No C^1-recurrence of iterations of symplectomorphisms

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Abstract

In this article, we study the behaviour of iterations of symplectomorphisms and Hamiltonian diffeomorphisms on symplectic manifolds. We prove that symplectomorphisms and Hamiltonian diffeomorphisms do not have C^1-recurrence on negatively monotone symplectic manifolds. This is a generalization of the results of the study by Polterovich, Ono, Atallah-Shelukhin. Hamiltonian group actions play very important roles in symplectic topology. We see that negatively monotone symplectic manifolds are far from being Hamiltonina G-maniofolds.

1 Introduction

In [10], Polterovich introduced "growth sequence" of a diffeomorphism f of a smooth compact manifold M as follows:

$$\Gamma_n(f) = \max\{ \max_{x\in M} |d_x f^n|,\ \max_{x\in M} |d_x^{-n}| \},\ n \in \mathbb{N}$$

Here, $|d_x f^n|$ is the operator norm of the differential map

$$d_x f^n : T_x M \longrightarrow T_{f^n(x)} M$$

caluculated with respect to some Riemannian metric on M. Polterovich studied the behabior of $\Gamma_n(f)$ as n goes to $+\infty$ for symplectomorphism f. Let (M,ω) be a closed symplectic manifold. We denote the group of symplectomorphisms by $\mathrm{Symp}(M,\omega)$.

$$\mathrm{Symp}(M,\omega) = \{\phi \in \mathrm{Diff}(M) \mid \phi^*\omega = \omega\}$$

$\mathrm{Symp}_0(M,\omega)$ is the identity component of $\mathrm{Symp}(M,\omega)$. In other words, any element of $\mathrm{Symp}_0(M,\omega)$ can be connected to the identity via an isotopy of symplectomorphisms.

$$\mathrm{Symp}_0(M,\omega) = \left\{ \phi \in \mathrm{Symp}(M,\omega) \;\middle|\; \begin{array}{c} \exists \text{ smooth isotopy}\{\phi^t\} \subset \mathrm{Symp}(M,\omega) \\ \text{s.t. } \phi^0 = \mathrm{Id}, \phi^1 = \phi \end{array} \right\}$$

Polterovich proved that if $\pi_2(M) = 0$ and $\phi \in \mathrm{Symp}_0(M, \omega)\backslash\{\mathrm{Id}\}$ has a fixed point of contractible type,

$$\Gamma_n(f) \to +\infty \quad (n \to +\infty)$$

holds. In particular, any Hamiltonian diffeomorphism does not have C^1-recurrence property if $\pi_2(M) = 0$ holds ([10]). A symplectic manifold (M, ω) is called negatively monotone if there is a negative constant $\kappa < 0$ such that

$$c_1|_{\pi_2(M)} = \kappa \cdot \omega|_{\pi_2(M)}$$

holds. Here, $c_1 \in H^2(M)$ is the first Chern class of the tangent bundle of M. Ono proved that there is no Hamiltonian S^1-action on any negatively monotone symplectic manifold ([9]). Recently, Atallah-Shelukhin proved that there is no Hamiltonian torsion on negatively monotone symplectic manifolds ([2]). In other words, there is no Hamiltonian diffeomorphism ϕ ($\phi \neq \mathrm{Id}$) such that $\phi^k = \mathrm{Id}$ holds for some positive integer $k > 1$. In this paper, we generalize these results and prove that there is no C^1-recurrence of iterations of symplectomorphisms and Hamiltonian diffeomorphisms on negatively monotone symplectic manifolds. In particular, the set $\{\phi^k\}_{k\in\mathbb{Z}}$ is discrete in C^1-topology. Moreover, (M, ω) is far from being a Hamiltonian G-manifold.

For any smooth function $H \in C^\infty(M)$, we define the Hamiltonian vector field X_H by the following relation:

$$\omega(X_H, \cdot) = -dH$$

We also consider an S^1-dependent (= 1-periodic) Hamiltonian function $H \in C^\infty(S^1 \times M)$ and an S^1-dependent Hamiltonian vector field X_H by the same formula. The time 1 flow of the vector field X_H is called Hamiltonian diffeomorphism generated by H. We denote this Hamiltonian diffeomorphism by ϕ_H. The set of all Hamiltonian diffeomorphisms is called the Hamiltonian diffeomorphism group and we denote it by $\mathrm{Ham}(M, \omega)$, i.e.,

$$\mathrm{Ham}(M, \omega) = \{\phi_H \mid H \in C^\infty(S^1 \times M)\}$$

$\mathrm{Ham}(M, \omega)$ is a subgroup of $\mathrm{Symp}_0(M, \omega)$. The composition of ϕ_H, ϕ_K is generated by a Hamiltonian function

$$H\sharp K(t, x) = H(t, x) + K(t, (\phi_H^t)^{-1}(x)).$$

Moreover, $\phi^t_{H\sharp K} = \phi^t_H \circ \phi^t_K$ holds. The inverse of ϕ_H is generated by a Hamiltonian function

$$\overline{H}(t, x) = -H(t, \phi^t_H(x)).$$

This $\overline{H}$ satisfies $\phi^t_{\overline{H}} = (\phi^t_H)^{-1}$. We also consider k-th power of Hamiltonian diffeomorphisms for any integer $k \in \mathbb{N}$. We define $H^{(k)} \in C^\infty(S^1 \times M)$ as follows:

$$H^{(k)}(t, x) = kH(kt, x)$$

It is straightforward to see that $\phi_{H^{(k)}} = (\phi_H)^k$ holds. In other words, $H^{(k)}$ generates the k-th power of ϕ_H.

