NO DIVISION IMPLIES CHAOS

BY

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ABSTRACT. Let I be a closed interval in R^1 and $f\colon I \to I$ be continuous. Let $x_0 \in I$ and
$$x_{i+1} = f(x_i) \quad \text{for } i > 0.$$
We say there is no division for $(x_1, x_2, \ldots, x_n)$ if there is no $a \in I$ such that $x_j < a$ for all j even and $x_j < a$ for all j odd. The main result of this paper proves the simple statement: no division implies chaos.

Also given here are some converse theorems, detailed estimates of the existing periods, and examples which show that, under our conditions, one cannot do any better.

1. Introduction. Let I be a closed interval in R^1 and $f\colon I \to I$ be a continuous function. A point $x \in I$ is said to be a point of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $1 \leq k < n$. For brevity, we will say property $P(k)$ holds if f has a periodic orbit with period k. The theorem by A. N. Sharkovski [5] answered the general question of when the existence of a point of period n implies the existence of points of peroid l, that is, when $P(n) \Rightarrow P(l)$. The proof of this theorem was considerably shortened in [2].

It is not always immediately apparent whether a periodic orbit of period n exists. In [4], this uncertainty was overcome by use of a more robust hypothesis. We say $(x_0, x_1, \ldots, x_n)$ is a *trajectory* if

$$(1.1) \qquad x_{i+1} = f(x_i)$$

for $i = 0, 1, \ldots, n - 1$. It was proved there [4] that if there is a trajectory $(x_0, \ldots, x_n)$ satisfying either

$$(1.2) \qquad x_n \leq x_0 < x_1 \quad \text{or} \quad x_n \geq x_0 > x_1$$

with n odd, then $P(k)$ holds for some k where n/k is an integer. In [4] we show that $P(k)$ for odd $k > 1$ implies "chaos" and we discuss the meaning of chaos.

The main purpose of this paper is to study situations for which n is even in (1.2) and yet there still must be an orbit of odd period. For this, we introduce the notion of 'no division'. For the n-tuple $(x_0, x_1, \ldots, x_n) \subset I$ we say that there is no division

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if there is no $a \in I$ such that either $x_j < a$ for all j even and $x_j > a$ for all j odd or $x_j > a$ for all j even and $x_j < a$ for all j odd. In §2, we prove that if $(x_0, \ldots, x_n)$ is a trajectory which has no division and satisfies (1.2), then $P(k)$ holds for some odd k. In §3, we obtain better estimates of the existing periods under the above conditions when n is even. In §4, we give examples to show that under the assumptions of our theorems we cannot obtain stronger results.

2. For future reference, we quote the Sharkovski's theorem in the following:

THEOREM 2.1. *Let f be continuous. Then* $P(3) \Rightarrow P(5) \Rightarrow P(7) \Rightarrow \cdots \Rightarrow P(2 \cdot 3) \Rightarrow P(2 \cdot 5) \Rightarrow \cdots \Rightarrow P(2^2 \cdot 3) \Rightarrow P(2^2 \cdot 5) \Rightarrow \cdots \Rightarrow P(2^3) \Rightarrow P(2^2) \Rightarrow P(2) \Rightarrow P(1).$

For what follows, we let $x_k = f^k(x_0)$ for all $k \geq 0$.

PROPOSITION 2.2. *Let $x_n \leq x_0 < x_1$, or $x_n \geq x_0 > x_1$, and let there be no division for the trajectory $(x_0, x_1, \ldots, x_n)$. Then there exists a periodic point of period*
   (1) n *if n is odd*,
   (2) $n - 1$ *if n is even.*

PROOF. For simplicity, we only consider $x_n \leq x_0 < x_1$. The other case follows by reversing signs in the argument. Let $X = \{x_0, \ldots, x_{n-1}\}$. Since $f(x_0) > x_0$ and $f(\max X) < \max X$, the set $S = \{x \in X: x > x_0, f(x) < x\}$ is nonempty. Let $x_i = \min S$ and $x_j = \max\{x \in X: x < x_i\}$. Then, for $J = [x_j, x_i]$ we have int $J \cap X = \varnothing$ and $f(J) \supset J$. Since $x_n = f^{n-j}(x_j)$ is in $f^{n-j}(J)$ and x_j is in $J \subset f^{n-j}(J)$, and $x_n \leq x_0 \leq x_j$, we have $x_0 \in f^{n-j}(J)$. Therefore, all elements of X, except perhaps x_{j-1} (or x_{n-1} if $j = 0$) are in $\bigcup_{i=0}^{n-2} f^i(J)$. Since $f(J) \supset J$, $f^{n-2}(J) \supset f^{n-1}(J) \supset \cdots \supset f(J) \supset J$, hence $\bigcup_{i=0}^{n-2} f^n(J) = f^{n-2}(J)$. We may replace x_j with x_i in the above argument, and conclude that every point of X except perhaps x_{i-1} is in $f^{n-2}(J)$. Since $i \neq j$, we have $f^{n-2}(J) \supset X$.

