



Nodal curves and polarizations with good properties

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Abstract

In this paper we deal with polarizations on a nodal curve C with smooth components. Our aim is to study and characterize a class of polarizations, which we call “good”, for which depth one sheaves on C reflect some properties that hold for vector bundles on smooth curves. We will concentrate, in particular, on the relation between the w -stability of \mathcal{O}_C and the goodness of w . We prove that these two concepts agree when C is of compact type and we conjecture that the same should hold for all nodal curves.

Keyword Polarizations, Stability, Nodal curves, Moduli spaces

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Introduction

Let C be a projective curve over the complex field. One of the most interesting problems in Algebraic Geometry is the construction of moduli spaces parametrizing line bundles or in general vector bundles on C . These moduli spaces have been studied first by Mumford [22] and Le Potier [21] in the smooth case. These spaces are interesting by themselves as higher dimensional varieties but also for important related constructions: just to mention some, one can consider higher-rank Brill-Noether theory, Theta divisors and Theta functions and the moduli spaces of coherent systems. For surveys on these topics see, for example [3,6,7]; for some results by the authors see [5,8,11–14]. When the curve is singular, these spaces are not in general complete. It is natural to study their possible compactifications and this has driven the attention of many authors since the

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'60s, who addressed the problem with different approaches (see, for instance, [4,18–20,23,24]). When C is a reducible nodal curve, that is it has only ordinary double points, we have more explicit results. In several of the constructions mentioned above, the objects of these compact moduli spaces are equivalence classes of depth one sheaves (i.e. torsion free) on the curve that are semistable with respect to a polarization (see [25,26]).

A polarization \underline{w} on C is given by rational weights on each irreducible component of C adding up to 1 or, equivalently, by an ample line bundle L on C (see [20,24]). Once a polarization on the curve is fixed, the notions of degree and rank can be generalized to the notions of \underline{w} -degree and \underline{w} -rank which are also defined for depth one sheaves. With these data Seshadri introduced the notion of \underline{w} -stability (or \underline{w} -semistability) for depth one sheaves allowing the construction of moduli spaces of such objects.

In this paper we are interested in studying polarizations on nodal reducible curves having nice properties, i.e. which allow us to generalize to nodal curves some natural properties of vector bundles on smooth curves and to simplify the study of stability of vector bundles and coherent systems on nodal reducible curves. As motivation, consider the following facts. On a smooth curve C , the sheaf \mathcal{O}_C is stable (as all line bundles) and any globally generated vector bundle has non-negative degree. This is not true anymore on reducible nodal curves. Moreover, in order to construct vector bundles on a reducible nodal curve, one can glue vector bundles on its components. In general, though, it is not true that glueing stable vector bundles yields a \underline{w} -stable sheaf: additional conditions on the polarization and on the degree of the restrictions are needed (see [9,10,27]).

This motivates our definition of a good polarization. Let C be a nodal curve with γ smooth irreducible components. For any depth one sheaf E on C , we denote by E_i the restriction (modulo torsion) of E to the component C_i . Note that if E is locally free, then the degree of E is actually the sum of the degrees of its restrictions E_i , but this is not true in general. We will say that \underline{w} is a *good polarization* if for any depth one sheaf E the difference $\Delta_{\underline{w}}(E)$ of the \underline{w} -degree of E and the sum of degrees of its restrictions E_i is non negative and it is zero if and only if E is locally free (see Definition 2.6). As anticipated, the first result of this paper is the following:

Theorem (Theorem 2.9) *Let C be a nodal curve and let \underline{w} be a good polarization on it. Let E be a depth one sheaf on C . Then we have the following properties:*

- (a) *Assume that E is locally free and, for $i = 1, \dots, \gamma$, E_i is stable with $\deg(E_i) = 0$. Then E is \underline{w} -stable.*
- (b) *If E is globally generated, then $\deg_{\underline{w}}(E) \geq 0$.*
- (c) *If E is \underline{w} -semistable and $\deg_{\underline{w}}(E) > 0$, then $h^0(E^*) = 0$.*

In particular, if $E = \mathcal{O}_C$ or, more generally, if E is a line bundle whose restrictions have degree 0, then E is \underline{w} -stable.

We will show that good polarizations exist on any stable nodal curve with $p_a(C) \geq 2$ (see Proposition 2.8 and Corollary 3.15). For nodal curves with $p_a(C) \leq 1$ we are able to characterize exactly which curves admit a good polarization (see Corollary 3.11).

The second result of this paper provides sufficient conditions in order to obtain a good polarization on a nodal curve. The method relies on the choice of particular paths

on the dual graph Γ_C of C which yields a finite collection of subcurves A_j of C . This allows us to get a rather technical description of $\Delta_{\underline{w}}(E)$, for any depth one sheaf E on C , and to obtain the mentioned sufficient conditions. These are stated in Theorem 3.9. More precisely, consider, for each non-empty subcurve A_j , the condition

$$(\star\star)_{A_j} : \quad \frac{1}{2}(\delta_{A_j} - 1) < \Delta_{\underline{w}}(\mathcal{O}_{A_j}) < \frac{1}{2}(\delta_{A_j} + 1)$$

where δ_{A_j} is the number of the nodes of C lying on A_j which are not nodes for the subcurve A_j (see Sect. 1 for details). Then we have the following:

Theorem (Theorem 3.9) *Let (C, \underline{w}) be a polarized nodal curve. If conditions $(\star\star)_{A_j}$ hold for all non empty A_j , then \underline{w} is a good polarization.*

Motivated by many examples (some of them have been reported in Sect. 4), we make this conjecture:

Conjecture (Conjecture 3.13) *Let (C, \underline{w}) be a polarized nodal curve. Then \mathcal{O}_C is \underline{w} -stable if and only if \underline{w} is a good polarization.*

In the third result of this paper we prove that this conjecture holds for curves of compact type:

Theorem (Theorem 3.10) *Let (C, \underline{w}) be a polarized nodal curve of compact type. Then \mathcal{O}_C is \underline{w} -stable if and only if \underline{w} is a good polarization.*

The idea is to prove that conditions $(\star\star)_{A_j}$ are always implied by stability of \mathcal{O}_C in the case of curves of compact type.

Finally, we wonder how being a good polarization reflects on the line bundle inducing the polarization. This turns out to be related to the notion of balanced line bundles, as defined in [15]. Balanced line bundles are important tools when one has to deal with reducible nodal curves. For example, for such line bundles, a generalization of Clifford’s Theorem holds. Our results can be summarized as (see Corollary 2.19 and Corollary 3.12):

Theorem *Let C be a stable nodal curve with $p_a(C) \geq 2$. Let L be a line bundle of degree $p_a(C) - 1$ and \underline{w} be the polarization induced by L . Then:*

- (1) *L is strictly balanced if and only if \mathcal{O}_C is \underline{w} -stable;*
- (2) *if C is of compact type, then L is strictly balanced if and only if \underline{w} is good.*

1 Notations and preliminary results on nodal curves

In this section we will introduce notations and we recall useful facts about nodal curves, their subcurves and polarizations.

Let C be a connected reduced nodal curve over the complex field (i.e. having only ordinary double points as singularities). We will denote by γ the number of irreducible components and by δ the number of nodes of C . We will assume that each irreducible

component C_i is a smooth curve of genus g_i . For the theory of nodal curves see [2, Ch X]. We will denote by

$$\nu: C^\nu = \bigsqcup_{i=1}^{\gamma} C_i \rightarrow C$$

the normalization map. If $p \in C$ is a node, we will denote by q_{p,i_1} and q_{p,i_2} the branch points over the node p , with $q_{p,i_k} \in C_{i_k}$. From the exact sequence:

$$0 \rightarrow \mathcal{O}_C \rightarrow \nu_* \nu^*(\mathcal{O}_C) \rightarrow \bigoplus_{p \in \text{Sing}(C)} \mathbb{C}_p \rightarrow 0,$$

we deduce that $\chi(\mathcal{O}_C) = \sum_{i=1}^{\gamma} \chi(\mathcal{O}_{C_i}) - \delta$, and we obtain the *arithmetic genus* of C :

$$p_a(C) = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1. \quad (1.1)$$

The *dual graph* of C is the graph Γ_C whose vertices are identified with the irreducible components of C and whose edges are identified with the nodes of C . An edge joins two vertices if the corresponding node is in the intersection of the corresponding irreducible components. So, Γ_C has δ edges and γ vertices, moreover it is connected since C is connected. Its first Betti number is $b_1(\Gamma_C) = \delta - \gamma + 1$. We recall that a connected nodal curve is said to be of *compact type* if every irreducible component of C is smooth and its dual graph is a tree. For a curve of compact type we have $\delta - \gamma + 1 = 0$ and the pull-back ν^* of the normalization map induces an isomorphism $\text{Pic}(C) \simeq \bigoplus_{i=1}^{\gamma} \text{Pic}(C_i)$ between the Picard groups.

Let B be a proper subcurve of C , the *complementary curve* of B is defined as the closure of $C \setminus B$ and it is denoted by B^c . We will denote by Δ_B the Weil divisor $\Delta_B = B \cdot B^c = \sum_{p \in B \cap B^c} p$, we will denote its degree by δ_B so $\delta_B = \#B \cap B^c$. In particular, when C_i is a component of C , Δ_{C_i} is given by the nodes on C_i . To simplify notations we set $\Delta_{C_i} = \Delta_i$ and $\delta_i = \#\Delta_i$.

As the only singularities of C are nodes, C can be embedded in a smooth projective surface, see [1]. This gives, for any proper subcurve B of C , the following fundamental exact sequence

$$0 \rightarrow \mathcal{O}_{B^c}(-\Delta_B) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_B \rightarrow 0, \quad (1.2)$$

from which we deduce

$$p_a(C) = p_a(B) + p_a(B^c) + \delta_B - 1. \quad (1.3)$$

We recall that a connected nodal curve C of arithmetic genus $p_a(C) \geq 2$ is called *stable* if each smooth rational component E of C meets E^c in at least three points, i.e. $\delta_E \geq 3$. A curve is stable if and only if ω_C is ample. The curve C is called *semistable* if $\delta_E \geq 2$. If C is semistable, a rational component E with $\delta_E = 2$ is said to be an

exceptional component. Finally, C is called *quasistable* if it is semistable and if any two exceptional components do not intersect each other. Good references for these topics are [15,16].