The main results of this paper are as follows:

Theorem 1 (no C^1-recurrence)**.** *Let (M, ω) be a closed negatively monotone symplectic manifold.*

1. *There is a C^1-small open neighborhood $\mathcal{U} \subset Ham(M, \omega)$ of the identity such that for any $\phi \in Ham(M, \omega) \backslash \{Id\}$, we can choose a positive integer $N_\phi > 0$ so that*
$$k \geq N_\phi \Longrightarrow \phi^k \notin \mathcal{U}$$
holds.

2. *Assume that the Euler number of M is not zero ($\chi(M) \neq 0$) or $\pi_1(M)$ has finite center. Then, there is a C^1-small open neighborhood $\mathcal{V} \subset Symp_0(M, \omega)$ of the identity such that for any $\psi \in Symp_0(M, \omega) \backslash \{Id\}$, we can choose a positive integer $M_\psi > 0$ so that*
$$k \geq M_\psi \Longrightarrow \psi^k \notin \mathcal{V}$$
holds.

2 Flux homomorphism and flux group

Let $\widetilde{\mathrm{Symp}}_0(M, \omega)$ be the universal cover of $\mathrm{Symp}_0(M, \omega)$. A point in $\widetilde{\mathrm{Symp}}_0(M, \omega)$ is a homotopy class of smooth paths $\{\phi^t\} \subset \mathrm{Symp}(M, \omega)$ with fixed endpoints $\phi^0 = \mathrm{Id}$ and $\phi^1 = \phi$. Let $\{\phi^t\}_{t \in [0,1]} \subset \mathrm{Symp}(M, \omega)$ be a smooth symplectic isotopy generated by a vector field X_t.

$$\frac{d}{dt}\phi^t(x) = X_t(\phi^t(x))$$

Note that $\iota_{X_t}\omega \in \Omega^1(M)$ is a closed 1-form because $\{\phi^t\}$ is a symplectic isotopy and

$$0 = L_{X_t}\omega = d(\iota_{X_t}\omega) + \iota_{X_t}(d\omega) = d(\iota_{X_t}\omega)$$

holds. The flux homomorphism of this isotopy is defined as follows:

$$\mathrm{Flux}(\{\phi^t\}) = \Big[\int_0^1 \iota_{X_t}\omega dt\Big] \subset H^1(M : \mathbb{R})$$

The flux homomorphism is invariant under any homotopy with fixed endpoints. So, it is defined on the universal cover.

$$\mathrm{Flux} : \widetilde{\mathrm{Symp}}_0(M, \omega) \longrightarrow H^1(M : \mathbb{R})$$

The flux group $\Gamma \subset H^1(M:\mathbb{R})$ of a symplectic manifold (M,ω) is the image of the fundamental group of $\mathrm{Symp}_0(M,\omega)$ under the flux homomorphism.

$$\Gamma = \mathrm{Flux}(\pi_1(\mathrm{Symp}_0(M,\omega)))$$

The most important property of Γ is that it is a discrete subgroup of $H^1(M:\mathbb{R})$ for any closed symplectic manifold (M,ω)([9]).

3 Floer homology and mean index

Let (M,ω) be a $2n$-dimensional closed symplectic manifold and we fix a 1-periodic Hamiltonian function

$$H : S^1 \times M \longrightarrow \mathbb{R}.$$

Definition 1. *A Hamiltonian function $H \in C^\infty(S^1 \times M)$ is called non-degenerate if the differential map*

$$(d\phi_H)_x : T_xM \longrightarrow T_xM$$

does not have 1 as an eigenvalue for any fixed point $x \in \mathrm{Fix}(\phi_H)$. In other words, the graph of ϕ_H intersects the diagonal $\Delta_M \subset M \times M$ transverally.

We denote the set of contractible 1-periodic orbits of ϕ_H^t by $\mathcal{P}(H)$.

$$\mathcal{P}(H) = \{x : S^1 \to M \mid \dot{x}(t) = X_{H_t}(x(t)),\ x : \text{contractible}\}$$

We define a covering space of $\mathcal{P}(H)$ as follows:

$$\widetilde{\mathcal{P}}(H) = \{(u,x) \mid u : D^2 \to M, x \in \mathcal{P}(H), \partial u = x\}/\sim$$

The equivalence relation $\sim$ is defined as follows:

$$(u,x) \sim (v,y) \Longleftrightarrow \begin{cases} x = y \\ \int_{S^2}(u\sharp\overline{v})^*\omega = 0 \\ \int_{S^2}(u\sharp\overline{v})^*c_1 = 0 \end{cases}$$

