NO SEMISTABILITY AT INFINITY FOR CALABI-YAU METRICS ASYMPTOTIC TO CONES

SONG SUN AND JUNSHENG ZHANG

ABSTRACT. We discover a "no semistability at infinity" phenomenon for complete Calabi-Yau metrics asymptotic to cones, which is proved by eliminating the possible appearance of an intermediate K-semistable cone in the 2-step degeneration theory developed by Donaldson and the first author. It is in sharp contrast to the setting of local singularities of Kähler-Einstein metrics. A byproduct of the proof is a polynomial convergence rate to the asymptotic cone for such manifolds, which bridges the gap between the general theory of Colding-Minicozzi and the classification results of Conlon-Hein.

1. INTRODUCTION

Throughout this paper we denote by $(X, p, g, J, \omega, \Omega)$ a pointed complete Calabi-Yau manifold of complex dimension n. This means that $p \in X$, (g, J, ω) is a complete Kähler metric, Ω is a holomorphic volume form and the following complex Monge-Ampère equation is satisfied

$$\omega^n = (\sqrt{-1})^{n^2} \Omega \wedge \overline{\Omega}.$$

We will always impose the following two conditions

- Euclidean volume growth: there exists a $\kappa > 0$ such that $\mathrm{Vol}(B_g(p, R)) \geqslant \kappa R^{2n}$ for all $R > 0$.
- Quadratic curvature decay: there exists a $C > 0$ such that $|Rm_g(q)| \leqslant C d_g(p, q)^{-2}$ for all $q \neq p$.

We also denote by $(\mathcal{C}, O, g_{\mathcal{C}}, J_{\mathcal{C}}, \omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$ a Calabi-Yau cone. This means that O is a distinguished point called the *vertex*, $\mathcal{C} \setminus \{O\}$ is a smooth manifold diffeomorphic to $\mathbb{R}_{>0} \times L$ for some compact manifold L, $(g_{\mathcal{C}}, J_{\mathcal{C}}, \omega_{\mathcal{C}}, \Omega_{\mathcal{C}})$ is a Calabi-Yau metric on $\mathcal{C} \setminus \{O\}$ and $g_{\mathcal{C}} = dr^2 + r^2 g_L$ is a Riemannian cone. A Calabi-Yau cone is naturally a normal affine algebraic cone.

It is known by [15, 24] that any X as above is asymptotic to a unique Calabi-Yau cone $\mathcal{C}$ at infinity. Moreover, X is naturally a quasi-projective variety and there is an algebro-geometric 2-step degeneration from X to $\mathcal{C}$, via a possible intermediate K-semistable cone W. The main result of this paper is

Theorem 1.1 (No semistability at infinity). *W is isomorphic to $\mathcal{C}$ as normal affine algebraic cones.*

This comes out of our expectation, since the 2-step degeneration theory of [15] works symmetrically with respect to asymptotic cones at infinity and local tangent cones at a singularity and it is known that W can *not* always be eliminated in the local setting. Theorem 1.1 reveals a sharp contrast between the two cases. Our proof makes a novel use of the Tian-Yau construction of complete Calabi-Yau metrics and the Bishop-Gromov volume monotonicity in Riemannian geometry. As a byproduct, we also show that X is polynomially asymptotic to $\mathcal{C}$ at infinity:

Theorem 1.2 (Polynomial rate to the asymptotic cone). *There exists a diffeomorphism Ψ from the open subset $\{r > 1\} \subset \mathcal{C}$ to the complement of a compact set in X such that for some $\delta > 0$ and for all $k \geqslant 0$,*

$$\left| \nabla^k_{g_C}(\Psi^* g - g_C) \right|_{g_C} + \left| \nabla^k_{g_C}(\Psi^* J - J_{\mathcal{C}}) \right|_{g_C} + \left| \nabla^k_{g_C}(\Psi^* \Omega - \Omega_{\mathcal{C}}) \right|_{g_C} = O(r^{-\delta-k}), \quad r \to \infty. \tag{1.1}$$

Theorem 1.2 bridges the gap between two previous results. On the one hand, on the level of Riemannian metrics the result of Colding-Minicozzi [6], which uses the Łojasiewicz-Simon theory, implies that X is asymptotic to a unique cone at a *logarithmic* rate. On the other hand, Conlon-Hein

[11] classified complete Calabi-Yau manifolds asymptotic to a Calabi-Yau cone, however under the stronger extra assumption of *polynomial* rate (as given exactly by (1.1)).

2. Outline of the proof

2.1. **2-step degeneration theory.** Fix $\lambda \equiv 1/\sqrt{2}$. Denote by $(X_j, J_j, g_j, \Omega_j, p_j)$ the rescaling of X by a factor λ^j. This means that we take $J_j = J, p_j = p, g_j = \lambda^{2j} g, \Omega_j = \lambda^{nj} \Omega$. It follows from the Bishop-Gromov volume monotonicity that after passing to a subsequence (X_j, p_j) converges in the pointed Gromov-Hausdorff sense to a Calabi-Yau cone. We summarize below the theory developed in [15] and [24]. In [15] a more general singular setting is considered but under a technical assumption that X is a rescaled Gromov-Hausdorff limit of *polarized* Kähler-Einstein manifolds. This assumption was later removed in [24] for complete Calabi-Yau manifolds with Euclidean volume growth and quadratic curvature decay, so the results of [15] apply in our setting.

One conclusion of [15, 24] is that the asymptotic cone is unique as a Calabi-Yau cone, i.e., it does not depend on the choice of subsequences involved in the definition. This may also follow from the more general result of Colding-Minicozzi [6] (which does not concern the complex structure), but the complex geometric proof in [15, 24] more importantly gives an algebro-geometric description of the asymptotic cone. This is crucial for us so below we review some relevant statements.

Let $(\mathcal{C}, g_\mathcal{C}, J_\mathcal{C}, \omega_\mathcal{C}, \Omega_\mathcal{C}, O)$ be the unique asymptotic Calabi-Yau cone of X. Since $\mathcal{C}$ is a Kähler cone, there is a Reeb vector field $\xi = J_\mathcal{C}(r\partial_r)$, which is holomorphic and Killing. It generates a holomorphic isometric action of a compact torus $\mathbb{T}$ on $\mathcal{C}$. The Kähler form has a simple expression given by $\omega_\mathcal{C} = \dfrac{\sqrt{-1}}{2} \partial_{J_\mathcal{C}} \bar{\partial}_{J_\mathcal{C}} r^2$. Furthermore, $\mathcal{C}$ is naturally a normal affine variety. The coordinate ring $R(\mathcal{C})$ can be intrinsically characterized as the space of holomorphic functions on $\mathcal{C} \setminus \{O\}$ that have polynomial growth at infinity. More explicitly, there is a holomorphic embedding $\Phi_\infty : (\mathcal{C}, O) \to (\mathbb{C}^N, 0)$ such that the Reeb vector field ξ extends to a linear vector field on $\mathbb{C}^N$ of the form $Re(\sqrt{-1} \sum_j d_j z_j \partial_{z_j})$, where $d_j \in \mathbb{R}_{>0}$ for all j. Let $\Lambda : \mathbb{C}^N \to \mathbb{C}^N$ be the diagonal linear transformation of $\mathbb{C}^N$ given by

$$\Lambda(z_1, \cdots, z_N) = (\lambda^{d_1} z_1, \cdots, \lambda^{d_N} z_N),$$

and let T_ξ be the one parameter group of linear transformations generated by ξ. Denote by G_ξ the subgroup of $GL(N; \mathbb{C})$ consisting of elements that commute with T_ξ and denote $K_\xi = G_\xi \cap U(N)$. Notice T_ξ naturally acts on $\mathcal{C}$.

Denote by B_j the unit ball around p_j in X_j, which we can identify with the ball of radius λ^{-j} around p in X. Denote by B_∞ the unit ball around the vertex O in $\mathcal{C}$. Then we know that B_∞ is the Gromov-Hausdorff limit of B_j as $j \to \infty$. Fix a distance function on $B_j \sqcup B_\infty$ that realizes the Gromov-Hausdorff convergence. We have the following

(a). The ring $R(X)$ of holomorphic functions on X with polynomial growth is finitely generated and $\mathrm{Spec}(R(X))$ defines a normal affine algebraic variety X' with isolated singularities. There is a natural map $\pi : (X, p) \to (X', p')$ which is a crepant resolution of singularities.

(b). For all $j \geqslant 1$, there are holomorphic embeddings $\Phi_j : (X', p') \to (\mathbb{C}^N, 0)$ and $F_j \in G_\xi$ satisfies $F_j = \Lambda + \tau_j$ for linear maps $\tau_j \to 0$, such that
  (1). $\Phi_{j+1} = F_{j+1} \circ \Phi_j$.
  (2). For any subsequence of $\{j\}$ tends to infinity, passing to a further subsequence $\Phi_j(\pi(B_j))$ converges to $h.\Phi_\infty(B_\infty)$ in the Hausdorff sense in $\mathbb{C}^N$ for some $h \in K_\xi$. This convergence

is compatible with the Gromov-Hausdorff convergence in the sense that given any sequence $x_j \in B_j$ converging to $x_\infty \in B_\infty$, we have $\Phi_j(\pi(x_j))$ converges to $h.\Phi_\infty(x_\infty)$.

(3). We will always identify (X', p') with $(\Phi_1(X'), 0)$. Denote $Y_j \equiv \Lambda^{j-1}.X'$. Then the weighted asymptotic cone $W = \lim_{j\to\infty} Y_j$ is a normal affine algebraic variety in $\mathbb{C}^N$. In particular, W is also invariant under the T_ξ action. This limit is to be understood in a strong sense: there exists a generating set $\{P_1, \cdots, P_l\}$ of the ideal $\mathcal{I}_{X'}$ of polynomials on $\mathbb{C}^N$ vanishing on X', such that the leading terms $\{\tilde{P}_1, \cdots, \tilde{P}_l\}$ generate the ideal $\mathcal{I}_W$. Here the leading term of a polynomial is defined via the weighted degree associated to Λ.

(4). The coordinate ring $R(W)$ can be intrinsically described as the graded ring associated to a filtration on $R(X)$ determined by a *degree* function associated to the Calabi-Yau metric g.

(5). $\mathcal{C}$ is in the closure of the G_ξ orbit of W in a suitable multi-graded Hilbert scheme. In particular, W has only one isolated singularity.

The motivation in [15] was to relate the metric scaling on X to the algebro-geometric scaling Λ. The main discovery there is that they are *almost but not exactly* the same, and one may realize the asymptotic cone $\mathcal{C}$ as a 2-step degeneration from X' through the intermediate cone W. From the technical aspect the central issue is that as $j \to \infty$, even though F_j converges to Λ the accumulated error in the composition $F_j \circ F_{j-1} \circ \cdots \circ F_2$ may still diverge away from Λ^{j-1}. Notice W does not inherit any canonical metric from X or $\mathcal{C}$. In [15] a parallel result is also proved for tangent cones at local singularities of Kähler-Einstein metrics. However there are important distinctions between the two settings.

One such distinction was already pointed out in [15], which shows that the local setting is more *rigid* and the asymptotic setting is more *flexible*. In the case of local tangent cones, it is conjectured in [15] that there is a local notion of (K-)stability which characterizes both (W, ξ) and $\mathcal{C}$ as *invariants* of the underlying algebraic singularity (by analogy with the Harder-Narasimhan-Seshadri filtration for sheaves, one should think of W as a *semistable* object and $\mathcal{C}$ as a *polystable* object). Later Li [19] related this picture to a generalized volume minimization in Sasaki geometry and reformulated this conjecture using more algebro-geometric terminologies. There has been active research on this topic. The uniqueness part of the conjecture in [15] was confirmed by Li-Wang-Xu [20]. By way of contrast as suggested at the end of [15], for asymptotic cones of complete Calabi-Yau metrics one should not expect either W or $\mathcal{C}$ is a canonical object associated to the underlying algebraic variety X. Explicit examples have been constructed on $\mathbb{C}^n$ if we allow W and $\mathcal{C}$ to have singular cross sections (see [18, 23, 28, 12]).

The main result of this paper shows that the asymptotic setting is however more *rigid* from a different perspective. In the case of tangent cones at singularities of Kähler-Einstein metrics, it is known that both of the 2 steps are necessary in order to degenerate the singularity to the metric tangent cone $\mathcal{C}$. In the case of asymptotic cones at infinity we will show that W is always isomorphic to $\mathcal{C}$, in other words, one can degenerate the variety to $\mathcal{C}$ in one step. Theorem 1.1 can be reformulated as

Theorem 1.1 (No semistability at infinity). *W is isomorphic to $\mathcal{C}$ as affine varieties with T_ξ action.*

2.2. **Outline of the proof.** Now we explain the key ideas in the proof of Theorem 1.1. Theorem 1.2 will be a consequence. Our strategy is to construct a Calabi-Yau metric ω_W on $W \setminus \{0\}$ so that

(1) the asymptotic cone of ω_W at infinity is given by $\mathcal{C}$;

(2) the metric completion of ω_W is homeomorphic to W and is a pointed Gromov-Hausdorff limit of complete Calabi-Yau metrics. Furthermore the argument in [15] can be extended to study the singular point $0 \in W$.

Now the Bishop-Gromov inequality appling on W yields that the volume ratio

$$\nu(R) = \frac{\text{Vol}(B(0, R))}{R^{2n}}$$

is a non-increasing function in R. On a Calabi-Yau cone $\mathcal{C}$ we denote by $\kappa(\mathcal{C}) = \mathrm{Vol}(B(O,1))$ the volume density of $\mathcal{C}$. Now as $R \to \infty$, $\nu(R)$ converges to $\kappa(\mathcal{C})$, while as $R \to 0$ it converges to the volume density of the metric tangent cone at 0. By [15] and the generalized volume minimization in [21] we know the latter is not bigger than $\kappa(\mathcal{C})$. So $\nu(R)$ is constant in R, which implies that (W, ω_W) must be a metric cone hence is isometric to $\mathcal{C}$. With more work one can show that W and $\mathcal{C}$ are T_ξ-equivariantly isomorphic.

The existence of ω_W is therefore the crux of the matter. To achieve this we will make use of the Tian-Yau construction [30] which produces complete Calabi-Yau metrics out of lower dimensional compact Kähler-Einstein metrics. In [30] one starts with a Fano manifold X and a smooth divisor D which is $\mathbb{Q}$-linearly equivalent to K_X^{-1}, then assuming that D admits a Kähler-Einstein metric with positive Ricci curvature, one can find a complete Calabi-Yau metric on $X \setminus D$. The rough idea is to first use the Calabi ansatz to write down a background complete Kähler metric which is approximately Ricci-flat and then solve the complex Monge-Ampère equation on the noncompact manifold $X \setminus D$. The work of [30] has been extended widely to produce more examples of Calabi-Yau metrics.

In the proof of Theorem 1.1 we will apply the Tian-Yau construction in a non-traditional manner. That is, we will use the unknown Calabi-Yau metric ω on X as a model at infinity to construct a Calabi-Yau metric ω_W on W. This is achieved by constructing a diffeomorphism between the ends of X and W, then grafting ω to a Kähler form on W that is slowly asymptotic to the cone $\mathcal{C}$, but satisfies the complex Monge-Ampère equation with polynomially decaying error. Most of the work in this paper is devoted to making this strategy rigorous. There is an extra difficulty caused by the fact that W has an isolated singularity. To get around the issue, we notice that W is the limit of Y_j in $\mathbb{C}^N$, and Y_j admits a crepant resolution $\hat{Y}_j$ (which is isomorphic to X). We will first construct a family of Calabi-Yau metrics $\hat{\omega}_{j,\epsilon}$ on $\hat{Y}_j$ with uniform estimates, where ϵ roughly measures the size of the exceptional set, and then take a double limit by first letting $\epsilon \to 0$ and then letting $j \to \infty$. The first limiting step follows similarly the line of argument in [8] which constructs certain asymptotically conical (in the sense of [10]) Calabi-Yau metrics with singularities, under more restrictive assumptions. In the course of the proof we will bring in more robust techniques, such as the Hörmander L^2 estimate, which is of independent interest.