Since there is no division, there exists an interval $K = [c, d]$ such that $c, d \in X$ and $(c, d) \cap X = \varnothing$ and c and d lie on the same side of J but $f(c)$ and $f(d)$ lie on the opposite sides of J. Therefore $f(K) \supset J$ and K and J have at most one point in common. We thus have the following:
   (1) $f(J) \supset J$,
   (2) $f(K) \supset J$,
   (3) $f^{n-2}(J) \supset K$.
To find periodic points we use the technique from [1] and [2].

From (2) and (3), there is a point $x \in J$, with $f^{n-1}(x) = x$ and $f^{n-2}(x) \in K$. From (1), (2) and (3), there is a point $y \in J$ with $f^n(y) = y$ and $f(y) \in J$ and $f^{n-1}(y) \in K$. Neither x nor y are fixed points of f, because they would then be common endpoints of J and K. Such a common endpoint (if it exists) is an element of X and lies to the right of (or is equal to) x_0. Some iterate of it is equal to x_n, and therefore, it cannot be a fixed point.

If n is odd (resp. even) then y (resp. x) is a peroidic point whose period is a factor of n (resp. $n - 1$). Hence, by Theorem 2.1, there exists a periodic point of period n (resp. $n - 1$).   □

REMARK. If n is odd and $x_n \leqslant x_0 < x_1$ or $x_n \geqslant x_0 > x_1$ then automatically there is no division for $(x_0, x_1, \ldots, x_n)$.

As a corollary to Proposition 2.2, we have

THEOREM 2.3. *Let $x_n \leqslant x_0 < x_1$ or $x_n \geqslant x_0 > x_1$ and assume there is no division for $(x_0, x_1, \ldots, x_n)$. Then there exists a periodic point of odd ($\neq 1$) period.*

The above results show it is sometimes possible to conclude there is a periodic orbit by knowing only the trajectory through one point. We can ask how much improvement is possible in our results. Suppose we have a bounded trajectory $Y = (y_0, y_1, \ldots)$. We assume for the remainder of this section that there is at least one continuous F which has Y as a trajectory. Is possible that Y implies the existence of an odd period orbit and yet our results are insufficient for us to be able to conclude this existence?

We say Y *implies the existence of an odd period orbit* if every continuous f that has Y as a trajectory also has an odd period orbit with period greater than one.

We now prove that the above results can be applied to every (infinite) trajectory Y that implies the existence of an odd period orbit.

THEOREM 2.4. *Let $(y_0, y_1, \ldots)$ be a sequence which implies the existence of an odd period orbit. Then there exist integers u, v such that if $x_0 = y_u$, $x_1 = y_{u+1}, \ldots, x_v = y_{u+v}$, then there is no division for $(x_0, \ldots, x_v)$ and (1.2) is satisfied for $n = v$.*

PROOF. Denote $B = \{y_0, y_1, \ldots\}$, $J = [\inf B, \sup B]$. It is easy to see that we may assume that f is monotone on components of $J \setminus B$ and constant to the left and to the right of J. By assumption, there is a periodic point of F with odd period n ($\neq 1$). Denote the orbit of this point by X. We shall consider the following two cases:

Case 1: $X \subset \overline{B}$.

Choose $x \in X \{\min X, \max X\}$. Without loss of generality we assume $f(x) > x$. We have $f^k(x) = f^{n+k}(x) = \min X < x$ for some $k > 1$. One of the integers k of $k + n$ is odd. Call this integer v. Then, $f^v(x) < x < f(x)$. Since $x \in \overline{B}$, we can choose $x_0 \in B$ with $f^v(x_0) < x_0 < f(x_0)$. By the remark, there is no division for $(x_0, f^1(x_0), f^2(x_0), \ldots, f^v(x_0))$. This ends the proof in this case.

Case 2: $X \not\subset \overline{B}$.