Let L be a line bundle on C . For all $i = 1, \dots, \gamma$, let L_i denote the restriction of L to the component C_i . It is a line bundle on C_i with $\deg(L_i) = d_i$. We will call (d_1, \dots, d_γ) the *multidegree* of L . Then the *degree* of L is $\deg(L) = \sum_{i=1}^\gamma d_i$. We have an exact sequence

$$0 \rightarrow L \rightarrow v_*v^*L \rightarrow \bigoplus_{p \in \text{Sing}(C)} \mathbb{C}_p \rightarrow 0,$$

from which we deduce $\chi(L) = \sum_{i=1}^\gamma \chi(L_i) - \delta$. In complete analogy with the smooth case, Riemann-Roch’s Theorem holds for any line bundle L on C : $\chi(L) = \deg(L) + 1 - p_a(C)$. We recall that L is ample if and only if $d_i > 0$ for all $i = 1, \dots, \gamma$. We will denote by $\text{Pic}^0(C) \subset \text{Pic}(C)$ the variety parametrizing the isomorphism classes of line bundles on C having multidegree $(0, \dots, 0)$.

There exists on C a dualizing sheaf ω_C , which is invertible. For simplicity, if L is a line bundle on C and B is a subcurve of C , we will denote by $\deg_B(L) = \deg_B(L|_B)$ the degree of $L|_B$ as line bundle on B . Then, we have $\omega_C|_B = \omega_B(B \cdot B^c)$, from which we obtain that the degree of $\omega_C|_B$ is $\deg_B(\omega_C|_B) = 2p_a(B) - 2 + \delta_B$. In particular, we have $\deg(\omega_C) = 2p_a(C) - 2$.

A central object in this paper will be the notion of polarization. One can refer to [23,24] for details about polarizations and their role in studying stability of depth one sheaves on reducible nodal curves.

Definition 1.1 A **polarization** on the curve C is a vector $\underline{w} = (w_1, \dots, w_\gamma) \in \mathbb{Q}^\gamma$ such that

$$0 < w_i < 1 \quad \sum_{i=1}^\gamma w_i = 1. \tag{1.4}$$

We will say that the pair (C, \underline{w}) is a **polarized curve**.

Remark 1.2 Let L be an ample line bundle on C , with $\deg(L) = d = \sum_{i=1}^\gamma d_i$. We can associate to L a polarization \underline{w}_L on C by setting $\underline{w}_L = \frac{1}{d}(d_1, \dots, d_\gamma)$. We will call \underline{w}_L the **polarization induced by L** . Note that for any polarization \underline{w} there exists a line bundle L which induces \underline{w} . Such a line bundle is not unique: many modifications of L (for instance, one can consider a multiple of L), lead to the same polarization.

We recall that a *depth one sheaf* on a curve is a coherent sheaf E with $\dim \text{Supp}(F) = 1$ for any subsheaf F of E . On a nodal curve this is equivalent to saying that E is torsion free. If E is a depth one sheaf on C and B is any proper subcurve of C , we denote by $E|_B$ the restriction of E to B and by E_B the restriction $E|_B$ modulo torsion. Then E_B is a depth one sheaf on B . If C_i is an irreducible component of C we define E_i to be E_{C_i} . We denote by d_i the degree of E_i and r_i the rank of E_i .

If \underline{w} is a polarization on C , we define the \underline{w} -rank and the \underline{w} -degree of E as $\text{rk}_{\underline{w}}(E) = \sum_{i=1}^r r_i w_i$ and $\text{deg}_{\underline{w}}(E) = \chi(E) - \text{rk}_{\underline{w}}(E)\chi(\mathcal{O}_C)$ respectively.

Definition 1.3 Let \underline{w} be a polarization on C and let E be a depth one sheaf on C . The \underline{w} -slope of E is defined as

$$\mu_{\underline{w}}(E) = \frac{\chi(E)}{\text{rk}_{\underline{w}}(E)} = \frac{\text{deg}_{\underline{w}}(E)}{\text{rk}_{\underline{w}}(E)} + \chi(\mathcal{O}_C).$$

E is said to be \underline{w} -semistable if for any proper subsheaf F of E we have $\mu_{\underline{w}}(F) \leq \mu_{\underline{w}}(E)$, i.e. if

$$\frac{\text{deg}_{\underline{w}}(F)}{\text{rk}_{\underline{w}}(F)} \leq \frac{\text{deg}_{\underline{w}}(E)}{\text{rk}_{\underline{w}}(E)}.$$

E is said to be \underline{w} -stable if the above inequality is strict.

We stress that in the case of depth one sheaves having rank 1 on each irreducible component of C , many different notions of semistability have been introduced. One can see for instance [17,23], for two different approaches which give equivalent stability conditions. In particular, we recall the following characterization of \underline{w} -semistability, see [23].

Proposition 1.4 Let (C, \underline{w}) be a polarized curve and let L be a depth one sheaf with $r_i = 1$ for all i . Then L is \underline{w} -semistable if and only if for any proper subcurve B of C

$$\text{deg}_{\underline{w}}(L_B) \geq \text{deg}_{\underline{w}}(L)\text{rk}_{\underline{w}}(L_B).$$

It is \underline{w} -stable if and only if the inequality is strict.

2 Polarizations with nice properties

From now on we will assume that C is a reducible nodal curve.

2.1 The function $\Delta_{\underline{w}}$ and its properties

Definition 2.1 Let \underline{w} be a polarization on C . Let E be a depth one sheaf on C and let E_i be the restriction of E to C_i modulo torsion. We define $\Delta_{\underline{w}}(E)$ as

$$\Delta_{\underline{w}}(E) = \text{deg}_{\underline{w}}(E) - \sum_{i=1}^{\gamma} \text{deg}(E_i).$$

Note that if $p_a(C) = 1$, then $\Delta_{\underline{w}}(E) = \chi(E) - \sum_{i=1}^{\gamma} \text{deg}(E_i)$, so it does not depend on the chosen polarization.

Let E be a depth one sheaf on C . Let $p \in C_{i_1} \cap C_{i_2}$ be a node of C , then $v^{-1}(p) = \{q_{p,i_1}, q_{p,i_2}\}$ with $q_{p,i_k} \in C_{i_k}$. The stalk of E in p can be written (see [24]) as

$$E_p = \mathcal{O}_p^{s_p} \oplus \mathcal{O}_{q_{p,i_1}}^{a_{p,i_1}} \oplus \mathcal{O}_{q_{p,i_2}}^{a_{p,i_2}} \tag{2.1}$$

where s_p is the rank of the free part of the stalk of E in p . Moreover, we have

$$r_{i_1} = s_p + a_{p,i_1} \quad r_{i_2} = s_p + a_{p,i_2}. \tag{2.2}$$

We set $t_p = a_{p,i_1} + a_{p,i_2}$.

Definition 2.2 Let E be a depth one sheaf on C and let p be a node, we will call t_p the **residual rank of E at p** .

Remark 2.3 A depth one sheaf E on C is locally free if and only all the residual ranks of E are zero.

In the following lemma we summarize some basic properties satisfied by $\Delta_{\underline{w}}$.

Lemma 2.4 Let C be a nodal curve with nodes p_1, \dots, p_δ and let \underline{w} be a polarization on it. Let E be a depth one sheaf on C . Then we have:

(a) set $\lambda_i = \Delta_{\underline{w}}(\mathcal{O}_{C_i}) = 1 - g_i - w_i \chi(\mathcal{O}_C)$. Then $\sum_{i=1}^\gamma \lambda_i = \delta$ and

$$\Delta_{\underline{w}}(E) = \sum_{i=1}^\gamma r_i \lambda_i - \sum_{j=1}^\delta s_{p_j}; \tag{2.3}$$

(b) if E is locally free, then $\Delta_{\underline{w}}(E) = 0$, i.e. $\deg_{\underline{w}}(E) = \sum_{i=1}^\gamma \deg(E_i)$;

(c) if $r_i = r$ for all $i = 1, \dots, \gamma$, then for any node p_j we have $a_{p_j,i_1} = a_{p_j,i_2} = t_{p_j}/2$.
 Moreover, $\Delta_{\underline{w}}(E) = \frac{1}{2} \sum_{j=1}^\delta t_{p_j} \geq 0$ and equality holds if and only if E is locally free;

(d) for any line bundle L we have $\Delta_{\underline{w}}(E \otimes L) = \Delta_{\underline{w}}(E)$;

(e) if $\text{Supp}(E)$ is a disjoint union of connected subcurves B_s for $s = 1, \dots, c$, then

$$\Delta_{\underline{w}}(E) = \sum_{s=1}^c \Delta_{\underline{w}}(E|_{B_s});$$

(f) if B is a proper subcurve of C , then $\Delta_{\underline{w}}(E_B) + \Delta_{\underline{w}}(E_{B^c}) = \Delta_{\underline{w}}(E) + \sum_{p_j \in B \cap B^c} s_{p_j}$;

(g) if E is locally free of rank r and B is a subcurve of C , then $\Delta_{\underline{w}}(E|_B) = r \Delta_{\underline{w}}(\mathcal{O}_B)$.

Proof (a) From [24] we have an exact sequence

$$0 \rightarrow E \rightarrow \bigoplus_{i=1}^\gamma E_i \rightarrow T \rightarrow 0, \tag{2.4}$$

where T is a torsion sheaf on C whose support is contained in the set of nodes. Hence we have $\chi(E) = \sum_{i=1}^{\gamma} \chi(E_i) - \chi(T)$. More precisely, if p_j is a node, we have $h^0(T_{p_j}) = s_{p_j}$ so $\chi(T) = \sum_{j=1}^{\delta} s_{p_j}$. Then, by definition, we have

$$\begin{aligned} \text{deg}_{\underline{w}}(E) &= \sum_{i=1}^{\gamma} \chi(E_i) - \chi(T) - \text{rk}_{\underline{w}}(E)\chi(\mathcal{O}_C) \\ &= \sum_{i=1}^{\gamma} [d_i + r_i(1 - g_i)] - \sum_{i=1}^{\gamma} w_i r_i \chi(\mathcal{O}_C) - \sum_{j=1}^{\delta} s_{p_j} \end{aligned}$$

so we get

$$\Delta_{\underline{w}}(E) = \sum_{i=1}^{\gamma} r_i [1 - g_i - w_i \chi(\mathcal{O}_C)] - \sum_{j=1}^{\delta} s_{p_j} = \sum_{i=1}^{\gamma} r_i \lambda_i - \sum_{j=1}^{\delta} s_{p_j}.$$

Finally, we have

$$\sum_{i=1}^{\gamma} \lambda_i = \sum_{i=1}^{\gamma} [1 - g_i - w_i \chi(\mathcal{O}_C)] = \gamma - (p_a(C) - \delta + \gamma - 1) - (1 - p_a(C)) = \delta.$$

(b) Let E be a locally free sheaf of rank r . By the previous formula we have

$$\Delta_{\underline{w}}(E) = \sum_{i=1}^{\gamma} r \lambda_i - \sum_{j=1}^{\delta} r = r \sum_{i=1}^{\gamma} \lambda_i - \delta r = r \delta - r \delta = 0.$$

(c) Assume that $r_i = r$ for all $i = 1, \dots, \gamma$. By Equation (2.2) we get $a_{p_j, i_1} = a_{p_j, i_2} := a_j$. As $s_{p_j} = r - a_j$, from (a) we have

$$\Delta_{\underline{w}}(E) = r \sum_{i=1}^{\gamma} \lambda_i - \sum_{j=1}^{\delta} (r - a_j) = r \delta - r \delta + \sum_{j=1}^{\delta} a_j = \sum_{j=1}^{\delta} a_j = \frac{1}{2} \sum_{j=1}^{\delta} t_j.$$

Then $\Delta_{\underline{w}}(E) = 0$ if and only if $t_j = 0$ for all j , that is E is locally free.