In the rest of this paper we will present a detailed proof of the main results. In Section 3 we construct a family of background Kähler metrics on $\hat{Y}_j$ which are asymptotic to the cone $\mathcal{C}$ and are approximate solutions to the complex Monge-Ampère equation at infinity. In Section 4 we solve the complex Monge-Ampère equation with respect to these background metrics and derive uniform estimates. In Section 5 we complete the proof of Theorem 1.1 and 1.2.

We make a convention on the notations throughout this paper: K will denote a compact set in some space; C will denote a positive constant; $C_\sharp$ will denote a constant that depends only on $\sharp$; in particular when j does not appear in the subscript it means that the constant is uniform in j. The precise meaning of these objects may vary between lines.

3. Construction of background metrics

3.1. Weak conical approximation at infinity.
We denote by $r = d(O, \cdot)$ the radial function on the asymptotic Calabi-Yau cone $\mathcal{C}$, by $\mathcal{Q}_R$ the subset $\{r \geqslant R\}$ and by $A_{a,b}$ the annulus $\{a \leqslant r \leqslant b\}$. The following lemma is standard so we only give a sketch of proof.

Lemma 3.1. *There exist an $R_0 > 0$, a compact set $K \subset X$ and a diffeomorphism $\Phi : \mathcal{Q}_{R_0} \to X \setminus K$ such that for all $l \geqslant 0$, $\displaystyle \lim_{s \to \infty} \sup_{\partial \mathcal{Q}_s} s^l \left(\left| \nabla_{g_\mathcal{C}}^l (\Phi^* g - g_\mathcal{C}) \right|_{g_\mathcal{C}} + \left| \nabla_{g_\mathcal{C}}^l (\Phi^* J - J_\mathcal{C}) \right|_{g_\mathcal{C}} \right) = 0.$*

Proof. Since $\mathcal{C}$ is the unique asymptotic cone of X, for i large we may find a diffeomorphism Φ_i from $A_{4^i, 8 \cdot 4^i}$ onto $B(p, 8 \cdot 4^i(1 + \epsilon_i)) \setminus B(p, 4^i(1 + \epsilon_i))$ such that

$$4^{il} \sup_{A_{4^i, 8 \cdot 4^i}} \left(\left| \nabla_{g_\mathcal{C}}^l (\Phi_i^* g - g_\mathcal{C}) \right|_{g_\mathcal{C}} + \left| \nabla_{g_\mathcal{C}}^l (\Phi_i^* J - J_\mathcal{C}) \right|_{g_\mathcal{C}} \right) \leqslant \epsilon_{i,l},$$

where $\epsilon_i \to 0$ and for each l fixed, $\epsilon_{i,l} \to 0$ as $i \to \infty$. Moreover, the rescaled maps $\Phi_i \circ 4^i$ realize the Gromov-Hausdorff convergence to the cone $\mathcal{C}$. It suffices to glue together Φ_i's to get the desired Φ. This can done in a straightforward way. By a contradiction argument one can prove that for $i \geq i_0$, in the intersection $\mathrm{Im}(\Phi_i) \cap \mathrm{Im}(\Phi_{i+1})$, the composition $4^{-i-1} \circ \Phi_{i+1}^{-1} \circ \Phi_i \circ 4^{i+1}$ is ϵ_i (for $\epsilon_i \to 0$) close to a holomorphic isometric embedding Ψ_i of the annulus $A_{1,2}$ into itself with $\Psi_i^* r = r$. It follows that Ψ_i preserves the Reeb vector field on $\mathcal{C}$, hence it extends to a global holomorphic isometry of $\mathcal{C}$ which fixes the vertex. Then we replace Φ_{i+1} by $\Phi_{i+1} \circ \Psi_i \circ \cdots \circ \Psi_{i_0}$ and perform an obvious interpolation between Φ_i and Φ_{i+1}. $\square$

Notice that at this point we know the metric $\Phi^* g$ is asymptotic to $g_{\mathcal{C}}$ at infinity, but without any quantitative *rate* of decay. Nevertheless, the difference of the Levi-Civita connections does decay at the order $o(r^{-1})$. The general result of Colding-Minicozzi [6] yields a logarithmic decay rate of the Riemannian metric but we will not need it in this paper. Using the diffeomorphism constructed above, we may naturally view r as a function on the end of X which satisfies

$$C^{-1} d_g(p, \cdot) \leqslant r(\cdot) \leqslant C d_g(p, \cdot).$$

Given a constant $\widetilde{R} > R_0 + 1$, we can further replace r by the regularized maximum $\widetilde{\max}\{r, \widetilde{R}\}$ (see [13, Chapter 1, Lemma 5.18]), so it can be viewed as a smooth function on X. Moreover, by Lemma 3.1, we can fix an $\widetilde{R}$ large so that r^2 is a smooth plurisubharmonic exhaustion function on X which is strictly plurisubharmonic outside a compact set. Then we know that for any sufficiently large number R, the sublevel set

$$X_R := \{r < R\} \tag{3.1}$$

is 1-convex. By the Grauert-Riemenschneider vanishing theorem [16] and triviality of the canonical bundle K_X, we get

$$H^1(X_R, \mathcal{O}) = H^1(X_R, K_{X_R}) = 0. \tag{3.2}$$

This will be used in the proof of Theorem 3.9.

3.2. Weighted elliptic estimates.
Here we collect some facts on weighted elliptic theory for ends of Riemannian manifolds asymptotic to cones. These are not sharp results but suffice for our applications in this paper.

For any $k \in \mathbb{Z}_{\geqslant 0}$, $\alpha \in (0,1)$, $\delta \in \mathbb{R}$, $R > 0$, we define the Banach space $C_\delta^{k,\alpha}(\mathcal{Q}_R)$ using the following weighted norm on a function f defined on $\mathcal{Q}_R \subset \mathcal{C}$:

$$\|f\|_{C_\delta^{k,\alpha}(Q_R)} \equiv \sup_{s \geqslant R} \Big\{ \sum_{m=0}^{k} s^{-\delta+m} \|\nabla_{g_{\mathcal{C}}}^m f\|_{C_{g_{\mathcal{C}}}^0(A_{s,2s})} + s^{-\delta+k+\alpha} [f]_{C_{g_{\mathcal{C}}}^{k,\alpha}(A_{s,2s})} \Big\},$$

where

$$[f]_{C^{k,\alpha}(A_{s,2s})} \equiv \sup_{y_1, y_2 \in A_{s,2s}, d_{g_{\mathcal{C}}}(y_1, y_2) < \mathrm{inj}_{g_{\mathcal{C}}}(y_1)} \Big\{ \frac{|\nabla_{g_{\mathcal{C}}}^k f(y_1) - \nabla_{g_{\mathcal{C}}}^k f(y_2)|}{d_{g_{\mathcal{C}}}(y_1, y_2)^\alpha} \Big\}.$$

Fix $k \geqslant 2n+1$ and $\alpha \in (0,1)$. The following can be proved in exactly the same way as [27, Proposition 6.7], so we omit the proof here.

Lemma 3.2. *There exists a finite set $\Gamma \subset (0,1)$ such that for all $\delta \in (0,1) \setminus \Gamma$ and for all $R \geqslant 1$, one can find a bounded linear map $\mathcal{S}_R : C_{-\delta}^{k,\alpha}(\mathcal{Q}_R) \to C_{-\delta+2}^{k+2,\alpha}(\mathcal{Q}_R)$ with $\Delta_{g_{\mathcal{C}}} \circ \mathcal{S}_R = \mathrm{Id}$ and with the operator norm $\|\mathcal{S}_R\| \leqslant C$ for some C depending only on $\mathcal{C}$, δ, k, α (but not on R).*

Given any $R_1 > 1$ and suppose there is another Riemannian metric g on $\mathcal{Q}_{R_1}$ such that for all $l \geqslant 0$ and $s \geqslant R_1$,

$$s^l \sup_{\partial \mathcal{Q}_s} |\nabla_{g_{\mathcal{C}}}^l (g - g_{\mathcal{C}})| \leqslant e_l(s) \tag{3.3}$$

for some function e_l defined on $[R_1, \infty)$ with $\lim_{s \to \infty} e_l(s) = 0$.

Proposition 3.3. *Given $\delta \in (0,1) \setminus \Gamma$, there exist $R_2 \geqslant R_1$ and $C > 0$ depending only on $\mathcal{C}$, R_1, k, δ, α and the functions $e_1, \cdots, e_{k+2}$, and a bounded linear map $\mathcal{T} : C_{-\delta}^{k,\alpha}(\mathcal{Q}_{R_2}) \to C_{-\delta+2}^{k+2,\alpha}(\mathcal{Q}_{R_2})$ such that $\Delta_g \circ \mathcal{T} = \mathrm{Id}$ and $\|\mathcal{T}\| \leqslant C$.*

Proof. One can write $\Delta_g = \Delta_{g_{\mathcal{C}}} + (g - g_{\mathcal{C}}) * \nabla_{g_{\mathcal{C}}}^2 + \nabla_{g_{\mathcal{C}}} g * \nabla_{g_{\mathcal{C}}}$. Then on $\mathcal{Q}_R$, $R \geqslant R_1$, we have

$$\|\Delta_g - \Delta_{g_{\mathcal{C}}}\| \leqslant C \sup_{s \geqslant R, \ 0 \leqslant l \leqslant k+2} e_l(s).$$

The right hand side is small when $R \gg 1$. Then the conclusion follows from standard functional analysis. $\qquad\square$

The following is an application of the standard elliptic regularity by a straightforward rescaling argument. We omit the proof.

Proposition 3.4. *Let g be a Riemannian metric on $\mathcal{Q}_{R_1}$ satisfying (3.3). Suppose u is a function defined on $\mathcal{Q}_{R_1}$ such that for some $s, s' \in \mathbb{R}$ and for all $l \geqslant 0$, $\left|\nabla_g^l(\Delta_g u)\right|_g \leqslant C_l r^{-l-2+s}$, and*

$$\int_{\mathcal{Q}_{R_1}} u^2 r^{-2s'-2n} d\,\mathrm{Vol}_g \leqslant \hat{C}. \ \textit{Then for any } l > 0 \textit{ and } \gamma > \max(s, s'), \textit{ there exist constants } A_{\gamma l} > 0$$

depending only on $\gamma, \mathcal{C}$, R_1, $C_0, C_1, \cdots, C_l, \hat{C}$ and the functions $e_1, \cdots, e_l$, such that

$$\left|\nabla_g^l u\right|_g \leqslant A_{\gamma l} r^{\gamma-l} \textit{ on } \mathcal{Q}_{R_1}.$$

3.3. **Rough algebraic approximation of the Calabi-Yau metric.** We continue to use the notations in Section 2. Recall that X' is the affine variety defined by the ring $R(X)$ of holomorphic functions on X with polynomial growth and we identify it with its image in $\mathbb{C}^N$ under the holomorphic embedding Φ_1. We also have the crepant resolution map $\pi : X \to X' \subset \mathbb{C}^N$. In the following outside a compact K of X containing the exceptional set of π, we also identify X with X'. In particular, $X \setminus K$ is holomorphically embedded in $\mathbb{C}^N$.

Since the coordinate functions on $\mathbb{C}^N$ have positive weight with respect to ξ, we can find a Kähler cone metric ω_ξ on $\mathbb{C}^N$ with Reeb vector field ξ. Fix a choice of such ω_ξ. Then we have $\Lambda^* \omega_\xi = \lambda^2 \omega_\xi$. Denote by $r_\xi = d_{\omega_\xi}(0, \cdot)$ the radial function on $\mathbb{C}^N$ defined by ω_ξ. Notice we are not assuming ω_ξ is Calabi-Yau. The following result shows that we may view the restriction of ω_ξ to X as a rough approximation of the unknown Calabi-Yau metric ω.

Proposition 3.5. *There exists a compact set $K \subset X$ such that for all $\delta > 0$ and $k \geqslant 1$, on $X \setminus K$ we have*

$$C_\delta^{-1} r^{-\delta} \omega \leqslant \omega_\xi \leqslant C_\delta r^\delta \omega, \tag{3.4}$$

$$\left|\nabla_\omega^k \omega_\xi\right|_\omega \leqslant C_{\delta,k} r^{\delta-k}, \tag{3.5}$$

$$C_\delta^{-1} r^{1-\delta} \leqslant r_\xi \leqslant C_\delta r^{1+\delta}. \tag{3.6}$$

Proof. Since we identify X' with $\Phi_1(X')$ and outside a compact set identify X with X' using π, we have $\Phi_j = F_j \circ \cdots F_2 : X \to \mathbb{C}^N$. By Item (b) in Section 2 we know that for all $j \geqslant 1$ and $k \geqslant 0$, the following holds on $B_{j+1} \setminus B_j$:

$$C^{-1} \lambda^{2j} r \leqslant \Phi_j^* r_\xi \leqslant C \lambda^{2j} r, \quad C^{-1} \lambda^{2j} \omega \leqslant \Phi_j^* \omega_\xi \leqslant C \lambda^{2j} \omega, \quad |\nabla_\omega^k(\lambda^{-2j} \Phi_j^* \omega_\xi)|_\omega \leqslant C_k \lambda^{kj}. \tag{3.7}$$

By construction we have $F_j^{-1} = \Lambda^{-1} \circ \sigma_j$ for a linear map $\sigma_j \in G_\xi$ with $\|\sigma_j - \mathrm{Id}\| \to 0$, where $\|\cdot\|$ is a fixed norm on the space of $N \times N$ matrices. Since σ_j commutes with Λ we may write

$$F_2^{-1} \circ \cdots \circ F_j^{-1} = \Lambda^{1-j} \sigma_2 \circ \cdots \circ \sigma_j.$$

For any $\sigma \in G_\xi$ we have $\mathcal{L}_{r_\xi \partial_{r_\xi}} \sigma^* \omega_\xi = 2\sigma^* \omega_\xi$. It follows that $\sigma^* \omega_\xi$ is homogeneous of degree 2 on $\mathbb{C}^N$. So one can find a small neighborhood U of Id such that for all $\sigma \in U$ and all $k \geqslant 0$, the following holds on $\mathbb{C}^N$:

$$|\nabla_{\omega_\xi}^k(\sigma^* \omega_\xi - \omega_\xi)|_{\omega_\xi} \leqslant C_k r_\xi^{-k} \|\sigma - \mathrm{Id}\|. \tag{3.8}$$

In particular we can also assume

$$|\nabla_{\omega_\xi}^k (\nabla_{\sigma^*\omega_\xi} - \nabla_{\omega_\xi})|_{\omega_\xi} \leqslant C_k r_\xi^{-k-1} \|\sigma - \mathrm{Id}\|, \tag{3.9}$$

where as usual we view the difference of the two connections as a tensor.

Since $\|\sigma_j - \mathrm{Id}\| \to 0$, for $j \gg 1$ we have $\sigma_j \in U$. We notice the elementary fact that for all $\delta > 0$,

$$\prod_{l=2}^{j} \left(1 + C\|\sigma_l - \mathrm{Id}\|\right) \leqslant C_\delta \lambda^{-j\delta}.$$

From this, iterating (3.8) and using (3.7) yield (3.4). (3.6) follows from a similar argument.