This $u\sharp\overline{v}$ is a sphere map $u\sharp\overline{v} : S^2 \to M$ obtained by gluing $u : D^2 \to M$ and $\overline{v} : D^2 \to M$ along the boundary. Here, $\overline{v}$ is the disc with the opposite orientation. $c_1 \in H^2(M)$ is the first Chern class of (M,ω). We denote the equivariance class of (u,x) by $[u,x]$. For any $[u,x]$, the action functional $A_H([u,x])$ is defined by

$$A_H([u,x]) = -\int_{D^2} u^*\omega + \int_0^1 H(t,x(t))dt.$$

We have the Conley-Zehnder index μ_{CZ} of $\widetilde{\mathcal{P}}(H)$ (see [13, 12]). We normalize μ_{CZ} so that the Conley-Zehnder index of a local maximum of a C^2-small Morse function is equal to n. $\pi_2(M)$ acts $\widetilde{\mathcal{P}}(H)$ naturally and it changes the Conley-Zehnder index as follows:

$$\mu_{CZ}([u\sharp A, x]) = \mu_{CZ}([u, x]) - 2c_1(A) \quad (\forall A \in \pi_2(M))$$

We also have the mean index $\mu([u, x]) \in \mathbb{R}$ ([13]). The mean index has the following properties.

1. $|\mu([u, x]) - \mu_{CZ}([u, x])| \leq n$
2. $\mu([u^{(k)}, x^{(k)}]) = k\mu([u, x])$

Here $[u^{(k)}, x^{(k)}] \in \widetilde{\mathcal{P}}(H^{(k)})$ is the natural k-th iteration of $[u, x]$. The Floer chain complex $CF_*(H)$ is defined as

$$CF_*(H) = \Big\{ \sum_{z \in \widetilde{\mathcal{P}}(H)} a_z \cdot z \mid a_z \in \mathbb{Q}, \forall C \in \mathbb{R}, \sharp\{z \in \widetilde{\mathcal{P}}(H) \mid a_z \neq 0, A_H(z) > 0\} < \infty \Big\}.$$

The boundary operator $d_F : CF_*(H) \to CF_{*-1}(H)$ is defined as follows:

$$d_F(z) = \sum_{w \in \widetilde{\mathcal{P}}(H)} n(z, w) \cdot w$$

This coefficient $n(z, w) \in \mathbb{Q}$ is the number of solutions of the following Floer equation modulo the natural $\mathbb{R}$-action (see [3, 6]). We choose an almost complex structure J on the tangent bundle TM.

$$\begin{gathered} z = [v_-, x_-], \ w = [v_+, x_+] \\ u : \mathbb{R} \times S^1 \longrightarrow M \\ \partial_s u(s,t) + J\big(\partial_t u(s,t) - X_{H_t}(u(s,t))\big) = 0 \\ \lim_{s \to \pm\infty} u(s,t) = x_\pm(t) \\ [v_- \sharp u, x_+] = [v_+, x_+] \end{gathered}$$

The Floer homology $HF_*(H)$ is the homology of the chain complex $(CF_*(H), d_F)$. It is known that $HF_*(H)$ is isomorphic to the quantum homology of (M, ω). Let $\Gamma_{(M,\omega)}$ be an abelian group defined as follows:

$$\Gamma_{(M,\omega)} = \frac{\pi_2(M)}{\ker\omega \cap \ker c_1}$$

Here, $\omega : \pi_2(M) \to \mathbb{R}$ is the integration of the symplectic form ω and $c_1 : \pi_2(M) \to \mathbb{Z}$ is the integration of the first Chern class c_1. The degree of $u \in \Gamma_{(M,\omega)}$ is $-2c_1(u)$. The Novikov ring $\Lambda_{(M,\omega)}$ is defined as the set of possibly infinite sum of $\Gamma_{(M,\omega)}$ with suitable convergence, i.e.,

$$\Lambda_{(M,\omega)} = \Big\{ \sum_{u \in \Gamma_{(M,\omega)}} a_u \cdot u \mid a_u \in \mathbb{Q}, \forall C \in \mathbb{R}, \sharp\{u \in \Gamma_{(M,\omega)}, \ a_u \neq 0, \omega(u) < C\} < \infty \Big\}.$$

The quantum homology of (M, ω) is the singular homology with $\Lambda_{(M,\omega)}$ coeffient, i.e.,

$$QH_*(M, \omega) = H_*(M : \mathbb{Q}) \otimes \Lambda_{(M,\omega)}.$$

It is known that there is a natural isomorphism (PSS-isomorphism) between $QH_*(M, \omega)$ and $HF_*(H)$ ([11, 3, 6]).

$$\mathrm{PSS} : QH_*(M, \omega) \longrightarrow HF_{*-n}(H)$$

The PSS-isomorphism is used to define the spectral invariant of the Floer homology. For any nonzero chain $c = \sum_{z \in \widetilde{\mathcal{P}}(H)} a_z \cdot z \in CF_*(H)$, the set $\{A_H(z) | a_z \neq 0\}$ is bounded above and discrete. We denote its supremum by $l(c)$. If $c = 0$, we define $l(0) = -\infty$.

$$l : CF_*(H) \longrightarrow \mathbb{R} \cup \{-\infty\}$$
$$c \mapsto \sup\{A_H(z) | a_z \neq 0\}$$