(d) Let L be a line bundle on C with $\text{deg}(L_i) = l_i, i = 1, \dots, \gamma$. Since $(E \otimes L)_i = E_i \otimes L_i$, we have $\text{deg}(E \otimes L)_i = d_i + r_i l_i$ and $\chi(E_i \otimes L_i) = \chi(E_i) + r_i l_i$. If we tensor the exact sequence (2.4) by L we obtain $\chi(E \otimes L) = \chi(E) - \sum_{i=1}^{\gamma} r_i l_i$, hence we get

$$\begin{aligned} \Delta_{\underline{w}}(E \otimes L) &= \chi(E \otimes L) - \text{rk}_{\underline{w}}(E \otimes L)\chi(\mathcal{O}_C) - \sum_{i=1}^{\gamma} (d_i + r_i l_i) \\ &= \chi(E) + \sum_{i=1}^{\gamma} r_i l_i - \text{rk}_{\underline{w}}(E)\chi(\mathcal{O}_C) - \sum_{i=1}^{\gamma} (d_i + r_i l_i) = \Delta_{\underline{w}}(E). \end{aligned}$$

(e) Let $B = \sqcup_{s=1}^c B_s$ be the disjoint union of $c \geq 1$ connected proper subcurves B_s . Since $\text{Supp}(E) = B$, then $E = \bigoplus_{s=1}^c E_{B_s}$ and E_{B_s} is a depth one sheaf too. The \underline{w} -degree is additive with respect to direct sum, so we have

$$\Delta_{\underline{w}}(E) = \text{deg}_{\underline{w}}(E) - \sum_{C_i \subseteq B} d_i = \sum_{s=1}^c \text{deg}_{\underline{w}}(E_{B_s}) - \sum_{s=1}^c \sum_{C_i \subseteq B_s} d_i = \sum_{s=1}^c \Delta_{\underline{w}}(E_{B_s}).$$

(f) Assume that B is a proper connected curve. By (a) we have:

$$\Delta_{\underline{w}}(E_B) = \sum_{C_i \subseteq B} r_i \lambda_i - \sum_{p_j \in B \setminus B^c} s_{p_j}.$$

If $B = \sqcup_{s=1}^c B_s$ is the disjoint union of $c \geq 1$ connected proper subcurves B_s . Then by (e)

$$\begin{aligned} \Delta_{\underline{w}}(E_B) &= \sum_{s=1}^c \Delta_{\underline{w}}(E_{B_s}) \\ &= \sum_{s=1}^c \left[\sum_{C_i \subseteq B_s} r_i \lambda_i - \sum_{p_j \in B_s \setminus B_s^c} s_{p_j} \right] = \sum_{C_i \subseteq B} r_i \lambda_i - \sum_{p_j \in B \setminus B^c} s_{p_j}. \end{aligned}$$

and a similar formula holds for B^c . So we have:

$$\begin{aligned} \Delta_{\underline{w}}(E_B) + \Delta_{\underline{w}}(E_{B^c}) &= \sum_{C_i \subseteq B} r_i \lambda_i - \sum_{p_j \in B \setminus B^c} s_{p_j} + \sum_{C_i \subseteq B^c} r_i \lambda_i - \sum_{p_j \in B^c \setminus B} s_{p_j} \\ &= \sum_{i=1}^{\gamma} r_i \lambda_i - \sum_{p_j \notin B \cap B^c} s_{p_j}. \end{aligned}$$

As $\Delta_{\underline{w}}(E) = \sum_{i=1}^{\gamma} r_i \lambda_i - \sum_{j=1}^{\delta} s_{p_j}$, we obtain:

$$\Delta_{\underline{w}}(E_B) + \Delta_{\underline{w}}(E_{B^c}) = \Delta_{\underline{w}}(E) + \sum_{p_j \in B \cap B^c} s_{p_j}.$$

(g) By (e) it is enough to prove the assertion for any connected subcurve B . Then $B = \bigcup_{k=1}^b C_{i_k}$. Then

$$\begin{aligned} \Delta_{\underline{w}}(E|_B) &= \text{deg}_{\underline{w}}(E|_B) - \sum_{k=1}^b d_{i_k} = \chi(E|_B) - \text{rk}_{\underline{w}}(E|_B)\chi(\mathcal{O}_C) - \sum_{k=1}^b d_{i_k} \\ &= \sum_{k=1}^b d_{i_k} + r(1 - p_a(B)) - r \sum_{k=1}^b w_{i_k} \chi(\mathcal{O}_C) - \sum_{k=1}^b d_{i_k} \\ &= r \text{deg}_{\underline{w}}(\mathcal{O}_B) = r \Delta_{\underline{w}}(\mathcal{O}_B). \end{aligned}$$

as claimed. \square

The following proposition gives a description of $\Delta_{\underline{w}}(E)$ as function of the residual ranks of E at its nodes and its multirank.

Proposition 2.5 *Let (C, \underline{w}) be a connected nodal polarized curve. Let E be a depth one sheaf on C , then we have*

$$\Delta_{\underline{w}}(E) = \sum_{i=1}^{\gamma} r_i \left(\lambda_i - \frac{\delta_i}{2} \right) + \frac{1}{2} \sum_{j=1}^{\delta} t_{p_j},$$

where t_j is the residual rank of E at the node p_j .

Proof Let p_j be a node, assume that $p_j \in C_{i_1} \cap C_{i_2}$. To avoid confusion we denote by r_{j,i_k} the rank of $E|_{C_{i_k}}$ so that

$$r_{j,i_1} + r_{j,i_2} = 2s_{p_j} + a_{p_j,i_1} + a_{p_j,i_2} = 2s_{p_j} + t_{p_j}.$$

We recall that we set $\lambda_i = 1 - g_i - w_i \chi(\mathcal{O}_C)$, so by Equation (2.3) we have

$$\begin{aligned} \Delta_{\underline{w}}(E) &= \sum_{i=1}^{\gamma} \lambda_i r_i - \sum_{j=1}^{\delta} \frac{r_{j,i_1} + r_{j,i_2} - t_{p_j}}{2} \\ &= \sum_{i=1}^{\gamma} \lambda_i r_i - \frac{1}{2} \sum_{j=1}^{\delta} (r_{j,i_1} + r_{j,i_2}) + \frac{1}{2} \sum_{j=1}^{\delta} t_{p_j}. \end{aligned} \quad (2.5)$$

We claim that the following relation holds:

$$\sum_{j=1}^{\delta} (r_{j,i_1} + r_{j,i_2}) = \sum_{i=1}^{\gamma} r_i \delta_i \quad (2.6)$$

We will proceed by induction on γ . If C has 2 components and δ nodes, we denote by r_1 and r_2 the ranks of the restrictions of E to the components. If p_j is a node, then $r_{j,i_1} + r_{j,i_2} = r_1 + r_2$ so

$$\sum_{j=1}^{\delta} (r_{j,i_1} + r_{j,i_2}) = \sum_{j=1}^{\delta} (r_1 + r_2) = \delta r_1 + \delta r_2$$

so Equation (2.6) holds when $\gamma = 2$.

Assume now, by induction hypothesis, that the same equation holds for nodal curves with at most $\gamma - 1$ components. Let C be a nodal curve with γ components. We claim that there exists a component of C whose complementary curve is connected.

This is true since the graph Γ_C is connected and every connected graph has a non-disconnecting vertex¹. Fix an ordering of the components of C in such a way that this non-disconnecting curve is C_γ . By assumption, its complementary curve C_γ^c is connected, with $\gamma' = \gamma - 1$ components with indices $i = 1, \dots, \gamma - 1$. Moreover, it has $\delta' = \delta - \delta_\gamma$ nodes and $r'_i = r_i$ for all $i = 1, \dots, \gamma - 1$. We can write

$$\sum_{j=1}^{\delta} (r_{j,i_1} + r_{j,i_2}) = \sum_{p_j \notin C_\gamma} (r_{j,i_1} + r_{j,i_2}) + \sum_{p_j \in C_\gamma} (r_{j,i_1} + r_{j,i_2}). \tag{2.7}$$

In the first summation on the right hand side of Equation (2.7), the sum is done over the nodes which are not on C_γ so they are exactly the nodes of C_γ^c as a nodal curve. Then, by induction hypothesis, we have

$$\sum_{p_j \notin C_\gamma} (r_{j,i_1} + r_{j,i_2}) = \sum_{j=1}^{\delta'} (r_{j,i_1} + r_{j,i_2}) = \sum_{i=1}^{\gamma'} (r'_i \delta'_i).$$

For all $i = 1, \dots, \gamma - 1$ we denote by ϵ_i the number of points of $C_i \cap C_\gamma$, i.e. the nodes common to C_i and C_γ . Then we have $\delta'_i = \delta_i - \epsilon_i$, as the nodes of $C_i \cap C_\gamma$ are not nodes of C_γ^c . If $\epsilon_i = 0$, C_i and C_γ are disjoint and $\delta'_i = \delta_i$. So we have:

$$\sum_{p_j \notin C_\gamma} (r_{j,i_1} + r_{j,i_2}) = \sum_{i=1}^{\gamma-1} r_i (\delta_i - \epsilon_i). \tag{2.8}$$

In the second summation on the right hand side of Equation (2.7), the sum is done over the δ_γ nodes which are on C_γ so we can write

$$\sum_{p_j \in C_\gamma} (r_{j,i_1} + r_{j,i_2}) = \sum_{C_i \mid C_i \cap C_\gamma \neq \emptyset} (r_\gamma + \epsilon_i r_i)$$

as $r_i = r_{j,i_k}$ for some j if and only if C_i is one of the components through p_j and this happens one times for each of the nodes which are on both C_i and C_γ , i.e. exactly ϵ_i times. Hence

$$\begin{aligned} \sum_{p_j \in C_\gamma} (r_{j,i_1} + r_{j,i_2}) &= [\dots] = \sum_{C_i \mid \epsilon_i > 0} r_\gamma + \sum_{C_i \mid \epsilon_i > 0} \epsilon_i r_i + \sum_{C_i \mid \epsilon_i = 0} \epsilon_i r_i \\ &= r_\gamma \delta_\gamma + \sum_{i=1}^{\gamma-1} \epsilon_i r_i. \end{aligned} \tag{2.9}$$

¹ Let $G = (\mathcal{V}, \mathcal{E})$ be a finite connected graph, with at least 3 vertices. Then one can fix $P \in \mathcal{V}$ and consider the distance $d_P(Q)$ of Q from P , i.e. the minimum number of edges that one needs to go through in order to make a path from P to Q . Let $R \in \mathcal{V}$ such that $d_P(R) = \max_{Q \in \mathcal{V}} d_P(Q)$. Then R is a non-disconnecting vertex of G . Indeed, if $Q \in \mathcal{V}$ different from R , the shortest path from P to Q cannot pass through R otherwise $d_P(R) < d_P(Q)$ and we get a contradiction.