Since $f(\overline{B}) \subset \overline{B}$, X is disjoint from $\overline{B}$. Also $f(I) \subset J$, so $X \subset J \setminus \overline{B}$. For any $x \in J \setminus \overline{B}$ let J_x denote the component of $J \setminus \overline{B}$ containing x. For $c \in X$, we say c is an increasing point if $f(c) > c$. Otherwise, we call c a decreasing point. The orbit of X forms a cycle of n points. Since n is odd, we cannot have alternating increasing and decreasing points. Therefore, there exists $c \in X$ with either $c < f(c) < f^2(c)$ or $c > f(c) > f^2(c)$. Without loss of generality, we may assume that $c < f(c) < f^2(c)$. Since f is monotone on J_c, the image of one of the endpoints of J_c is to the left of $f(c)$. Denote this endpoint by a. Therefore $f(a) < f(c)$. Similarly, we can find an endpoint b of $J_{f(c)}$ such that $f(b) > f^2(c)$. Clearly, $b \leqslant \sup J_{f(c)} \leqslant \sup J_{f^2(c)} \leqslant f(b)$. If $b = f(b)$, then $J_{f(c)} = J_{f^2(c)}$ and $b = \sup J_{f(c)}$. It follows that $f([f(c), b]) = [f^2(c), b] \subset [f(c), b]$, and $\lim_{n \to \infty} f^n(c) = b$. This contradicts the periodicity of c.

Therefore,

(2.1) $b < f(b)$.

Suppose that $f(a) \leqslant a \leqslant b$. Since $a \geqslant f(a)$ and $b < f(b)$, we have $a \neq b$. Hence, $f(a) \leqslant a < b < f(b)$.

If $f(a) < a$, then since $a \in \bar{B}$, we can find x and $x' \in B$ with $f(x) < x < (a + b)/2$ and $(a + b)/2 < x' < f(x')$. We have $x = y_k$, $x' = y_m$ for some k, m. Therefore $y_{k+1} < y_k < y_m < y_{m+1}$. Since the first pair of subscripts contain one odd and one even and the same is true of the final pair, there is no division. The conclusion of the theorem is proved.

We now consider the sole remaining situation: $f(a) = a$.

If a is an isolated point of $\bar{B}$, then $a \in B$ and consequently $a = y_q$ for all q sufficiently large. Hence, $y_q < y_m < y_{m+1}$ for some $m < q$ with $q - m$ being odd. The conclusion of the theorem thus follows. If a is not an isolated point of $\bar{B}$, then there exists y_q with arbitrarily large q, arbitrarily close to a. We may require $q > m$ and y_q, $f(y_q) < b$. But then one of the integers $q - m$ and $q + 1 - m$ is odd and again we have the conclusion of the theorem.

The remaining situation is either

(2.2) $f(a) > a$ or $a > b$.

Suppose that $J_c = J_{f(c)}$. Since $c < f(c)$ and $f(c) < f^2(c)$, f is increasing in J_c. Therefore, $f(a) \leqslant a = \inf J_c < \sup J_c = b \leqslant f(b)$ which contradicts (2.2). Hence $J_c \neq J_{f(c)}$, and $a, f(a) \leqslant \inf J_{f(c)} \leqslant b$. In view of (2.1) and (2.2), we have

(2.3) $a < f(a) \leqslant b < f(b)$.

Since a, $b \in \bar{B}$, we can choose x, $x' \in B$ such that $x < f(x) \leqslant x' < f(x')$. We have $x = y_k$, $x' = y_m$ for some k, m. If we can choose x, x' in such a way that $k > m$, then we are done since $y_k < y_{k+1} \leqslant y_m < y_{m+1}$ and one of the integers $k - m$, $k + 1 - m$ is odd. There are only two cases in which we cannot choose such x and x': either x and x' cannot be chosen independently, i.e., $f(a) = b$, or there is no y_i with large i in some neighborhood of a.

Suppose $a \notin B$. Then the second possibility cannot occur, and consequently we are left with $f(a) = b$. We may take three elements of B arbitrarily close to a: y_{m-1}, y_k, y_p such that $m - 1 < k < p$. Then we have $y_p < y_m < y_{m+1}$ and $y_k < y_{k+1} < y_{k+2}$, and for $y_m = x_0, \ldots, y_p = x_v$ there is no division.

If $a \in B$, then we take $x = y_k = a$. If $f^2(a) \leqslant f(a)$, then $a < b$. We may find $x' = y_m$ and $y_{k+2} \leqslant y_{k+1} < y_m < y_{m+1}$, which has no division.

