

No-Forcing and No-Matching Theorems for Classical Probability Applied to Quantum Mechanics

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Abstract Correlations of spins in a system of entangled particles are inconsistent with Kolmogorov’s probability theory (KPT), provided the system is assumed to be non-contextual. In the Alice–Bob EPR paradigm, non-contextuality means that the identity of Alice’s spin (i.e., the probability space on which it is defined as a random variable) is determined only by the axis α_i chosen by Alice, irrespective of Bob’s axis β_j (and vice versa). Here, we study contextual KPT models, with two properties: (1) Alice’s and Bob’s spins are identified as A_{ij} and B_{ij} , even though their distributions are determined by, respectively, α_i alone and β_j alone, in accordance with the no-signaling requirement; and (2) the joint distributions of the spins A_{ij}, B_{ij} across all values of α_i, β_j are constrained by fixing distributions of some subsets thereof. Of special interest among these subsets is the set of probabilistic connections, defined as the pairs $(A_{ij}, A_{i'j'})$ and $(B_{ij}, B_{i'j'})$ with $\alpha_i \neq \alpha_{i'}$ and $\beta_j \neq \beta_{j'}$ (the non-contextuality assumption is obtained as a special case of connections, with zero probabilities of $A_{ij} \neq A_{i'j'}$ and $B_{ij} \neq B_{i'j'}$). Thus, one can achieve a complete KPT characterization of the Bell-type inequalities, or Tsirelson’s inequalities, by specifying the distributions of probabilistic connections compatible with those and only those spin pairs (A_{ij}, B_{ij}) that are subject to these inequalities. We show, however, that quantum-mechanical (QM) constraints are special. No-forcing theorem says that if a set of probabilistic connections is not compatible with correlations violating QM, then it is compatible only with the classical–mechanical correlations. No-matching theorem says that there

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are no subsets of the spin variables A_{ij}, B_{ij} whose distributions can be fixed to be compatible with and only with QM-compliant correlations.

Keywords CHSH inequalities · Contextuality · EPR/Bohm paradigm · Probabilistic couplings · Random variables · Tsirelson inequalities

1 Introduction

Half a century ago John Bell [1] posed and answered in the negative the question of whether probability distributions of spins in entangled particles could be accounted for by a model written in the language of classical probability. These distributions being among the most basic predictions of QM, Bell's theorem and its subsequent elaborations [2–4] seem to mathematically isolate quantum determinism from the probabilistic forms of classical determinism, and establish the necessity for a quantum probability theory that is not reducible to the classical one. However, Bell-type theorems do not engage the full potential of the classical probability theory, if the latter is understood as the theory adhering to Kolmogorov's conceptual framework [5]. The use of probability theory in Bell-type theorems is constrained by the following assumption:

(*Non-Contextuality*, NC) A spin of a given particle is a random variable whose *identity* does not depend on measurement settings (axes) chosen for other particles.

This meaning of NC differs from the descriptions and definitions found in the literature [6–12], but all of them agree that Bell-type theorems are predicated on NC. To understand our definition and why it leads to the Bell-type theorems, we need to recapitulate basic facts about random variables in KPT. Although the discussion can be conducted on a very high level of generality [13–15], in this paper we confine it to the simplest Bohmian version of the EPR paradigm [16], depicted in Fig. 1. For each of the four combined settings (α_i, β_j) , the recorded spins form a random pair (A, B) . *No-signaling requirement* (forced by special relativity if the two particles are separated by a space-like interval, but usually assumed to hold even if they are not) means that the distribution of A does not depend on β_j , nor the distribution of B on α_i : then, by observing successive realizations of spin A for a given value of α_i , Alice should never be able to guess that Bob exists.

The no-signaling requirement is trivially satisfied if both A and B are binary $(+1/-1)$ random variables with

$$\Pr[A = 1] = \Pr[B = 1] = \frac{1}{2}, \quad (1)$$

for all settings (α_i, β_j) . This is the case we confine our analysis to. The joint distribution of A and B for a given (α_i, β_j) is then uniquely determined by the joint probability

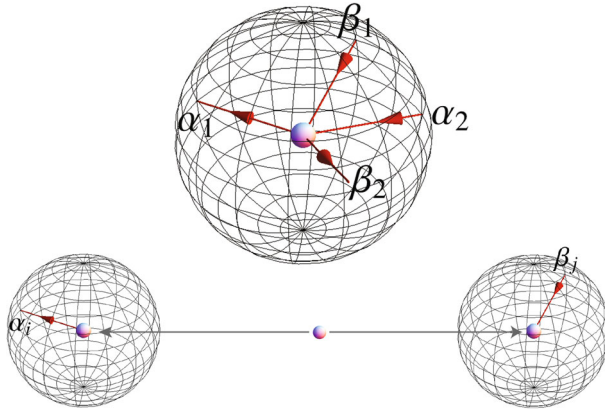


Fig. 1 The experimental paradigm considered in this paper. Two spin-half particles created in a singlet state are running away from each other. Each particle has its spin measured along one of two axes (measurement settings): α_1 or α_2 for “Alice’s” particle (*left*), and β_1 or β_2 for “Bob’s” particle (*right*). Each measurement results in a random variable attaining one of two values, +1 (spin-up, shown by *outward-pointing cones*) or -1 (spin-down, *inward-pointing cones*). We confine the consideration to the case when these values are equiprobable (so the no-signaling requirement is satisfied trivially)

$$p_{ij} = \Pr [A = 1, B = 1]. \tag{2}$$

The distribution of a random variable, however, does not determine its identity. In KPT, a binary (+1/ - 1) random variable X is identified with a mapping $f : S \rightarrow \{-1, 1\}$, measurable with respect to some probability space (S, Σ, μ) , where S is a set, Σ is a set of events included in S , and μ is a probability measure. The measurability of f means that $f^{-1}(1) \in \Sigma$, and the distribution of X is defined by $\Pr [X = 1] = \mu (f^{-1}(1))$. We say in this case that this random variable is *defined on* (S, Σ, μ) . If two +1/-1 random variables, X and Y , are *jointly distributed*, then they are identified with mappings $f : S \rightarrow \{-1, 1\}$ and $g : S \rightarrow \{-1, 1\}$, measurable with respect to one and the same probability space (S, Σ, μ) . The joint distribution of these random variables is then determined by

$$\Pr [X = \pm 1, Y = \pm 1] = \mu \left(f^{-1}(\pm 1) \cap g^{-1}(\pm 1) \right). \tag{3}$$

The NC assumption says that the identity of A at a given α_i does not depend on β_j , nor does the identity of B at a given β_j depend on α_i . If so, for any given (α_i, β_j) , the output pair should be indexed (A_i, B_j) . Being jointly distributed, A_1 and B_1 , are defined on the same probability space, and so are A_1, B_2 , and A_2, B_1 , and A_2, B_2 . It follows that all four random variables A_1, A_2, B_1, B_2 are defined on one and the same (S, Σ, μ) . That is, they are jointly distributed, even though the joint distribution of A_1, A_2 is not observable in the sense of physical co-occurrence of their values, and analogously for B_1, B_2 . The complete version of the Bell theorem [4] says that A_1, A_2, B_1, B_2 with known distributions of (A_i, B_j) for $i, j \in \{1, 2\}$ can be imposed a joint distribution upon (i.e., A_1, A_2, B_1, B_2 can be presented as measurable functions on the same probability space) if and only if these marginal distributions satisfy the

CHSH inequalities,

$$0 \leq p_{11} + p_{12} + p_{21} + p_{22} - 2p_{ij} \leq 1, \quad \text{for all } i, j \in \{1, 2\}, \quad (4)$$

with p_{ij} defined in (2). In the general case we have to replace the CHSH inequalities in the previous statement with the conjunction of these inequalities and the no-signaling condition, but we deal here with the special case (1), where no-signaling is satisfied “automatically.”