Then, using Equations (2.8) and (2.9), we can rewrite Equation (2.7) as

$$\sum_{j=1}^{\delta} (r_{j,i_1} + r_{j,i_2}) = \sum_{i=1}^{\gamma-1} r_i (\delta_i - \epsilon_i) + r_{\gamma} \delta_{\gamma} + \sum_{i=1}^{\gamma-1} \epsilon_i r_i = \sum_{i=1}^{\gamma} r_i \delta_i$$

which concludes the proof of the claim. From Equations (2.5) and (2.6) one obtains easily the desired result. \square

2.2 Good polarizations and main properties

Now we will deal with a class of polarizations which will allow us to extend some properties that hold for locally free sheaves on smooth curves to depth one sheaves on polarized nodal curves (see Theorem 2.9). In order to do this we will use the function $\Delta_{\underline{w}}$ that we have studied in Sect. 2.1.

Definition 2.6 Let (C, \underline{w}) be a polarized nodal curve. We say that \underline{w} is a **good polarization** if $\Delta_{\underline{w}}(E) \geq 0$ for all depth one sheaves E on C and equality holds if and only if E is locally free.

By Lemma 2.4 (b), for any polarization \underline{w} we have $\Delta_{\underline{w}}(E) = 0$ for all locally free sheaves on C . Nevertheless, it can happen that $\Delta_{\underline{w}}(E) < 0$ for a depth one sheaf which is not locally free, as the next example shows.

Example 2.7 Let (C, \underline{w}) be a polarized nodal curve with two smooth components C_1 and C_2 of genus 2 and a single node. Then $\Delta_{\underline{w}}(\mathcal{O}_{C_1}) = -1 + 3w_1$. If we consider the polarization $\underline{w} = \left(\frac{1}{6}, \frac{5}{6}\right)$, we have $\Delta_{\underline{w}}(\mathcal{O}_{C_1}) = -1/2 < 0$. Moreover, by Proposition 1.4, this also implies that \mathcal{O}_C is a \underline{w} -unstable sheaf on C .

First of all we will see that on all stable nodal curves with $p_a(C) \geq 2$ there exists a good polarization (we will see in Remark 2.13 that is not true in general).

Proposition 2.8 Let C be a stable connected nodal curve with $p_a(C) \geq 2$ and let be $\underline{\eta}$ be the polarization induced by ω_C (this is often called canonical polarization). Then, $\underline{\eta}$ is a good polarization on C .

Proof First of all, since C is a stable curve, we have that ω_C is an ample line bundle so the definition of $\underline{\eta}$ makes sense. As recalled in Sect. 1 we have $\omega_C|_{C_i} = \omega_{C_i} \otimes (\Delta_i)$ so we have

$$\eta_i = \frac{g_i - 1 + \delta_i/2}{p_a(C) - 1}, \quad i = 1, \dots, \gamma.$$

In order to see that $\underline{\eta}$ is good we will compute $\Delta_{\underline{\eta}}(E)$ for a depth one sheaf E . For the canonical polarization we have

$$\lambda_i = 1 - g_i - \eta_i \chi(\mathcal{O}_C) = \delta_i/2,$$

so, by Proposition 2.5 we can conclude that

$$\Delta_{\underline{\eta}}(E) = \frac{1}{2} \sum_{j=1}^{\delta} t_{p_j}.$$

In particular, $\Delta_{\underline{\eta}}(E) \geq 0$ and equality holds if and only if $t_{p_j} = 0$ for all j . By Remark 2.3, this happens if and only if E is locally free. \square

The following theorem summarizes some important properties which hold when we deal with good polarizations. Recall that $\text{Pic}^0(C)$ is the variety parametrizing line bundles having degree 0 on each component (see Sect. 1).

Theorem 2.9 *Let C be a nodal curve and \underline{w} a good polarization on it. Let E be a depth one sheaf on C . Then we have the following properties:*

- (a) *Assume that E is locally free and, for $i = 1, \dots, \gamma$, E_i is stable with $\deg(E_i) = 0$. Then E is \underline{w} -stable.*
- (b) *If E is globally generated, then $\deg_{\underline{w}}(E) \geq 0$.*
- (c) *If E is \underline{w} -semistable and $\deg_{\underline{w}}(E) > 0$, then $h^0(E^*) = 0$.*

In particular, if $E = \mathcal{O}_C$ or more generally $E \in \text{Pic}^0(C)$ then, E is \underline{w} -stable.

Proof (a) Let E be a locally free sheaf such that E_i is stable and $\deg(E_i) = 0$ for all $i = 1, \dots, \gamma$. Then, by Lemma 2.4 (b) we have $\deg_{\underline{w}}(E) = 0$. In order to prove that E is \underline{w} -stable it is enough to show that for any proper subsheaf F of E we have $\deg_{\underline{w}}(F) < 0$.

Let F be a proper subsheaf of E and let's consider the quotient $Q = E/F$. If $\text{rk}_{\underline{w}}(Q) = 0$, then Q is a torsion sheaf with finite support. Then $\deg_{\underline{w}}(Q) = \sum_{P \in \text{Supp}(Q)} l(Q_P) > 0$ and then $\deg_{\underline{w}}(F) < 0$ as claimed.

Assume now that $\text{rk}_{\underline{w}}(Q) > 0$. Since F is a proper subsheaf of E we also have $\text{rk}_{\underline{w}}(Q) < \text{rk}_{\underline{w}}(E)$. We define $Q' = Q/\text{Tors}(Q)$ which is a depth one sheaf with $\text{rk}_{\underline{w}}(Q') = \text{rk}_{\underline{w}}(Q)$ and $\deg_{\underline{w}}(Q) \geq \deg_{\underline{w}}(Q')$. Moreover, as Q' is a quotient of Q we have that Q' is a proper quotient of E . So for all $i = 1, \dots, \gamma$, we have a surjective map $q_i : E_i \rightarrow Q'_i$. If Q'_i is not zero, then either q_i is an isomorphism (this cannot occur for all i) or Q'_i is a proper quotient, in this case $\deg(Q'_i) > \deg(E_i) = 0$ by the stability assumption on E_i . Hence $\sum_i \deg(Q'_i) > 0$. Then, as \underline{w} is a good polarization, we have

$$\Delta_{\underline{w}}(Q') = \deg_{\underline{w}}(Q') - \sum_i \deg(Q'_i) \geq 0$$

which implies $\deg_{\underline{w}}(Q') > 0$. Then $\deg_{\underline{w}}(Q) > 0$ and we can conclude as in the previous case.

From (a), if E is a line bundle with $\deg(E_i) = 0$ for all $i = 1, \dots, \gamma$, we have that E is \underline{w} -stable. One can also prove this fact directly using Proposition 1.4 by checking that $\deg_{\underline{w}}(L|_B) > 0$ for any proper subcurve B . Indeed, we have

$$\deg_{\underline{w}}(L|_B) = \Delta_{\underline{w}}(L|_B) > 0,$$

as \underline{w} is a good polarization and $L|_B$ is not locally free on C .

- (b) Assume that E is a depth one sheaf on C which is generated by $k \geq 1$ global sections. Then we have a surjective map $V \otimes \mathcal{O}_C \rightarrow E$, where $V \subseteq H^0(E)$ is a vector space of dimension k . Since by (a), \mathcal{O}_C is \underline{w} -stable, then $V \otimes \mathcal{O}_C$ is \underline{w} -semistable. So we have $\frac{\deg_{\underline{w}}(E)}{\text{rk}_{\underline{w}}(E)} \geq 0$ and then $\deg_{\underline{w}}(E) \geq 0$.
- (c) Assume that $H^0(E^*) = \text{Hom}(E, \mathcal{O}_C) \neq 0$. Then, there exists a non zero homomorphism $\varphi: E \rightarrow \mathcal{O}_C$. We will show that $\deg_{\underline{w}}(E) < 0$. If φ is surjective or injective, we conclude by \underline{w} -semistability of E and by \underline{w} -stability of \mathcal{O}_C (which holds by (a), since \underline{w} is good) respectively. We can assume then, that $\text{Im}(\varphi)$ is a proper subsheaf of \mathcal{O}_C and a proper quotient of E . In this case we have

$$\deg_{\underline{w}}(E)/\text{rk}_{\underline{w}}(E) \leq \deg_{\underline{w}}(\text{Im}(\varphi))/\text{rk}_{\underline{w}}(\text{Im}(\varphi)) < 0$$

where we used the \underline{w} -semistability of E and the \underline{w} -stability of \mathcal{O}_C respectively. \square

Remark 2.10 In point (a) of Theorem 2.9 if E_i is only semistable then, with the same arguments, one obtain that E is \underline{w} -semistable.

Another interesting consequence of the previous theorem is the following corollary.

Corollary 2.11 *Let C be a nodal curve and \underline{w} a good polarization. If $\underline{w} = \underline{w}_L$ for some ample line bundle L , \underline{w} -(semi)stability is preserved by tensoring with L . In particular, L is \underline{w} -stable.*

Proof Let L be a line bundle which induces the polarization \underline{w}_L , with $L_i \in \text{Pic}^{d_i}(C_i)$. Since $w_i = d_i/d$ then we have

$$w_i d_j = w_j d_i.$$

This implies, by [24], that \underline{w}_L -stability is preserved by tensoring with L . In particular, since \mathcal{O}_C is \underline{w}_L -stable by Theorem 2.9, then L is \underline{w}_L -stable too. \square

2.3 Polarizations and \underline{w} -stability of \mathcal{O}_C

In this subsection we investigate polarized nodal curves (C, \underline{w}) with \underline{w} -stable \mathcal{O}_C .