We now prove the case $k = 1$ in (3.5). The case $k > 1$ can be proved similarly by induction. Notice for any $\sigma \in U$ and 2-form γ, we have

$$\begin{aligned}
\nabla_{\omega_\xi}\sigma^*\gamma &= \nabla_{\sigma^*\omega_\xi}\sigma^*\gamma + (\nabla_{\omega_\xi} - \nabla_{\sigma^*\omega_\xi})\sigma^*\gamma \\
&= \sigma^*(\nabla_{\omega_\xi}\gamma) + (\nabla_{\omega_\xi} - \nabla_{\sigma^*\omega_\xi})\sigma^*\gamma.
\end{aligned}$$

Therefore,

$$|\nabla_{\omega_\xi}(\sigma_j^* \cdots \sigma_2^*\omega_\xi)|_{\omega_\xi} = |\sum_{l=2}^{j} \sigma_j^* \cdots \sigma_{l+1}^*(\nabla_{\omega_\xi} - \nabla_{\sigma_l^*\omega_\xi})\sigma_{l-1}^* \cdots \sigma_2^*\omega_\xi|_{\omega_\xi}.$$

Using (3.8) and (3.9) we obtain that on the annulus $\{\lambda \leqslant r_\xi \leqslant \lambda^{-1}\} \subset \mathbb{C}^N$ for all $\delta > 0$,

$$|\nabla_{\omega_\xi}(F_2^{-1} \circ \cdots \circ F_j^{-1})^*\omega_\xi|_{\omega_\xi} \leqslant C\lambda^{-2j}\prod_{l=2}^{j}(1 + C\|\sigma_l - \mathrm{Id}\|) \leqslant C_\delta \lambda^{-(2+\delta)j}. \tag{3.10}$$

By definition $\omega_\xi = \Phi_j^* \left((F_2^{-1} \circ \cdots \circ F_j^{-1})^*\omega_\xi\right)$, so we have

$$|\nabla_\omega\omega_\xi|_\omega = |\Phi_j^*(\nabla_{\omega_\xi}(F_2^{-1} \circ \cdots \circ F_j^{-1})^*\omega_\xi)|_\omega + |(\nabla_\omega - \nabla_{\Phi_j^*\omega_\xi})\Phi_j^*(F_2^{-1} \circ \cdots \circ F_j^{-1})^*\omega_\xi|_\omega.$$

The first term of the right hand side can be handled using (3.10) and (3.7), and the second term can be dealt with using (3.7) and (3.4). Together we get (3.5) when $k = 1$. □

Although not needed in this paper, we remark that the above ω_ξ is not unique and we can find one with explicit algebraic formula. We emphasize that from the proof we can not get rid of the δ in the estimates. In particular, we can *not* claim the uniform equivalence between ω and ω_ξ. Nonetheless we do know the error is smaller than any polynomial order which suffices for our applications. Moreover, the differences between the covariant derivatives do decay at infinity. Namely,

Corollary 3.6. *There exists a compact set $K \subset X$ such that on $X \setminus K$, for all $\delta > 0$ and $k \geqslant 0$ we have*

$$|\nabla_\omega^k(\nabla_\omega - \nabla_{\omega_\xi})|_\omega \leqslant C_{\delta,k} r^{\delta-k-1}. \tag{3.11}$$

Proof. Two Riemannian metrics g_1 and g_2 on a manifold determine a section T of $\mathrm{End}(TM)$ via

$$g_2(\cdot, \cdot) = g_1(T\cdot, \cdot),$$

which is positive and self-adjoint with respect to both g_1 and g_2. Using Koszul formula, one can check directly that

$$\nabla_{g_2} - \nabla_{g_1} = T^{-1} * \nabla_{g_1}T$$

where we view both sides as (1,2)-tensors and $*$ denotes some algebraic operations. Therefore to estimate $|\nabla_{g_1}^k(\nabla_{g_2} - \nabla_{g_1})|_{g_1}$ it suffices to estimate T^{-1} and $\nabla_{g_1}^k T$. Then (3.11) follows from Proposition 3.5. □

3.4. **Plurisubharmonic weight functions.** We construct here some plurisubharmonic functions that will be used later. As mentioned in Section 2 we always identify (X', p') with $(\Phi_1(X'), 0)$ in $\mathbb{C}^N$. Then we can naturally restrict functions and forms defined on $\mathbb{C}^N$ to X' and then pull-back to X via the crepant resolution map π. When there is no confusion we will omit the various pull-back notations.

Lemma 3.7. *(1). Let $(\mathcal{K}, \omega_{\mathcal{K}})$ be a Kähler cone and let ρ denote the distance function from the vertex of the cone. Then the function $\varphi = \log(\rho^2 + 1) - \frac{1}{100}(\log(\rho^2 + 3))^{\frac{1}{2}}$ is a strictly plurisubharmonic function and there exists a compact set $K \subset \mathcal{K}$ such that on $\mathcal{K} \setminus K$,*

$$\sqrt{-1}\partial\bar{\partial}\varphi \geqslant C\rho^{-2}(\log \rho)^{-\frac{3}{2}}\omega_{\mathcal{K}}.$$

(2). The function on X defined by $\varphi = \log(r_\xi^2 + 1) - \frac{1}{100}(\log(r_\xi^2 + 3))^{\frac{1}{2}}$ satisfies $\sqrt{-1}\partial\bar{\partial}\varphi > 0$ away from the exceptional set of π. Moreover outside a compact set K it satisfies that for all $\epsilon > 0$,

$$(1 - \epsilon)\log(r^2 + 1) - C_\epsilon \leqslant \varphi \leqslant (1 + \epsilon)\log(r^2 + 1) + C_\epsilon,$$
$$\sqrt{-1}\partial\bar{\partial}\varphi \geqslant C_\epsilon^{-1}r^{-2-\epsilon}\omega,$$
$$|\nabla_\omega\varphi|_\omega \leqslant C_\epsilon r^{-1+\epsilon}.$$

Proof. (1) follows from a straightforward computation: the first term is strictly plurisubharmonic but is decaying at the rate ρ^{-4} along the radial direction; the second term is used to strengthen the positivity along the radial direction. Applying (1) to the Kähler cone $(\mathbb{C}^N, \omega_\xi)$ and then using Proposition 3.5, we obtain (2). $\qquad\square$

3.5. **Decomposition of the Kähler form.** We will use the following version of Hörmander's L^2 estimate, which follows from for example [13, Chapter VIII, Theorem 6.1].

Theorem 3.8. *Let M be an n-dimensional complex manifold admitting a complete Kähler metric and ω_M be a Kähler metric on M which is not necessarily complete. Suppose φ_M is a smooth function with $\sqrt{-1}\partial\bar{\partial}\varphi_M \geqslant \Upsilon\omega_M$ for a continuous non-negative function Υ. Let q be a positive integer. Then we have*

(1). for any (n, q) form η on M with $\bar{\partial}\eta = 0$ and $\int_M \Upsilon^{-1}|\eta|_{\omega_M}^2 e^{-\varphi_M}\omega_M^n < \infty$, there exists an $(n, q - 1)$ form ζ satisfying $\bar{\partial}\zeta = \eta$ and with estimate

$$\int_M |\zeta|_{\omega_M}^2 e^{-\varphi_M}\omega_M^n \leqslant \int_M q^{-1}\Upsilon^{-1}|\eta|_{\omega_M}^2 e^{-\varphi_M}\omega_M^n;$$

(2). if the Ricci curvature of ω_M is non-negative, then the conclusion in (1) holds for $(0, q)$ forms.

The main result of this subsection is

Theorem 3.9. *For every $\epsilon > 0$, there exist a Kähler form β and a real-valued smooth function ψ on X such that*

(1). $\omega = \beta + \sqrt{-1}\partial\bar{\partial}\psi$,
(2). $|\nabla_\omega^k\beta|_\omega = O(r^{-2+\epsilon-k})$ for all $k \geqslant 0$,
(3). $|\nabla_\omega^k\psi|_\omega = O(r^{2+\epsilon-k})$ for all $k \geqslant 0$.

Proof. Let φ be the plurisubharmonic function on X obtained in Lemma 3.7-(2). Fix $\epsilon > 0$.

Step 1. Using the diffeomorphism Φ constructed in Lemma 3.1, for R large we can identify $X \setminus X_R$ with $\mathcal{Q}_R$ and we may assume X_R is 1-convex and contains the exceptional set of π. Then one can write $\omega = \tilde{\beta} + d\tilde{\eta}$ with $|\nabla_\omega^k\tilde{\beta}|_\omega = O(r^{-2-k})$ for all $k \geqslant 0$ and $\tilde{\eta} = 0$ on X_R.

Step 2. We decompose $\tilde{\beta}$ according to types: $\tilde{\beta} = \tilde{\beta}^{2,0} + \tilde{\beta}^{1,1} + \tilde{\beta}^{0,2}$. Since $d\tilde{\beta} = 0$, we have $\bar{\partial}\tilde{\beta}^{0,2} = 0$. Notice by **Step 1**, we know that for all $\tau > 0$,

$$\int_X |\tilde{\beta}^{0,2}|_\omega^2 e^{-(n-2+\tau)\varphi}\omega^n < \infty.$$

Notice also that $\widetilde{\beta}^{0,2} = 0$ on X_R, so we may apply Theorem 3.8 (with $\varphi_M = (n-1+\tau)\varphi$) to obtain that for every $\tau > 0$, there exists a $(0,1)$-form ζ on X satisfying $\bar{\partial}\zeta = \tilde{\beta}^{0,2}$ and $\int_X |\zeta|^2_\omega e^{-(n-1+\tau)\varphi}\omega^n < \infty$. It follows from Proposition 3.4 that $|\nabla^k_\omega \zeta|_\omega = O(r^{-1+2\tau-k})$ for all $k \geqslant 0$. Choose $\tau = \dfrac{\epsilon}{4}$. Then we obtain $\omega = \beta_2 + d\eta_2$, where

$$\beta_2 = \tilde{\beta}^{1,1} - \partial\zeta - \bar{\partial}\bar{\zeta}, \quad \eta_2 = \zeta + \bar{\zeta} + \tilde{\eta},$$

and $|\nabla^k_\omega \beta_2|_\omega = O(r^{-2+\frac{\epsilon}{2}-k})$ for all $k \geqslant 0$. Notice β_2 and $d\eta_2$ are both real-valued $(1,1)$ forms.

Step 3. Let χ be a smooth compactly supported function on X which equals 1 in a neighborhood of X_R. Then $\eta_2 = \chi\eta_2 + \eta'_2$, where η'_2 vanishes in a neighborhood of X_R. Writing $d\eta'_2 = dr \wedge \alpha_1 + \alpha_2$, where $\partial_r \lrcorner \alpha_1 = \partial_r \lrcorner \alpha_2 = 0$, we define $\lambda = \int_R^r \alpha_1 dr$. Then λ vanishes in a neighborhood of X_R so extends to a 1-form on X. Moreover, one can check that $d\eta'_2 = d\lambda$. The decay of β_2 implies $|d\eta_2|_\omega \leqslant C$ on X, so we have $|\lambda| = O(r^{1+\delta})$ for all $\delta > 0$. It follows that $d\eta_2 = d\gamma$, where $\gamma = \chi\eta_2 + \lambda$, with $|\gamma| = O(r^{1+\delta})$ for all $\delta > 0$.

Step 4. Since $d\gamma = \omega - \beta_2$ is of type $(1,1)$, we have $\bar{\partial}\gamma^{0,1} = 0$. Since $H^1(X_R, \mathcal{O}) = 0$ by (3.2), there exists a smooth function f on X_R such that $\bar{\partial}f = \gamma^{0,1}$ on X_R. Choose a cut off function χ_0 which equals 1 in a neighborhood of the exceptional set of π and 0 outside $X_{R-\epsilon'}$ for some small positive number ϵ'. We now apply Theorem 3.8 (with $\varphi_M = (n+2+\dfrac{\epsilon}{2})\varphi$) to obtain a function u satisfying $\bar{\partial}u = \gamma^{0,1} - \bar{\partial}(\chi_0 f)$ and

$$\int_X |u|^2 e^{-(n+2+\frac{\epsilon}{2})\varphi}\omega^n < \infty.$$

Then $d\gamma = \sqrt{-1}\partial\bar{\partial}\psi_1$, where $\psi_1 = \sqrt{-1}(u + \chi_0 f - \bar{u} - \bar{\chi}_0\bar{f})$. Again by Proposition 3.4 we see $|\nabla^k_\omega \psi_1|_\omega = O(r^{2+\epsilon-k})$ for all $k \geqslant 0$.

Step 5. We have achieved that $\omega = \beta_2 + \sqrt{-1}\partial\bar{\partial}\psi_1$ with estimates on both β_2 and ψ_1. It remains to modify β_2 to be a Kähler form. By the asymptotics for the complex structure and the Riemannian metric obtained in Lemma 3.1, we know that there exists a large constant $\overline{R} > R$ such that on the region $\{r > \overline{R}\}$ we have $\sqrt{-1}\partial\bar{\partial}r^\epsilon \geqslant cr^{-2+\epsilon}\omega$ for some positive number $c > 0$. Choose a smooth non-decreasing convex function $u : \mathbb{R} \to \mathbb{R}$ such that $u(t) = 1$ for $t \leqslant 2\overline{R}^\epsilon$ and $u(t) = t$ for $t \gg 1$ and let $h = u \circ r^\epsilon$. Then h can be naturally viewed as a smooth plurisubharmonic function on X and outside a compact set K' we have

$$\sqrt{-1}\partial\bar{\partial}h \geqslant cr^{-2+\epsilon}\omega$$

for some positive number $c > 0$. Choose a smooth compactly supported function χ on X which equals 1 on a neighborhood of K'. Then we define

$$\beta = \beta_2 + \sqrt{-1}\partial\bar{\partial}(\chi\psi_1 + Ah),$$

$$\psi = (1 - \chi)\psi_1 - Ah,$$

where A is a constant. One can choose A large so that β is Kähler. Then $\omega = \beta + \sqrt{-1}\partial\bar{\partial}\psi$. It follows from the above construction that we have the desired estimates. $\square$

3.6. **Construction of background Kähler forms.** Recall some notations in Section 2: $X' \subset \mathbb{C}^N$ is the affine variety defined by the ring $R(X)$ of holomorphic functions on X with polynomial growth, W is the weighted asymptotic cone of X' with respect to Λ and $Y_j = \Lambda^{j-1}.X'$. So in particular away from 0 we know that Y_j converges to W as smooth submanifolds in $\mathbb{C}^N$. For notational convenience we also denote $Y_\infty \equiv W$. We denote by $\text{Sing}(Y_j)$ the singular set of Y_j, which consists of finitely many points, and denote $Y_j^\circ = Y_j \setminus \text{Sing}(Y_j)$. We also define $\mathfrak{S} \equiv \cup_{1 \leqslant j \leqslant \infty} \text{Sing}(Y_j)$.

Proposition 3.10. *There exist a constant $\delta_0 > 0$, a compact set $K \subset \mathbb{C}^N$ which contains $\mathfrak{S}$, and for each $j \in \mathbb{Z}_{\geqslant 1} \cup \{\infty\}$ a diffeomorphism $p_j : X' \setminus K \to Y_j \setminus K$ such that for all $k \geqslant 0$*

$$|\nabla^k_{\omega_\xi}(p_j - \mathrm{Id})|_{\omega_\xi} \leqslant C_k r_\xi^{-\delta_0-k},$$

where we view both p_j and Id as maps into $\mathbb{C}^N$, and

$$\left|\nabla^k_{\omega_\xi}\left(p_j^*(J_{Y_j}) - J_{X'}\right)\right|_{\omega_\xi} \leqslant C_k r_\xi^{-\delta_0-k}.$$

Remark 3.11. *Here the definite rate δ_0 arises from the algebraicity of W. It is of crucial importance in our argument as it beats all the (arbitrarily small polynomial order) error when we compare X to the asymptotic cone $\mathcal{C}$ using the 2-step degeneration theory (Proposition 3.5).*

Proof. By definition we can find finitely many holomorphic polynomials $F_1, \cdots, F_m$ on $\mathbb{C}^N$ which generate the ideal of X', such that $G_1, \cdots, G_m$ generate the ideal of W. Here G_α is the leading term of F_α with respect to the Λ action. Then we know that for all $l \in \mathbb{N}$ we have

$$F_\alpha(\Lambda^{-l}.z) = \lambda^{-lc_\alpha}(G_\alpha(z) + \lambda^{l\delta_\alpha} E_{\alpha l}(z))$$

for some constants $\delta_\alpha, c_\alpha > 0$ and some polynomials $E_{\alpha l}$ whose coefficients are uniformly bounded independent of l. Denote $F_{\alpha,l} = (\Lambda^{-l})^* F_\alpha$, then $F_{\alpha,1} \cdots, F_{\alpha,l}$ generate the ideal of Y_l. Let $\delta_0 = \min_\alpha \delta_\alpha$.

We define p_∞ to be the normal projection map from X' to W, i.e., for $x \in X'$, we let $p_\infty(x)$ to the unique point in W that is closest to X' with respect to the metric ω_ξ. We need to show this is well-defined outside a compact set in X and satisfies the desired properties. For $l \geqslant 1$ we denote by A_l the annulus in $\mathbb{C}^N$ defined by $\lambda^{-l+1} \leqslant r_\xi \leqslant \lambda^{-l-1}$. Then Λ^l maps $X' \cap A_l$ to $Y_l \cap A_1$. By the conical nature of ω_ξ, it suffices to consider the normal projection from $Y_l \cap A_1$ to W for $l \gg 1$.