To relate this to the original language of the EPR discussions, the statement that A_1, A_2, B_1, B_2 are jointly distributed (which is equivalent to NC) is equivalent to the “hidden-variable” assumption. Indeed, A_1, A_2, B_1, B_2 are respectively representable by functions f_1, f_2, g_1, g_2 defined on the same (S, Σ, μ) if and only if

$$A_i = f_i(R), \quad B_j = g_j(R), \quad \text{for all } i, j \in \{1, 2\}, \quad (5)$$

where R is the random variable represented by the identity mapping $\iota : S \rightarrow S, \iota(x) = x$, obviously measurable with respect to (S, Σ, μ) . Moreover [17], insofar as only binary A_1, A_2, B_1, B_2 are concerned, the “hidden variable” R can always be chosen to be a random variable whose values are 16 possible combinations $(a_1, a_2, b_1, b_2) = (\pm 1, \pm 1, \pm 1, \pm 1)$, such that

$$\Pr[R = (a_1, a_2, b_1, b_2)] = \Pr[A_1 = a_1, A_2 = a_2, B_1 = b_1, B_2 = b_2]. \quad (6)$$

The functions f_i, g_j in (5) are then simply coordinate-wise projections of the vectors (a_1, a_2, b_1, b_2) :

$$f_i(a_1, a_2, b_1, b_2) = a_i, \quad g_j(a_1, a_2, b_1, b_2) = b_j, \quad i, j \in \{1, 2\}. \quad (7)$$

Summarizing, we have the equivalences

$$\begin{aligned} & \text{NC} \\ & \Updownarrow \\ & A_1, A_2, B_1, B_2 \text{ are single-indexed} \\ & \Updownarrow \\ & A_1, A_2, B_1, B_2 \text{ are jointly distributed} \\ & \Updownarrow \\ & A_1, A_2, B_1, B_2 \text{ are functions of a “hidden” } R \\ & \Updownarrow \\ & \text{CHSH inequalities (4),} \end{aligned} \quad (8)$$

of which the equivalence of the first four statements holds in KPT essentially by definition. We know that the QM prediction for p_{ij} in (2) is, for $i, j \in \{1, 2\}$,

$$p_{ij} = 1/4 - 1/4\langle\alpha_i|\beta_j\rangle, \quad (9)$$

where $\langle \alpha_i | \beta_j \rangle$ is the cosine of the angle between axes α_i and β_j , and we know that for some choices of the axes these values of p_{ij} violate (4). There are only two ways of dealing with this situation: to reject KPT (replace it with a different, QM probability theory) or, if one wishes to remain within the confines of KPT, to reject NC. The standard QM theory has chosen the first way, we in this paper (following others, e.g., [18, 19]) explore the limits of the second one.

It is clear from (8) that to reject NC means to double-index A or B (by symmetry, A and B). In the Alice–Bob-paradigm considered this yields eight random variables A_{ij} and B_{ij} , $i, j \in \{1, 2\}$, with known joint distributions for four pairs (A_{ij}, B_{ij}) ,

$$\begin{array}{|c|c|c|}
 \hline
 (\alpha_i, \beta_j) & B_{ij} = +1 & B_{ij} = -1 \\
 \hline
 A_{ij} = +1 & p_{ij} & 1/2 - p_{ij} \\
 \hline
 A_{ij} = -1 & 1/2 - p_{ij} & p_{ij} \\
 \hline
 \end{array} \tag{10}$$

Since A_{i1} and A_{i2} are different (even if identically distributed) random variables, and so are B_{1j} and B_{2j} , we are no longer forced to assume, as we were under NC, that all the random variables in play are defined on one and the same probability space. In fact, if we interpret a joint distribution as implying that values of jointly distributed random variables can be observed “together” (e.g., simultaneously, in some inertial frame of reference), then A_{i1} and A_{i2} are not jointly distributed (and neither are B_{1j} and B_{2j}). Our denial of NC, therefore, does not amount to admission of a “spooky action at a distance.” Rather it is based on our acknowledging, as a general principle,

(*Contextuality-by-Default*, CbD) No two spins recorded under different, mutually exclusive measurement settings (across all particles involved) ever *co-occur*, because of which they are *stochastically unrelated* (i.e., defined on different probability spaces, possess no joint distribution) [13–15].

The concept of four stochastically unrelated to each other random pairs (A_{ij}, B_{ij}) is well within the framework of KPT. Put differently, A_{ij} and B_{ij} , being jointly distributed for any given (α_i, β_j) , can be considered functions of one and the same “hidden” random variable R_{ij} , but KPT does not compel the four R_{ij} ’s to be viewed as jointly distributed. This seems to be the essence of Gudder’s analysis [20] of the hidden-variable theories. The existence of a single probability space for all random variables imaginable is often mistakenly taken as one of the tenets of Kolmogorov’s theory. The untenability of this view is apparent by cardinality considerations alone. Also, it is easy to see that for any class of random variables of a given type (e.g., binary $+1 / -1$ ones) one can construct a variable of the same type stochastically independent of each of them. The idea that all such variables can be defined on a single space therefore leads to a contradiction [14].

CbD does not mean, of course, that stochastically unrelated (i.e., possessing no joint distribution) spins cannot be mathematically *imposed* a joint distribution on (in the same way as dealing with spatially disparate points on a sheet of paper does not prevent one from variously grouping them in one’s mind). In fact this can generally be done in a variety of ways, referred to as different *probabilistic couplings* [21] (p-couplings):

(All-Possible-p-Couplings, APpC) The stochastically unrelated spins recorded under different, mutually exclusive measurement settings can be p-coupled arbitrarily, insofar as the joint distribution imposed on them is consistent with the observable joint distributions of the spins recorded under the same measurement settings [13–15].