Lemma 2.12 *Let (C, \underline{w}) be a polarized nodal curve. Then \mathcal{O}_C is \underline{w} -stable if and only if*

$$0 < \Delta_{\underline{w}}(\mathcal{O}_B) < \delta_B \tag{2.10}$$

for any proper subcurve B of C . If equality holds for some subcurve B then \mathcal{O}_C is \underline{w} -semistable. Moreover we can specialize the result in the following cases:

- If $p_a(C) = 0$, then \mathcal{O}_C is always \underline{w} -stable;
- If $p_a(C) = 1$, then \mathcal{O}_C is always \underline{w} -semistable and it is \underline{w} -stable if and only if C is a cycle of rational curves;

- If $p_a(C) \geq 2$, then \mathcal{O}_C is \underline{w} -stable if and only the conditions

$$(\star)_B : \frac{p_a(B) - 1}{p_a(C) - 1} < \text{rk}_{\underline{w}}(\mathcal{O}_B) < \frac{p_a(B) - 1 + \delta_B}{p_a(C) - 1} \tag{2.11}$$

hold for all proper subcurves B of C .

Actually, it is enough to check the the Inequalities (2.10) and (2.11) only for connected subcurves.

Proof By Proposition 1.4 we have that \mathcal{O}_C is \underline{w} -stable if and only if $\text{deg}_{\underline{w}}(\mathcal{O}_B) > 0$ for any proper subcurve B of C . Moreover, since $\text{deg}_{\underline{w}}(\mathcal{O}_B) = \Delta_{\underline{w}}(\mathcal{O}_B)$, by Lemma 2.4(e), it is enough to check the condition $\text{deg}_{\underline{w}}(\mathcal{O}_B) > 0$ only for connected subcurves.

Let B be a proper subcurve of C and B^c its complementary curve. Then \mathcal{O}_B and \mathcal{O}_{B^c} are two depth one sheaves on C . We have

$$\text{deg}_{\underline{w}}(\mathcal{O}_B) = \chi(\mathcal{O}_B) - \text{rk}_{\underline{w}}(\mathcal{O}_B)\chi(\mathcal{O}_C) = 1 - p_a(B) - \text{rk}_{\underline{w}}(\mathcal{O}_B)\chi(\mathcal{O}_C).$$

From Equation (1.2) we have $\chi(\mathcal{O}_{B^c}) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_B) + \delta_B$, so

$$\begin{aligned} \text{deg}_{\underline{w}}(\mathcal{O}_{B^c}) &= \chi(\mathcal{O}_{B^c}) - \text{rk}_{\underline{w}}(\mathcal{O}_{B^c})\chi(\mathcal{O}_C) \\ &= \chi(\mathcal{O}_C) - \chi(\mathcal{O}_B) + \delta_B - (1 - \text{rk}_{\underline{w}}(\mathcal{O}_B))\chi(\mathcal{O}_C) \\ &= \text{rk}_{\underline{w}}(\mathcal{O}_B)\chi(\mathcal{O}_C) + p_a(B) - 1 + \delta_B. \end{aligned}$$

Hence \mathcal{O}_C is \underline{w} -stable if and only if both the above values are strictly positive, we obtain Inequality (2.10). If $p_a(C) \geq 2$, solving the inequalities we get condition $(\star)_B$.

Assume now $p_a(C) = 0$. Then C is a curve of compact type whose components are rational. Then, if B is a proper connected subcurve of C , we have that B is also of compact type. In particular $p_a(B) = 0$ too. By Inequality (2.10) we get $1 - \delta_B < \text{rk}_{\underline{w}}(\mathcal{O}_B) < 1$, so \mathcal{O}_C is \underline{w} -stable.

Assume now $p_a(C) = 1$. Then Inequality (2.10) is equivalent to $1 - \delta_B < p_a(B) < 1$. Since $p_a(B) \leq 1$ and $p_a(B) \geq 1 - \delta_B$ we have that \mathcal{O}_C is always \underline{w} -semistable. Now we investigate the \underline{w} -stability of \mathcal{O}_C . As $p_a(C) = 1$, we have either C is of compact type whose components consist of an elliptic curve C_1 and $\gamma - 1$ rational curves or the dual graph has a single cycle and all components are rationals. In the first case, $p_a(C_1) = 1$ so \mathcal{O}_C is never \underline{w} -stable. In the second case, if we can find a proper connected subcurve B of C which contains a cycle then $p_a(B) = 1$ and \mathcal{O}_C is never \underline{w} -stable. This happens exactly when C is not a cycle. If C is a cycle and B is a proper connected subcurve, then $\delta_B = 2$ and $p_a(B) = 0$ so \mathcal{O}_C is \underline{w} -stable. \square

Remark 2.13 Let C be a nodal curve with $p_a(C) = 1$ which is not a cycle. Then good polarizations do not exist on C .

Remark 2.14 Assume that (C, \underline{w}) is a polarized nodal curve of compact type. We can translate the conditions of \underline{w} -stability for \mathcal{O}_C , given by Teixidor i Bigas in [25], using our notation as follows: \mathcal{O}_C is \underline{w} -stable if and only if $0 < \Delta_{\underline{w}}(\mathcal{O}_{A_i}) < 1$ for a suitable family of connected subcurves $A_i \subset C$.

Corollary 2.15 *Let \underline{w} be a good polarization on a nodal curve C with $p_a(C) \geq 2$. Then \underline{w} satisfies $(\star)_B$ for all B subcurve of C . In particular, we have*

$$\frac{g_i - 1}{p_a(C) - 1} < w_i < \frac{g_i - 1 + \delta_i}{p_a(C) - 1}.$$

An interesting question is then the following:

Question 2.16 *Are all polarizations for which \mathcal{O}_C is \underline{w} -stable also good?*

We will give a complete answer for curves of compact type in Sect. 3.

2.4 Polarizations and balanced line bundles

In this subsection we deal with polarized curves (C, \underline{w}_L) where \underline{w}_L is induced by a line bundle L . We highlight the relation between the \underline{w}_L -stability of \mathcal{O}_C and a particular class of line bundles: balanced line bundles (for details one can see [15,16]).

Definition 2.17 Let C be a quasistable curve of arithmetic genus $p_a(C) \geq 2$. A line bundle L on C is said to be **balanced** if the following properties hold:

- (1) for every exceptional component E of C we have $\deg_E(L) = 1$;
- (2) for any proper subcurve B we have

$$\left| \deg_B(L) - \frac{\deg(L)}{2p_a(C) - 2} \deg_B(\omega_C) \right| \leq \frac{1}{2} \delta_B. \quad (2.12)$$

L is said to be **strictly balanced** if the inequality is strict for every subcurve B such that $B \cap B^c$ is not contained in the exceptional locus of C .

Proposition 2.18 *Let C be a quasistable nodal curve with $p_a(C) \geq 2$. Let $L \in \text{Pic}^d(C)$ be an ample line bundle and let $\underline{w} = \underline{w}_L$ be the polarization induced by L .*

- (a) *If $d \geq p_a(C) - 1$ and L is balanced, then \mathcal{O}_C is \underline{w} -semistable and it is \underline{w} -stable when $d > p_a(C) - 1$;*
- (b) *if $d \leq p_a(C) - 1$ and \mathcal{O}_C is \underline{w} -stable then C is stable and L is strictly balanced.*

Proof Let $L \in \text{Pic}^d(C)$ be an ample line bundle. Then $d_i = \deg(L_i) > 0$ for all i and $d = \sum_{i=1}^{\gamma} d_i$. As \underline{w} is induced by L , we have $w_i = \frac{d_i}{d}$, for all $i = 1, \dots, \gamma$. Let B be a subcurve of C . Then $B = \bigcup_{k=1}^b C_{i_k}$. Since $L|_B$ is a line bundle on B , we have:

$$\deg_B(L) = \sum_{k=1}^b d_{i_k} = \sum_{k=1}^b w_{i_k} d = \text{drk}_{\underline{w}}(\mathcal{O}_B),$$

moreover we recall that

$$\deg_B(\omega_C) = 2p_a(B) - 2 + \delta_B.$$

We have:

$$\begin{aligned} & \left| \deg_B(L) - \frac{d}{2p_a(C) - 2} \deg_B(\omega_C) \right| \tag{2.13} \\ &= \left| \operatorname{rk}_{\underline{w}}(\mathcal{O}_B) - \frac{d}{p_a(C) - 1} (p_a(B) - 1 + \delta_B/2) \right| \\ &= \frac{d}{p_a(C) - 1} \left| (p_a(C) - 1) \operatorname{rk}_{\underline{w}}(\mathcal{O}_B) - (p_a(B) - 1 + \delta_B/2) \right|. \end{aligned}$$

Note that condition $(\star)_B$ in Lemma 2.12 can be also written as

$$p_a(B) - 1 < (p_a(C) - 1) \operatorname{rk}_{\underline{w}}(\mathcal{O}_B) < p_a(B) - 1 + \delta_B,$$

which is equivalent to

$$\left| (p_a(C) - 1) \operatorname{rk}_{\underline{w}}(\mathcal{O}_B) - (p_a(B) - 1 + \delta_B/2) \right| < \delta_B/2.$$

(a) Let $d \geq p_a(C) - 1$ and assume that L is balanced. Then Equations (2.12) and (2.13) imply

$$\left| (p_a(C) - 1) \operatorname{rk}_{\underline{w}}(\mathcal{O}_B) - (p_a(B) - 1 + \delta_B/2) \right| \leq \frac{\delta_B}{2} \frac{p_a(C) - 1}{d}.$$

If $d > p_a(C) - 1$, we get

$$\left| (p_a(C) - 1) \operatorname{rk}_{\underline{w}}(\mathcal{O}_B) - p_a(B) + 1 - \delta_B/2 \right| < \delta_B/2,$$

which is equivalent to $(\star)_B$. This implies that \mathcal{O}_C is \underline{w} -stable. If $d = p_a(C) - 1$, we get

$$\left| (p_a(C) - 1) \operatorname{rk}_{\underline{w}}(\mathcal{O}_B) - p_a(B) + 1 - \delta_B/2 \right| \leq \delta_B/2,$$

so we can conclude that \mathcal{O}_C is \underline{w} -semistable.

(b) Let $d \leq p_a(C) - 1$ and assume that \mathcal{O}_C is \underline{w} -stable. Then \underline{w} satisfies $(\star)_B$ for all subcurve B . Let R be a rational component of C , since $(\star)_R$ holds, we have:

$$-1 < (p_a(C) - 1)w_R < \delta_R - 1.$$

We recall that $w_R = \frac{d_R}{d}$ and $d_R \geq 1$ since L is ample. So we have:

$$1 \leq d_R < \frac{d}{p_a(C) - 1} (\delta_R - 1),$$

as $d \leq p_a(C) - 1$ we obtain $1 \leq d_R < \delta_R - 1$. This implies $\delta_R \geq 3$, so C is a stable curve.