Claim. *there is a covering of $W \cap A_1$ by finitely many open sets of the form $U_\gamma \cap W$ where $U_\gamma \subset \mathbb{C}^N$ such that for l large there are holomorphic functions $h_{\gamma,l} : U_\gamma \cap W \to \mathbb{C}^N$ with $|\nabla^k_{\omega_\xi}(h_{\gamma,l} - \mathrm{Id})| \leqslant C_k \lambda^{l\delta_0}$ for all $k \geqslant 0$ and $Y_l \cap A_1 \subset \cup_\gamma Im(h_{\gamma,l})$.*

To prove the **Claim**, we fix a point $w \in W \cap A_1$. Then we can find a neighborhood $U \subset \mathbb{C}^N$, such that $W \cap U$ is given by the zero set of $N - n$ number of G_α's (in the scheme-theoretic sense). For simplicity of notation we may assume these are $G_1, \cdots G_{N-n}$. Shrinking U if necessary we may find local holomorphic coordinates $\{\zeta_1, \cdots, \zeta_N\}$ such that $\zeta_\alpha = G_\alpha$ for $\alpha = 1, \cdots N - n$. Now we have $F_{\alpha,l} = \lambda^{-lc_\alpha}(G_\alpha(z) + \lambda^{l\delta_\alpha} E_{\alpha l}(z))$. It follows that for l large the common zero set of $F_{\alpha,l}(\alpha = 1, \cdots, N - n)$ is a smooth complex submanifold in U, so in particular it agrees with $Y_l \cap U$. Using the local coordinates $\{\zeta_1, \cdots, \zeta_N\}$ it is easy to see that $Y_l \cap U$ is contained in the image of a holomorphic function $h_l : U \to \mathbb{C}^N$ such that $|\nabla^k_{\omega_\xi}(h_l - \mathrm{Id})|_{\omega_\xi} \leqslant C_k \lambda^{l\delta_0}$ for all $k \geqslant 0$. Since A_1 is compact the **Claim** follows.

Since the normal injectivity radius of $W \cap A_1$ is uniformly bounded, there is a tubular neighborhood $\mathcal{N}$ of $W \cap A_1$ such that the normal projection map $\Pi : \mathcal{N} \to W$ is smooth. It follows from the **Claim** that for l large, $Y_l \cap A_1 \subset \mathcal{N}$. So p_∞ is well-defined and smooth outside a compact set in X'. It is straightforward to check that it satisfies the desired derivative bounds. The estimates on the complex structures follow from the fact that $h_{\gamma,l}$ is holomorphic.

Now for each $j < \infty$ we can perform the above arguments uniformly (noticing that the closeness between Y_j and W gets improved as j gets larger) to construct the corresponding projection maps $q_j : Y_j \setminus K \to W$ with uniform estimates. Then we let $p_j = q_j^{-1} \circ p_\infty$. $\qquad\qquad\square$

Now we fix $\epsilon_0 \ll \delta_0$ and apply Theorem 3.9 with $\epsilon = \epsilon_0$ to get a decomposition on X

$$\omega = \beta + \sqrt{-1}\partial\bar{\partial}\psi. \tag{3.12}$$

Let $\psi'_j = \psi \circ p_j^{-1}$ and $r_j = r \circ p_j^{-1}$. Denote $\gamma_j \equiv \sqrt{-1}\partial\bar{\partial}\psi'_j$. By Lemma 3.1, Proposition 3.5, Corollary 3.6 and Proposition 3.10, we have

Corollary 3.12. *There exists a compact set $K \subset \mathbb{C}^N$ such that for all $j \in \mathbb{Z}_{\geqslant 1} \cup \{\infty\}$, γ_j is a Kähler form on $Y_j \setminus K$. Moreover,*

(1). for all $k \geqslant 0$ and $\delta > 0$,

$$C_\delta^{-1} r_j^{1-\delta} \leqslant r_\xi \leqslant C_\delta r_j^{1+\delta} \quad and \quad |\nabla_{\gamma_j}^k (\gamma_j - \omega_\xi)|_{\gamma_j} \leqslant C_{\delta,k} r_j^{\delta-k},$$

(2). for all $k \geqslant 0$ and $\delta < \delta_0 - \epsilon_0$,

$$|\nabla_\omega^k (p_j^* \gamma_j - \omega)|_\omega \leqslant C_{\delta,k} r_j^{-\delta-k}. \tag{3.13}$$

Next we extend γ_j to a global Kähler form on Y_j° in a uniformly controlled way. In the following, we denote by $\omega_0 = \dfrac{\sqrt{-1}}{2} \partial\bar{\partial} r_0^2$ the Kähler form corresponding to the standard Euclidean metric on $\mathbb{C}^N$, where r_0 is the radius function for the Euclidean metric.

Lemma 3.13. *There exist compact sets $K_0 \subset K_1 \subset \mathbb{C}^N$ containing a neighborhood of $\mathfrak{S}$, a constant A_0 and Kähler forms $\omega_j = \sqrt{-1}\partial\bar{\partial}\psi_j$ on Y_j°, such that for all $j \in \mathbb{Z}_{\geqslant 1} \cup \{\infty\}$,*

(1). $\omega_j = A_0 \omega_0$ on $K_0 \cap Y_j^\circ$,
(2). $\psi_j = \psi_j'$ on $Y_j \setminus K_1$.

Proof. By Proposition 3.5 and Proposition 3.10, we know that there exist constants $c, T_1 > 0$ such that $\sqrt{-1}\partial\bar{\partial}\psi_j' \geqslant c r_\xi^{-\frac{1}{2}} \omega_\xi$ on $\{r_\xi \geqslant T_1/2\} \cap Y_j$. Using a cut-off function on $\mathbb{C}^N$ we may extend ψ_j' smoothly to Y_j°, so that it vanishes when $r_\xi \leqslant T_1/2$, equals $\psi \circ p_j^{-1}$ when $r_\xi \geqslant T_1$ and satisfies $\sqrt{-1}\partial\bar{\partial}\psi_j' \geqslant -\dfrac{A}{2}\omega_0$ on $\{r_\xi \leqslant T_1\} \cap Y_j$ for some constant $A > 0$ independent of j.

Choose a smooth increasing and convex function $u : \mathbb{R} \to \mathbb{R}$ such that $u(t) = T_1^2/3$ for $t \leqslant T_1^2/2$ and $u(t) = t$ for $t \geqslant T_1^2$. Define the function $r_1^2 = u \circ r_\xi^2$ on $\mathbb{C}^N$. Then $\sqrt{-1}\partial\bar{\partial} r_1^2$ is globally non-negative and equals ω_ξ on $\{r_\xi \geqslant T_1\}$. Choose a cutoff function $\rho : \mathbb{R} \to \mathbb{R}$ which equals 1 when $t \leqslant 2T_1^2$ and equals 0 when $t \geqslant 4T_1^2$. Then it is straightforward to check that there exists a small positive constant ϵ such that $\phi = \epsilon(\rho \circ r_\xi^2) \cdot r_0^2 + r_1^2$ satisfies that $\sqrt{-1}\partial\bar{\partial}\phi$ is a Kähler form on $\mathbb{C}^N$. By construction there exists a constant $T > 1$ such that $\{r_\xi \leqslant 2T_1\} \subseteq \{\phi \leqslant T\}$.

It is easy to find a smooth function $\chi : [0, \infty) \to \mathbb{R}$ such that the following hold

- $\chi(t) = A_0 t$ if $t \leqslant T$, where A_0 is a constant bigger than $A\epsilon^{-1}$;
- $\chi(t) = 0$ for $t \gg 1$;
- for all $t > 0$ we have $\chi'' \geqslant -\dfrac{c}{8} t^{-\frac{3}{2}}$ and $\chi' \geqslant -\dfrac{c}{8} t^{-\frac{1}{2}}$.

Then $\psi_j = \chi \circ \phi + \psi_j'$ satisfies the desired properties. $\qquad\square$

Recall that we have the crepant resolution map $\pi : X \to X'$, thus for $j \in \mathbb{Z}_{\geqslant 1}$ the composition map $\Lambda^j \circ \pi$ is a crepant resolution of Y_j. We denote the latter by $\pi_j : \hat{Y}_j = X \to Y_j$. For all $j \in \mathbb{Z}_{\geqslant 1}$ and $\epsilon \in (0, \lambda^{4j})$ (we emphasize here that the domain of ϵ depends on j), we have the following family of Kähler forms on $\hat{Y}_j$

$$\widetilde{\omega}_{j,\epsilon} \equiv \epsilon\beta + \pi_j^*\omega_j = \epsilon\beta + \sqrt{-1}\partial\bar{\partial}\pi_j^*\psi_j,$$

where β is given in (3.12). Notice that β and $\widetilde{\omega}_{j,\epsilon}$ can also be naturally viewed as forms on Y_j° via $(\pi_j^{-1})^*$. The following lemma (together with Corollary 3.12-(2)) shows that they are uniformly asymptotic to the cone $\mathcal{C}$.

Lemma 3.14. *There exists a compact set $K \subset \mathbb{C}^N$ such that for all $j \in \mathbb{Z}_{\geqslant 1}$ and $k \geqslant 0$ on $Y_j \setminus K$ we have*

$$\lambda^{4j} \left| \nabla_{\omega_j}^k (\pi_j^{-1})^* \beta \right|_{\omega_j} \leqslant C_k r_j^{-\frac{1}{2}-k}.$$

Proof. Outside a compact set, we identify X with X' and therefore we may just view π_j as Λ^j. Let us first prove the case $k = 0$. By Corollary 3.12 and the fact $\Lambda^*\omega_\xi = \lambda^2\omega_\xi$, we know that for any

$\epsilon_1, \epsilon_2 > 0$, there exists constants C_{ϵ_1} and C_{ϵ_2} such that

$$
\begin{aligned}
\left| (\Lambda^{-j})^* \beta \right|_{\omega_j} &\leqslant C_{\epsilon_1} \left| (\Lambda^{-j})^* \beta \right|_{\omega_\xi} r_\xi^{\epsilon_1} \leqslant C_{\epsilon_1} (\Lambda^{-j})^* (|\beta|_{(\Lambda^j)^* \omega_\xi}) r_\xi^{\epsilon_1} \\
&\leqslant C_{\epsilon_1} \lambda^{-2j} (\Lambda^{-j})^* (|\beta|_{\omega_\xi}) r_\xi^{\epsilon_1} \leqslant C_{\epsilon_1} C_{\epsilon_2} \lambda^{-2j} (\Lambda^{-j})^* (|\beta|_\omega r_\xi^{\epsilon_2}) r_\xi^{\epsilon_1} \\
&\leqslant C_{\epsilon_1} C_{\epsilon_2} \lambda^{-2j - \epsilon_2 j} (\Lambda^{-j})^* (r^{-1}) r_\xi^{\epsilon_1 + \epsilon_2},
\end{aligned}
\tag{3.14}
$$

where for the last inequality we used the estimate of β with respect to the metric ω obtained in Theorem 3.9. By choosing ϵ_1 and ϵ_2 small enough, one can directly show that there exists a positive constant C_0 such that

$$
\lambda^{4j} \left| (\Lambda^{-j})^* \beta \right|_{\omega_j} \leqslant C_0 r_j^{-\frac{1}{2}}.
$$

This proves the case $k = 0$. Note that suppose we have two Riemannian metrics g_1 and g_2 then for any tensor η and another fixed background metric g_0, we have

$$
\begin{aligned}
\left| \nabla_{g_1}^k \eta \right|_{g_0} \leqslant &C_k \sum_{\substack{k_1 + k_2 = k \\ k_i \geqslant 0}} \left| \nabla_{g_1} - \nabla_{g_2} \right|_{g_0}^{k_1} \left| \nabla_{g_2}^{k_2} \eta \right|_{g_0} \\
&+ C_k \sum_{\substack{k_1 + k_2 + k_3 = k-1 \\ k_i \geqslant 0, k_1 \geqslant 1}} \left| \nabla_{g_2}^{k_1} (\nabla_{g_1} - \nabla_{g_2}) \right|_{g_0} \left| \nabla_{g_1} - \nabla_{g_2} \right|_{g_0}^{k_2} \left| \nabla_{g_2}^{k_3} \eta \right|_{g_0},
\end{aligned}
\tag{3.15}
$$

where we view $\nabla_{g_1} - \nabla_{g_2}$ as a tensor and make a convention that $\nabla_{g_2}^0 \eta = \eta$. Then the proof for $k \geqslant 1$ follows from the same argument as we did in (3.14) by using (3.15) together with Corollary 3.6 and Theorem 3.9. $\qquad\square$

Recall that Ω is the given holomorphic volume form on X. Since Ω is holomorphic and satisfies the equation $\omega^n = (\sqrt{-1})^{n^2} \Omega \wedge \overline{\Omega}$, we obtain $\nabla_\omega \Omega = 0$. We can view Ω naturally as a holomorphic volume form on $X' = Y_1$ (notice the canonical bundle of each Y_j is trivial).

Lemma 3.15. *For any $j \in \mathbb{Z}_{\geqslant 1}$, there exists a nowhere vanishing holomorphic volume form Ω_j on Y_j satisfying:*

(1). there exist a constant $\delta_1 > 0$ and a compact set $K \subset \mathbb{C}^N$ such that on $Y_j \setminus K$, for all $k \geqslant 0$ the function $h_j = \log \left(\dfrac{\Omega_j \wedge \overline{\Omega}_j}{(p_j^{-1})^* (\Omega \wedge \overline{\Omega})} \right)$ satisfies

$$
|\nabla_{\omega_j}^k h_j| \leqslant C_k r_j^{-\delta_1 - k}.
$$

(2). for every compact set $K_1 \subset \mathbb{C}^N$ we have

$$
\sup_{j \in \mathbb{Z}_{\geqslant 1}} \int_{K_1 \cap Y_j} (\sqrt{-1})^{n^2} \Omega_j \wedge \overline{\Omega}_j < \infty.
$$

Remark 3.16. *It follows from the proof that for $j \in \mathbb{Z}_{\geqslant 1}$, Ω_j coincides with $(\Lambda^{-j})^* \Omega$ up to a nonzero multiplicative constant depending on j. For our purpose we need the precise asymptotics of Ω_j and the stated properties of h_j essentially determine this constant. Moreover from the proof, we also obtain*

$$
\left| \nabla_{\omega_j}^k ((p_j^{-1})^* \Omega - \Omega_j) \right|_{\omega_j} \leq C_k r_j^{-\delta_1 - k} \text{ on } Y_j \setminus K, \text{ for all } k \geqslant 0.
\tag{3.16}
$$

Note that (1) and (2) in the proposition are invariant if we multiply Ω_j by a complex number with norm 1 and the estimate (3.16) can be used to eliminate this freedom.

Proof. Let $\varphi = \log(r_\xi^2 + 1) - \dfrac{1}{100} (\log(r_\xi^2 + 3))^{\frac{1}{2}}$ and denote $\varphi_j = \varphi|_{Y_j}$. Then by Lemma 3.7-(1) and Corollary 3.12, we know that outside a compact set K for every $\epsilon > 0$, there is a positive constant c_ϵ such that $\sqrt{-1} \partial \bar{\partial} \varphi_j \geqslant c_\epsilon r_j^{-2-\epsilon} \omega_j$. Let p_j be given in Proposition 3.10. For all $\delta \in (0, \delta_0 - \epsilon_0)$ we have

$$
\left| ((p_j^{-1})^* \Omega)^{n,0} \right|_{\omega_j} - 1 = O(r_j^{-\delta}),
\tag{3.17}
$$

and for all $k \geqslant 1$, $\left| \nabla^k_{\omega_j} \left((p_j^{-1})^* \Omega \right)^{n,0} \right|_{\omega_j} \leqslant C_{\delta,k} r_j^{-\delta-k}$ since $\nabla_\omega \Omega = 0$.