To p-couple the eight random variables $A_{ij}, B_{ij}, i, j \in \{1, 2\}$, means to specify 2^8 probabilities (nonnegative and summing to one)

$$\Pr [A_{11} = \pm 1, \dots, A_{22} = \pm 1, B_{11} = \pm 1, \dots, B_{22} = \pm 1]. \tag{11}$$

Each p-coupling is a way of designing a scheme of grouping realizations of the eight random variables, *as if* they co-occurred in an imaginary experiment. Equivalently, to p-couple all A_{ij}, B_{ij} means to find a “hidden” random variable R^* which all A_{ij}, B_{ij} can be presented as functions of,

$$A_{ij} = f_{ij}(R^*), B_{ij} = g_{ij}(R^*), \quad i, j \in \{1, 2\}. \tag{12}$$

In particular, once the 2^8 probabilities in (11) have been assigned to all vectors

$$(a_{11}, \dots, a_{22}, b_{11}, \dots, b_{22}) = (\pm 1, \dots, \pm 1, \pm 1, \dots, \pm 1), \tag{13}$$

one can choose R^* to be the random variable with these values and these probabilities [17], the functions f_{ij}, g_{ij} being the coordinate-wise projections,

$$\begin{aligned} f_{ij}(a_{11}, \dots, a_{22}, b_{11}, \dots, b_{22}) &= a_{ij}, \\ g_{ij}(a_{11}, \dots, a_{22}, b_{11}, \dots, b_{22}) &= b_{ij}, \quad i, j \in \{1, 2\}. \end{aligned} \tag{14}$$

Therefore, the “hidden-variable” meaning for a p-coupling of the double-indexed A, B is precisely the same as in the case of the single-indexed ones, cf. (5)–(7).

In fact, a p-coupling of the single-indexed A_1, A_2, B_1, B_2 is merely a special case of p-couplings for the double-indexed $A_{11}, \dots, A_{22}, B_{11}, \dots, B_{22}$. One obtains this special case by constraining (11) not only by the four empirically observable marginal distributions of (A_{ij}, B_{ij}) , but also by the additional assumption

$$\Pr [A_{i1} \neq A_{i2}] = 0, \Pr [B_{1j} \neq B_{2j}] = 0, \quad i, j \in \{1, 2\}. \tag{15}$$

Without this additional assumption a p-coupling for $A_{11}, \dots, A_{22}, B_{11}, \dots, B_{22}$ always exists, whatever the observed distributions of (A_{ij}, B_{ij}) . For instance, one can always construct a p-coupling in which all stochastically unrelated random pairs (A_{ij}, B_{ij}) are considered stochastically independent. This is not particularly interesting, precisely because this scheme of p-coupling is compatible with any distributions of (A_{ij}, B_{ij}) , QM-compliant and QM-contravening alike.

An interesting question is whether there is a scheme by which the stochastically unrelated random pairs (A_{ij}, B_{ij}) can be p-coupled so as to “match” QM precisely, in the sense of allowing for all QM-compliant correlations and no other. At the end of

this paper we answer this question in the negative for all Kolmogorovian models in which the p-couplings mentioned in APpC are constrained by fixing distributions of some subsets of all spins involved (no-matching theorem).

Prior to that, however, we consider a case when these constraining distributions are those of certain spin pairs, called *connections* [13]. This special class of models is the most straightforward generalization of the models compatible with NC. For the models with connections we prove a stronger result (no-forcing theorem): such a model either allows for correlations forbidden by QM or it only allows for the correlations of classical mechanics, those satisfying the CHSH inequalities (4).

2 No-Forcing and No-Matching Theorems for QM

The results of the Alice–Bob experiment depicted in Fig. 1 are uniquely described by the *outcome vector*

$$p = (p_{11}, p_{12}, p_{21}, p_{22}). \tag{16}$$

We say that p is *QM-compliant* if there exists some choice of the settings $\alpha_1, \alpha_2, \beta_1, \beta_2$ under which p satisfies (9). The following inequality is known to be a necessary and sufficient condition for p being QM-compliant [22–24]:

$$|r_{11}r_{12} - r_{21}r_{22}| \leq \sqrt{1 - r_{11}^2}\sqrt{1 - r_{12}^2} + \sqrt{1 - r_{21}^2}\sqrt{1 - r_{22}^2}, \tag{17}$$

where

$$r_{ij} = 4p_{ij} - 1 \tag{18}$$

is correlation between A_{ij} and B_{ij} ($i, j \in \{1, 2\}$). For geometric reasons obvious from Fig. 1, it is referred to as the *cosphericity inequality* [24].

In accordance with CbD, the spin pairs recorded under mutually exclusive settings, say (A_{11}, B_{11}) and (A_{12}, B_{12}) , should generally be treated as stochastically unrelated, possessing no joint distribution. In accordance with APpC, one can consider all possible eight-component random vectors

$$H = (A_{11}, B_{11}, A_{12}, B_{12}, A_{21}, B_{21}, A_{22}, B_{22}) \tag{19}$$

(see Fig. 2) whose empirically observable two-component parts (A_{ij}, B_{ij}) have the distributions shown in the matrices (10).¹ A subset of any $k \leq 8$ components of H is referred to as its *k-marginal*, and its distribution is referred to as a *k-marginal distribution* (the number k being omitted if clear from the context). So far we considered 1-marginals, that we posited to have equiprobable $+1/ -1$ values, and certain 2-marginals. Of the latter, the marginal distributions of (A_{ij}, B_{ij}) are the mandatory constraints imposed on H , in fact underlying the definition of H .

¹ A rigorous formulation [13, 27–29] requires that H be defined as $(A'_{ij}, B'_{ij} : i, j \in \{1, 2\})$ such that each pair (A'_{ij}, B'_{ij}) has the same distribution as (rather than is identical to) (A_{ij}, B_{ij}) for $i, j \in \{1, 2\}$. Our lax notation is unlikely to cause confusion in the present paper.

We know that, in this conceptual framework, NC is equivalent to the choice of the p-coupling scheme in which (15) is satisfied. Under the assumption that all 1-marginal probabilities are 1/2, this means that we have the following joint probabilities for the four (empirically unobservable) 2-marginals (A_{i1}, A_{i2}) and (B_{1j}, B_{2j}) :

	$A_{i2} = +1$	$A_{i2} = -1$
$A_{i1} = +1$	1/2	0
$A_{i1} = -1$	0	1/2

(20)

	$B_{2j} = +1$	$B_{2j} = -1$
$B_{1j} = +1$	1/2	0
$B_{1j} = -1$	0	1/2

We know that these joint probabilities have neither empirical nor theoretical justification, because de facto A_{i1} and A_{i2} (or B_{1j} and B_{2j}) are not jointly distributed. There is therefore no prohibition against p-coupling them differently, so that generally,

	$A_{i2} = +1$	$A_{i2} = -1$
$A_{i1} = +1$	$1/2 - \varepsilon_i^1$	ε_i^1
$A_{i1} = -1$	ε_i^1	$1/2 - \varepsilon_i^1$

(21)

	$B_{2j} = +1$	$B_{2j} = -1$
$B_{1j} = +1$	$1/2 - \varepsilon_j^2$	ε_j^2
$B_{1j} = -1$	ε_j^2	$1/2 - \varepsilon_j^2$

In other words, a general p-coupling H allows us to replace (15) with

$$\Pr [A_{i1} \neq A_{i2}] = 2\varepsilon_i^1, \quad \Pr [B_{1j} \neq B_{2j}] = 2\varepsilon_j^2,$$

where $0 \leq \varepsilon_i^1, \varepsilon_j^2 \leq 1/2, i, j \in \{1, 2\}$.