Then, $CF_*^{<a}(H) = \{c \in CF_*(H) | l(c) < a\}$ is a subcomplex of $(CF_*(H), d_F)$ for any $a \in \mathbb{R}$. We denote the homology of $(CF_*^{<a}(H), d_F)$ by $HF_*^{<a}(H)$. The inclusion $CF_*^{<a}(H) \to CF_*(H)$ induces a natural map

$$\iota_a : HF_*^{<a}(H) \longrightarrow HF_*(H).$$

For any $\alpha \in QH_*(M, \omega)$, the spectral invariant $c(\alpha, H)$ is defined as follows ([8, 14, 20]):

$$c(\alpha, H) = \inf\{a \in \mathbb{R} \mid \mathrm{PSS}(\alpha) \in \mathrm{Im}(\iota_a)\}$$

Usher proved that $c(\alpha, H)$ is achieved by some cycle in $\mathrm{PSS}(\alpha) \in HF_*(H)$.

Lemma 1 ([19])**.** *Let $H \in C^\infty(S^1 \times M)$ be a non-degenerate Hamiltonian function. For any $\alpha \in QH_*(M, \omega) \backslash \{0\}$, there is a cycle $c \in CF_*(H)$ such that*

$$l(c) = c(\alpha, H)$$
$$[c] = \mathrm{PSS}(\alpha)$$

holds.

Note that $c(\alpha, H)$ is defined for possibly degenerate Hamiltonian function H because $c(\alpha, H)$ is continuous with respect to the Hofer norm (=L^∞-norm) of the function H.

$$|c(\alpha, H) - c(\alpha, K)| \leq \int_0^1 \max_{x \in M} |H(t, x) - K(t, x)| dt$$

This continuity enable us to extend $c(\alpha, \cdot)$ to continuous functions. The spectral norm of H is defined by the spectral invariant of H and $\overline{H}$ with respect to the funcamental class $[M] \in QH_{2n}(M, \omega)$.

$$\rho(H) = c([M], H) + c([M], \overline{H})$$

It is known that $\rho(H) \geq 0$ holds and moreover, $\rho(H) > 0$ holds if $\phi_H \neq \mathrm{Id}$. This is a consequence of the proof of the energy-capacity inequality between the Hofer-Zehnder capacity and the displacement energy (see [18, 15]). More precisely, we have the following estimate of $\rho(H)$. Let $(\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic space.

$$\omega_0(x_1, \cdots, x_n, y_1, \cdots, y_n) = \sum_i x_i \wedge y_i$$

We say that $\phi \in \mathrm{Ham}(M, \omega)$ displaces a symplectically embedded r-ball ($r > 0$) if there is a symplectic embedding ι

$$\iota : \big(B(r), \omega_0|_{B(r)}\big) \longrightarrow (M, \omega)$$
$$B(r) = \big\{(x_1, \cdots, x_n, y_1, \cdots, y_n) \in \mathbb{R}^{2n} \mid \sum_i x_i^2 + \sum_i y_i^2 \leq r^2\big\}$$

such that $\phi(\iota(B(r))) \cap \iota(B(r)) = \emptyset$ holds. If ϕ_H displaces a symplectically embedded r-ball,

$$\rho(H) \geq \pi r^2$$

holds. The Poincare duality of the spectral invariant (see [5]) implies that if H is non-degenerate,

$$c([M], \overline{H}) = -\inf\{c(\beta, H) \mid \beta \in QH_0(M, \omega), \beta = [pt] + \sum_{b \in H_{2k}(M), k \geq 1} a_b \cdot b\}$$

holds. In particular, this equality and Lemma 1 implies that there exists $z, w \in \widetilde{\mathcal{P}}(H)$ such that

$$\mu_{CZ}(z) = n$$
$$\mu_{CZ}(w) = -n$$
$$A_H(z) = c([M], H), A_H(w) = -c([M], \overline{H})$$

holds. In particular, if ϕ_H displaces a symplectically embedded r-ball, there exists $z, w \in \widetilde{\mathcal{P}}(H)$ such that

$$0 \leq \mu(z) \leq 2n$$
$$-2n \leq \mu(w) \leq 0$$
$$A_H(z) - A_H(w) \geq \pi r^2$$

holds. The same conclusion holds for possibly degenerate Hamiltonian function H.

Lemma 2. *Let $H \in C^\infty(S^1 \times M)$ be a possibly degenerate Hamiltonian function such that ϕ_H displaces a symplectically embedded r-ball. Then, there are capped* *periodic orbits $z, w \in \widetilde{\mathcal{P}}(H)$ such that*

$$0 \le \mu(z) \le 2n$$
$$-2n \le \mu(w) \le 0$$
$$A_H(z) - A_H(w) \ge \pi r^2$$

holds.