Now we prove that L is strictly balanced. Since $d \leq p_a(C) - 1$ we have

$$\left| \deg_B(L) - \frac{d}{2p_a(C) - 2} \deg_B(\omega_C) \right| < \frac{d}{(p_a(C) - 1) 2} \leq \frac{\delta_B}{2}$$

by Inequality (2.13). This proves that L is strictly balanced. \square

Corollary 2.19 *Let C be a stable nodal curve with $p_a(C) \geq 2$. Let L be an ample line bundle of degree $p_a(C) - 1$ and \underline{w}_L be the polarization induced by L on C . Then L is strictly balanced if and only if \mathcal{O}_C is \underline{w}_L -stable.*

Proof Since C is stable, the exceptional locus of C is empty. Moreover, as we assumed $\deg(L) = p_a(C) - 1$, Condition (2.12) is equivalent to $(\star)_B$. This implies the claim. \square

3 Good polarizations and \underline{w} -Stability of \mathcal{O}_C

Let (C, \underline{w}) be a polarized nodal curve. In this section we will obtain sufficient conditions for a polarization \underline{w} to be good (see Theorem 3.9). Recall that, by Corollary 2.15, any good polarization satisfies properties $(\star)_B$ of Lemma 2.12, or equivalently, is such that \mathcal{O}_C is \underline{w} -stable. We will show that for curves of compact type, \underline{w} -stability of \mathcal{O}_C is also sufficient in order to have \underline{w} good (see Theorem 3.10).

With this aim, we will give a description of $\Delta_{\underline{w}}(E)$ as a function depending only on the residual ranks and on the contribution of the non-free part of the stalks of E at nodes of C . We will get this description by considering paths on the dual graph of C , as follows.

Assume that C has γ irreducible components and δ nodes. Let C_1, \dots, C_γ denote the smooth components of C and p_1, \dots, p_δ denote the nodes of C . Let $\Gamma_C = (\mathcal{V}, \mathcal{E})$ be the dual graph of C . It is a finite graph with $\gamma = \#\mathcal{V}$ vertices and $\delta = \#\mathcal{E}$ edges. Since C is connected the same holds for Γ_C .

Notations 3.1

Given a path γ in Γ_C , we will denote by $L(\gamma) \in \mathbb{N}$ the **length** of γ i.e. the number of edges which are part of γ . A path has length 0 if and only if it is the trivial path. A path joining C_i with C_j is said **minimal** if it has minimal length among all the paths joining C_i and C_j . As the graph Γ_C is connected and finite, minimal paths exist for each pair of vertices. Two edges of Γ_C are said **equivalent** if and only if the corresponding nodes lie on the same two components, i.e. if they connect the same vertices of Γ_C .

A **marking** \mathcal{M} is a subset of \mathcal{E} which is a transversal for the above equivalence relation, i.e. every edge of Γ_C is equivalent to exactly one edge in \mathcal{M} . The subgraph $\Gamma_C^{\mathcal{M}} = (\mathcal{V}, \mathcal{M})$ has the same vertices of Γ_C , is connected and it is also simple (i.e. for each pair of vertices there is at most one edge).

For our construction we will need to fix arbitrarily a component of C . For simplicity, we will use C_γ . We define \mathcal{P} as any set satisfying the following properties:

- (1) the elements of \mathcal{P} are minimal paths in $\Gamma_C^{\mathcal{M}}$ connecting a vertex C_i to C_γ ;

- (2) for each C_i there exists exactly one path in \mathcal{P} starting from C_i , which we will be denoted by γ_i ;
- (3) if $\gamma_i \in \mathcal{P}$ and C_j is a vertex on γ_i , then γ_j is a restriction of γ_i .

We will call \mathcal{P} a **set of minimal paths** of Γ_C . In order to simplify the notations, if $C_j \in \mathcal{V}$, $p_k \in \mathcal{E}$ we will write $p_k \subseteq \gamma_i$ if and only if p_k is an edge on γ_i and $C_j \in \gamma_i$ if and only if C_j is a vertex on γ_i . We set \mathcal{M}' the subset of \mathcal{M} which consists of all the edges on some path in \mathcal{P} .

If $\gamma_i \in \mathcal{P}$ and $p_j \subseteq \gamma_i$ is a node in $C_{k_1} \cap C_{k_2}$, we say that C_{k_1} **precedes** C_{k_2} **with respect to** γ_i if and only if, compared to C_{k_2} , C_{k_1} is closer to C_i along the path γ_i .

Indeed, this does not depend on the choice of $\gamma_i \in \mathcal{P}$ passing through p_j as the next lemma shows.

Lemma 3.2 Assume that γ_{i_1} and γ_{i_2} are two minimal paths ending in C_γ , which pass through $p_j \in \mathcal{M}$ with $p_j \in C_{k_1} \cap C_{k_2}$. Then the curve C_{k_1} precedes C_{k_2} with respect to γ_{i_1} if and only if the same happens with respect to γ_{i_2} .

Proof Assume, by contradiction, that C_{k_1} precedes C_{k_2} with respect to γ_{i_1} and follows C_{k_2} with respect to γ_{i_2} . For all $l = 1, 2$, we denote by γ'_{i_l} the path obtained by γ_{i_l} by removing all the edges before p_j and by γ''_{i_l} the path obtained by γ'_{i_l} were we have removed also p_j . Hence, γ'_{i_1} and γ''_{i_2} are both minimal paths (since minimality is preserved by restriction) which start from C_{k_1} and end in C_γ . Similarly, γ'_{i_2} and γ''_{i_1} are both minimal paths connecting C_{k_2} and C_γ . As two minimal path joining the same vertices must have the same length we have

$$\begin{cases} L(\gamma'_{i_1}) = L(\gamma''_{i_2}) = L(\gamma'_{i_2}) - 1 \\ L(\gamma'_{i_2}) = L(\gamma''_{i_1}) = L(\gamma'_{i_1}) - 1 \end{cases}$$

which is clearly impossible. □

Definition 3.3 Let $p_j \in \mathcal{E}$ corresponding to a node in $C_{k_1} \cap C_{k_2}$. If p_j is equivalent to an edge which is on a path $\gamma_i \in \mathcal{P}$ we say that C_{k_1} **precedes** C_{k_2} if and only if C_{k_1} precedes C_{k_2} with respect to γ_i . If p_j is not equivalent to any edge on a path $\gamma_i \in \mathcal{P}$, we choose arbitrarily one of the two possible cases (C_{k_1} precedes C_{k_2} or C_{k_2} precedes C_{k_1}) making the same choice for equivalent edges.

Lemma 3.2 ensures that the above definition is well posed. This gives the structure of oriented graph to Γ_C and to its subgraph $\Gamma_C^{\mathcal{M}}$.

Notations 3.4

Let E be a depth one sheaf on C . Let p_j be a node with $p_j \in C_{k_1} \cap C_{k_2}$. Denote by q_{j,k_1} and q_{j,k_2} the points of C_{k_1} and C_{k_2} respectively on the normalization of C which are glued together in order to obtain p_j . We recall that we have integers s_j, a_{j,k_1} and a_{j,k_2} such that

$$E_{p_j} = \mathcal{O}_{p_j}^{s_j} \oplus \mathcal{O}_{q_{j,k_1}}^{a_{j,k_1}} \oplus \mathcal{O}_{q_{j,k_2}}^{a_{j,k_2}},$$

and satisfying $r_{k_l} = s_j + a_{j,k_l}$ for $l = 1, 2$. We set

$$a_j := a_{j,k_1} \text{ and } b_j := a_{j,k_2} \iff C_{k_1} \text{ precedes } C_{k_2} \quad (3.1)$$

and the opposite in the other case. In particular, we have that $a_j + b_j = t_{p_j}$.

Lemma 3.5 *Let E be any depth one sheaf on C . Then*

- (a) if p_l and p_j are equivalent edges, we have $b_l - a_l = b_j - a_j$;
 (b) if $\gamma_i \in \mathcal{P}$ then we have $\sum_{p_j \subseteq \gamma_i} (b_j - a_j) = r_\gamma - r_i$.

Proof (a) Let E be a depth one sheaf. Let p_j and p_l be two equivalent edges. Then $p_j, p_l \in C_{k_1} \cap C_{k_2}$. Without loss of generality we can assume that C_{k_1} precedes C_{k_2} . Then

$$r_{k_1} = s_j + a_j = s_l + a_l \quad r_{k_2} = s_j + b_j = s_l + b_l,$$

so $a_l - a_j = s_j - s_l = b_l - b_j$ and then $b_l - a_l = b_j - a_j$ as claimed.

- (b) Let $\gamma_i \in \mathcal{P}$. We will prove the formula by induction on the length of γ_i . If $L(\gamma_i) = 1$ then γ_i is a single edge (say p_j) joining the vertices C_i and C_γ . Then $r_i = s_j + a_j, r_\gamma = s_j + b_j$ so $r_\gamma - r_i = b_j - a_j$ as claimed. Now assume that the formula is true for any minimal path of length at most L and consider a minimal path γ_i of length $L + 1$. Let p_l be the first edge, and denote by C_k the second vertex on the path (the first is C_i). If we remove p_l from the path we get, by the definition of \mathcal{P} the minimal path γ_k joining C_k to C_γ which has length L . So, by induction, we have

$$r_\gamma - r_k = \sum_{p_j \subseteq \gamma_k} (b_j - a_j).$$

On the other hand we have $r_i = s_l + a_l, r_k = s_l + b_l$ so $r_k - r_i = b_l - a_l$ and we have

$$r_\gamma - r_i = (r_k - r_i) + (r_\gamma - r_k) = (b_l - a_l) + \sum_{p_j \subseteq \gamma_k} (b_j - a_j) = \sum_{p_j \subseteq \gamma_i} (b_j - a_j)$$

as claimed. □

By Lemma 3.5(a) it follows that the choice of the marking \mathcal{M} does not influence the relation in Lemma 3.5(b).

Definition 3.6 Assume that a marking \mathcal{M} and a set \mathcal{P} of minimal paths on Γ_C (as in Notation 3.1) have been chosen. Then, for any $p_j \in \mathcal{M}$, we define A_j to be the subcurve of C with the following property: C_i is a component of A_j if and only if $p_j \subseteq \gamma_i$.

Note that A_j could be empty for same j : this occurs exactly when $p_j \notin \mathcal{M}'$.

Before stating the main result of this section, we will need the following technical result:

Lemma 3.7 *Let $A_j \subseteq C$ be as in Definition 3.6 and assume that A_j is not empty. Then*

- (a) A_j is a proper connected subcurve of C ;
- (b) A_j^c is connected;
- (c) $\sum_{C_i \subseteq A_j} \left(\lambda_i - \frac{\delta_i}{2} \right) = 1 - p_a(A_j) + (p_a(C) - 1) \text{rk}_{\underline{w}}(\mathcal{O}_{A_j}) - \frac{1}{2} \delta_{A_j} = \Delta_{\underline{w}}(\mathcal{O}_{A_j}) - \frac{1}{2} \delta_{A_j}$;
- (d) if C is of compact type, then $\delta_{A_j} = 1$.