Fix T large so that $K \subset \{r_\xi \leqslant T\}$, and fix a cut off function χ on $\mathbb{C}^N$ which equals 0 when $r_\xi \leqslant T$, equals 1 when $r_\xi > 2T$. Recall that we use $Y_j^\circ = Y_j \setminus \mathrm{Sing}(Y_j)$ to denote the smooth locus of Y_j. Since Y_j is affine and $\mathrm{Sing}(Y_j)$ consists of finitely many points, we know Y_j° admits a complete Kähler metric. Then for any $\tau > 0$ small, we can apply Theorem 3.8 on $M = Y_j^\circ$, with $\omega_M = \omega_j$ and $\varphi_M = (n-1+\tau)\varphi_j$, to obtain an $(n,0)$ form v_j satisfying $\bar{\partial} v_j = \bar{\partial} \left(\chi \cdot ((p_j^{-1})^* \Omega)^{n,0} \right)$, such that

$$\int_{Y_j^\circ} e^{-(n-1+\tau)\varphi_j} (\sqrt{-1})^{n^2} v_j \wedge \bar{v}_j \leqslant C.$$

By Proposition 3.4, when $r_\xi \geqslant 2T$, we have $|\nabla^k_{\omega_j} v_j| \leqslant C_k r_j^{-\delta_1 - k}$ for some constant $\delta_1 > 0$ and all $k \geqslant 0$.

Now let $\Omega_j = \chi((p_j^{-1})^* \Omega)^{n,0} - v_j$. It is holomorphic on Y_j° and is non-zero by (3.17) and the decay of v_j. Notice that the function $\dfrac{(\Lambda^j)^* \Omega_j}{\Omega}$ is a holomorphic function on X which is $O(r^\epsilon)$ for all $\epsilon > 0$, so must be a nonzero constant. In particular, Ω_j is a nowhere vanishing holomorphic volume form on Y_j. The desired estimates follow from straightforward computation. $\qquad\square$

In the following proposition, we omit the notation $(\pi_j^{-1})^*$ and view $\widetilde{\omega}_{j,\epsilon}$ as a Kähler form on $Y_j^\circ \subset \mathbb{C}^N$. We also view r_j (which is a priori only defined on the end of Y_j using $r \circ p_j^{-1}$) as a positive smooth function on Y_j which is a constant on $K_0 \cap Y_j$ where $K_0 \subset \mathbb{C}^N$ is a fixed compact set independent of j, containing $\mathfrak{S} = \cup_{1 \leqslant j \leqslant \infty} \mathrm{Sing}(Y_j)$.

Proposition 3.17. *There exist a $\delta_2 > 0$, a compact set $K \subset \mathbb{C}^N$ containing a neighborhood of $\mathfrak{S}$ and for all $j \in \mathbb{Z}_{\geqslant 1}$ and $\epsilon \in (0, \lambda^{4j})$ a smooth function $u_{j,\epsilon}$ on Y_j° satisfying*

(1). for all $k \geq 0$, $\left| \nabla^k_{\widetilde{\omega}_{j,\epsilon}} u_{j,\epsilon} \right|_{\widetilde{\omega}_{j,\epsilon}} \leqslant C_k r_j^{-\delta_2 + 2 - k}$ on $Y_j \setminus K$,

(2). let $e^{-f_{j,\epsilon}} = \dfrac{\left(\widetilde{\omega}_{j,\epsilon} + \sqrt{-1}\partial\bar{\partial} u_{j,\epsilon} \right)^n}{(\sqrt{-1})^{n^2} \Omega_j \wedge \overline{\Omega}_j}$, then on $Y_j \setminus K$ for all $k \geq 0$, $\left| \nabla^k_{\widetilde{\omega}_{j,\epsilon}} f_{j,\epsilon} \right|_{\widetilde{\omega}_{j,\epsilon}} \leqslant C_k r_j^{-\delta_2 - 2 - k}$,

(3). $u_{j,\epsilon}$ is supported on $Y_j \setminus K$,

(4). $\omega_j + \sqrt{-1}\partial\bar{\partial} u_{j,\epsilon} \geqslant \dfrac{1}{2}\omega_j$ and for all $k \geqslant 0$, $\left| \nabla^k_{\omega_j} u_{j,\epsilon} \right|_{\omega_j} \leq C_k (r_j + 1)^{-\delta_2 + 2 - k}$ on Y_j°.

Proof. We first identify the end of X' differentiably with $\mathcal{Q}_{R_0}$ via Lemma 3.1. Then using the maps p_j constructed in Proposition 3.10 we may identify the end of all Y_j with $\mathcal{Q}_{R_1}$ for some $R_1 > R_0$. In particular we may view the metrics $\widetilde{g}_{j,\epsilon}$ (associated to $\widetilde{\omega}_{j,\epsilon}$) on Y_j° as a family of Riemannian metrics on $\mathcal{Q}_{R_1}$. By Lemma 3.1, Corollary 3.12 and Lemma 3.14 we know that for all j,

$$s^l \sup_{\partial\mathcal{Q}_s} |\nabla^l_{gc}(\widetilde{g}_{j,\epsilon} - g_{\mathcal{C}})|_{gc} \leqslant e_l(s)$$

for some $e_l(s)$ with $\lim\limits_{s \to \infty} e_l(s) = 0$. Then we can apply Proposition 3.3 to solve the Poisson equation with uniform estimates for the metric $\widetilde{g}_{j,\epsilon}$ on $\mathcal{Q}_{R_2}$, where $R_2 \geqslant R_1$ is given in Proposition 3.3. More precisely, we solve the equation in a fixed Hölder space for k large and then apply elliptic regularity as Proposition 3.4 to gain the decay for all derivatives. Next we can argue as in [10, Section 2.4] to obtain functions $u_{j,\epsilon}$ defined only on the end of Y_j satisfying item (1) and (2) in the proposition. Fix a smooth function $H(t) : [0,\infty) \to [0,1]$, which equals 0 for $t \leqslant 1$ and equals 1 for $t \geqslant 2$. Combining the estimate in (1) with Lemma 3.13 and Lemma 3.14, it is direct to show that we can choose R_3 sufficiently large independent of j and ϵ, such that $H(\frac{r_j}{R_3})u_{j,\epsilon}$ is a global defined function on Y_j satisfying all (1)-(4) in the proposition. $\qquad\square$

For all $j \in \mathbb{Z}_{\geqslant 1}$ and $\epsilon \in (0, \lambda^{4j})$, we define Kähler forms on $\hat{Y}_j$

$$\omega_{j,\epsilon} \equiv \widetilde{\omega}_{j,\epsilon} + \sqrt{-1}\partial\bar{\partial}\pi_j^* u_{j,\epsilon},$$

and we denote by $g_{j,\epsilon}$ and $J_{j,\epsilon}$ the corresponding Riemannian metric and complex structure. It follows from Lemma 3.1, Proposition 3.10, Corollary 3.12, Lemma 3.14 and Proposition 3.17 that $(g_{j,\epsilon}, J_{j,\epsilon})$ are still uniformly asymptotic the cone $\mathcal{C}$. More precisely, we have

Proposition 3.18. *There exist a compact set $K \subset \mathbb{C}^N$ and a large positive number R_2 such that for all $j \in \mathbb{Z}_{\geqslant 1}$, the diffeomorphism $P_j = p_j \circ \Phi : \mathcal{Q}_{R_2} \to Y_j \setminus K$ satisfies that for all $l \geqslant 0$,*

$$\lim_{s \to \infty} \sup_{j \geqslant 1} \sup_{\partial \mathcal{Q}_s} s^l \left(|\nabla^l_{g_{\mathcal{C}}}(P_j^* g_{j,\epsilon} - g_{\mathcal{C}})|_{g_{\mathcal{C}}} + |\nabla^l_{g_{\mathcal{C}}}(P_j^* J_{j,\epsilon} - J_{\mathcal{C}})|_{g_{\mathcal{C}}} \right) = 0,$$

where we omit $(\pi_j^{-1})^*$ again and view $g_{j,\epsilon}$ and $J_{j,\epsilon}$ as tensors on Y_j.

4. Uniform estimates for the complex Monge-Ampère equation

From now on for $j \in \mathbb{Z}_{\geqslant 1}$ we will abuse notations and identify ω_j, Ω_j and r_j with their pull-backs under $\pi_j : \hat{Y}_j \to Y_j$. We now look for Calabi-Yau metrics on $\hat{Y}_j$ which are asymptotic to $\omega_{j,\epsilon}$. To do this we need to solve on $\hat{Y}_j$ the equation

$$\left(\omega_{j,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_{j,\epsilon}\right)^n = (\sqrt{-1})^{n^2}\Omega_j \wedge \overline{\Omega}_j. \tag{4.1}$$

Let μ_j denote the volume form $(\sqrt{-1})^{n^2}\Omega_j \wedge \overline{\Omega}_j$ which is also equal to $e^{f_{j,\epsilon}}\omega_{j,\epsilon}^n$. It is proved in Proposition 3.17 that there exist a $\delta_2 > 0$ and a compact set $K \subset \mathbb{C}^N$ such that on $\hat{Y}_j \setminus \pi_j^{-1}(K \cap Y_j)$ for all $k \geqslant 0$,

$$\left|\nabla^k_{\omega_{j,\epsilon}} f_{j,\epsilon}\right|_{\omega_{j,\epsilon}} \leqslant C_k r_j^{-\delta_2 - 2 - k}. \tag{4.2}$$

The following theorem can be derived from Tian-Yau [30, Section 4]. See also Hein [17, Section 4].

Theorem 4.1 ([30, 17]). *For fixed j and ϵ, (4.1) admits a smooth solution $\varphi_{j,\epsilon}$ on $\hat{Y}_j$ such that*

    (1) there exists a constant p_0 depending only on δ_2 such that $\|\varphi_{j,\epsilon}\|_{L^{p_0}(\mu_j)} < \infty$ and $\varphi_{j,\epsilon}(x) \to 0$ as $r_j(x) \to \infty$.

    (2) $\left\|\nabla^k_{\omega_{j,\epsilon}}\varphi_{j,\epsilon}\right\|_{L^\infty(\omega_{j,\epsilon})} < \infty$ for all $k \geqslant 0$.

Remark 4.2. *The proof of the L^∞ bound of $\varphi_{j,\epsilon}$ is via the Moser iteration, thus indeed we know that $\|\varphi_{j,\epsilon}\|_{L^p(\mu_j)} < \infty$ for all $p \gg 1$. In the following, we fix a large p_0 such that $q_0 = \dfrac{n(p_0 + 1)}{n + p_0}$ satisfies*

$$(2 + \delta_2)q_0 > 2n. \tag{4.3}$$

Since the background metrics $\omega_{j,\epsilon}$ are asymptotic to the cone metric on $\mathcal{C}$, by the method in [30, Section 5] and using $r_j^{-2\delta}$ as a barrier to apply the maximum principle, we can prove the following decay estimates for $\varphi_{j,\epsilon}$. We remark that alternatively, one can also use the method of [10, 8] to solve the complex Monge-Ampère equation in a suitable weighted space (since we have Lemma 3.2), then the decay property will be a direct by-product.

Lemma 4.3. *There exist a constant $\delta_3 > 0$ depending only on δ_2, compact sets $K_{j,\epsilon} \subset \mathbb{C}^N$ depending only on $\|\varphi_{j,\epsilon}\|_{L^\infty}$ and constants $Q_{k,j,\epsilon}$ depending only on k and $\|\varphi_{j,\epsilon}\|_{L^\infty}$ such that on $Y_j \setminus K_{j,\epsilon}$ for all $k \geqslant 0$,*

$$\left|\nabla^k_{\omega_{j,\epsilon}}\varphi_{j,\epsilon}\right|_{\omega_{j,\epsilon}} \leqslant Q_{k,j,\epsilon} r_j^{-\delta_3 - k}.$$

Proof. We focus on the C^0 decay property since the higher order estimates follow from a standard rescaling argument and elliptic estimates (see [30, Proposition 5.1]). Fix a positive constant $\delta < \min\left\{\dfrac{\delta_2}{2}, \dfrac{1}{2}\right\}$ and then we can prove the lemma holds for $\delta_3 = 2\delta$.

Claim. *there is a large constant R_* independent of j and ϵ such that for every $R \geqslant R_*$, on the region $\{r_j \geqslant R\}$ we have*

$$\left(\omega_{j,\epsilon} + R\sqrt{-1}\partial\bar{\partial}r_j^{-2\delta}\right)^n < e^{f_{j,\epsilon}}\omega_{j,\epsilon}^n \text{ and } \left(\omega_{j,\epsilon} - R\sqrt{-1}\partial\bar{\partial}r_j^{-2\delta}\right)^n > e^{f_{j,\epsilon}}\omega_{j,\epsilon}^n.$$

The decay of $\varphi_{j,\epsilon}$ follows from this claim by applying the maximal principle to the region $\{r_j \geqslant R_{j,\epsilon}\}$ with $R_{j,\epsilon} = \max\{R_*, \|\varphi_{j,\epsilon}\|_{L^\infty}^{\frac{1}{1-2\delta}}\}$. Let us prove the **Claim**. We only prove the first inequality since the second inequality follows from the same argument. By (4.2) and Proposition 3.18, we know that outside a compact set K_0, there exists a constant $C_0 > 0$ such that

$$1 - C_0 r^{-2-\delta_2} \leqslant e^{f_{j,\epsilon}} \leqslant 1 + C_0 r^{-2-\delta_2},$$

$$\left|\sqrt{-1}\partial\bar{\partial}r_j^{-2\delta}\right|_{\omega_{j,\epsilon}} \leqslant C_0 r_j^{-2(\delta+1)}, \tag{4.4}$$

$$\Delta_{\omega_{j,\epsilon}} r_j^{-2\delta} \leqslant -\delta r_j^{-2(\delta+1)}.$$

The last inequality here is due to the fact that on the cone $\mathcal{C}$, for the radial function r we have

$$\Delta_{\omega_{\mathcal{C}}} r^{-2\delta} = 4\delta(-n + \delta + 1)r^{-2(\delta+1)}.$$

Choose R_* large such that $K_0 \subseteq \{r_j \leqslant R_*\}$ and by the second inequality in (4.4) and the choice of δ we may also assume that for all $R \geqslant R_*$, on $\{r_j \geqslant R\}$ we have

$$\frac{\left(\omega_{j,\epsilon} + R\sqrt{-1}\partial\bar{\partial}r_j^{-2\delta}\right)^n}{\omega_{j,\epsilon}^n} \leqslant 1 + R\Delta_{\omega_{j,\epsilon}} r_j^{-2\delta} + \frac{\delta}{2}r_j^{-2(\delta+1)}, \tag{4.5}$$

and

$$\left(-R\delta + \frac{\delta}{2}\right) r_j^{-2(\delta+1)} \leqslant -2C_0 r_j^{-2-\delta_2}. \tag{4.6}$$

Then the claim follows from (4.4), (4.5) and (4.6). $\qquad\square$

Denote

$$\hat{\omega}_{j,\epsilon} = \omega_{j,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_{j,\epsilon}.$$

Below we will derive uniform estimates on $\varphi_{j,\epsilon}$. Then we will take a double limit, first as $\epsilon \to 0$ and then as $j \to \infty$, to obtain a Calabi-Yau metric on $W \setminus \{0\}$. The following arguments are essentially standard, given the large body of literature in this field; see for example the recent work by Collins-Guo-Tong [8] which deals with the case when the background metric is asymptotically conical in the stronger sense (with polynomial decay rate). Notice that we do not need uniform estimates for all j and $\epsilon \in (0, \lambda^{4j})$. Instead we first fix j and derive uniform estimates as $\epsilon \to 0$, which allows us to take a limit φ_j. Then we will derive uniform estimates for φ_j as $j \to \infty$. Notice we are free to pass to subsequences when taking limits and we do not need uniqueness of the limits.

Let

$$A_{j,\epsilon} = \sup_{q \geqslant q_0} \|e^{-f_{j,\epsilon}} - 1\|_{L^q(\mu_j)}, \qquad A_j = \limsup_{\epsilon \to 0} A_{j,\epsilon}$$

By (4.2) and (4.3) we know for each j, ϵ, $A_{j,\epsilon} < \infty$. The following lemma shows that $A_j < \infty$ and moreover we have a uniform bound on A_j.