First, we choose a family of Hamiltonian functions $\{H_m\}_{m \in \mathbb{N}}$ such that every H_m is non-degenerate and H_m converges to H in C^∞-topology. We assume that every ϕ_{H_m} displaces a symplectically embedded r-ball. Then, we can choose $z_m = [u_m, x_m]$, $w_m = [v_m, y_m]$ in $\widetilde{\mathcal{P}}(H_m)$ so that

$$0 \le \mu(z_m) \le 2n,\ -2n \le \mu(w_m) \le 0$$
$$A_{H_m}(z_m) = c([M], H_m),\ A_{H_m}(w_m) = c([M], \overline{H_m})$$
$$A_{H_m}(z_m) - A_{H_m}(w_m) \ge \pi r^2$$

Without loss of generality, we assume that x_m and y_m converges to contractible periodic orbits of H.

$$x_m \to x,\ y_m \to y$$

Note that the continuity of the spectral invariant implies that $A_{H_m}(z_m)$ converges to $c([M], H)$ and $A_{H_m}(w_m)$ converges to $c([M], \overline{H})$. Let C_m and D_m be small cylinders $[0,1] \times S^1 \to M$ which connect x_m and x, y_m and y respectively.

$$C_m(0,t) = x_m(t),\ C_m(1,t) = x(t)$$
$$D_m(0,t) = y_m(t),\ D_m(1,t) = y(t)$$

Here, "small" means that $\{C_m(s,t)\}_{s \in [0,1]}$ and $\{D_m(s,t)\}_{s \in [0,1]}$ are shortest geodesics with respect to some Riemanninan metric. We define $\widetilde{z}_m$ and $\widetilde{w}_m$ in $\widetilde{\mathcal{P}}(H)$ as follows:

$$\widetilde{z}_m = [u_m \sharp C_m, x],\ \widetilde{w}_m = [v_m \sharp D_m, y]$$

Then, $\{A_H(\widetilde{z}_m)\}$ converges to $c([M], H)$ and $A_H(\widetilde{w}_m)$ converges to $c([M], \overline{H})$. Our monotonicity assumption implies that the differences are contained in the discrete subgroup of $\mathbb{R}$.

$$A_H(\widetilde{z}_m) - A_H(\widetilde{z}_{m'}) \in \frac{1}{\kappa}\mathbb{Z}$$
$$A_H(\widetilde{w}_m) - A_H(\widetilde{w}_{m'}) \in \frac{1}{\kappa}\mathbb{Z}$$

This implies that $A_H(\widetilde{z}_m)$ and $A_H(\widetilde{w}_m)$ stabilize for sufficiently large m. In other words, we can choose $N \in \mathbb{N}$ so that

$$A_H(\widetilde{z}_m) = A_H(\widetilde{z}_{m'}),\ A_H(\widetilde{w}_m) = A_H(\widetilde{w}_{m'})$$

holds for any $m, m' \geq N$. Our monotonicity assumption implies that the mean index also stabilizes. So,

$$\mu(\widetilde{z}_m) = \mu(\widetilde{z}_{m'}),\ \mu(\widetilde{w}_m) = \mu(\widetilde{w}_{m'})$$

holds for any $m, m' \geq N$. This also implies that $\widetilde{z}_m = \widetilde{z}_{m'}$ and $\widetilde{w}_m = \widetilde{w}_{m'}$ hold. We denote these capped periodic orbits by z and w.

$$z = \widetilde{z}_m,\ w = \widetilde{w}_m \quad m \geq N$$

This z and w satisfies the following relations:

$$0 \leq \mu(z) \leq 2n,\ -2n \leq \mu(w) \leq 0$$
$$A_H(z) = c([M], H),\ A_H(w) = c([M], \overline{H})$$
$$A_H(z) - A_H(w) \geq \pi r^2$$

So we proved the lemma.

4 Proof of the main theorem

In this section, we prove the main theorem. First, we prove the following proposition.

Proposition 1. *Let (M, ω) be a closed symplectic manifold. We fix a sufficiently small C^1-open neighborhood of the identity $\mathcal{U} \subset \mathrm{Ham}(M, \omega)$. For any $\psi \in \mathcal{U}$, we can construct a C^1-small Hamiltonian isotopy between the identity and ψ.*

Remark 1. *It is almost trivial that we can construct a C^1-small symplectic isotopy between the identity and ψ. The important point is that we can construct a Hamiltonian isotopy.*

We apply the following correspondence between the set of symplectomorphisms which are C^1-close to the identity and the set of small closed 1-forms on M (see [7]). Let $\sigma \in \Omega(M \times M)$ be a symplectic form on $M \times M$.

$$\sigma = -\pi_1^*\omega + \pi_2^*\omega$$

Here, $\pi_i : M \times M \to M$ is the projection from the i-th factor

$$\pi_i(x, y) = \begin{cases} x & i = 1 \\ y & i = 2. \end{cases}$$

We denote the canonical Liouville 1-form on the cotangent bundle T^*M by λ.

$$\lambda(X) = w(d\pi_*(X)) \quad X \in T_w T^*M$$

This $d\pi : TT^*M \to TM$ is the differentail of the natural projection $\pi : T^*M \to M$. Then, $d\lambda \in \Omega(T^*M)$ is the canonical symplectic form on the cotangent bundle.