Proof (a) Consider a component C_i of A_j . Then the path γ_i passes through p_j . Assume that $p_j \in C_{k_1} \cap C_{k_2}$ and that C_{k_1} precedes C_{k_2} . Let C_l be a vertex on γ_i which is between C_i and C_{k_1} (included). Then γ_l is the restriction of γ_i and p_j is an edge in γ_l . In particular, C_l is a component of A_j . This shows that C_i is connected to C_{k_1} using only curves in A_j so A_j is connected. Properness follows as C_{k_2} cannot be a component of A_j .

- (b) It is enough to show that if C_i is a component not in A_j then there is a path in Γ_C from C_i to C_γ which only passes through vertices which correspond to components not in A_j . The path γ_i connects C_i with C_γ . Assume, by contradiction, that one of the vertex on the path γ_i , say C_k , is a component of A_j . Then, the restriction of γ_i from C_k to C_γ is γ_k . Since C_k is a component of A_j we have that $p_j \subset \gamma_k$, so the same is true for γ_i . But this is impossible as we assumed that $C_i \notin A_j$.
- (c) We denote by $C(A_j)$ and $N(A_j)$ the number of components and of nodes respectively of the curve A_j . We recall that $\delta_{A_j} = A_j \cdot A_j^c$ is the number of nodes of C lying on A_j which are not nodes of A_j . Then we have

$$\begin{aligned} \sum_{C_i \subseteq A_j} \left(\lambda_i - \frac{\delta_i}{2} \right) &= \sum_{C_i \subseteq A_j} [1 - g_i + w_i(p_a(C) - 1)] - \frac{1}{2} \sum_{C_i \subseteq A_j} \delta_i \\ &= C(A_j) - \sum_{C_i \subseteq A_j} g_i \\ &\quad + \text{rk}_{\underline{w}}(\mathcal{O}_{A_j})(p_a(C) - 1) - N(A_j) - \frac{1}{2} \delta_{A_j} \\ &= 1 - p_a(A_j) + (p_a(C) - 1) \text{rk}_{\underline{w}}(\mathcal{O}_{A_j}) - \frac{1}{2} \delta_{A_j} \quad (3.2) \end{aligned}$$

as A_j is connected and $p_a(A_j) = \sum_{C_i \subseteq A_j} g_i + N(A_j) - C(A_j) + 1$. Finally, we recall that $1 - p_a(A_j) + (p_a(C) - 1) \text{rk}_{\underline{w}}(\mathcal{O}_{A_j}) = \Delta_{\underline{w}}(\mathcal{O}_{A_j})$.

- (d) Since C is of compact type, by (a) and (b) it follows that A_j and A_j^c are both curves of compact type too. From Equation (1.3) we have:

$$\sum_{i=1}^{\gamma} g_i = \sum_{C_i \subseteq A_j} g_i + \sum_{C_i \subseteq A_j^c} g_i + \delta_{A_j} - 1,$$

which implies $\delta_{A_j} = 1$. \square

Remark 3.8 We point out that, if C is of compact type, the family of connected curves $\{A_j\}$, defined in Definition 3.6, can be used to obtain the conditions of \underline{w} -stability in [25] (see also Remark 2.14).

We are now able to state our first result of this section:

Theorem 3.9 *Let (C, \underline{w}) be a polarized nodal curve. Fix a marking \mathcal{M} on the dual graph Γ_C and a set of minimal path \mathcal{P} as in Notations 3.1. Then for any depth one sheaf E we have:*

$$\Delta_{\underline{w}}(E) = \sum_{p_j \in \mathcal{M}'} \left[a_j \left(\frac{1}{2}(1 - \delta_{A_j}) + \Delta_{\underline{w}}(\mathcal{O}_{A_j}) \right) + b_j \left(\frac{1}{2}(1 + \delta_{A_j}) - \Delta_{\underline{w}}(\mathcal{O}_{A_j}) \right) \right] + \frac{1}{2} \sum_{p_j \notin \mathcal{M}'} (a_j + b_j).$$

In particular, if the conditions

$$(\star\star)_{A_j} : \quad \frac{1}{2}(\delta_{A_j} - 1) < \Delta_{\underline{w}}(\mathcal{O}_{A_j}) < \frac{1}{2}(\delta_{A_j} + 1) \quad (3.3)$$

hold for all the non-empty subcurves A_j then \underline{w} is a good polarization.

Proof We start from the expression of $\Delta_{\underline{w}}(E)$ given by Proposition 2.5. Then, using Lemma 3.5(b) we have

$$\begin{aligned} \Delta_{\underline{w}}(E) &= \sum_{i=1}^{\gamma} r_i \left(\lambda_i - \frac{\delta_i}{2} \right) + \frac{1}{2} \sum_{j=1}^{\delta} t_{p_j} \\ &= \sum_{i=1}^{\gamma} \left(r_{\gamma} + \sum_{p_j \subseteq \gamma_i} (a_j - b_j) \right) \left(\lambda_i - \frac{\delta_i}{2} \right) + \frac{1}{2} \sum_{j=1}^{\delta} t_{p_j} \\ &= r_{\gamma} \sum_{i=1}^{\gamma} \left(\lambda_i - \frac{\delta_i}{2} \right) + \sum_{i=1}^{\gamma} \sum_{p_j \subseteq \gamma_i} (a_j - b_j) \left(\lambda_i - \frac{\delta_i}{2} \right) + \frac{1}{2} \sum_{j=1}^{\delta} t_{p_j}. \end{aligned}$$

By Lemma 2.4(a) we have that the coefficient of r_{γ} in the last equality is 0 so $\Delta_{\underline{w}}(E)$ is equal to

$$\begin{aligned} &\sum_{i=1}^{\gamma} \sum_{p_j \subseteq \gamma_i} (a_j - b_j) \left(\lambda_i - \frac{\delta_i}{2} \right) + \frac{1}{2} \sum_{j=1}^{\delta} (a_j + b_j) \\ &= \sum_{j=1}^{\delta} (a_j - b_j) \sum_{\gamma_i \supseteq p_j} \left(\lambda_i - \frac{\delta_i}{2} \right) + \frac{1}{2} \sum_{j=1}^{\delta} (a_j + b_j) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{p_j \in \mathcal{M}'} \left[a_j \left(\frac{1}{2} + \sum_{\gamma_i \supseteq p_j} \left(\lambda_i - \frac{\delta_i}{2} \right) \right) + b_j \left(\frac{1}{2} - \sum_{\gamma_i \supseteq p_j} \left(\lambda_i - \frac{\delta_i}{2} \right) \right) \right] \\
 &\quad + \frac{1}{2} \sum_{p_j \notin \mathcal{M}'} (a_j + b_j)
 \end{aligned}$$

since, if $p_j \notin \mathcal{M}'$ the sum over the path passing through p_j is trivial. If $p_j \in \mathcal{M}'$, the condition $\gamma_i \supseteq p_j$ is equivalent to $C_i \in A_j$ so, by Lemma 3.7(c) we have

$$\begin{aligned}
 \Delta_{\underline{w}}(E) &= \sum_{p_j \in \mathcal{M}'} \left[a_j \left(\frac{1}{2} + \Delta_{\underline{w}}(\mathcal{O}_{A_j}) - \frac{\delta_{A_j}}{2} \right) + b_j \left(\frac{1}{2} - \Delta_{\underline{w}}(\mathcal{O}_{A_j}) + \frac{\delta_{A_j}}{2} \right) \right] \\
 &\quad + \frac{1}{2} \sum_{p_j \notin \mathcal{M}'} (a_j + b_j)
 \end{aligned}$$

which is equal to the expression in the statement of the Theorem.

Finally, if Conditions (3.3) hold, we have that all the coefficients of a_j and b_j in the last expression of $\Delta_{\underline{w}}(E)$ are strictly positive. This proves that $\Delta_{\underline{w}}(E) \geq 0$. Moreover, if at least one among a_j and b_j for $j = 1, \dots, \delta$ is not zero we have $\Delta_{\underline{w}}(E) > 0$. Hence we have that $\Delta_{\underline{w}}(E) > 0$ if and only if E is locally free, i.e. \underline{w} is a good polarization. \square

With the expression given in Theorem 3.9 we are able to give a (positive) answer to Question 2.16 for curves of compact type.

Theorem 3.10 *Let (C, \underline{w}) be a polarized nodal curve of compact type. Then the collection $\{A_j \mid j \in \mathcal{E}\}$ depends only on the choice² of C_γ , for all $p_j \in \mathcal{E}$ the curve A_j is non-empty and we have*

$$\Delta_{\underline{w}}(E) = \sum_{p_j \in \mathcal{E}} [a_j (\Delta_{\underline{w}}(\mathcal{O}_{A_j})) + b_j (1 - \Delta_{\underline{w}}(\mathcal{O}_{A_j}))].$$

Moreover, we have that \mathcal{O}_C is \underline{w} -stable if and only if \underline{w} is good.

Proof As C is of compact type we have that $\mathcal{E} = \mathcal{M}$ and also that $\mathcal{M} = \mathcal{M}'$. In fact, assume that there exists an edge $p_j \in \mathcal{M} \setminus \mathcal{M}'$, then $p_j = C_{k_1} \cap C_{k_2}$ and $p_j \not\subseteq \gamma_{k_i}$, with $\gamma_{k_i} \in \mathcal{P}$. Then $\gamma_{k_1} \cup \gamma_{k_2} \cup p_j$ is the support of a cycle in Γ_C , which is impossible. The set \mathcal{P} is uniquely determined by the curve fixed at the beginning, i.e. on the component we have labeled C_γ . Then, the collection $\{A_j \mid j \in \mathcal{E}\}$ is also uniquely determined by C_γ . Finally, since Γ_C does not have any cycles, then A_j is non-empty for all $p_j \in \mathcal{E}$.

² It is the arbitrary curve which we fix when we define the set of minimal paths \mathcal{P} .