If $f^2(a) > f(a)$, then we have $a = y_k < y_{k+1} < y_{k+2}$. Let $K = [\inf_{i \geqslant k+1} y_i$, $\sup_{i \geqslant k+1} y_i]$. Since $y_{k+1} = f(a) < f(c) < f^2(c) = f(x') = y_{m+1}$, and $m + 1 > k + 1$, we have $X \cap K$ is nonempty. Hence $c \in f^p(K)$ for some p. Since f is monotone on components of $J \setminus \bar{B}$, there exists $z \in B \cap K$ such that $f^p(z) \leqslant \inf J_c \leqslant a$. We have $z = y_s$ for some s. If $s \geqslant k + 1$, then $f^p(z) = y_{s+p}$ and $s + p > k + 1$. It follows that $y_{s+p} \leqslant y_k < y_{k+1} < y_{k+2}$ which has no division because of the last three terms. If $s < k + 1$ then $y_t < y_s < y_r$ for some t, $r \geqslant k + 1 > s$. The result of the theorem is proved no matter where y_{s+1} is, unless either $t = k + 1$ and $y_s < y_{s+1}$, or $r = k + 1$

and $y_{s+1} < y_s$. In the first case, we get $y_k < y_{k+1} < y_s < y_{s+1}$ and $s < k$, and in the second case $y_{s+1} < y_s < y_k < y_{k+1}$ and $s < k$, which are cases where there is no division. $\square$

3. Let $f: I \to I$ be continuous and $x_i = f^i(x_0)$ for all $i \geq 1$. In this section, we obtain better estimates of the periods for the case where $x_n \leq x_0 < x_1$ or $x_n \geq x_0 > x_1$ with even n and there is no division for $(x_0, x_1, \ldots, x_n)$. We first prove the following lemma:

LEMMA 3.1. *Let* $x_{n+1} \leq x_n < x_0 \leq x_1$ *or* $x_{n+1} \geq x_n > x_0 \geq x_1$ *for some* $n > 1$. *Then f has periodic points of all periods.*

PROOF. Notice that $x_0 \neq x_1$ since otherwise, $x_n = x_0$. As usual we shall only consider the case $x_{n+1} \leq x_n < x_0 < x_1$. Without loss of generality, we may assume that n is the smallest positive integer for which $x_{n+1} \leq x_n < x_0$. Let

$$m = \min\{r \geq 0, \; x_{r+1} \leq x_n\}.$$

Clearly, $m < n$. If $x_m < x_0$, then $x_{m+1} \leq x_n < x_m < x_0$, which contradicts the definition of n. If $x_m = x_0$ then $x_{m+1} = x_1$, but $x_{m+1} \leq x_n < x_1$, a contradiction. Consequently, $x_m > x_0$. Let $k = \min\{r \geq 0: x_{r+1} \geq x_m\}$. Then $k < m$, $x_k < x_m$ and $x_{k+1} \geq x_m$. Since $k < m$, we have $x_k > x_n$. Let $K = [x_n, x_k]$ and $M = [x_k, x_m]$. Then, $f(K) \supset [x_{n+1}, x_{k+1}] \supset K \cup M$ and $f(M) \supset [x_{m+1}, x_{k+1}] \supset K \cup M$. Since $K \cap M = \{x_k\}$ and x_k is not a fixed point of f, f has periodic points of all periods. $\square$

COROLLARY 3.2. *If f does not have periodic points of all periods, then there exists a fixed point z of f such that if $x_i < z$ then $x_{i+1} > x_i$, and if $x_i > z$ then $x_{i+1} < x_i$.*

PROOF. Suppose that f does not have periodic points of all periods. Then from Lemma 3.1, it follows that if $x_{m+1} \leq x_m$ and $x_{k+1} \geq x_k$ then $x_m \leq x_k$. Let $a = \sup\{x_i: x_{i+1} \geq x_i\}$, $b = \inf\{x_i: x_{i+1} \leq x_i\}$. Then $a \leq b$. If $f(a) = a$, we may take $z = a$; if $f(b) = b$, we may take $z = b$. If not, then $f(a) \geq b$ and $f(b) \leq a$ and there is a fixed point z between a and b. Clearly, such z has the required properties. $\square$

LEMMA 3.3. *Let* $X = (x_0, x_1, \ldots, x_{n-1}, x_n)$ *have no division and assume n is even and* $x_n = x_0$. *Then, f has a periodic orbit of period $n/2$ if $n/2$ is odd, period $(n/2) + 1$ if $n/2$ is even.*

PROOF. Suppose the conclusion of the lemma is false. Then, by Corollary 3.2, there is a fixed point z of f, such that for any $x \in X$, if $x < z$ then $f(x) > x$ and if $x > z$ then $f(x) < x$. We consider the two cases separately. Since $x_n = x_0$, we have $x_{i+n} = x_i$ for all i.