Let $M_0 \subset T^*M$ be the image of the zero section. We can construct a symplectomorphism Ψ between an open neighborhood of the zero section $\mathcal{N}(M_0) \subset T^*M$ and an open neighborhood of the diagonal $\mathcal{N}(\Delta) \subset M \times M$. This symplectomorphism Ψ satisfies the following properties.

$$\Psi : \mathcal{N}(\Delta) \longrightarrow \mathcal{N}(M_0)$$
$$\Psi(\Delta) = M_0$$
$$\pi(\Psi(q, q)) = q \in M_0$$

We fix a Hamiltonian diffeomorphism $\psi \in \mathrm{Ham}(M, \omega)$ which is sufficiently C^1-close to the identity. We define a closed 1-form σ_ψ by

$$\sigma_\psi = \Psi(\mathrm{graph}(\psi)).$$

Next we fix a 1-periodic Hamiltonian function $H \in C^\infty(S^1 \times M)$ so that $\phi_H = \psi$ holds. We define a path of symplectomorphisms $\{\psi^t\}_{0 \le t \le 1}$ as follows:

$$\{(q, \psi^t(q))\}_{q \in M} = \Psi^{-1}(t\sigma_\psi)$$

In particular, $\sigma_{\psi^t} = t\sigma_\psi$ holds. Our purpose is to prove that $\{\psi^t\}$ is a Hamiltonian isotopy. It sufficies to prove that σ_ψ is an exact 1-form because the flux of the path $\{\psi^t\}_{0 \le t \le T}$ is equal to $T[\sigma_\psi]$ for any $T \in [0, 1]$. Let γ_t be a loop of symplectomorphisms defined as follows:

$$\gamma_t = \begin{cases} \psi_{2t} & 0 \le t \le \frac{1}{2} \\ \phi_H^{2(1-t)} & \frac{1}{2} \le t \le 1 \end{cases}$$

The flux of the loop $\{\gamma^t\}$ is equal to $[\sigma_\psi] \subset \Gamma$. Note that Γ is a discrete subgroup of $H^1(M : \mathbb{R})$. This implies that $[\sigma_\phi] = 0$ holds if σ_ψ is a sufficiently small closed 1-form on M. In particular, $\{\psi^t\}$ is a Hamiltonian path. So, we proved the proposition.

Remark 2. *This argument can be used to prove that C^1-topology is stronger than the topology induced from Hofer's metric ([17]).*

Henceforth, we assume that (M, ω) is a $2n$-dimensional closed negatively monotone symplectic manifold. Let $H \in C^\infty(S^1 \times M)$ be a non-degenerate Hamiltonian function such that ϕ_H displaces a symplectically embedded r-ball. By lemma 2, we can choose two capped periodic orbits $\bar{x}, \bar{y} \in \widetilde{\mathcal{P}}(H)$ so that

$$-n \le \mu(\bar{x}) \le 0$$
$$0 \le \mu(\bar{y}) \le n$$
$$A_H(\bar{y}) - A_H(\bar{x}) \ge \pi r^2$$

holds. The k-th power $\bar{x}^{(k)}, \bar{y}^{(k)} \in \widetilde{\mathcal{P}}(H^{(k)})$ satisfies

$$\mu(\bar{x}^{(k)}) \le 0 \le \mu(\bar{y}^{(k)})$$
$$A_{H^{(k)}}(\bar{y}^{(k)}) - A_{H^{(k)}}(\bar{x}^{(k)}) = k\{A_H(\bar{y}) - A_H(\bar{x})\} \ge k\pi r^2.$$

We choose $A_k, B_k \in \pi_2(M)$ so that

$$0 \le \mu(\bar{x}^{(k)} \sharp A_k) \le 2N$$
$$0 \le \mu(\bar{y}^{(k)} \sharp B_k) \le 2N$$

holds. This $N \in \mathbb{N} \cup \{+\infty\}$ is the minimal Chern number of (M, ω). In other words, N is the positive generator of the image of $c_1 : \pi_2(M) \to \mathbb{Z}$. We assume that B_k is trivial if

$$0 \le \mu(\bar{y}^{(k)}) \le 2N$$

holds. This implies that

$$A_{H^{(k)}}(\bar{y}^{(k)} \sharp B_k) - A_{H^{(k)}}(\bar{x}^{(k)} \sharp A_k) \ge A_{H^{(k)}}(\bar{y}^{(k)}) - A_{H^{(k)}}(\bar{x}^{(k)})$$
$$= k(A_H(\bar{y}) - A_H(\bar{x})) \ge k\pi r^2$$

holds because

$$c_1(A_k) \le 0, \ \omega(A_k) \ge 0$$

holds. Next, we assume that $(\phi_H)^k \subset \mathcal{U}$ holds where $\mathcal{U} \subset \mathrm{Ham}(M, \omega)$ is a C^1-small open neighborhood of the identity. Proposition 1 implies that we can connect $(\phi_H)^k$ and Id by a C^1-small Hamiltonian isotopy generated by a Hamiltonian function $K \in C^\infty(S^1 \times M)$. In particular,

$$\phi_{H^{(k)} \sharp K} = \mathrm{Id}$$

holds. By concatenating this small isotopy, $\bar{x}^{(k)} \sharp A_k$ and $\bar{y}^{(k)} \sharp B_k$ become capped periodic orbits $z_k, w_k \in \widetilde{\mathcal{P}}(H^{(k)} \sharp K)$ such that

