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NODAL DOMAIN THEOREMS FOR GENERAL ELLIPTIC EQUATIONS[1]

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1. **Introduction.** Let Λ be a formally selfadjoint elliptic operator defined by

$$\Lambda u \equiv - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial u}{\partial x_i}\right) + \gamma u \tag{1.1}$$

for x in a sufficiently regular bounded domain $G \subset R^n$, and let Λ_σ be the selfadjoint realization defined by

$$\begin{aligned} \Lambda_\sigma v &\equiv -\sum \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial v}{\partial x_i}\right) + \gamma v; && x \in G, \\ \frac{\partial v}{\partial \nu} &\equiv \sum \alpha_{ij}\frac{\partial v}{\partial x_i}\cos(\nu, x_j) = -\sigma(x)v; && x \in \partial G, \end{aligned} \tag{1.2}$$

where ν denotes the exterior normal to G and $\sigma(x)$ is a piecewise continuous function (which is allowed to take on the value $+\infty$ to denote the boundary condition $v = 0$). A well-known theorem of Courant [1] asserts that the nodal lines of the kth eigenfunction of Λ_σ divide G into at most k nodal domains. That is, if N is the number of nodal domains of the kth eigenfunction of Λ_σ, then $N \leqq k$. (While Courant's Theorem is formulated for a somwhat narrower class of operators, his method of proof applies equally well to the class of operators defined by (1.1).)

From a slightly different point of view Courant's Theorem establishes an upper bound for the number of nodal domains of a solution of $\Lambda u = 0$ in terms of the boundary behavior of u and the spectrum of a boundary value problem of the form (1.2). For if the domain G and the coefficients of Λ are sufficiently regular, then every solution of $\Lambda u = 0$ determines a function

$$\sigma(x) = -\frac{1}{u(x)}\frac{\partial u}{\partial \nu}(x)$$

defined on ∂G. Thus every nontrivial solution of $\Lambda u = 0$ becomes an eigenfunction of Λ_σ corresponding to the eigenvalue $\lambda = 0$. In this

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context Courant's Theorem can be formulated as follows: if N denotes the number of nodal domains of $u(x)$ and if Λ_σ has k nonpositive eigenvalues $\lambda_1 < \lambda_2 \leqq \lambda_3 \leqq \cdots \leqq \lambda_k = 0$, then $N \leqq k$.

As an immediate generalization of this result, consider a real valued function $u(x)$ which is a solution of

$$(1.3) \qquad Lu \equiv -\sum \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + cu = 0; \qquad x \in G,$$

and satisfies

$$(1.4) \qquad \frac{\partial u}{\partial \nu} \equiv \sum a_{ij}\frac{\partial u}{\partial x_i}\cos(\nu, x_j) = -s(x)u; \qquad x \in \partial G.$$

If

$$(1.5) \qquad \begin{array}{lll} \text{(i)} & \sum(a_{ij} - \alpha_{ij})\xi_i\xi_j \geqq 0 & \text{for all } x \in G \text{ and all real } n\text{-tuples } (\xi_1, \cdots, \xi_n); \\ \text{(ii)} & c(x) - \gamma(x) \geqq 0 & \text{for all } x \in G; \\ \text{(iii)} & s(x) \geqq \sigma(x) & \text{for all } x \in \partial G, \end{array}$$

then the kth eigenvalue of L_s is at least as large as the kth eigenvalue of Λ_σ. Furthermore, $u(x)$ is an eigenfunction of L_s corresponding to the eigenvalue $l = 0$. If Λ_σ has k nonpositive eigenvalues, then L_s has at most k nonpositive eigenvalues and the number N of nodal domains of $u(x)$ satisfies $N \leqq k$.

Proceeding along similar lines, we shall extend Courant's nodal domain theorem to more general second order elliptic equations than (1.3). In particular, estimates for N will be given when L is nonlinear and nonselfadjoint. These estimates will be given in terms of the spectrum of a "smaller" selfadjoint linear operator of the form Λ_σ.