Case 1: $n/2$ *is odd*. Suppose that for some k, x_k and x_{k_*} where $k_* = k + n/2$ are on the same side of z. If x_k is closer to z, then we have $x_{k_*} < x_k < x_{k+1}$ or $x_{k_*} > x_k > x_{k+1}$ and we get a periodic point of period $n/2$, a contradiction. If x_{k_*} is closer to z, then we can use the same type of argument because $x_k = x_{n+k}$ and also get a contradiction. Hence, for all k, x_k and x_{k_*} are on opposite sides of z. We may assume, without loss of generality, that there are no elements of X in $(x_0, z]$. Since

there is no division for X, there exists d odd, $0 < d < n$, such that $x_d < z$. Hence, $x < x_0 < z < x_{n/2}$. We know that $z < x_{d_*}$ where $d_* = d + n/2$. If $x_{d_*} < x_{n/2}$ then either $d \leqslant n/2$ and then $x_d < x_0 < x_1$ gives a contradiction, or $d > n/2$ and then (writing $t = d - n/2$) $x_{t+1} < x_t < x_{n/2}$ gives a contradiction. It follows that $x_d < x_0 < z < x_{n/2} < x_{d_*}$. Since $x_0 < z$, we have $x_1 > x_0$ and consequently $x_1 > z$. Hence, $x_{n/2+1} < z$, and $x_{n/2+1} < x_0$. Since $x_d < x_0 < x_1$, we must have $d > n/2$. If $x_1 < x_t$, then $x_2 < x_1 < x_t$ we get a periodic point of odd period $d - n/2 - 1 \leqslant n/2$, a contradiction ($t \neq 1$ since $x_1 < x_t$). Therefore $x_t < x_1$ (there is no equality since $d - n/2$ is even). Hence, $x_{1+n/2} < x_0 < x_{n/2} < x_t < x_1$. Since $n/2$ is odd and t is even, there is no division for $\{x_0, x_1, \ldots, x_{n/2+1}\}$. By Proposition 2.1, there exists a periodic point of period $n/2$.

Case 2: $n/2$ *is even*. By Theorem 2.1, if there is no periodic point of period $n/2 + 1$ then there is no periodic point of period $n/2 - 1$. By an argument similar to the one used in Case 1, we establish that every k, x_k and x_{k_*-1} are on the opposite sides of z. Also, x_k and x_{k_*+1} are on the opposite sides of z. Hence, for every m, x_m and x_{m+2} are on the same side of z. Consequently, either all elements of x are on the same side of z, or there is a division for X. In both cases we obtain a contradiction.   □

PROPOSITION 3.4. *Let* $x_n \leqslant x_0 < x_1$ *or* $x_n \geqslant x_0 > x_1$ *with* n *even, and assume there is no division for* $(x_0, x_1, \ldots, x_n)$. *Then there exists a periodic point of period* $n/2$ *if* $n/2$ *is odd, and* $n/2 + 1$ *if* $n/2$ *is even.*

PROOF. Suppose the conclusion of the proposition is false. Write $X = \{x_0, \ldots, x_n\}$. By Corollary 3.2, there is a fixed point z of f such that for $x \in X$, if $x < z$ then $f(x) > x$ and if $x > z$ then $f(x) < x$. It is obvious that z is not an element of X. We will first find a periodic orbit of period n with no division and then apply Lemma 3.3. For simplicity, we only examine the case $x_n \leqslant x_0 < x_1$.

Let J_k denote the interval: $[x_k, z]$ if $x_k < z$ and $[z, x_k]$ if $x_k > z$, $k = 0, 1, \ldots, n$. Since $x_1 > x_0$, we have $x_0 < z$. Hence, $J_n \supset J_0$. It follows that $f(J_k) \supset J_{k+1}$ for $k = 0, 1, \ldots, n - 2$ and $f(J_{n-1}) \supset J_0$. Since there is no division for X, there exists $q \leqslant n - 1$ such that x_q, x_{q+1} are on the same side of z. Look subsequently at the points $x_{q+2}, x_{q+3}, \ldots, x_{n-1}, x_0, x_1, \ldots, x_{q-1}$. Some of them have to lie on the opposite side of z from x_q. Let x_p be the first such point. Then x_{p-2}, x_{p-1} lie on one side of z and x_p is on the opposite side (the indices are interpreted mod n). Suppose $x_{p-2} = x_{p-1}$. If $p - 2 = n - 2$ and $x_{p-1} = x_{p-2}$, then x_{p-1} and x_p are on the same side of z. If $p - 2 = n - 1$ and $x_{p-2} = x_{p-1}$ (note: $x_{p-1} = x_0$), then $x_n = x_1$. If $p - 2 < n - 2$ then $x_p \neq x_{p-1}$. In all cases, we get a contradiction. Consequently, $x_{p-2} \neq x_{p-1}$. Let P denote the interval $[x_{p-2}, x_{p-1}]$ if $x_{p-2} < x_{p-1}$ and $[x_{p-1}, x_{p-2}]$ if $x_{p-1} > x_{p-2}$.