We use the term *connection* to refer to the 2-marginals (A_{i1}, A_{i2}) and (B_{1j}, B_{2j}) . The four connections are uniquely characterized by the *connection vector*

$$\varepsilon = (\varepsilon_1^1, \varepsilon_2^1, \varepsilon_1^2, \varepsilon_2^2), \tag{22}$$

or the corresponding correlations

$$r = (r_1^1, r_2^1, r_1^2, r_2^2), \tag{23}$$

where $r_l^k = 1 - 4\varepsilon_l^k, k, l \in \{1, 2\}$.

An outcome vector p and a connection vector ε are *mutually compatible* [13] if they can be embedded in one and the same p-coupling H , as shown in Fig. 2. In other words, ε and p are mutually compatible if they can be computed as marginal

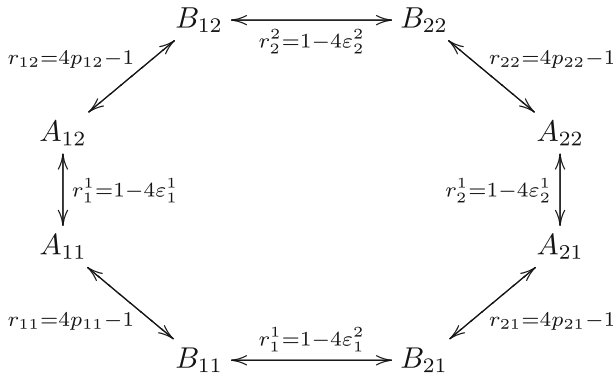


Fig. 2 A p-coupling H for pairs (A_{ij}, B_{ij}) (see footnote 1). The number at a double-arrow connecting two random variables is their correlation. Horizontal and vertical arrows correspond to a connection vector $\varepsilon = (\varepsilon_1^1, \varepsilon_2^1, \varepsilon_1^2, \varepsilon_2^2)$; diagonal arrows correspond to an outcome vector $p = (p_{11}, p_{12}, p_{21}, p_{22})$

probabilities from the probabilities assigned to the 2^8 values of H . That is, p_{ij} is the sum of the probabilities for all values of H with $A_{ij} = B_{ij} = 1$, and $\varepsilon_i^1, \varepsilon_j^2$ are the sums of the probabilities for all values of H with, respectively, $A_{i1} = -A_{i2} = 1$ and $B_{1j} = -B_{2j} = 1$. The set of all compatible pairs (p, ε) forms an 8-dimensional polytope described by Lemma 1. But we need some notation first.

Given a connection vector $\varepsilon = (\varepsilon_1^1, \varepsilon_2^1, \varepsilon_1^2, \varepsilon_2^2)$, consider the sums

$$\frac{1}{4} \left(\pm r_1^1 \pm r_2^1 \pm r_1^2 \pm r_2^2 \right) \tag{24}$$

where each \pm is replaced with either $+$ or $-$. Let $s_0(\varepsilon)$ denote the largest of the eight such sums with even numbers of plus signs,

$$s_0(\varepsilon) = \max \left\{ \frac{1}{4} \left(\pm r_1^1 \pm r_2^1 \pm r_1^2 \pm r_2^2 \right) : \text{the number of } +\text{'s is } 0, 2, \text{ or } 4 \right\}. \tag{25}$$

Let

$$s_1(\varepsilon) = \max \left\{ \frac{1}{4} \left(\pm r_1^1 \pm r_2^1 \pm r_1^2 \pm r_2^2 \right) : \text{the number of } +\text{'s is } 1 \text{ or } 3 \right\}. \tag{26}$$

Since the components of ε belong to $[0, 1/2]$, the pairs $(s_0(\varepsilon), s_1(\varepsilon))$ fill in the triangular area connecting $(0, 0)$, $(1/2, 1)$, and $(1, 1/2)$. In particular,

$$\begin{aligned} s_0(\varepsilon) + s_1(\varepsilon) &\leq 3/2, \\ 0 &\leq s_0(\varepsilon) \leq 1, \\ 0 &\leq s_1(\varepsilon) \leq 1. \end{aligned} \tag{27}$$

We define $s_0(p)$ and $s_1(p)$ for any outcome vector $p = (p_{11}, p_{12}, p_{21}, p_{22})$ analogously (using $r_{ij} = 4p_{ij} - 1$ in place of r_j^i , see Fig. 2):

$$\begin{aligned}
 s_0(p) &= \max \left\{ \frac{1}{4} (\pm r_{11} \pm r_{12} \pm r_{21} \pm r_{22}) : \text{the number of +’s is 0, 2, or 4} \right\}, \\
 s_1(p) &= \max \left\{ \frac{1}{4} (\pm r_{11} \pm r_{12} \pm r_{21} \pm r_{22}) : \text{the number of +’s is 1 or 3} \right\}.
 \end{aligned}
 \tag{28}$$

Since the components of p also belong to $[0, 1/2]$, the pairs $s_0(p), s_1(p)$ have precisely the same properties as $s_0(\varepsilon), s_1(\varepsilon)$.

Lemma 1 p and ε are mutually compatible if and only if

$$\begin{aligned}
 s_0(\varepsilon) + s_1(p) &\leq 3/2, \\
 s_1(\varepsilon) + s_0(p) &\leq 3/2.
 \end{aligned}
 \tag{29}$$

For the proof of this lemma see [13].

The following observation that we need later on is proved by simple algebra.

Lemma 2 The set E_0 of connection vectors ε with $s_0(\varepsilon) = 1$ consists of the null vector $\varepsilon_0 = (0, 0, 0, 0)$ and seven vectors obtained by replacing any two of or all four zeros in ε_0 with $1/2$. For all ε in $E_0, s_1(\varepsilon) = 1/2$.

The null (or identity) connection vector ε_0 plays a special role, as it corresponds to (15) and (20): this is the choice implicitly made in all Bell-type theorems. It also plays a central role in the no-forcing theorem below. Note that according to Lemmas 1 and 2, an outcome vector p is compatible with ε_0 if and only if it is compatible with all connection vectors in E_0 .

It is easy to see that a connection vector can be chosen so that it is compatible with all QM-compliant outcome vectors. The simplest example is the connection vector $\varepsilon_{ind} = (1/4, 1/4, 1/4, 1/4)$ corresponding to the p-couplings (19) with all components pairwise independent, except, possibly, for pairs (A_{ij}, B_{ij}) . Since $s_0(\varepsilon_{ind}) = s_1(\varepsilon_{ind}) = 0$, it follows from Lemma 1 that ε_{ind} is compatible with any p , whether QM-compliant or not.