Lemma 4.4. *There exists a constant $C > 0$ such that $A_j \leqslant C$ for all j.*

Proof. Let K be the union of the compact sets obtained in Lemma 3.15-(1) and Proposition 3.17, then by (4.2), (4.3) and Lemma 3.15, we obtain there exists a constant C independent of j and ϵ such that

$$A_{j,\epsilon} \leqslant C\left(\|e^{-f_{j,\epsilon}}\|_{L^\infty(\pi_j^{-1}(K))} + 1\right). \tag{4.7}$$

By the construction in Lemma 3.13 and Proposition 3.17, $\omega_j + \sqrt{-1}\partial\bar{\partial}u_{j,\epsilon} \leqslant C_1\omega_j \leqslant C_2\omega_0$ on $Y_j^\circ \cap K$, where ω_0 denotes the Kähler form for the Euclidean metric on $\mathbb{C}^N$, for some constants C_1 and C_2 independent of j and ϵ. Therefore by the definition of $f_{j,\epsilon}$ we obtain

$$\limsup_{\epsilon \to 0} \|e^{-f_{j,\epsilon}}\|_{L^\infty(\pi_j^{-1}(K))} \leqslant C_2^n \left\|\frac{\omega_0^n}{(\sqrt{-1})^{n^2}\Omega_j \wedge \bar{\Omega}_j}\right\|_{L^\infty(K \cap Y_j^\circ)}. \tag{4.8}$$

By the Gauss-Codazzi equation we know the Ricci curvature of $\omega_0|_{Y_j}$ is non-positive, it follows that the function $\dfrac{\omega_0^n}{(\sqrt{-1})^{n^2}\Omega_j \wedge \bar{\Omega}_j}$ is plurisubharmonic on $Y_j^\circ = Y_j \setminus \text{Sing}(Y_j)$. Pulling back to $\hat{Y}_j$ it is a

smooth plurisubharmonic function. By the maximal principle and Lemma 3.15, we conclude that the supremum of $\dfrac{\omega_0^n}{(\sqrt{-1})^{n^2}\Omega_j \wedge \overline{\Omega}_j}$ on $K \cap Y_j^\circ$ has a uniform upper bound independent of j. Then the lemma follows from (4.7) and (4.8). $\square$

As observed by [8] (following an earlier idea of Tosatti) it is more convenient to use the Calabi-Yau metrics $\hat{\omega}_{j,\epsilon}$ as background metrics when we do Moser iteration. By Lemma 4.3 each $\hat{\omega}_{j,\epsilon}$ is weakly asymptotic to the cone $\mathcal{C}$, so by Bishop-Gromov inequality we know the volume of a metric ball of radius R in $(\hat{Y}_j, \hat{\omega}_{j,\epsilon})$ is bounded below by $\kappa(\mathcal{C})R^{2n}$, where $\kappa(\mathcal{C})$ is the volume density of $\mathcal{C}$. So we have a uniform Sobolev inequality, namely, there exists $S > 0$ independent of j and ϵ such that for all $v \in C_0^\infty(\hat{Y}_j)$,

$$\|v\|^2_{L^{\frac{2n}{n-1}}(\hat{Y}_j, \hat{\omega}_{j,\epsilon})} \leqslant S \left\|\nabla_{\hat{\omega}_{j,\epsilon}} v\right\|^2_{L^2(\hat{Y}_j, \hat{\omega}_{j,\epsilon})}.$$

Lemma 4.5. *There exists a continuous function* $\mathcal{P} : (0, \infty) \to (0, \infty)$ *such that for all* $j \in \mathbb{Z}_{\geqslant 1}$ *and* $\epsilon \in (0, \lambda^{4j})$,

(1). $\|\varphi_{j,\epsilon}\|_{L^\infty} \leqslant \mathcal{P}(A_{j,\epsilon})$.

(2). $\mathcal{P}(A_{j,\epsilon})^{-1}\omega_j \leqslant \hat{\omega}_{j,\epsilon} \leqslant \mathcal{P}(A_{j,\epsilon})\dfrac{\mu_j}{\omega_j^n}\omega_j$.

Proof. The proof is standard, see for example [8, Section 4]. For completeness, we include some details.

(1). We have $\hat{\omega}_{j,\epsilon}^n - \omega_{j,\epsilon}^n = \sqrt{-1}\partial\bar{\partial}\varphi_{j,\epsilon} \wedge \sum_{k=0}^{n-1} \left(\hat{\omega}_{j,\epsilon}^k \wedge \omega_{j,\epsilon}^{n-1-k}\right)$. For any exponent $p \geqslant p_0$, multiplying both sides by $|\varphi_{j,\epsilon}|^{p-1}\varphi_{j,\epsilon}$ and integrating by parts (the estimates obtained in Theorem 4.1 justify this), we get

$$\int_X |\varphi_{j,\epsilon}|^{p-1}\varphi_{j,\epsilon}(e^{-f_{j,\epsilon}} - 1)\mu_j = \frac{4p}{(p+1)^2}\int_X \sqrt{-1}\partial|\varphi_{j,\epsilon}|^{\frac{p+1}{2}} \wedge \bar{\partial}|\varphi_{j,\epsilon}|^{\frac{p+1}{2}} \wedge \sum_{k=0}^{n-1}\left(\hat{\omega}_{j,\epsilon}^k \wedge \omega_{j,\epsilon}^{n-1-k}\right).$$

In the following C always denote a constant depending only on n and p_0. By Sobolev inequality, we get

$$\left(\int_X |\varphi_{j,\epsilon}|^{(p+1)\frac{n}{n-1}}\mu_j\right)^{\frac{n-1}{n}} \leqslant CSp\int_X |\varphi_{j,\epsilon}|^p|e^{-f_{j,\epsilon}} - 1|\mu_j.$$

Applying the Hölder's inequality to the right hand side, we obtain

$$\|\varphi_{j,\epsilon}\|_{L^{\frac{n(p_0+1)}{n-1}}(\mu_j)} \leqslant CS\left\|e^{-f_{j,\epsilon}} - 1\right\|_{L^{q_0}(\mu_j)},$$

$$\|\varphi_{j,\epsilon}\|^{p+1}_{L^{\frac{n(p+1)}{n-1}}(\mu_j)} \leqslant CSp\left\|e^{-f_{j,\epsilon}} - 1\right\|_{L^{p+1}(\mu_j)}\|\varphi_{j,\epsilon}\|^p_{L^{p+1}(\mu_j)} \text{ for all } p \geqslant p_0.$$

Then a standard Moser iteration argument completes the proof.

(2). Note that the second inequality follows from the first one and the elementary inequality that $\mathrm{tr}_\beta(\alpha) \leqslant \dfrac{\alpha^n}{\beta^n}\left(\mathrm{tr}_\alpha(\beta)\right)^{n-1}$ for any two positive $(1,1)$-forms α and β. The first inequality here is a consequence of Chern-Lu's inequality. More precisely, by construction the bisectional curvature of ω_j has a uniform upper bound independent of j. Then Chern-Lu's inequality implies that there is a constant B independent of j and ϵ such that,

$$\Delta_{\hat{\omega}_{j,\epsilon}} \log(\mathrm{tr}_{\hat{\omega}_{j,\epsilon}} \omega_j) \geqslant -B\,\mathrm{tr}_{\hat{\omega}_{j,\epsilon}} \omega_j.$$

By the definition of $\hat{\omega}_{j,\epsilon}$ and the estimates for $u_{j,\epsilon}$ in Proposition 3.17, we have

$$\mathrm{tr}_{\hat{\omega}_{j,\epsilon}}(\sqrt{-1}\partial\bar{\partial}\varphi_{j,\epsilon}) \leqslant n - \frac{1}{2}\,\mathrm{tr}_{\hat{\omega}_{j,\epsilon}} \omega_j.$$

Therefore,

$$\Delta_{\hat{\omega}_{j,\epsilon}}\left(\log(\mathrm{tr}_{\hat{\omega}_{j,\epsilon}} \omega_j) - 2(B+1)\varphi_{j,\epsilon}\right) \geqslant \mathrm{tr}_{\hat{\omega}_{j,\epsilon}} \omega_j - 2(B+1)n.$$

Now we apply the maximum principle to the function $F = \log(\mathrm{tr}_{\hat{\omega}_{j,\epsilon}} \omega_j) - 2(B+1)\varphi_{j,\epsilon}$. Since $\varphi_{j,\epsilon}$ tends to zero at infinity, F converges to $\log n$ at infinity. Notice $\pi_j^* \omega_j$ is degenerate along the exceptional set, so F tends to $-\infty$ on $\pi_j^{-1}(\mathrm{Sing}(Y_j))$. If the supremum of F is attained at infinity, then we automatically have a uniform upper bound for $\log(\mathrm{tr}_{\hat{\omega}_{j,\epsilon}}(\omega_j))$. If the maximum is achieved on $\hat{Y}_j$, Then $\log(\mathrm{tr}_{\hat{\omega}_{j,\epsilon}}(\omega_j)) \leqslant C$, where C depends only on $\|\varphi_{j,\epsilon}\|_{L^\infty}$. $\qquad\square$

Using this we can also improve Lemma 4.3, namely, we can make the dependence of $K_{j,\epsilon}$ and $Q_{k,j,\epsilon}$ on j and ϵ to be through $A_{j,\epsilon}$. Since we are free to pass to subsequences when taking limits and we do not need uniqueness of the limits, we may assume in the following that $A_j = \lim_{\epsilon \to 0} A_{j,\epsilon}$ for every j. For fixed j since as $\epsilon \to 0$ we have $A_{j,\epsilon} \to A_j$, passing to a subsequence $\epsilon_{j,l}$, as $l \to \infty$ we can make $\varphi_{j,\epsilon_{j,l}}$ converge locally smoothly to a limit φ_j and $u_{j,\epsilon_{j,l}}$ converge locally smoothly to a limit u_j on Y_j°. The metric $\hat{\omega}_j = \omega_j + \sqrt{-1}\partial\bar{\partial}(u_j + \varphi_j)$ defines a Calabi-Yau metric on $Y_j \setminus \mathrm{Sing}(Y_j)$, i.e. $\hat{\omega}_j^n = (\sqrt{-1})^{n^2}\Omega_j \wedge \overline{\Omega}_j$. Moreover, we have

- On Y_j°,
$$|\varphi_j| \leqslant \mathcal{P}(A_j). \tag{4.9}$$
  This implies $\hat{\omega}_j$ extends as a closed positive $(1,1)$ current on Y_j.
- On Y_j°,
$$\hat{\omega}_j \geqslant \mathcal{P}(A_j)^{-1}\omega_j. \tag{4.10}$$
- There exist compact sets K_j depending only on A_j and continuous functions $Q_k : (0,\infty) \to (0,\infty)$ for all $k \geqslant 0$ such that on $Y_j \setminus K_j$
$$|\nabla_{\omega_j}^k \varphi_j| \leqslant Q_k(A_j) r_j^{-\delta_3 - k}.$$
- there exists a compact set $K \subset \mathbb{C}^N$ such that u_j is a constant on $K \cap Y_j$ and on Y_j°
$$\left|\nabla_{\omega_j}^k u_j\right|_{\omega_j} \leq C_k(r_j + 1)^{-\delta_2 + 2 - k}.$$

Moreover, a standard argument using the Bishop-Gromov volume comparison (see [8, Lemma 4.14]) implies that for any compact set $K \subset \mathbb{C}^N$, there is a constant $C_K > 0$ independent of j and ϵ such that
$$\mathrm{diam}(\pi_j^{-1}(K \cap Y_j), \hat{\omega}_{j,\epsilon}) \leqslant C_K.$$
So the diameter of $K \cap Y_j^\circ$ with respect to $\hat{\omega}_j$ is also uniformly bounded independent of j.

By construction we know that as $j \to \infty$, ω_j converges smoothly to $\omega_\infty = \sqrt{-1}\partial\bar{\partial}\psi_\infty$ on $W \setminus \{0\}$, and ψ_∞ is continuous across 0. Using the above estimates and Lemma 4.4 by passing to a subsequence we may assume that φ_j converges locally smoothly to a smooth function φ_∞ and u_j converges locally smoothly to a smooth function u_∞ and on $W \setminus \{0\}$, such that $\hat{\omega}_\infty = \omega_\infty + \sqrt{-1}\partial\bar{\partial}(u_\infty + \varphi_\infty)$ is a Kähler Ricci flat metric on $W \setminus \{0\}$. Moreover u_∞ is a constant near 0, φ_∞ is globally bounded, $\hat{\omega}_\infty \geqslant C^{-1}\omega_\infty$ and there exists a compact set K such that on $W \setminus K$
$$|\nabla_{\omega_\infty}^k u_\infty| \leqslant C_k r_\infty^{-\delta_2 + 2 - k}, \quad |\nabla_{\omega_\infty}^k \varphi_\infty| \leqslant C_k r_\infty^{-\delta_3 - k}, \tag{4.11}$$
where we recall $r_\infty = r \circ p_\infty^{-1}$. Notice that since Ω_j is parallel with respect to $\hat{\omega}_j$, we may assume Ω_j converges to a holomophric volume form Ω_∞ on $W \setminus \{0\}$ such that $\hat{\omega}_\infty^n = (\sqrt{-1})^{n^2}\Omega_\infty \wedge \overline{\Omega}_\infty$. Moreover the metric $\hat{\omega}_\infty$ has finite diameter on $K \cap W$, and it is asymptotic to the cone $\mathcal{C}$ at infinity.

5. PROOF OF THE MAIN RESULTS

We now investigate the singular behavior of $\hat{\omega}_\infty$ at 0. To this end we need to study the singular behavior of $\hat{\omega}_j$, so we first fix j and consider the family $\hat{\omega}_{j,l} \equiv \hat{\omega}_{j,\epsilon_{j,l}}$ as $l \to \infty$. Passing to a further subsequence we let (Z_j, d_{Z_j}) be a pointed Gromov-Hausdorff limit of $(\hat{Y}_j, \hat{\omega}_{j,l}, p_j)$ (where p_j is a point on the exceptional set of π_j) as $l \to \infty$. As mentioned before, the metrics $\hat{\omega}_{j,\epsilon}$ are uniformly volume non-collapsing, so the general convergence theory leads to a decomposition $Z_j = \mathcal{R} \cup \mathcal{S}$, where the

regular set $\mathcal{R}$ is a smooth open manifold with $d_{Z_j}|_{\mathcal{R}}$ induced by a Calabi-Yau metric $(\omega_{\mathcal{R}}, J_{\mathcal{R}})$, and the singular set $\mathcal{S}$ is closed and has Hausdorff codimension at least 4.

Proposition 5.1. *Z_j agrees with the metric completion of $(Y_j^\circ, \hat{\omega}_j)$ and is naturally homeomorphic to Y_j.*

This would likely follow from the arguments of J. Song [26] (and its generalization in [8]). We outline a different and more direct proof here, which may be of independent interest. It is not hard to see that we can naturally view $(Y_j^\circ, \hat{\omega}_j)$ as embedded in $(\mathcal{R}, \omega_{\mathcal{R}})$ as an open Kähler manifold. Moreover, since $\omega_j \leqslant \mathcal{P}(A_j)\hat{\omega}_{j,l}$, by passing to a further subsequence, one can take the limit of the holomorphic maps $\pi_j : (\hat{Y}_j, \hat{\omega}_{j,l}) \to Y_j \subset \mathbb{C}^N$ and obtain a surjective continuous map $\pi_\infty : Z_j \to Y_j \subset \mathbb{C}^N$, which is the identity on Y_j° and holomorphic on $\mathcal{R}$. It then follows that $\mathcal{R} \setminus Y_j^\circ$ is a complex analytic set in $\mathcal{R}$, given by $\pi_\infty^{-1}(\mathrm{Sing}(Y_j)) \cap \mathcal{R}$. By the discussion in Section 4, on Y_j° we may write $\hat{\omega}_j = \sqrt{-1}\partial\bar{\partial}\hat{\varphi}_j$, where $\hat{\varphi}_j$ is locally bounded across $\mathrm{Sing}(Y_j)$. A key point is the following

Lemma 5.2. *The function $\hat{\varphi}_j$ extends smoothly to $\mathcal{R}$. Moreover, for any compact set $K \subset Z_j$ we have*

$$\sup_{K \cap \mathcal{R}} |\nabla_{\omega_{\mathcal{R}}} \hat{\varphi}_j|_{\omega_{\mathcal{R}}} < \infty.$$

Proof. On Y_j°, the form $\sqrt{-1}\partial\bar{\partial}\hat{\varphi}_j = \hat{\omega}_j = \omega_{\mathcal{R}}$ is smooth. Since $\mathcal{R} \setminus Y_j^\circ$ is a closed pluripolar set and $\hat{\varphi}_j$ is locally bounded across $\mathrm{Sing}(Y_j)$, by well-known removable singularity theorem for plurisubharmonic functions (see [13, Chapter I, Theorem 5.24]), we obtain that the same equation holds on $\mathcal{R}$ in the sense of currents, so elliptic regularity gives a smooth extension of $\hat{\varphi}_j$ to $\mathcal{R}$.