We have $P \subset J_{p-2}$ and therefore $f(J_{p-3}) \supset P$. If $p - 2 < n - 2$ then $f(P) \supset [x_{p-1}, x_p]$ (or $[x_p, x_{p-1}]$), and since x_{p-1} and x_p are on the opposite sides of z, $f(P) \supset J_{p-1}$. If $p - 2 = n - 2$ then $f(P) \supset [x_n, x_{n-1}] \supset [z, x_{n-1}] = J_{p-1}$. If $p - 2 = n - 1$ then $f(P) \supset [x_n, x_1] \supset J_0 = J_{p-1}$. In all cases, we thus have $f(P) \supset J_{p-1}$. It follows that $\ldots, f(J_{p-4}) \supset J_{p-3}, f(J_{p-3}) \supset P, f(P) \supset J_{p-1}, f(J_{p-1}) \supset J_p, \ldots$ and the

cycle closes. The length of this cycle is n. There is a periodic point $y \in P$ such that $f(y) \in J_{p-1}, f^2(y) \in J_p, \ldots, f^{n-2}(y) \in J_{p-4}, f^{n-1}(y) \in J_{p-3}$, and $f^n(y) = y$. Since the intervals P, J_{p-1}, J_p have pairwise disjoint interiors and J_{p-1} lies between P and J_p, we have either $y < f(y) < f^2(y)$ or $y > f(y) > f^2(y)$. Hence, for the orbit of y there is no division. This orbit has a period $k \geqslant 3$ and k divides n. If k is odd then we have $k \leqslant n/2$ and we use Theorem 2.1. If k is even, then we use Lemma 3.3 and then (if necessary) Theorem 2.1. $\square$

We now investigate what happens if the lack of division occurs "too late" (i.e., after n with $x_n \leqslant x_0 < x_1$).

PROPOSITION 3.5. *Let either* $x_n \leqslant x_0 < x_1$ *and* $x_d \leqslant x_p$ *or* $x_n \geqslant x_0 > x_1$ *and* $x_d \geqslant x_p$, *for some* d, p, n *where* $d \leqslant n$ *and* d *is odd and* p *is even. Then there exists a periodic orbit of some odd* ($\neq 1$) *period* $k \leqslant \max\{n, p - d\}$.

PROOF. As usual we only prove the case for $x_n \leqslant x_0 < x_1$ and $x_d \leqslant x_p$. If n is odd we get, by Proposition 2.2 and Remark after it, a periodic point of period n. So, we only consider the case where n is even.

Case 1: $x_d \leqslant x_0$. Then, $x_d \leqslant x_0 < x_1$ and we get a periodic point of period $d \leqslant n$ with odd d.

Case 2: $x_d > x_0$. If $x_{d+1} = x_d$, then $x_n = x_d > x_0$, a contradiction. If $x_{d+1} > x_d$ then $x_n < x_d < x_{d+1}$ and there is a periodic point of period $n - d < d$ with $n - d$ odd. If $x_{d+1} < x_d$ then $x_{d+1} < x_d \leqslant x_p$. If $p > n$, we get a point of period $p - d$. If $p \leqslant n$, there is no division for $(x_0, \ldots, x_n)$. By Proposition 2.2, there is a periodic point of period $n - 1$. $\square$

Dropping the hypothesis that $d \leqslant n$, we lose the existence of an odd period.

PROPOSITION 3.6. *Let either* $x_n \leqslant x_0 < x_1$ *and* $x_d \leqslant x_p$ *or* $x_n \geqslant x_0 > x_1$ *and* $x_d \geqslant x_p$ *for some* d, p, n *with* d *odd and* p *even. Then, there exists a periodic point of period* 6.