$$-\epsilon \le \mu(z_k) \le 2N + \epsilon$$
$$-\epsilon \le \mu(w_k) \le 2N + \epsilon$$
$$A_{H^{(k)} \sharp K}(w_k) - A_{H^{(k)} \sharp K}(z_k) \ge k\pi r^2 - \epsilon$$

holds for some $\epsilon > 0$. $\epsilon > 0$ is determined by $\mathcal{U}$. Note that $\phi_{H^{(k)} \sharp K} = \mathrm{Id}$ holds. We compare $\widetilde{\mathcal{P}}(H^{(k)} \sharp K)$ and $\widetilde{\mathcal{P}}(0) = \Gamma_{(M,\omega)}$. More generally, we can construct a one to one correspondence between $\widetilde{\mathcal{P}}(G_1)$ and $\widetilde{\mathcal{P}}(G_2)$ if $\phi_{G_1} = \phi_{G_2}$ holds (see [16], section 3.1). We fix a Hamiltonian function L so that

$$\phi^t_{G_2} = \phi^t_L \circ \phi^t_{G_1}$$

holds. Note that L generates a Hamiltonian loop. Let $\mathcal{L}(M)$ be the space of contractible loops in M.

$$\mathcal{L}(M) = \{x : S^1 \to M \mid x : \text{contractible}\}$$

We define a covering $\pi : \widetilde{\mathcal{L}}(M) \to \mathcal{L}(M)$ as follows:

$$\widetilde{\mathcal{L}}(M) = \{(u,x) \mid x \in \mathcal{L}(M), u : D^2 \to M, \partial u = x\}/\sim$$
$$(u,x) \sim (v,y) \iff \begin{cases} x = y \\ \int_{S^1} (u\sharp\overline{v})^*\omega = 0 \\ \int_{S^1} (u\sharp\overline{v})^* c_1 = 0 \end{cases}$$

The Hamiltonian loop $\{\phi_L^t\}$ acts on $\mathcal{L}(M)$ as follows:

$$f : \mathcal{L}(M) \longrightarrow \mathcal{L}(M)$$
$$f(x)(t) = \phi_L^t(x(t))$$

Note that $f(\mathcal{P})(G_1) = \mathcal{P}(G_2)$ holds. Let $\widetilde{f}$ be a covering transformation such that the following diagram is commutative.

$$\begin{array}{ccc} \widetilde{\mathcal{L}}(M) & \xrightarrow{\widetilde{f}} & \widetilde{\mathcal{L}}(M) \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathcal{L}(M) & \xrightarrow{f} & \mathcal{L}(M) \end{array}$$

We see that $\widetilde{f}$ restricted to $\widetilde{\mathcal{P}}(G_1)$ makes shifts of the action functional and the mean index.

Lemma 3. *For any $z, w \in \widetilde{\mathcal{P}}(G_1)$, the following equations holds:*

$$A_{G_1}(z) - A_{G_1}(w) = A_{G_2}(\widetilde{f}(z)) - A_{G_2}(\widetilde{f}(w))$$
$$\mu(z) - \mu(w) = \mu(\widetilde{f}(z)) - \mu(\widetilde{f}(w))$$

For the first equation, we prove that $A_{G_1}(z) - A_{G_2}(\widetilde{f}(z))$ does not depend on $z \in \widetilde{\mathcal{L}}(M)$. It suffices to prove that the differential

$$T_z\widetilde{\mathcal{L}}(M) \longrightarrow \mathbb{R}$$
$$X \mapsto D(A_{G_1}(z) - A_{G_2}(\widetilde{f}(z)))X$$

vanishes for any $z = [u, x]$. Note that $T_z\widetilde{\mathcal{L}}(M) = \Gamma(x^*TM)$ holds. So X is a section of the vector bundle $x^*TM \to S^1$.

$$D(A_{G_1}(z))X = -\int_{S^1} \omega(X(t), \dot{x}(t))dt + \int_{S^1} d(G_1)_t \cdot X(t)dt$$

$$
\begin{aligned}
&D(A_{G_2}(\widetilde{f}(z)))X \\
=&-\int_{S^1}\omega(d\phi_L^t(X(t)),\frac{\partial}{\partial t}(\phi_L^t(x(t))))dt+\int_{S^1}d(G_2)_t\cdot(d\phi_L^t\cdot X(t))dt \\
=&-\int_{S^1}\omega(d\phi_L^t(X(t)),X_{L_t}+d\phi_L^t(\dot{x}(t)))dt \\
&+\int_{S^1}(dL_t+d(G_1)_t\circ d(\phi_L^t)^{-1})\circ(d\phi_L^t\cdot X(t))dt \\
=&-\int_{S^1}\Big\{\omega(X(t),\dot{x}(t))+dL_t\circ d\phi_L^t(X(t))\Big\}dt \\
&+\int_{S^1}\Big\{dL_t\circ d\phi_L^t(X(t))+d(G_1)_t\cdot X(t)\Big\}dt \\
=&-\int_{S^1}\omega(X(t),\dot{x}(t))dt+\int_{S^1}d(G_1)_t\cdot X(t)dt \\
=&\,D(A_{G_1}(z))X
\end{aligned}
$$