As C is of compact type we have, by Lemma 3.7(d) that $\delta_{A_j} = 1$ for all subcurve A_j . With this information we can write the formula of Theorem 3.9 as follows:

$$\Delta_{\underline{w}}(E) = \sum_{p_j \in \mathcal{E}} [a_j (\Delta_{\underline{w}}(\mathcal{O}_{A_j})) + b_j (1 - \Delta_{\underline{w}}(\mathcal{O}_{A_j}))].$$

In order to conclude the proof, by Theorem 2.9, we only need to show that if \mathcal{O}_C is \underline{w} -stable then \underline{w} is a good polarization. Assume that \mathcal{O}_C is \underline{w} -stable. This, by Lemma 2.12, is equivalent to saying $0 < \Delta_{\underline{w}}(\mathcal{O}_B) < \delta_B$ for all proper subcurves B of C . In particular, for all j we have $0 < \Delta_{\underline{w}}(\mathcal{O}_{A_j}) < 1$, which are the Conditions (3.3) stated in Theorem 3.9. \square

In particular, for nodal curves of arithmetic genus $p_a(C) \leq 1$ we have a complete picture of the situation:

Corollary 3.11 *Let C be a nodal curve with $p_a(C) \leq 1$.*

- (a) *If $p_a(C) = 0$, then any polarization on C is good;*
- (b) *if $p_a(C) = 1$ and C is a cycle of rational curves, then any polarization is good;*
- (c) *if $p_a(C) = 1$ and C is of compact type, then a good polarization on C does not exist.*

In particular, if (C, \underline{w}) is any polarized nodal curve with $p_a(C) \leq 1$, then \mathcal{O}_C is \underline{w} -stable if and only if \underline{w} is good.

Proof (a) Let C be a nodal curve with $p_a(C) = 0$. Then C is of compact type and by Lemma 2.12, \mathcal{O}_C is \underline{w} -stable for any polarization \underline{w} . By Theorem 3.10 we have that any \underline{w} is a good polarization.

(b) Let C be a cycle of rational curves and \underline{w} a polarization. Fix a marking \mathcal{M} and a set \mathcal{P} of minimal paths on Γ_C and let $\{A_j\}$ be the subcurves defined in Definition 3.6. By Lemma 3.7, for any j for which A_j is not empty, A_j and A_j^c are both proper connected subcurves of C , so $p_a(A_j) = p_a(A_j^c) = 0$. From Equation (1.3), we get $\delta_{A_j} = 2$. Since $\Delta_{\underline{w}}(\mathcal{O}_{A_j}) = 1 - p_a(A_j) = 1$, we have $\frac{1}{2} < \Delta_{\underline{w}}(\mathcal{O}_{A_j}) < \frac{3}{2}$ which are the sufficient conditions $(\star\star)_{A_j}$ stated in Theorem 3.9. This implies that \underline{w} is good.

(c) It follows by Remark 2.13. \square

Finally as an immediate consequence of Corollary 2.19 and Theorem 3.9, we have the following:

Corollary 3.12 *Let C be a stable nodal curve of compact type with $p_a(C) \geq 2$. Let L be a line bundle on C with degree $p_a(C) - 1$. Then \underline{w}_L is a good polarization if and only if L is strictly balanced.*

As we have seen in the proof of Theorem 3.10, Conditions (3.3) are really useful as they allow us to prove that, on a curve of compact type, a polarization \underline{w} is good if and only if \mathcal{O}_C is \underline{w} -stable. Nevertheless, it can happen that the notion of good polarization

is equivalent to the \underline{w} -stability of \mathcal{O}_C also for curves which are not of compact type (see Corollary 3.11 and the examples in Sect. 4). The reason for this is that Conditions (3.3) are, in general, only sufficient. Moreover, to the authors' knowledge, there is no example of a polarized curve (C, \underline{w}) with \mathcal{O}_C which is \underline{w} -stable but for which \underline{w} is not good. This suggests the following conjecture:

Conjecture 3.13 *Let (C, \underline{w}) be a polarized nodal curve. Then \mathcal{O}_C is \underline{w} -stable if and only if \underline{w} is a good polarization.*

Finally, Theorem 3.9 allows us to produce an open subset of good polarizations on a stable nodal curve C with $p_a(C) \geq 2$. For any nodal curve C we will denote by $\mathcal{W}_C \subset \mathbb{Q}^{\mathcal{Y}}$ the variety parametrizing polarizations on C .

Lemma 3.14 *Let C be a nodal curve, fix a marking \mathcal{M} and a set of minimal paths \mathcal{P} . Then Conditions (3.3) are open in \mathcal{W}_C .*

Proof Let $\{A_j\}$ be the curves constructed starting from \mathcal{P} . Consider \underline{w} and \underline{w}' in \mathcal{W}_C and set $\epsilon_i = w'_i - w_i$. Then we have

$$\Delta_{\underline{w}'}(\mathcal{O}_{A_j}) = \Delta_{\underline{w}}(\mathcal{O}_{A_j}) + (p_a(C) - 1) \sum_{C_i \subseteq A_j} \epsilon_i.$$

If \underline{w} satisfies Conditions $(\star\star)_{A_j}$ for all non-empty A_j then one can take ϵ_i to be small enough so that $(\star\star)_{A_j}$ hold also for \underline{w}' . □

Corollary 3.15 *Let C be a stable nodal curve with $p_a(C) \geq 2$. Then there is a non-empty open subset of \mathcal{W}_C whose elements are good polarizations.*

Proof Since C is stable we can consider the canonical polarization $\underline{\eta}$ (see 2.8). From its definition it follows that $\Delta_{\underline{\eta}}(\mathcal{O}_{A_j}) = \frac{1}{2}\delta_{A_j}$, so $\underline{\eta}$ satisfies condition $(\star\star)_{A_j}$. One can then conclude by using Lemma 3.14. □

4 Some examples

In this section we propose some examples of curves (not of compact type) which we have analyzed in order to study the relation between \underline{w} -stability of \mathcal{O}_C and the fact that \underline{w} is a good polarization. We underline that we always obtain an equivalence between these two concept. So these are motivating examples for Conjecture 3.13.

Example 4.1 Let (C, \underline{w}) be a polarized nodal curve with two smooth irreducible components and δ nodes.

Let C_1 and C_2 be the components of C and p_1, \dots, p_δ the nodes. If $\delta = 1$ then C is of compact type and the assertion follows from Theorem 3.10, so we will assume $\delta \geq 2$. We fix $\mathcal{M} = \{p_1\}$ so that $\mathcal{P} = \{\gamma_1, \gamma_2\}$, where γ_1 has support on the edge corresponding to p_1 and γ_2 is trivial. We have $A_1 = C_1$ and $A_2 = \dots = A_\delta = \emptyset$. Let

E be any depth one sheaf on C , by Theorem 3.9 we have:

$$\Delta_{\underline{w}}(E) = a_1 \left(\frac{1}{2}(1 - \delta) + \lambda_1 \right) + b_1 \left(\frac{1}{2}(1 + \delta) - \lambda_1 \right) + \frac{1}{2} \sum_{j=2}^{\delta} (a_j + b_j)$$

where, as in Lemma 2.4, $\lambda_i = \Delta_{\underline{w}}(\mathcal{O}_{C_i})$. Assume that \mathcal{O}_C is \underline{w} -stable, then for $i = 1, 2$, we have $0 < \lambda_i < \delta$, with $\lambda_1 + \lambda_2 = \delta$. If $(\star\star)_{C_1}$ holds, i.e. if

$$\frac{1}{2}(\delta - 1) < \lambda_1 < \frac{1}{2}(1 + \delta) \quad (4.1)$$

then by Theorem 3.9 \underline{w} is good. If $(\star\star)_{C_1}$ does not hold we have either $\lambda_1 \in (0, \frac{1}{2}(\delta - 1)) = I_1$ or $\lambda_1 \in (\frac{1}{2}(1 + \delta), 1) = I_2$. In the second case we have $\lambda_2 \in I_1$ so up to changing the label to C_1 and C_2 we can assume $\lambda_1 \in I_1$. Then we have

$$\Delta_{\underline{w}}(E) = (b_1 - a_1) \left(\frac{1}{2}(\delta - 1) - \lambda_1 \right) + b_1 + \frac{1}{2} \sum_{j=2}^{\delta} (a_j + b_j).$$

If $b_1 \geq a_1$, then we are done. Assume now that $a_1 \geq b_1$. By Lemma 3.5, we have $b_j - a_j = b_1 - a_1$ for all j so we can write

$$\begin{aligned} \Delta_{\underline{w}}(E) &= (b_1 - a_1) \left(\frac{1}{2}(\delta - 1) - \lambda_1 \right) + \frac{1}{2} \sum_{j=1}^{\delta} b_j + \frac{1}{2} (a_1 - b_1)(\delta - 1) \\ &= \lambda_1 (a_1 - b_1) + \sum_{j=1}^{\delta} b_j. \end{aligned}$$

Hence, also in this case we have that \underline{w} is good.

Example 4.2 Let (C, \underline{w}) be a polarized nodal curve which is a cycle with 3 smooth irreducible components.

Let C_1, C_2 and C_3 be the components and let p_1, p_2 and p_3 the nodes. The dual graph is a triangle with edge p_i opposite to the node C_i . In this case $\mathcal{E} = \mathcal{M}, \mathcal{P} = \{\gamma_1, \gamma_2, \gamma_3\}$, where γ_1 and γ_2 have support on the edge corresponding to p_2 and p_1 respectively and γ_3 is trivial. Then $A_1 = C_2, A_2 = C_1$ and $A_3 = \emptyset$. Let E be any depth one sheaf on C , by Theorem 3.9 we have:

$$\begin{aligned} \Delta_{\underline{w}}(E) &= a_1 \left(\lambda_2 - \frac{1}{2} \right) + b_1 \left(\frac{3}{2} - \lambda_2 \right) + a_2 \left(\lambda_1 - \frac{1}{2} \right) \\ &\quad + b_2 \left(\frac{3}{2} - \lambda_1 \right) + \frac{1}{2} (a_3 + b_3) \end{aligned}$$

where, as above $\lambda_i = \Delta_{\underline{w}}(\mathcal{O}_{C_i})$. Assume that \mathcal{O}_C is \underline{w} -stable, we have $0 < \lambda_i < 2$ with $\lambda_1 + \lambda_2 + \lambda_3 = 3$. If conditions $(\star\star)_{A_i}$ hold, i.e. if $\frac{1}{2} < \lambda_1, \lambda_2 < \frac{3}{2}$ we can

conclude. If $(\star\star)_{A_i}$ do not hold, one can prove that by exchanging the labels to C_1, C_2 and C_3 one can assume $0 < \lambda_1 < \frac{1}{2}$ and $\frac{1}{2} < \lambda_2 < \frac{3}{2}$. We can write

$$\Delta_{\underline{w}}(E) = a_1 \left(\lambda_2 - \frac{1}{2} \right) + b_1 \left(\frac{3}{2} - \lambda_2 \right) + \left(\frac{1}{2} - \lambda_1 \right) (b_2 - a_2) + b_2 + \frac{1}{2}(a_3 + b_3).$$

The cycle in the dual graph yields the following relation

$$b_2 - a_2 = b_1 - a_1 + b_3 - a_3.$$

As in the previous example using the above relation, one can prove that $\Delta_{\underline{w}}(E) \geq 0$ and equality holds if and only if E is locally free, i.e. that \underline{w} is good.

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