What is less obvious is the answer to the question: what are all the connection vectors that are compatible *only* with QM-compliant outcome vectors? In other words, we are interested in the set Force_{QM} of connection vectors, defined as follows:

Force_{QM} : all connection vectors ε such that if p is compatible with ε , then p is QM-compliant, i.e., satisfies the cosphericity inequality (17).

The name of the set is to indicate that $\varepsilon \in \text{Force}_{QM}$ “forces” every p compatible with it to be QM-compliant. The set is not empty, because, as the next lemma shows, it includes E_0 of Lemma 2.

Lemma 3 $E_0 \subset \text{Force}_{QM}$.

Proof By Fine’s theorem [4],² p is compatible with ε_0 (hence, by Lemmas 1 and 2, also with other members of E_0) if and only if it satisfies the CHSH inequalities (4). Since the cosphericity inequality is a necessary condition for the compatibility of p with ε_0 [24], QM-compliance follows from the CHSH inequalities. \square

We are thus led to the following questions:

- (Q1) What is the entire set Force_{QM} (what connection vectors it contains beside E_0)?
- (Q2) What is the set P_{QM} of the outcome vectors p each of which is compatible with at least one of the connection vectors in Force_{QM} ?

The questions are significant for the following reason. If P_{QM} turned out to coincide with the set of all QM-compliant p , we would have a hope of constructing a KPT model that would match QM in the sense of allowing all those p that are possible in QM and forbidding all those p that QM forbids. Quantum determinism could then be “explained” by pointing out that the multiple probability spaces corresponding to different measurement settings can be p-coupled by using appropriately chosen connection vectors for different settings. However, this hope should be abandoned, because Force_{QM} in fact coincides with E_0 , whence P_{QM} includes only those p that satisfy the CHSH inequalities. It is well known that the CHSH inequalities do not describe all QM-compliant vectors: e.g., they are violated if we use in (9) coplanar vectors at the angles $\alpha_1 = 0, \alpha_2 = \pi/2, \beta_1 = \pi/4, \beta_2 = -\pi/4$.

The proof makes use of the following observation.

Lemma 4 *If p belong to the set P_0 described by*

$$\begin{aligned} s_0(p) + s_1(p) &= 3/2, \\ s_0(p) &< 1, \end{aligned}$$

then p is not QM-compliant (violates the cosphericity inequality).

Proof One easily checks that

$$\begin{aligned} s_i(p) &= 1/4 (|r_{11}| + |r_{12}| + |r_{21}| + |r_{22}|) \\ s_{1-i}(p) &= s_i(p) - 1/2 \min(|r_{11}|, |r_{12}|, |r_{21}|, |r_{22}|), \end{aligned}$$

where i is 0 or 1 according as the number of positive correlations r_{ij} is even or odd. Without loss of generality, let the minimum in the second expression equal $|r_{22}|$. Then

$$s_0(p) + s_1(p) = 1/2 (|r_{11}| + |r_{12}| + |r_{21}|),$$

² The theorem states that the single-indexed A_1, A_2, B_1, B_2 are jointly distributed if and only if p satisfies the CHSH inequalities. We use the fact that the single-indexation means that the connection vector for the double-indexed A ’s and B ’s is ε_0 , and that the existence of the joint distribution of these A ’s and B ’s means, by definition, that ε_0 and p are compatible.

and this can only equal $3/2$ if each of the three correlations equals ± 1 . The cosphericity inequality (17) can only be satisfied then if

$$r_{22} = \pm 1 \quad \text{and} \quad r_{11}r_{12} = r_{21}r_{22}.$$

It is easy to see that the latter is possible only if the number of $+1$'s among the four ± 1 correlations is even. It follows that $s_0(p) = 1$. □

Theorem 5 (no-forcing) *The answer to Q1 is: $\text{Force}_{QM} = E_0$ (whence the answer to Q2 is: P_{QM} is the set of all p satisfying CHSH inequalities).*

Proof We know from (27) that $s_0(\varepsilon) \leq 1$, and from Lemmas 2 and 3 we know that $s_0(\varepsilon) = 1$ describes the set $E_0 \subset \text{Force}_{QM}$. The theorem is proved by showing that Force_{QM} does not contain any ε with $s_0(\varepsilon) < 1$. From the definition of Force_{QM} , if there is a p with which ε is compatible but which does not satisfy the cosphericity inequality (17), then $\varepsilon \notin \text{Force}_{QM}$. By Lemma 1, if for a given ε one chooses a p such that

$$\begin{aligned} s_0(\varepsilon) + s_1(p) &\leq s_0(p) + s_1(p) = 3/2, \\ s_1(\varepsilon) + s_0(p) &\leq s_0(p) + s_1(p) = 3/2, \end{aligned}$$

then p and ε are compatible. If $s_0(\varepsilon) \leq 1/2$, then choose p with $s_1(p) = 1$ and $s_0(p) = 1/2$ to satisfy this system. If $1/2 < s_0(\varepsilon) < 1$, then choose p with $s_1(p) = s_1(\varepsilon)$ and $s_0(p) = 3/2 - s_1(\varepsilon)$ to satisfy this system. By Lemma 4, all these choices of p belong to P_0 and therefore violate the cosphericity inequality. It follows that all ε with $s_0(\varepsilon) < 1$ do not belong to Force_{QM} . □

One consequence of this theorem is that KPT in which p -couplings are constrained by connections cannot match QM precisely: if it allows *only* for QM-compliant p (as does the choice of $\varepsilon = \varepsilon_0$), then it only allows for a proper subset thereof; and if it allows for *all* QM-compliant p , then it also allows for some p that are QM-contravening (as does the connection vector $\varepsilon_{ind} = (1/4, 1/4, 1/4, 1/4)$ that is compatible with all possible outcome vectors p). We now generalize this no-matching statement for connection vectors to arbitrary marginal distributions imposed on p -couplings H in (19).

The empirically inaccessible 2-marginals (A_{i1}, A_{i2}) and (B_{1j}, B_{2j}) constrain H by specifying a connection vector ε . There are, however, other empirically inaccessible marginals, such as (A_{11}, A_{22}) , $(A_{11}, A_{12}, A_{21}, A_{22})$, (A_{11}, B_{22}, B_{12}) , etc. Each of them, once its distribution is specified, constrains the possible distributions of H .