To prove the gradient bound we fix an open subset $U \subset Y_j$ containing $\mathrm{Sing}(Y_j)$ and with ∂U smooth. Denote $\hat{U} = \pi_j^{-1}(U)$. Its boundary $\partial\hat{U}$ is naturally identified with ∂U. We can solve the Poisson equation $\Delta_{\hat{\omega}_{j,l}}\eta_l = n$ on $\hat{U}$ with boundary condition given by $\hat{\varphi}_j|_{\partial U}$. Then by the standard Moser iteration and Cheng-Yau gradient estimate we have $|\nabla_{\hat{\omega}_{j,l}}\eta_l| \leqslant C$ for all l. So passing to a subsequence η_l converges uniformly to a limit η on $\pi_\infty^{-1}(U) \subset Z_j$, with $|\nabla_{\hat{\omega}_j}\eta| \leqslant C$, $\Delta_{\hat{\omega}_j}\eta = n$ on $\pi_\infty^{-1}(U) \cap \mathcal{R}$ and $\eta|_{\partial U} = \hat{\varphi}_j|_{\partial U}$.

Now consider the function $f = \hat{\varphi}_j - \eta$ on $\pi_\infty^{-1}(U) \cap \mathcal{R}$. It is harmonic and $f|_{\partial U} = 0$. One can conclude that f is identically zero from the standard removable singularity theorem for harmonic functions on Ricci limit spaces. For example, using the fact that the Hausdorff codimension of $\mathcal{S}$ is at least 4, for $\epsilon > 0$ small, one can construct (see [14, Proposition 3.5]) a cut-off function $\chi_\epsilon : Z_j \to \mathbb{R}$ which is equal to 1 when $d_{Z_j}(x, \mathcal{S}) \geqslant \epsilon$ and vanishes in a neighborhood of $\mathcal{S}$, such that $\int_{Z_j} d_{Z_j}(\cdot, \mathcal{S})^{-1}|\nabla_{\omega_{\mathcal{R}}}\chi_\epsilon|\omega_{\mathcal{R}}^n < \epsilon$. By the local Cheng-Yau gradient estimate we have $|\nabla_{\omega_{\mathcal{R}}}\hat{\varphi}_j| \leqslant Cd_{Z_j}(\cdot, \mathcal{S})^{-1}$. Then by a straightforward integration by parts argument one sees that $f \equiv 0$. $\qquad\square$

Given this Lemma, one can follow the strategy of [14, 15] to construct holomorphic functions by the Hörmander L^2 technique (see a related observation in [29] for studying the asymptotic cone of complete $\sqrt{-1}\partial\bar{\partial}$-exact Calabi-Yau manifolds with Euclidean volume growth). Here one can work on the trivial holomorphic line bundle L over Y_j° endowed with the hermitian metric $|\cdot|_{\hat{\varphi}_j} \equiv e^{-\hat{\varphi}_j}$. Notice Y_j° admits a complete Kähler metric, so one can apply Theorem 3.8 here. Let ∇_L be the corresponding Chern connection on L. For $k \geqslant 1$, denote by $\mathcal{V}_k$ the space of holomorphic sections f of L^k over Y_j° with

$$\|f\|_k^2 \equiv \int_{Y_j^\circ} |f|_{k\hat{\varphi}_j}^2 (k\hat{\omega}_j)^n < \infty.$$

Since Y_j is normal it is clear that any holomorphic function f on Y_j° extends to a holomorphic function on Y_j. In particular, for any compact set $K \subset \mathbb{C}^N$, $|f|$ and $|df|_{\omega_j}$ are uniformly bounded over $Y_j^\circ \cap K$. By (4.9), (4.10) and Lemma 5.2, we know that $|f|_{k\hat{\varphi}_j}$ and $|\nabla_L f|_{k\hat{\varphi}_j}$ are also uniformly bounded over $Y_j^\circ \cap K$. Viewed in Z_j, it follows that f extends to a holomorphic function on $\mathcal{R}$ and the above norms are locally bounded near $\mathcal{S}$.

To apply the idea in [14] we need to show that for any compact set $K \subset Y_j$, there exist constants $K_0, K_1 > 0$ depending on K such that for all $k \geqslant 1$ and any $f \in \mathcal{V}_k$, we have

$$\sup_K |f|_{k\hat{\varphi}_j} \leqslant K_0 \|f\|_k, \quad \sup_K |\nabla_L f|_{k\hat{\varphi}_j} \leqslant K_1 \|f\|_k. \tag{5.1}$$

These were proved by applying the Moser iteration to the corresponding differential inequalities

$$\Delta_{k\hat{\omega}_j} |f|_{k\hat{\varphi}_j} \geqslant -C|f|_{k\hat{\varphi}_j}, \quad \Delta_{k\hat{\omega}_j} |\nabla_L f|_{k\hat{\varphi}_j} \geqslant -C|\nabla_L f|_{k\hat{\varphi}_j},$$

which a priori only hold on $\mathcal{R}$, but again using the existence of a good cut-off function as in [14, Proposition 3.5], and noticing that there is a Sobolev inequality on Z_j (since Z_j is a volume non-collapsing Ricci limit space), one can make the usual arguments go through.

Using the existence of tangent cones at points in Z_j, it is straightforward to adapt the arguments in [14, 15] to construct holomorphic functions in $\mathcal{V}_k$ that separate points. Namely, given any $x_1 \neq x_2 \in Z_j$, one can find $k \geqslant 1$ and $f_1, f_2 \in \mathcal{V}_k$ such that $|f_\alpha(x_\alpha)|_{k\hat{\varphi}_j} \geqslant 1$, and $|f_\alpha(x_\beta)|_{k\hat{\varphi}_j} \leqslant 1/10$ for $\alpha \neq \beta$.

Proof of Proposition 5.1. We first prove π_∞ is injective (hence is a homeomorphism). Suppose $x_1 \neq x_2 \in Z_j$. Then we can construct the functions f_1, f_2 as above. But both can be viewed as holomorphic functions on Y_j, so it follows that $\pi_\infty(x_1) \neq \pi_\infty(x_2)$.

Since $\mathrm{Sing}(Y_j)$ consists of finitely many points, we see $Z_j \setminus Y_j^\circ = \pi_\infty^{-1}(\mathrm{Sing}(Y_j))$ is also a finite set. Clearly this implies that Z_j is indeed the metric completion of $(Y_j^\circ, \hat{\omega}_j)$. $\square$

Now we further let $j \to \infty$. Passing to a subsequence we may take a pointed Gromov-Hausdorff limit $(Z_j, d_{Z_j}) \to (Z_\infty, d_\infty)$. The same argument as above combined with the uniform estimates in Section 4 gives

Proposition 5.3. Z_∞ *agrees with the metric completion of* $(W \setminus \{0\}, \hat{\omega}_\infty)$ *and is naturally homeomorphic to* W.

In particular, we may naturally identify Z_∞ with W.

Proof of Theorem 1.1. As is proved in Section 4, Z_∞ has a unique asymptotic cone given by $\mathcal{C}$. From the above discussion it is straightforward to extend the arguments of [15] to show that there is a unique metric tangent cone $\mathcal{C}'$ at 0. Furthermore, by the minimizing property of K-semistable valuations proved by Li-Xu [21], we know the volume densities satisfy $\kappa(\mathcal{C}') \leqslant \kappa(\mathcal{C})$, where the right hand side can be interpreted as the normalized volume associated to the natural valuation of W at 0. Notice Z_∞ is a pointed Gromov-Hausdorff limit of complete Ricci-flat manifolds, so the Bishop-Gromov inequality applies to Z_∞, which gives that $\kappa(\mathcal{C}') \geqslant \kappa(\mathcal{C})$. Hence the equality holds and then Z_∞ must be a Kähler cone itself, hence is isomorphic to $\mathcal{C}$ as a Kähler cone. Denote by ξ' the Reeb vector field on Z_∞ and by $\tilde{r}$ the radial function on Z_∞.

It remains to show that $\xi' = \xi$. For this we need to use some results in [15]. The algebraic structure of Z_∞ can be intrinsically characterized in terms of the ring $R(Z_\infty)$ holomorphic functions on Z_∞ of polynomial growth (measured using the Kähler cone metric). The cone structure yields a decomposition

$$R(Z_\infty) = \bigoplus_{\mu \geqslant 0} R_\mu(Z_\infty), \tag{5.2}$$

where $R_\mu(Z_\infty)$ consists of f which are of homogeneous of degree μ, i.e., $\mathcal{L}_{J\xi'} f = -\mu f$. Conversely, ξ' can be recovered from this decomposition.

The above decomposition is determined by a filtration associated with a *degree* function. For any nonzero $f \in R(Z_\infty)$, the following is well-defined (see [15])

$$d_{Z_\infty}(f) \equiv \lim_{R \to \infty} \frac{\sup_{\tilde{r}(x)=R} \log |f(x)|}{\log R} < \infty.$$

Then the decomposition in (5.2) is determined by the filtration of $R(W)$ with respect to d_{Z_∞}.

On the other hand, the coordinate ring $R(W)$ of W consists of restrictions of polynomial functions on $\mathbb{C}^N$. We define

$$d_W(f) \equiv \lim_{R \to \infty} \frac{\sup_{r_\xi(x)=R} \log |f(x)|}{\log R}.$$

There is a decomposition $R(W) = \bigoplus_{\mu \geqslant 0} R_\mu(W)$, where $R_\mu(W)$ is the space of polynomials f with $\mathcal{L}_{J\xi} f = -\mu f$. It can be checked that this decomposition is also determined by the filtration of $R(W)$ with respect to d_W.

Notice that by the construction of $\hat{\omega}_\infty$ we know for all $\epsilon > 0$, $C_\epsilon^{-1} r_\xi^{1-\epsilon} \leqslant \tilde{r} \leqslant C_\epsilon r_\xi^{1+\epsilon}$ on $W \setminus K$ for a fixed compact set K. It follows that $R(Z_\infty) = R(W)$ and $d_W(f) = d_{Z_\infty}(f)$ for any $f \in R(W)$. So we must have $R_\mu(Z_\infty) = R_\mu(W)$ for all $\mu \geqslant 0$. In particular, $\xi' = \xi$. Hence we have constructed a T_ξ equivariant isomorphism between W and $\mathcal{C}$. This completes the proof of Theorem 1.1.

$\square$

Proof of Theorem 1.2. We simply notice that by the estimate of φ_∞ and u_∞ in (4.11), the background metric ω_∞ is polynomially asymptotic to the Calabi-Yau cone metric $\hat{\omega}_\infty$ on W (which is the same as $\mathcal{C}$). Moreover by the construction of ω_∞ (see Corollary 3.12 and Lemma 3.13), it is polynomially asymptotic to ω under the diffeomorphism p_∞ . Then p_∞ provides the diffeomorphism that shows ω and J are asymptotic to $\hat{\omega}_\infty$ and J_W in a polynomial rate. The asymptotics of Ω follows from (3.16) by letting $j \to \infty$.

$\square$

Remark 5.4. *Using the same proof one can weaken the Calabi-Yau condition in Theorem 1.1 and 1.2 to be Ricci-flat Kähler. Alternatively, one can show this by working on the universal cover since it is well known [2, 22] that Riemannian manifolds with non-negative Ricci curvature and Euclidean volume growth have finite fundamental groups.*

6. Discussions

We point out some further related directions that one can explore. The guiding problem is

Problem 6.1 (Algebraization for complete Calabi-Yau metrics)**.** Give an algebro-geometric characterization of all complete Calabi-Yau metrics with Euclidean volume growth.

Remark 6.2. *Without the volume growth condition, the situation is more complicated and extra assumptions are needed in order to make connections with algebraic geometry. In complex dimension 2 under a natural finite energy condition, the models at infinity are completely classified by Sun-Zhang [27], and there is a complete classification of the Calabi-Yau metrics in terms of algebro-geometric data by the work of many people; see [27] for references.*

Given Theorem 1.2, the results of Conlon-Hein [11] provide an answer to Problem 6.1 under the extra quadractic curvature decay condition. Namely, such Calabi-Yau metrics are always constructed as follows (ignoring uniqueness issues for the moment), where each step is essentially algebro-geometric.

(1) Choose a Calabi-Yau cone $(\mathcal{C}, \xi)$. This is equivalent to choosing a K-polystable Fano cone [7].
(2) Choose a nomal affine variety X' with $(\mathcal{C}, \xi)$ as a weighted asymptotic cone at infinity.
(3) Choose a crepant Kähler resolution $\pi : X \to X'$.
(4) Choose a Kähler class on X. This is a numerical condition according to [9, 11].

We want to ask how much of the above picture holds in general, if we allow singularities in the relevant objects. There are several technical obstacles to realize this program, mostly related to the possible appearance of singularities on the asymptotic cone. First one needs to extend the 2-step degeneration theory of [15] to this setting. In particular, we would like to construct abundant holomorphic functions with polynomial growth on X. This is related to a special case of Yau's compactification conjecture, which we reiterate as follows

Conjecture 6.3 (Yau's Compactification conjecture). *A complete Calabi-Yau manifold with Euclidean volume growth is naturally a quasi-projective variety.*

There has been partial progress towards this. For example, it is known that any asymptotic cone is naturally a normal affine variety [25]. It is also observed in [29] that the results of [15] automatically extend to the setting when the Calabi-Yau metric is $\partial\bar{\partial}$-exact so Conjecture 6.3 holds in this case. Now suppose the 2-step degeneration theory works. The following is natural to expect

Conjecture 6.4 (No semistability at infinity, the general version). *Theorem 1.1 holds without the quadratic curvature decay assumption.*

Remark 6.5. *One can also formulate a conjectural general version of Theorem 1.2, where one uses the Gromov-Hausdorff distance to measure the rate of convergence to the asymptotic cone.*

One difficulty in proving Conjecture 6.4 using the strategy of this paper is related to the generalization of the Tian-Yau construction, which is itself an interesting question. A first step would be

Question 6.6 (Prescribing asymptotic cone). Given a singular Calabi-Yau cone $(\mathcal{C}, \xi)$ and a normal affine variety X' with $(\mathcal{C}, \xi)$ as a weighted asymptotic cone at infinity, when does X' admit a (possibly singular) Calabi-Yau metric asymptotic to $(\mathcal{C}, \xi)$ in the Gromov-Hausdorff sense?

There are some recent extensions of the Tian-Yau construction with singular asymptotic cones for special examples; see [18, 23, 28, 12, 4].

As mentioned in Section 2 the classification of complete Calabi-Yau metrics with Euclidean volume growth on a fixed underlying algebraic variety is a subtle problem. Even on $\mathbb{C}^n (n \geqslant 3)$ the situation can be complicated; the recent works by Székelyhidi [29] and Chiu [4] make progress in this direction. We make an attempt here to formulate some general questions. To minimize technical issues we restrict to the case of $\partial\bar{\partial}$-exact metrics on smooth affine varieties. As mentioned above the results of [15] apply here. Fix a smooth affine variety X with the coordinate ring $R(X)$. A complete $\partial\bar{\partial}$-exact Calabi-Yau metric ω on X with Euclidean volume growth is said to be a *compatible* metric if the space of holomorphic functions on X with polynomial growth with respect to ω coincides with $R(X)$. A compatible Calabi-Yau metric ω defines a degree function $d_\omega : R(X) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying

- $d_\omega(f) = \infty$ if and only if $f = 0$;
- $d_\omega(f) = 0$ if and only if f is a nonzero constant;
- $d_\omega(fg) = d_\omega(f) + d_\omega(g)$;
- $d_\omega(f + g) \leqslant \max(d_\omega(f), d_\omega(g))$.