PROOF. If there exists a periodic point of an odd ($\neq 1$) period then we can use Theorem 2.1. Suppose that there is no periodic point of any odd period bigger than 1 (and, in particular, n must be even). We only consider the case where $x_n \leqslant x_0 < x_1$ and $x_d \leqslant x_p$. We shall use the induction argument on n.

Case 1: $n = 2$. Since there are no periodic points of odd period, there exists, by Corollary 3.2, a fixed point z of f such that if $x_i < z$ then $x_{i+1} > x_i$ and if $x_i > z$ then $x_{i+1} < x_i$. Hence, $x_2 \leqslant x_0 < z < x_1$. Since $x_d \leqslant x_p$, we have either $x_d \leqslant z$ or $x_p \geqslant z$. Hence z does not split the even indices from the odd indices. That is, there exists k such that either $x_k \leqslant x_{k+1} \leqslant z$ or $x_k \geqslant x_{k+1} \geqslant z$. Take the smallest such k. Now we shall use induction on k.

Case 1A: $k = 2$. In this case we have $x_2 < x_0 < z < x_1$ and $x_3 \leqslant z$. Consequently, $f([x_2, x_0]) \supset [z, x_1]$ and $f([x_0, z]) \supset [z, x_1]$ and $f([z, x_1]) \supset [x_2, x_0] \cup [x_0, z]$. Therefore, $f^2([x_2, x_0]) \supset [x_2, x_0] \cup [x_0, z]$ and $f^2([x_0, z]) \supset [x_2, x_0] \cup [x_0, z]$. This gives us the existence of periodic points of all periods for f^2 and hence the existence of periodic points of all even periods for f.

Case 1B: $k \geqslant 3$. We claim we only need to consider

(3.1) $$x_2 \leqslant x_j \leqslant x_1 \quad \text{for all } j = 0, 1, \ldots, k + 1.$$

Suppose (3.1) is not true. Then, there is $j \in [3, k+1]$ such that $x_j = \min\{x_0, \ldots, x_{k+1}\}$ or $x_j = \max\{x_0, \ldots, x_{k+1}\}$. For the first case, we have $x_j \leqslant x_{j-2} < x_{j-1}$, and, for the second, $x_j \geqslant x_{j-2} > x_{j-1}$. We then use the induction hypothesis for $y_0 = x_{j-2}, y_1 = x_{j-1}, \ldots, y_{k-j+2} = x_k$, with $k - j + 2 < k$.

Let $x_r = \max\{x_i : 0 \leqslant i \leqslant k, x_i < x_0\}$, $x_s = \min\{x_i : 0 \leqslant i \leqslant k, x_i > x_0\}$. Let $t = r$ if $x_{r+1} \geqslant x_{s+1}$ and $t = s$ if $x_{s+1} > x_{r+1}$. Suppose first that $t = k$. If $t = s$ and $x_s < z$, then $x_{s+1} \leqslant z < x_{r+1}$ and $t = r$, a contradiction. If $t = s$ and $x_s > z$ then $x_0 < z < x_{r+1} < x_{s+1} < x_s$. This contradicts the definition of s. If $t = r$, then $x_{r+1} \leqslant z < x_{s+1}$ and $t = s$. This is again a contradiction. Hence $t < k$. If $x_{t+1} \leqslant z$ then $x_t > z$ and consequently $t = s$. But then, by the definition of r and s, $x_{t+1} \leqslant x_r$ and therefore $x_{r+1} > x_{t+1}$, a contradiction. Hence, $x_{t+1} > z$. Suppose $t = k - 1$. Then $x_{t+2} \geqslant z$, and $f([x_r, x_0]) \supset [x_{t+1}, x_1]$, $f([x_0, z]) \supset [x_{t+1}, x_1]$ and $f([x_{t+1}, x_1]) \supset [x_r, x_0] \cup [x_0, z]$. Hence, $f^2([x_r, x_0]) \supset [x_r, x_0] \cup [x_0, z]$ and $f^2([x_0, z]) \supset [x_r, x_0] \cup [x_0, z]$. This gives the existence of period points of all periods for f^2 and hence, periodic points of all even periods for f. Suppose $t < k - 1$. Then, by definition of r and s, $x_{t+2} \notin (x_r, x_s)$. If $x_{t+2} \leqslant x_r$, then $x_{t+2} \leqslant x_t < x_{t+1}$ and we may use the induction hypothesis for $y_0 = x_t, \ldots, y_{k-t} = x_k$ with $k - t < k$. If $x_{t+2} \geqslant x_s$ then $f([x_r, x_0]) \supset [x_{t+1}, x_1]$ and $f([x_0, x_s]) \supset [x_{t+1}, x_1]$ and $f([x_{t+1}, x_1]) \supset [x_r, x_0] \cup [x_0, x_s]$. This gives the existence of periodic points of all even periods.