Theorem 6 (no-matching) *There is no set of marginal distributions imposed on H such that outcome vectors p are compatible with this set if and only if they are QM-compliant.*

Proof Observe first that a distribution of any k -marginal (X_1, \dots, X_k) of H can be presented by 2^k probabilities

$$\Pr [X_{i_1} = 1, \dots, X_{i_{k'}} = 1], \tag{30}$$

with all values equated to one, for all k' -(sub)marginals of X_1, \dots, X_k ($0 \leq k' \leq k$). This includes the empty subset, for which we put $\Pr [] = 1$. If we fix distributions of several marginals, with the numbers of components k_1, \dots, k_m , then the total number of different probabilities is $N < 2^{k_1} + \dots + 2^{k_m}$. This set of probabilities constrains the set of possible outcome vectors p to those for which one can find a 2^8 -component vector Q with the following properties:

$$\begin{aligned} M_1 Q &= p \\ \text{subject to} & \\ M_2 Q &= P, \quad Q \geq 0. \end{aligned} \tag{31}$$

Here, Q is the vector of probabilities assigned to all possible values of H (and $Q \geq 0$ is understood componentwise), M_1, M_2 are Boolean (0/1) matrices with dimensions 4×2^8 and $N \times 2^8$, respectively, and P is the vector of all probabilities of the form (30) that define the distributions of the marginals chosen. The entries of the matrices are defined by the following rule: (1) choose the row of the matrix corresponding to the probability $\Pr [X_1 = 1, \dots, X_k = 1]$; (2) choose the column of M corresponding to values

$$(A_{ij} = a_{ij}, B_{ij} = b_{ij} : i, j \in \{1, 2\})$$

of H ($a_{ij}, b_{ij} \in \{-1, 1\}$); (3) put 1 in the intersection of this row and this column if and only if a_{ij} and b_{ij} equal 1 for all A_{ij} and B_{ij} that belong to (X_1, \dots, X_k) ; (4) for the 0-marginal (empty set), all entries are 1. In matrix M_1 the marginals for its four rows are $(A_{ij}, B_{ij}), i, j \in \{1, 2\}$, and the row corresponding to, e.g., (A_{11}, B_{11}) contains 1 in each cell whose column corresponds to H -values with

$$\begin{pmatrix} A_{11} & B_{11} & A_{12} & \dots & B_{22} \\ 1 & 1 & \text{any} & \dots & \text{any} \end{pmatrix}.$$

To illustrate the structure of matrix M_2 and vector P , assume that one of the marginals chosen is $(A_{11}, A_{12}, A_{21}, A_{22})$. Then P includes the 16 probabilities

$$\begin{aligned} &\Pr [A_{11} = 1, A_{12} = 1, A_{21} = 1, A_{22} = 1] \\ &\Pr [A_{11} = 1, A_{12} = 1, A_{21} = 1], \dots, \Pr [A_{12} = 1, A_{21} = 1, A_{22} = 1] \\ &\Pr [A_{11} = 1, A_{12} = 1], \dots, \Pr [A_{21} = 1, A_{22} = 1] \\ &\Pr [A_{11} = 1] = 1/2, \dots, \Pr [A_{22} = 1] = 1/2 \\ &\Pr [] = 1. \end{aligned}$$

The row of M_2 corresponding to, say, (A_{11}, A_{12}, A_{21}) , contains one for all columns with H -values

$$\begin{pmatrix} A_{11} & B_{11} & A_{12} & B_{12} & A_{21} & B_{21} & A_{22} & B_{22} \\ 1 & \text{any} & 1 & \text{any} & 1 & \text{any} & \text{any} & \text{any} \end{pmatrix}.$$

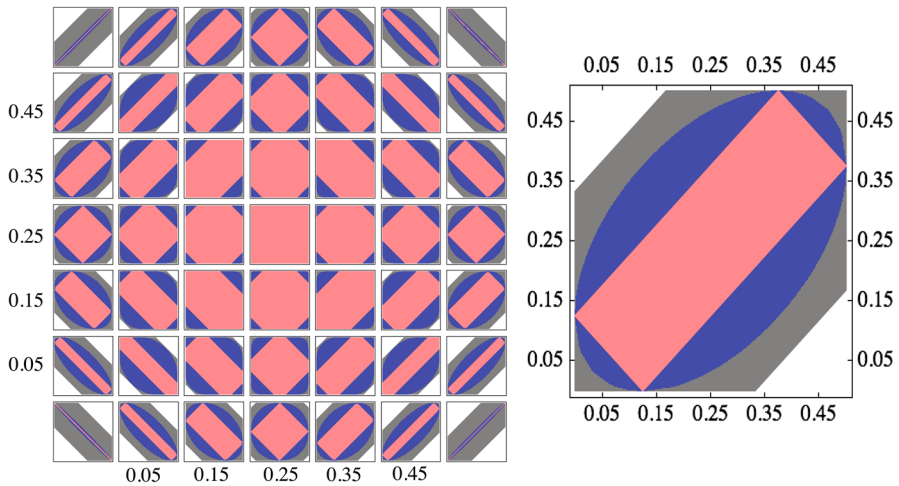


Fig. 3 Areas of outcome vectors $p = (p_{11}, p_{12}, p_{21}, p_{22})$, two components of which (no matter which) define panels on the *left*, and the remaining two form the axes of each panel, as shown on the *right*. The *pink area* contains all p satisfying the CHSH inequalities (4). One can constrain the p -couplings H by marginal distributions (notably, by putting $\varepsilon = (0, 0, 0, 0)$), so that the set of all p compatible with them coincides with the *pink area* [1–4]. The *gray area* contains all p satisfying the Tsirelson inequalities (32). One can constrain the p -couplings H by marginal distributions (e.g., by putting $\varepsilon = ((\sqrt{2} - 1)/8, (\sqrt{2} - 1)/8, (\sqrt{2} - 1)/8, (\sqrt{2} - 1)/8)$), so that the set of all p compatible with them coincides with the *gray area* [13]. The *blue area* (that includes the *pink area* and is included in the *gray one*) contains all p satisfying the cosphericity inequality (17), i.e., all QM-compliant p . The no-matching theorem says that, for any set of marginal distributions, the set of all p compatible with them never coincides with the *blue area* precisely

Now, the set of all vectors p for which a Q exists satisfying (31) forms a polytope confined within a $[0, 1/2]^4$ cube. This polytope can be empty (if the distributions for the marginals chosen are not compatible), consist of a single point (e.g., if the marginals chosen include H itself), or have any dimensionality between 1 and 4. The statement of the theorem follows from the fact that the set of QM-compliant p , those satisfying (17), is not a polytope. Figure 3 makes this fact obvious, by showing the curvilinear shape of the two-dimensional cross-sections of the set of QM-compliant p . □

3 Conclusion

We have seen that the Bell-type theorems do not allow one to gauge the ability of KPT for dealing with QM-compliant spin distributions in entangled particles. The Bell-type theorems are confined to the NC assumption, and the latter is not an integral part of KPT. The power of KPT is much greater if one uses its basic conceptual apparatus to systematically distinguish the distribution of spins (which, due to the no-signaling requirement, can never be affected by the measurement settings chosen in distant particles) and the identity of spins as random variables—which, in accordance with the Cbd

principle, may very well depend on the settings chosen across all particles (without violating any known laws of physics). Our results show, however, that QM preserves its special status even at this, much greater level of generality. The contextual KPT models (with marginal constraints) are not able to match QM predictions precisely.