So, $D(A_{G_1}(z)-A_{G_2}(\widetilde{f}(z)))$ is zero and the difference $A_{G_1}(z)-A_{G_2}(\widetilde{f}(z))$ is a constant function on $\widetilde{\mathcal{L}}(M)$. Next, we prove that $\mu(z)-\mu(\widetilde{f}(z))$ does not depend on $z\in\widetilde{\mathcal{P}}(G_1)$. We fix $z=[u,x]$ and $w=[v,y]$ in $\widetilde{\mathcal{P}}(G_1)$. We choose a cylinder $C:[0,1]\times S^1\to M$ which connects x and y and $w=[u\sharp C,y]$ holds. Let $D:[0,1]\times S^1\to M$ be a cylinder defined as follows:

$$D(s,t)=\phi_L^t(C(s,t))$$

We also fix a trivialization of the symplectic vector bundle C^*TM. Then x and y determines two paths of symplectomorphisms on the symplectic vecor space $(\mathbb{R}^{2n},\omega_0)$. $\mu(z)-\mu(w)$ is a difference of the mean index of these two paths. Similary, $\mu(\widetilde{f}(z))-\mu(\widetilde{f}(w))$ is caluculated by fixing a trivialization of D^*TM. One such trivialization is obtained by the trivialization of C^*TM and the Hamiltonian loop $\{\phi_L^t\}$. So the equality

$$\mu(z)-\mu(w)=\mu(\widetilde{f}(z))-\mu(\widetilde{f}(w))$$

holds and we proved Lemma 3.

Lemma 3 and $\phi_{H^{(k)}\sharp K}=\mathrm{Id}$ implies that there is a transformation

$$\widetilde{f}:\widetilde{\mathcal{P}}(H^{(k)}\sharp K)\longrightarrow\widetilde{\mathcal{P}}(0)=\Gamma_{(M,\omega)}$$

such that

$$
\begin{aligned}
A_{H^{(k)}\sharp K}(z)-A_{H^{(k)}\sharp K}(w)&=A_0(\widetilde{f}(z))-A_0(\widetilde{f}(w)) \\
&=\kappa(\mu(\widetilde{f}(z)-\mu(\widetilde{f}(w)))=\kappa(\mu(z)-\mu(w))
\end{aligned}
$$

holds for any capped periodic orbits $z,w\in\widetilde{\mathcal{P}}(H^{(k)}\sharp K)$. However, this is a contradiction for sufficiently large $k\in\mathbb{N}$ because

$$|\mu(z_k)-\mu(w_k)|\leq 2N+2\epsilon$$
$$|A_{H^{(k)}\sharp K}(z_k)-A_{H^{(k)}\sharp K}(w_k)|\geq k\pi r^2-\epsilon\to+\infty$$

holds. So, we proved Theorem 1 (1). Next, we prove Theorem 1 (2). We need the following Proposition.

Proposition 2 ([1] Proposition 1.1)**.** *Let (M, ω) be a negatively monotone symplectic manifold whose Euler number is not zero or $\pi_1(M)$ has finite center. Then, the flux group Γ is trivial.*

We fix a symplectomorphism $\psi \in \mathrm{Symp}_0(M, \omega) \backslash \{\mathrm{Id}\}$. We choose a path of symplectomorphisms $\{\psi^t\}$ which connects Id and ψ ($\psi^0 = \mathrm{Id}$, $\psi^1 = \psi$). We extend $\{\psi^t\}_{0 \leq t \leq 1}$ to an isotopy $\{\psi^t\}_{t \in \mathbb{R}}$ periodically. Let $\mathcal{V} \subset \mathrm{Symp}_0(M, \omega)$ be a sufficiently C^1-small open neighborhood of the identity so that any element in $\mathcal{V}$ can be connected to Id by a C^1-small isotopy of symplectomorphisms. Assume that $\psi^k \in \mathcal{V}$ holds. We connect Id and ψ^k by a C^1-small isotopy of symplectomorphisms $\{\phi^t\}$. Let $\{\gamma^t\}$ be a loop of symplectomorphisms defined as follows:

$$\gamma^t = \begin{cases} \psi^{2kt} & 0 \leq t \leq \frac{1}{2} \\ \phi^{2(1-t)} & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Proposition 2 implies that

$$0 = [\mathrm{Flux}(\{\gamma^t\})] = k[\mathrm{Flux}(\{\psi^t\}_{0 \leq t \leq 1})] - [\mathrm{Flux}(\{\phi^t\})]$$

holds. Note that $[\mathrm{Flux}(\{\phi^t\})] \in H^1(M : \mathbb{R})$ is contained in a small neighborhood of 0 because $\{\phi^t\}$ is a C^1-small isotopy of symplectomorphisms. So $[\mathrm{Flux}(\{\psi^t\}_{0 \leq t \leq 1})]$ is zero if k is sufficiently large. In particular, ψ is a Hamiltonian diffeomorphism. This implies that Theorem 1 (2) follows from Theorem 1 (1).

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