It is assumed throughout that the coefficients $a_{ij}(x)$ and $\alpha_{ij}(x)$ are of class C'' and that $c(x)$ and $\gamma(x)$ are continuous. The domain G is to be bounded with a boundary which is piecewise of bounded curvature.

2. **Nonlinear equations.** Let L be a nonlinear elliptic operator defined by

$$(2.1) \qquad Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + c\left(x, u, \frac{\partial u}{\partial x_i}\right)u,$$

and let $u(x)$ be a solution of $Lu = 0$ having N nodal domains in G and satisfying

$$(2.2) \qquad \sum a_{ij}(\partial u/\partial x_i)\cos(\nu, x_j) + s(x)u = 0$$

for $x \in \partial G$. We shall obtain an upper estimate for N in terms of the operator Λ_σ defined by (1.2) under the assumption that

$$\text{(i)}\quad \sum (a_{ij} - \alpha_{ij})\xi_i\xi_j \geqq 0 \quad \text{for all } x \in G \text{ and all real } n\text{-tuples } (\xi_1, \cdots, \xi_n);$$

$$\text{(2.3)}\qquad \text{(ii)}\quad c(x, u, \partial u/\partial x_i) \geqq \gamma(x) \quad \text{for all } x \in G \text{ and all values of } u \text{ and } \partial u/\partial x_i;$$

$$\text{(iii)}\quad s(x) \geqq \sigma(x) \quad \text{for all } x \in \partial G.$$

2.1. **Lemma.** *Let G_i be a nodal domain for $u(x)$ and let $\lambda_1{}^i$ denote the first eigenvalue of*

$$\begin{aligned} \Lambda v &= \lambda^i v; & x &\in G_i, \\ v &= 0; & x &\in \partial G_i \cap G, \\ \partial v/\partial \nu + \sigma v &= 0; & x &\in \partial G_i \cap \partial G. \end{aligned} \tag{2.4}$$

If (2.3) *is satisfied, then* $\lambda_1{}^i \leqq 0$.

Proof. Suppose $\lambda_1{}^i > 0$ and that $v_1{}^i(x)$ is the eigenfunction of (2.4) corresponding to $\lambda_1{}^i$. It is well known that $\lambda_1{}^i$ is a strictly decreasing continuous function of G_i in the following sense: if G_i is enlarged along any part of ∂G_i where $v_1{}^i(x) = 0$ and if the boundary condition $v(x) = 0$ is imposed on the new boundary so obtained, then $\lambda_1{}^i$ will be reduced continuously. Therefore it is possible to expand G_i along that portion of the boundary where $v_1{}^i = 0$ and still retain the inequality $\lambda_1{}^i > 0$ for the first eigenvalue of (2.4) in this slightly enlarged domain. The first eigenfunction of this perturbed problem yields a function $w(x)$ which is positive in $\bar{G}_i$, satisfies $\partial w/\partial \nu + \sigma w = 0$ on that part of ∂G where $v_1{}^i(x) \neq 0$, and satisfies $\Lambda w - \delta w = 0$ for some $\delta > 0$.

Since $w(x) > 0$ for $x \in \bar{G}_i$, a direct calculation [2] yields the following generalized Picone identity:

$$\begin{aligned} \sum_j \frac{\partial}{\partial x_j} \left[\frac{u}{w} \left(w \sum_i a_{ij} \frac{\partial u}{\partial x_i} - u \sum_i \alpha_{ij} \frac{\partial w}{\partial x_i} \right) \right] &= u \sum_{i,j} \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) - \frac{u^2}{w} \sum_{i,j} \frac{\partial}{\partial x_j} \left(\alpha_{ij} \frac{\partial w}{\partial x_i} \right) \\ &\quad + \sum_{i,j} \left(a_{ij} - \alpha_{ij} \right) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \\ &\quad + \sum_{i,j} \alpha_{ij} \left(\frac{\partial u}{\partial x_i} - u \frac{\partial w}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} - u \frac{\partial w}{\partial x_j} \right). \end{aligned} \tag{2.5}$$

