eigenvalue computation. Ph.D. thesis, Department of Mathematics, University of California at Davis.

Mackey, D. S., Mackey, N., Mehl, C., and Mehrmann, V. 2006. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl. 28,* 4, 1029–1051.

Manocha, D. 1994. Solving systems of polynomial equations. *IEEE Comput. Graphics Appl. 14,* 2, 46–55.

Markus, A. S. 1988. *Introduction to the Spectral Theory of Polynomial Operator Pencils*. American Mathematical Society, Providence, RI.

Matrix Market. http://math.nist.gov/MatrixMarket/.

Mehrmann, V. and Voss, H. 2004. Nonlinear eigenvalue problems: A challenge for modern eigenvalue methods. *GAMM-Mitteilungen (GAMM-Reports) 27*, 121–152.

Mehrmann, V. and Watkins, D. 2002. Polynomial eigenvalue problems with Hamiltonian structure. *Electron. Trans. Numer. Anal. 13*, 106–118.

Meijaard, J. P., Papadopoulos, J. M., Ruina, A., and Schwab, A. L. 2007. Linearized dynamics equations for the balance and steer of a bicycle: A benchmark and review. *Proc. Roy. Soc. London Ser. A 463,* 2084, 1955–1982.

Mesirov, J. P. 2010. Accessible reproducible research. *Science 327*, 415–416.

Mičušík, K. B. and Pajdla, T. 2003. Estimation of omnidirectional camera model from epipolar geometry. In *Peoceedings of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR'03)*. IEEE.

Mičušík, K. B. and Pajdla, T. 2010. Simultaneous surveillance camera calibration and foot-head homology estimation from human detections. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*.

Orszag, S. A. 1971. Accurate solution of the Orr–Sommerfeld stability equation. *J. Fluid Mech. 50,* 4, 689–703.

Petráček, J. and Singh, K. 2002. Determination of leaky modes in planar multilayer waveguides. *IEEE Photonics Technol. Lett. 14,* 6, 810–812.

Ruhe, A. 1973. Algorithms for the nonlinear eigenvalue problem. *SIAM J. Numer. Anal. 10*, 674–689.

Solov′ëv, S. I. 2006. Preconditioned iterative methods for a class of nonlinear eigenvalue problems. *Linear Algebra Appl. 415*, 210–229.

Stowell, D. 2010. Computing eigensolutions for singular Sturm-Liouville problems in photonics. Ph.D. thesis, AAI3404032, Southern Methodist University.

Stowell, D. and Tausch, J. 2010. Variational formulation for guided and leaky modes in multilayer dielectric waveguides. *Comm. Comput. Phys. 7,* 3, 564–579.

Su, Y. and Bai, Z. 2011. Solving rational eigenvalue problems via linearization. *SIAM J. Matrix Anal. Appl. 32,* 1, 201–216.

Taylor, A. and Higham, D. J. 2009. CONTEST: A controllable test matrix toolbox for MATLAB. *ACM Trans. Math. Softw. 35,* 4, 26:1–26:17.

Thaller, B. 1992. *The Dirac Equation*. Springer-Verlag, Berlin.

Tisseur, F. 2000. Backward error and condition of polynomial eigenvalue problems. *Linear Algebra Appl. 309*, 339–361.

Tisseur, F. and Higham, N. J. 2001. Structured pseudospectra for polynomial eigenvalue problems, with applications. *SIAM J. Matrix Anal. Appl. 23,* 1, 187–208.

Tisseur, F. and Meerbergen, K. 2001. The quadratic eigenvalue problem. *SIAM Rev. 43,* 2, 235–286.

Tisseur, F., Garvey, S. D., and Munro, C. 2011. Deflating quadratic matrix polynomials with structure preserving transformations. *Linear Algebra Appl. 435,* 3, 464–479.

Tokar, M. Z., Kelly, F. A., and Loozen, X. 2005. Role of thermal instabilities and anomalous transport in threshold of detachment and mulitfacetted asymmetric radiation from the edge (MARFE). *Phys. Plasmas 12,* 052510.

Van Dooren, P. M. and Dewilde, P. 1983. The eigenstructure of an arbitrary polynomial matrix: Computational aspects. *Linear Algebra Appl. 50*, 545–579.

Wei, S. and Kao, I. 2000. Vibration analysis of wire and frequency response in the modern wiresaw manufacturing process. *J. Sound and Vib. 231,* 5, 1383–1395.

Wortelboer, P. M. R., Steinbuch, M., and Bosgra, O. H. 1996. Closed-loop balanced reduction with application to a compact disc mechanism. In *Selected Topics in Identification, Modeling and Control*, Vol. 9, Delft University Press, 47–58.

Zhang, B. and Li, Y. F. 2008. A method for calibrating the central catadioptric camera via homographic matrix. In *Proceedings of the IEEE International Conference on Information and Automation*. 972–977.

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