Furthermore, d_ω gives rise to a filtration of $R(X)$ whose associated graded ring is the coordinate ring $R(W)$ of the intermediate K-semistable cone W, and W degenerates to the unique asymptotic cone $\mathcal{C}$ as affine algebraic cones. The result of Li-Wang-Xu [20] implies that $(\mathcal{C}, \xi)$ is uniquely determined by d_ω. Clearly, if ω is compatible, then for any $\lambda > 0$ and any algebraic automorphism F of X, $\lambda F^*\omega$ is also compatible. Furthermore, $d_\omega = d_{\lambda\omega}$ and $d_{F^*\omega} = F^*d_\omega$.

In terms of algebro-geometric language one notices that $v_\omega \equiv -d_\omega$ is a valuation on $R(X)$ with values in $\mathbb{R}_{\leqslant 0} \cup \{-\infty\}$. We call such a valuation a *negative valuation*. We say a negative valuation v is *semistable* (resp. *polystable*) if it defines a filtration whose associated graded ring is finitely generated and defines a K-semistable (resp. K-polystable) Fano cone $\mathcal{C}_v$ (in the sense of [7]). Geometrically, the cone is a weighted asymptotic cone at infinity of X under some affine embedding. Conjecture 6.4 implies that the negative valuation induced by a compatible Calabi-Yau metric on X is always polystable.

Given a semistable valuation v on X, we define G_v as the group of algebraic automorphisms of X that preserve v and define $\tilde{G}_v$ as the group of algebraic automorphisms of $\mathcal{C}_v$ that fixes the vertex and preserves the cone structure. Both groups are finite-dimensional. Notice that $\tilde{G}_v$ contains the algebraic torus $\mathbb{T}_v$ generated by the cone vector field. There is a natural homomorphism $\varphi : G_v \to \tilde{G}_v$.

To classify the space of all compatible Calabi-Yau metrics on X, assuming Conjecture 6.4, one can proceed as follows:

Step 1. Determine the space $\mathcal{V}$ of all possible polystable negative valuations on X up to algebraic automorphism. This is essentially an algebro-geometric question. One can also study the related question with polystability replaced by semistability. In general there may be infinitely many polystable negative valuations on a given X. Indeed, given any polystable cone V in $\mathbb{C}^n$ which is a hypersurface, one can find a polystable negative valuation v on $X = \mathbb{C}^n$ with $\mathcal{C}_v = \mathbb{C} \times V$. Indeed, the result of Székelyhidi [28] implies that if V has only isolated singularities, then the associated negative valuation actually arises from a compatible Calabi-Yau metric on $\mathbb{C}^n$. Still, it seems interesting to investigate the structure of the space $\mathcal{V}$.

Step 2. Now we fix a polystable negative valuation v on X. Denote by $\mathcal{M}_v$ the space of all compatible Calabi-Yau metrics ω on X with $v = v_\omega$. Is $\mathcal{M}_v$ nonempty? This is related to Question 6.6.

Step 3. Suppose $\mathcal{M}_v \neq \emptyset$. We know that all $\omega \in \mathcal{M}_v$ have the same asymptotic cone at infinity given by $\mathcal{C}_v$. But we should expect more and one would like to have

Conjecture 6.7 (Uniform equivalence). *There exists a constant $C > 0$ such that for all $\omega_1, \omega_2 \in \mathcal{M}_v$, we have $C^{-1}\omega_1 \leqslant \omega_2 \leqslant C\omega_1$.*

It is easy to see the converse is true: if two compatible Calabi-Yau metrics ω_1 and ω_2 are uniformly equivalent, then $v_{\omega_1} = v_{\omega_2}$. We remark that one can also ask an analogous question for local singularities of Kähler-Einstein metrics (on algebraic varieties with klt singularities), that is, whether any two such metrics are indeed locally uniformly equivalent. The answer to this question, currently not known, is also related to Conjecture 6.7. The recent work by Chiu-Székelyhidi [5] made progress in understanding local singular behavior of polarized Kähler-Einstein metrics.

Step 4. Denote by $\mathcal{N}_v$ the space of Calabi-Yau cone metrics on $\mathcal{C}_v$ (with the given Reeb vector field determined by v). Notice $\mathcal{C}_v$ can be realized as a weighted asymptotic cone at infinity for some embedding $X \subset \mathbb{C}^N$. One is attempted to define a natural map $\mathfrak{C} : \mathcal{M}_v \to \mathcal{N}_v$ given by taking the (appropriate) rescaled limit of the Kähler form under the weighted asymptotic cone construction.

Conjecture 6.8 (Generalized uniqueness). *The map $\mathfrak{C}$ is well-defined and is bijective.*

We explain Conjecture 6.7 and 6.8 indeed hold if we assume $\mathcal{C}_v$ has only an isolated singularity at the vertex. In this case any Calabi-Yau metric in $\mathcal{M}_v$ has quadratic curvature decay so we may apply the arguments of this paper. Since we have fixed the negative valuation v, this fixes the algebro-geometric degeneration from X to $\mathcal{C}_v$, in terms of a weighted asymptotic cone. Then by Section 3.6 we can fix the diffeomorphism between the end of X and the end of $\mathcal{C}_v$. For every $\omega_\infty \in \mathcal{N}_v$, by [10] (or the arguments in Section 4 of this paper) one gets a compatible Calabi-Yau metric on X which is asymptotic to ω_∞ at a polynomial rate. Conversely, given a compatible Calabi-Yau metric $\omega \in \mathcal{M}_v$, using the arguments in this paper, we can get a Calabi-Yau cone metric on $\mathcal{C}_v$ which is the asymptotic cone of ω at a polynomial rate. One can see this is exactly the $\mathfrak{C}$ map that we want. Furthermore, it is independent of the auxiliary data chosen. The uniqueness theorem [10, Theorem 3.1] proved by Conlon-Hein implies that $\mathfrak{C}$ is bijective.

Notice even in this case it is not clear that the action of G_v is transitive on $\mathcal{M}_v$ (modulo scaling): we know that the action of $\widetilde{G}_v$ on $\mathcal{N}_v$ is transitive by the generalized Bando-Mabuchi theorem (see for example, [15, Proposition 4.8]). Indeed, $\mathcal{N}_v$ can be identified with the homogeneous space $\widetilde{G}_v/K_v$ for a maximal compact subgroup K_v whose complexification is $\widetilde{G}_v$. But the map $\varphi : G_v \to \tilde{G}_v$ could be complicated in general: neither the injectivity nor the surjectivity seems obvious. It seems to us that Conjecture 6.8 is a more natural formulation than a naive uniqueness statement. We mention that

the above conjecture is compatible with the result and conjecture in [29, 4] for complete Calabi-Yau metrics on $\mathbb{C}^3$ with specific asymptotic cones.

The above four steps altogether would lead to a satisfactory classification of compatible Calabi-Yau metrics on a fixed smooth affine variety. More generally, when the metrics are not necessarily $\partial\bar{\partial}$-exact, one needs to further involve Kähler classes in the discussion.

From another perspective, one can also try to classify complete Calabi-Yau metrics asymptotic to a given Calabi-Yau cone, without fixing the underlying algebraic variety. Notice again this should contrast with the local situation. As is well known (see [15, 18, 28]), there is no boundedness of local singularities of Calabi-Yau metrics with a given tangent cone if the latter has nonisolated singularities. One example is given by the A_k singularities in dimension $n \geqslant 3$ for $k \geqslant 4$, whose local tangent cones are all given by the product of an $n-1$ dimensional A_1 singularity with $\mathbb{C}$. However, for asymptotic cones, there are more constraints. For example, the Bishop-Gromov inequality ensures a strong uniform non-collapsing condition, which gives a preliminary compactness in this situation. In particular, one would expect a well-behaved moduli space when the volume density of the asymptotic cone is bounded away from zero.

Problem 6.9 (Moduli problem). Given a positive number $\lambda > 0$, study the structure of the moduli space of (possibly singular) $\partial\bar{\partial}$-exact Calabi-Yau metrics on affine varieties whose asymptotic cones have volume density at least λ.

It is straightforward to formulate an algebro-geometric counterpart of the above problem. That is, one can study the moduli space $\mathfrak{M}_{\geqslant\lambda}$ of pairs (X, v), where X is an affine variety and v is a polystable negative valuation on X with $\mathcal{C}_v$ having volume density at least λ. It seems to be an approachable question given recent advances in birational algebraic geometry. However, as pointed out by Yuji Odaka there are some complications. For simplicity we fix a K-polystable cone $\mathcal{C}$, then the deformation space $\mathrm{Def}(\mathcal{C})$ of $\mathcal{C}$ as affine varieties is in general infinite dimensional, but it is natural to expect that the subspace $\mathrm{Def}_-(\mathcal{C})$ which consists of those deformations to the above pairs (X, v) with $\mathcal{C}_v = \mathcal{C}$ is finite dimensional, and there is an action of $\widetilde{G}_v$ on $\mathrm{Def}_-(\mathcal{C})$. Now since $\widetilde{G}_v$ is noncompact the quotient is in general not Hausdorff, so any meaningful construction of such moduli space will have to address this issue. On the other hand, if we accept the above picture, then the space of $\partial\bar{\partial}$-exact Calabi-Yau metrics asymptotic to $\mathcal{C}$ (up to isomorphism) is given by the quotient $\mathrm{Def}_-(\mathcal{C})/K_v$. It seems an interesting question to us to explore the structure here in more detail.

Finally, we point out that while Theorem 1.2 is a Riemannian geometric statement, our proof in this paper hinges on complex geometry. One can ask whether the polynomial convergence rate is a more general phenomenon in geometric analysis, or is it only special in the complex geometric world. Notice this is a question involving *global* geometry, not only the end at infinity. Indeed, we expect that the technique of Adams-Simon [1] can be used to show that given a Calabi-Yau cone $\mathcal{C}$ with obstructed deformations in an appropriate sense, one can produce an end of Calabi-Yau metric which is asymptotic to $\mathcal{C}$ at only a logarithmic rate; see [3] for a related work on isolated singularities of G_2 metrics. Such an end is also interesting in terms of complex geometry since it can *not* be filled in as a complete Kählerian manifold – otherwise using the Tian-Yau construction one would get a contradiction with Theorem 1.2.

Question 6.10 (Polynomial rate to the asymptotic cone). Is a complete Ricci-flat Riemannian manifold with Euclidean volume growth always asymptotic to a unique metric cone at a polynomial rate (in the Gromov-Hausdorff distance)?

Even the uniqueness of asymptotic cones is not known to date without the quadratic curvature decay condition. An answer to Question 6.10, either in the positive or the negative, would be interesting. A special setting is given by manifolds with special holonomy, i.e., G_2 and $Spin(7)$ manifolds. One

can also explore similar questions for global solutions of other geometric PDEs, such as minimal submanifolds and Yang-Mills connections. These will be important in classifying singularity models.

References

[1] David Adams and Leon Simon. "Rates of asymptotic convergence near isolated singularities of geometric extrema". In: *Indiana University mathematics journal* 37.2 (1988), pp. 225–254.

[2] Michael T. Anderson. "On the topology of complete manifolds of non-negative Ricci curvature". In: *Topology* 29.1 (1990), pp. 41–55.

[3] Gao Chen. "Rate of Asymptotic Convergence Near an Isolated Singularity of G_2 Manifold". In: *The Journal of Geometric Analysis* 28.4 (2018), pp. 3139–3170.

[4] Shih-Kai Chiu. "Nonuniqueness of Calabi-Yau metrics with maximal volume growth". In: *arXiv preprint arXiv:2206.08210* (2022).

[5] Shih-Kai Chiu and Gábor Székelyhidi. "Higher regularity for singular Kähler-Einstein metrics". In: *arXiv preprint arXiv:2202.11083* (2022).

[6] Tobias Holck Colding and William P Minicozzi. "On uniqueness of tangent cones for Einstein manifolds". In: *Inventiones mathematicae* 196.3 (2014), pp. 515–588.

[7] Tristan Collins and Gábor Székelyhidi. "Sasaki–Einstein metrics and K–stability". In: *Geometry & Topology* 23.3 (2019), pp. 1339–1413.

[8] Tristan C Collins, Bin Guo, and Freid Tong. "On the degeneration of asymptotically conical Calabi–Yau metrics". In: *Mathematische Annalen* (2021), pp. 1–53.

[9] Tristan C Collins and Valentino Tosatti. "A singular Demailly–Păun theorem". In: *Comptes Rendus Mathematique* 354.1 (2016), pp. 91–95.

[10] Ronan J Conlon and Hans-Joachim Hein. "Asymptotically conical Calabi–Yau manifolds, I". In: *Duke Mathematical Journal* 162.15 (2013), pp. 2855–2902.

[11] Ronan J Conlon and Hans-Joachim Hein. "Classification of asymptotically conical Calabi-Yau manifolds". In: *arXiv preprint arXiv:2201.00870* (2022).

[12] Ronan J Conlon and Frédéric Rochon. "New examples of complete Calabi-Yau metrics on $\mathbb{C}^n$ for $n \geqslant 3$". In: *arXiv preprint arXiv:1705.08788* (2017).

[13] Jean-Pierre Demailly. *Complex analytic and differential geometry.* Citeseer, 1997.

[14] Simon Donaldson and Song Sun. "Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry". In: *Acta Mathematica* 213.1 (2014), pp. 63–106.

[15] Simon Donaldson and Song Sun. "Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry, II". In: *Journal of Differential Geometry* 107.2 (2017), pp. 327–371.

[16] O. Riemenschneider H. Grauert. "Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen." In: *Inventiones mathematicae* 11 (1970), pp. 263–292.

[17] Hans-Joachim Hein. *On gravitational instantons.* Phd thesis, Princeton University, 2010.

[18] Hans-Joachim Hein. "Ricci-flat metrics on A_k singularities". In: *Oberwolfach report* 31 (2015).

[19] Chi Li. "Minimizing normalized volumes of valuations". In: *Mathematische Zeitschrift* 289.1-2 (2018), pp. 491–513.

[20] Chi Li, Xiaowei Wang, and Chenyang Xu. "Algebraicity of the metric tangent cones and equivariant K-stability". In: *Journal of the American Mathematical Society* 34.4 (2021), pp. 1175–1214.

[21] Chi Li and Chenyang Xu. "Stability of valuations: higher rational rank". In: *Peking Mathematical Journal* 1.1 (2018), pp. 1–79.

[22] Peter Li. "Large time behavior of the heat equation on complete manifolds with non-negative Ricci curvature". In: *Annals of mathematics* 124.1 (1986), pp. 1–21.

[23] Yang Li. "A new complete Calabi–Yau metric on $\mathbb{C}^3$". In: *Inventiones mathematicae* 217.1 (2019), pp. 1–34.

[24] Gang Liu. "Compactification of certain Kähler manifolds with nonnegative Ricci curvature". In: *Advances in Mathematics* 382 (2021), p. 107652.

[25] Gang Liu and Gábor Székelyhidi. "Gromov-Hausdorff Limits of Kähler Manifolds with Ricci Curvature Bounded Below II". In: *Communications on Pure and Applied Mathematics* 74.5 (2021), pp. 909–931.

[26] Jian Song. "Riemannian geometry of Kähler-Einstein currents". In: *arXiv:1404.0445* (2014).

[27] Song Sun and Ruobing Zhang. "Collapsing geometry of hyperkaehler 4-manifolds and applications". In: *arXiv preprint arXiv:2108.12991* (2021).

[28] Gábor Székelyhidi. "Degenerations of $\mathbf{C}^n$ and Calabi–Yau metrics". In: *Duke Mathematical Journal* 168.14 (2019), pp. 2651–2700.

[29] Gábor Székelyhidi. "Uniqueness of some Calabi–Yau metrics on $\mathbf{C}^n$". In: *Geometric and Functional Analysis* 30.4 (2020), pp. 1152–1182.

[30] Gang Tian and Shing Tung Yau. "Complete Kähler manifolds with zero Ricci curvature II". In: *Inventiones mathematicae* 106.1 (1991), pp. 27–60.

Department of Mathematics, University of California, Berkeley, CA 94720
Email address: sosun@berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720
Email address: jszhang@berkeley.edu