From (2.2) and the definition of $w(x)$ it follows that

$$\sum_j \frac{\partial}{\partial x_j}\left[\frac{u}{w}\left(w\sum_i a_{ij}\frac{\partial u}{\partial x_i} - u\sum_i \alpha_{ij}\frac{\partial w}{\partial x_i}\right)\right] \geqq \left[c\left(x, u, \frac{\partial u}{\partial x_i}\right) - \gamma(x) + \delta\right]u^2. \tag{2.6}$$

Integrating over G_i and applying Green's Theorem yields

$$\int_{\partial G_i \cap \partial G} (\sigma - s)u^2 dS \geqq \int_G (c - \gamma + \delta)u^2 dx, \tag{2.7}$$

where the boundary integral in (2.7) is limited to that portion of $\partial G_i \cap \partial G$ where $s(x) < \infty$. However, our hypotheses assure that the left side of (2.7) is nonpositive while the right side is positive, and this contradiction proves that $\lambda_1{}^i \leqq 0$.

2.2. THEOREM. *Let $u(x)$ be a solution of $Lu = 0$ having N nodal domains in G. Let Λ_σ be defined by* (1.2) *and satisfy* (2.3). *If Λ_σ has k nonpositive eigenvalues, then $N \leqq k$.*

PROOF. Suppose $N > k$ and that $G_1, G_2, \cdots, G_{k+1}$ are nodal domains of $u(x)$. By the lemma, the first eigenvalue of (2.4) satisfies $\lambda_1{}^i \leqq 0$ for $i = 1, \cdots, k+1$. Therefore it is possible to choose a subdomain $\tilde{G}_i \subset G_i$ such that $\partial\tilde{G}_i \cap \partial G = \partial G_i \cap \partial G$ and such that the first eigenvalue of (2.4) in $\tilde{G}_i$ satisfies $\tilde{\lambda}_1{}^i = 0$. It then follows from the original argument used by Courant [1, p. 393] that Λ_σ has at least $k + 1$ nonpositive eigenvalues.

REMARK. If in Theorem 2.2 $u(x) = 0$ on ∂G, then condition (iii) of (2.3) is not required.

While Theorem 2.2 yields bounds for the number of nodal domains of any solution of $Lu = 0$, it does not assure the existence of any such solution. In connection with the question of existence, it is of interest to consider a special case of (2.1) where L is defined by

$$Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + p(x, u) \tag{2.8}$$

and the principal part of L is denoted by K, so that

$$Ku \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right). \tag{2.9}$$

We assume that the a_{ij} and the domain G are sufficiently regular so that the selfadjoint operator K_s defined by (2.9) and the boundary condition

$$\sum a_{ij} \frac{\partial u}{\partial x_i} \cos(\nu, x_j) + su = 0 \tag{2.10}$$

has an inverse defined by

$$K_s^{-1}f = -\int_G R_s(x, \xi) f(\xi) d\xi,$$

where K_s^{-1} is a positive definite compact selfadjoint operator with eigenvalues $\kappa_1^{-1} > \kappa_2^{-1} \geqq \cdots$. Under these assumptions (2.8), (2.10) become equivalent to the nonlinear Hammerstein equation

$$u(x) = \int_G R_s(x, \xi) p[\xi, u(\xi)]\, d\xi.$$

Such equations were studied by Dolph [3] and shown to have a unique solution if there exist numbers μ_{k-1} and μ_k such that

$$\kappa_{k-1} < \mu_{k-1} \leqq \frac{p(x, u_2) - p(x, u_1)}{u_2 - u_1} \leqq \mu_k < \kappa_k \tag{2.11}$$

for all $x \in G$ and all u_1, u_2. This fact leads to the following result.