Figure 3 serves an additional purpose of demonstrating that this failure of the contextual KPT models with marginal constraints is not due to its general inability to match theories outside the scope of classical mechanics. It is the nonlinearity of the area of QM-compliant outcome vectors rather than their non-classicality that is responsible for the no-matching theorem. Thus, consider the *Tsirelson inequalities* [25,26] for spin-1/2 particles (with equiprobable spin-up and spin-down in all directions),

$$\frac{1 - \sqrt{2}}{2} \leq p_{11} + p_{12} + p_{21} + p_{22} - 2p_{ij} \leq \frac{1 + \sqrt{2}}{2} \quad (i, j \in \{1, 2\}). \quad (32)$$

They are known to be satisfied by all QM-compliant outcome vectors p , and they impose the lower and upper bound on the QM-permitted violations of the CHSH inequalities: a linear combination $p_{11} + p_{12} + p_{21} + p_{22} - 2p_{ij}$ can achieve the values $\frac{1-\sqrt{2}}{2}$ and $\frac{1+\sqrt{2}}{2}$ by appropriate choices of the directions $\alpha_1, \alpha_2, \beta_1, \beta_2$ in (9).

Our approach allows one to offer a KPT account for the Tsirelson bounds by postulating that, in the Alice–Bob system depicted in Fig. 1, the connection vectors ε satisfy

$$s_0(\varepsilon) = \frac{3 - \sqrt{2}}{2}, \quad s_1(\varepsilon) \leq \frac{1}{2}, \quad (33)$$

where $s_0(\varepsilon)$ and $s_1(\varepsilon)$ are defined in (25) and (26). It has been shown [13] that such a connection vector is compatible with those and only those outcome vectors p that satisfy the Tsirelson inequalities (32).

Recall that a connection vector ε is compatible with those and only those p that satisfy the CHSH inequalities (4) if and only if $s_0(\varepsilon) = 1$. The latter means that A_{i1} and A_{i2} in the classical–mechanical system are either always equal or always opposite, and the same is true for B_{1j} and B_{2j} . With ε satisfying (33), A_{i1} and A_{i2} , as well as B_{1j} and B_{2j} , may be unequal and non-opposite with some small probabilities, e.g., $(\sqrt{2} - 1)/8$, if one assumes that these probabilities are the same for all four connections $(A_{i1}, A_{i2}), (B_{1j}, B_{2j}), i, j \in \{1, 2\}$. If these probabilities were larger, the connection vector would be compatible with outcome vectors exceeding the Tsirelson bounds.

This is not, of course, a physical explanation, but a principled way of embedding the Tsirelson bounds within the framework of KPT. The connection probabilities themselves do not have an interpretation within a physical theory. Recall, however, that connections are merely 2-marginals of a p-coupled eight-component vector H in (19), and that we use the connections to delineate a class of such p-couplings. As we explained in Introduction, a p-coupling H allows for the same interpretation in terms of “hidden variables” as the one traditionally used (whether or not one calls it “physical”) in the derivation and analysis of the Bell-type theorems.

Recently, Cabello [12] attempted to find an account for the Tsirelson bounds using another principle. In his analysis of the Alice–Bob paradigm, he considers sequences

of stochastically independent events (using our notation)

$$\begin{aligned}
 & (A_{i_1 j_1} = a_1, B_{i_1 j_1} = b_1), (A_{i_2 j_2} = a_2, B_{i_2 j_2} = b_2), \dots, (A_{i_n j_n} = a_n, B_{i_n j_n} = b_n), \\
 & (A_{i'_1 j'_1} = a'_1, B_{i'_1 j'_1} = b'_1), (A_{i'_2 j'_2} = a'_2, B_{i'_2 j'_2} = b'_2), \dots, (A_{i'_n j'_n} = a'_n, B_{i'_n j'_n} = b'_n), \\
 & \dots
 \end{aligned} \tag{34}$$

Two such sequences are called (mutually) exclusive if, for at least one $k \in \{1, 2, \dots, n\}$, either $i_k = i'_k$ and $a_k \neq a'_k$ or $j_k = j'_k$ and $b_k \neq b'_k$. Cabello postulates then that

$$\begin{aligned}
 & \sum \Pr [A_{i_1 j_1} = a_1, B_{i_1 j_1} = b_1] \Pr [A_{i_2 j_2} = a_2, B_{i_2 j_2} = b_2] \\
 & \times \dots \times \Pr [A_{i_n j_n} = a_n, B_{i_n j_n} = b_n] \leq 1
 \end{aligned} \tag{35}$$

if the sum is taken over any set of pairwise exclusive sequences, for any n . This postulate allows him to successfully derive a certain QM inequality [30], and he conjectures that the Tsirelson bounds follow from this postulate too. This conjecture, however, remains unproven, and the relation of Cabello’s postulate to KPT in general and to our characterization of the Tsirelson bounds by means of compatible connections remains unclear.

It may be useful to compare our approach to constructing a Kolmogovian account of the EPR paradigm to the only other systematic way of doing this known to us. We call it *conditionalization*. It consists in considering the settings (α_i, β_j) as values of a random variable C , and treating the spins as random variables whose distributions are conditioned upon the values of C . Avis et al. [31] implement this approach by considering the system of jointly distributed $(C, A'_1, A'_2, B'_1, B'_2)$ such that

$$\Pr [A'_i = \pm 1, B'_j = \pm 1 \mid C = (\alpha_i, \beta_j)] = \Pr [A_{ij} = \pm 1, B_{ij} = \pm 1]. \tag{36}$$

They describe two ways of achieving this. In one of them A'_i and B'_j have three possible values, ± 1 and 0 (that can be interpreted as “no value”), and

$$\begin{aligned}
 & \Pr [A'_i = a, B'_j = b, A'_{3-i} = a', B'_{3-j} = b' \mid C = (\alpha_i, \beta_j)] \\
 & = \begin{cases} \Pr [A_{ij} = a, B_{ij} = b] & \text{if } a \neq 0, b \neq 0, a' = b' = 0, \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned} \tag{37}$$

In another implementation, A'_i and B'_j have two possible values, ± 1 , and

$$\begin{aligned}
 & \Pr [A'_i = a, B'_j = b, A'_{3-i} = a', B'_{3-j} = b' \mid C = (\alpha_i, \beta_j)] \\
 & = \frac{1}{4} \Pr [A_{ij} = a, B_{ij} = b].
 \end{aligned} \tag{38}$$

In both cases the joint distribution of $(C, A'_1, A'_2, B'_1, B'_2)$ is well-defined for any distribution C with non-zero values of $\Pr[C = (\alpha_i, \beta_j)]$, $i, j \in \{1, 2\}$.