Case 2: $n > 2$. We may assume that $x_2 \geqslant x_0$. If $x_3 \leqslant x_2$ then there is no division for $(x_0, x_1, \ldots, x_n)$ and we get a periodic point of an odd ($\neq 1$) period. Hence we may assume $x_2 < x_3$. If $d < 2$ then we can use Proposition 3.5. If $p < 2$ then $p = 0$ and we have $x_d \leqslant x_0 < x_1$ and there is a periodic point of period d. For $p, d \geqslant 2$ we have $x_n \leqslant x_2 < x_3$. The induction hypothesis can be applied for $y_0 = x_2, \ldots, y_{n-2} = x_n$ using $n - 2$ instead of n. $\square$

4. Examples. In this section, we shall give some examples to show that under the assumptions of our theorems we cannot obtain stronger results. In every example we shall specify only one trajectory. To be more precise, we only specify the ordering of points of this trajectory. We assume that the interval on which f acts is the smallest interval containing that trajectory, and that f is monotone on the components of the complement of the trajectory.

The proofs of nonexistence of certain periodic points will not be given here. The reader may check these by himself by using the method of [2], that is, by making a related graph and investigating its loops. The methods of [2] also allow us to compute the topological entropy of these examples.

EXAMPLE 1. $x_{n-1} < x_{n-3} < \cdots < x_4 < x_2 < x_0 = x_n < x_1 < x_3 \cdots < x_{n-4} < x_{n-2}$.

This is a classical example given by Sharkovski [5]. Here x_0 is a periodic point of period n, with $n \geqslant 5$ and odd, and there is no periodic point of period $n - 2$. The topological entropy of this example is well known (see [2]).

EXAMPLE 2. Writing $n_* = n/2$, $x_{n-1} <_{n_*-1} < x_{n-3} < x_{n_*-3} < \cdots < x_{n_*+2} < x_2 < x_{n_*} < x_0 = x_n < x_1 < x_{n_*+1} < x_3 < x_{n_*+3} < \cdots < x_{n_*-4} < x_{n-4} < x_{n_*-2} < x_{n-2}$.

Here x_0 is a periodic point of period n, with n even and $n/2$ odd and $n \geqslant 10$. There is no division for $(x_0, x_1, \ldots, x_n)$. For this example, there is no periodic point

of period $n/2 - 2$. The topological entropy is the same as in Example 1 for n_* instead of n.

EXAMPLE 3. Writing n_* for $n/2$, $x_{n_*} < x_{n-1} < x_{n_*-2} < x_{n-3} < \cdots < x_4 < x_{n_*+3} < x_2 < x_{n_*+1} < x_0 = x_n < x_1 < x_{n_*+2} < x_3 < \cdots < x_{n-4} < x_{n_*-3} < x_{n-2} < x_{n_*-1}$.

Here x_0 is a periodic point of period n, with $n \geqslant 8$ and even, and $n/2$ is even. There is no division for $(x_0, x_1, \ldots, x_n)$. For this example, there is no periodic point of period $n/2 - 1$. The topological entropy is $\log \lambda_n$ where λ_n is the largest root of the polynomial $x^n - 2x^{n-2} - 2x^{n/2-1} - 1$.

EXAMPLE 4. $x_2 < x_0 < x_3 = x_4 < x_1$.

This is an example of $x_n \leqslant x_0 < x_1$ and $x_d \leqslant x_p$ with d odd and p even and there is no periodic point of any odd ($\neq 1$) period. The topological entropy is $\frac{1}{2} \log 2$.

The topological entropies in Examples 1, 2 and 4 are the smallest possible for such examples. It is not known whether it is so with Example 3.

Notice that no special example is needed for Proposition 3.5. An example can be derived from Example 1 by choosing appropriate n, d, p.

In [4], we prove that if $x_n \leqslant x_0 < x_1$ or $x_n \geqslant x_0 > x_1$ for n odd ($\neq 1$), then there exists a periodic point of odd period ($\neq 1$). One can ask if it is possible to prove that this result covers all sequences (that imply there is an odd period) in the sense that Theorem 2.4 shows Theorem 2.3 covers all sequences. The following examples shows that the answer is negative and shows there are sequences covered by Theorem 2.3 that are not covered by the results in [4].

EXAMPLE 5. $x_6 < x_0 < x_4 < x_3 = x_7 < x_5 < x_2 < x_1$.

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