An even simpler implementation of conditionalization would be to introduce just two binary (+1/−1) random variables A', B' , and to construct a joint distribution of (C, A', B') by positing

$$\Pr[A' = a, B' = b | C = (\alpha_i, \beta_j)] = \Pr[A_{ij} = a, B_{ij} = b]. \quad (39)$$

Conditionalization is universally applicable, and it indeed achieves the goal of embedding all imaginable distributions of (A_{ij}, B_{ij}) , $i, j \in \{1, 2\}$, into the framework of KPT. In fact, it veridically describes the experiment in which the settings (α_i, β_j) are chosen randomly according to some distribution. As we argue in greater detail elsewhere [32], however, this approach has its weakness: it is not only universal, it is also indiscriminate. Conditionalization applies in precisely the same way to the distributions of (A_{ij}, B_{ij}) whether they are subject to classical–mechanical constraints, to QM constraints, or anything else, the choice of the distribution for C being irrelevant. The conditionalization approach therefore can be compared to saying that the four pairs (A_{ij}, B_{ij}) can always be p-coupled as stochastically independent pairs: this is true, but not elucidating. By contrast, our contextual approach is aimed at characterizing different constraints imposed on the distributions of (A_{ij}, B_{ij}) by their compatibility with different distributions of the connections $(A_{i1}, A_{i2}), (B_{1j}, B_{2j})$, $i, j \in \{1, 2\}$ (or other marginals). This allows us, in particular, to characterize the classical–mechanical and Tsirelson constraints, and to identify the QM constraint as falling beyond the reach of such characterization.

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References

1. Bell, J.: On the Einstein–Podolsky–Rosen paradox. *Physics* **1**, 195–200 (1964)
2. Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.* **23**, 880–884 (1969)
3. Clauser, J.F., Horne, M.A.: Experimental consequences of objective local theories. *Phys. Rev. D* **10**, 526–535 (1974)
4. Fine, A.: Hidden variables, joint probability, and the Bell inequalities. *Phys. Rev. Lett.* **48**, 291–295 (1982)
5. Kolmogorov, A.: *Foundations of the Theory of Probability*. Chelsea, New York (1956)
6. Kochen, S., Specker, F.: The problem of hidden variables in quantum mechanics. *J. Math. Mech.* **17**, 59–87 (1967)
7. Laudisa, F.: Contextualism and nonlocality in the algebra of EPR observables. *Philos. Sci.* **64**, 478–496 (1997)
8. Spekkens, R.W., Buzacott, D.H., Keehn, A.J., Toner, B., Pryde, G.J.: Universality of state-independent violation of correlation inequalities for noncontextual theories. *Phys. Rev. Lett.* **102**, 010401 (2009)
9. Kirchmair, G., Zhringer, F., Gerritsma, R., Kleinmann, M., Ghne, O., Cabello, A., Blatt, R., Roos, C.F.: State-independent experimental test of quantum contextuality. *Nature* **460**, 494–497 (2009)
10. Badziąg, P., Bengtsson, I., Cabello, A., Pitowsky, I.: Universality of state-independent violation of correlation inequalities for noncontextual theories. *Phys. Rev. Lett.* **103**, 050401 (2009)
11. Khrennikov, A.Y.: Contextual approach to quantum formalism. In: *Fundamental Theories of Physics*, vol. 160. Springer, Dordrecht (2009)

12. Cabello, A.: Simple explanation of the quantum violation of a fundamental inequality. *Phys. Rev. Lett.* **110**, 060402 (2013)
13. Dzhafarov, E.N., Kujala, J.V.: All-possible-couplings approach to measuring probabilistic context. *PLoS ONE* **8**(5), e61712 (2013). doi:[10.1371/journal.pone.0061712](https://doi.org/10.1371/journal.pone.0061712)
14. Dzhafarov, E.N., Kujala, J.V.: A qualified Kolmogorovian account of probabilistic contextuality. In: *Lecture Notes in Computer Science* (in press). [arXiv:1304.4546](https://arxiv.org/abs/1304.4546)
15. Dzhafarov, E.N., Kujala, J.V.: Random variables recorded under mutually exclusive conditions: contextuality-by-Default. *Adv. Cogn. Neurodyn. IV* (in press). [arXiv:1309.0962](https://arxiv.org/abs/1309.0962)
16. Bohm, D., Aharonov, Y.: Discussion of experimental proof for the paradox of Einstein. Rosen Podolski. *Phys. Rev.* **108**, 1070–1076 (1957)
17. Dzhafarov, E.N., Kujala, J.V.: On selective influences, marginal selectivity, and Bell/CHSH inequalities. *Topics Cognit. Sci.* **6**, 121–128 (2014)
18. Khrennikov, A.Y.: Bell–Boole inequality: nonlocality or probabilistic incompatibility of random variables? *Entropy* **10**, 19–32 (2008)
19. Khrennikov, A.Y.: EPR–Bohm experiment and Bell’s inequality: quantum physics meets probability theory. *Theor. Math. Phys.* **157**, 1448–1460 (2008)
20. Gudder, S.: On hidden-variable theories. *J. Math. Phys.* **2**, 431–436 (1970)
21. Thorisson, H.: *Coupling, Stationarity, and Regeneration*. Springer, New York (2000)
22. Landau, L.J.: Empirical two-point correlation functions. *Found. Phys.* **18**, 449–460 (1988)
23. Cabello, A.: How much larger quantum correlations are than classical ones. *Phys. Rev. A* **72**, 12113 (2005)
24. Kujala, J.V., Dzhafarov, E.N.: Testing for selectivity in the dependence of random variables on external factors. *J. Math. Psychol.* **52**, 128–144 (2008)
25. Tsirelson, B.S.: Quantum generalizations of Bell’s inequality. *Lett. Math. Phys.* **4**, 93–100 (1980)
26. Landau, L.J.: On the violation of Bell’s inequality in quantum theory. *Phys. Lett. A* **120**, 54–56 (1987)
27. Dzhafarov, E.N., Kujala, J.V.: Selectivity in probabilistic causality: where psychology runs into quantum physics. *J. Math. Psychol.* **56**, 54–63 (2012)
28. Dzhafarov, E.N., Kujala, J.V.: Quantum entanglement and the issue of selective influences in psychology: an overview. In: *Lecture Notes in Computer Science*, vol. 7620, pp. 184–195 (2012)
29. Dzhafarov, E.N., Kujala, J.V.: Order-distance and other metric-like functions on jointly distributed random variables. *Proc. Am. Math. Soc.* **141**, 3291–3301 (2013)
30. Klyachko, A.A., Can, M.A., Binicioglu, S., Shumovsky, A.S.: Simple test for hidden variables in spin-1 systems. *Phys. Rev. Lett.* **101**, 020403 (2008)
31. Avis, D., Fischer, P., Hilbert, A., Khrennikov, A.: Single, complete, probability spaces consistent with EPR–Bohm–Bell experimental data. In: Khrennikov, A. (ed.) *Foundations of Probability and Physics-5*, AIP Conference Proceedings, vol. 750, pp. 294–301. AIP, Melville, New York (2009)
32. Dzhafarov, E.N., Kujala, J.V.: Embedding quantum into classical: contextualization vs conditionalization. [arXiv:1312.0097](https://arxiv.org/abs/1312.0097) (2013)