2.3. Theorem. *Let L be defined by* (2.8) *and suppose that $p(x, u_2 - u_1) \leqq p(x, u_2) - p(x, u_1)$. Suppose further that* (2.11) *is satisfied so that $L_s u = 0$ has a unique solution $u(x)$. If N denotes the number of nodal domains of $u(x)$, then $N < k$.*

Proof. We shall apply Theorem 2.2 with Λ_σ replaced by $K_s - \mu_k I$. Since

$$p(x, u_2) - p(x, u_1) \geqq p(x, u_2 - u_1)$$

we have

$$\mu_k \geqq \frac{p(x, u_2) - p(x, u_1)}{u_2 - u_1} \geqq \frac{p(x, u_2 - u_1)}{u_2 - u_1}$$

so that $K_s - \mu_k I$ is "smaller" than L_s in the sense of (2.3). Since $K_s - \mu_k I$ has less than k nonpositive eigenvalues, it follows from Theorem 2.2 that $N < k$.

3. **Nonselfadjoint equations.** In this section we shall consider a nonselfadjoint elliptic operator L defined by

$$Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u}{\partial x_i}\right) + 2\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu, \tag{3.1}$$

and let $u(x)$ denote a solution of the boundary value problem

$$Lu = 0; \quad x \in G, \qquad u = 0; \quad x \in \partial G. \tag{3.2}$$

In order to obtain estimates for N, the number of nodal domains of $u(x)$, we shall consider a "smaller" selfadjoint operator Λ defined by

$$\Lambda v \equiv -\sum_{i,j=1} \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial v}{\partial x_i}\right) + \gamma v; \qquad x \in G.$$

The sense in which Λ is to be "smaller" than L is given by the conditions

(3.3)
(i) $\sum (a_{ij} - \alpha_{ij})\xi_i\xi_j \geqq 0$ for all $x \in G$ and all real n-tuples $(\xi_1, \cdots, \xi_n)$;

(ii) $c - \gamma - \sum_i \frac{\partial b_i}{\partial x_i} \geqq 0$ for all $x \in G$.

3.1. Lemma. *Let G_i be a nodal domain for $u(x)$ and let $\lambda_1{}^i$ denote the first eigenvalue of*

(3.4) $$\Lambda v = \lambda^i v; \quad x \in G_i, \qquad v = 0; \quad x \in \partial G_i.$$

If (3.3) *is satisfied, then* $\lambda_1{}^i \leqq 0$.

Proof. Suppose $\lambda_1{}^i > 0$ and that $v_1{}^i(x)$ is the eigenfunction of (3.4) corresponding to $\lambda_1{}^i$. Since $\lambda_1{}^i$ is a continuous strictly decreasing function of the domain G_i, it is possible to expand G_i slightly and still retain the inequality $\lambda_1{}^i > 0$ for the first eigenvalue of (3.4) in this enlarged domain. The first eigenfunction of this perturbed problem yields a function $w(x)$ which is positive in $\bar{G}_i$ and satisfies $\Lambda w - \delta w = 0$ for some $\delta > 0$. However, a comparison theorem due to Swanson [4] asserts that under the conditions (3.3), every solution of $\Lambda w - \delta w = 0$ must have a zero in $\bar{G}_i$, and this contradiction proves that $\lambda_1{}^i \leqq 0$.

3.2. Theorem. *Let $u(x)$ be a solution of*

$$Lu = 0; \quad x \in G, \qquad u = 0; \quad x \in \partial G,$$

having N nodal domains in G. Let Λ_∞ be defined by (3.4). *If Λ_∞ has k nonpositive eigenvalues, then $N \leqq k$.*

Proof. The proof is an exact analogy of the proof of Theorem 2.2, utilizing Lemma 3.1 in place of Lemma 2.1.

Remark. Since an arbitrary homogeneous boundary condition in (3.4) would not decrease the number k, we could replace the condition $v = 0$ by $\partial v/\partial \nu + \sigma v = 0$, for arbitrary $\sigma(x)$.

References

1. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*. Vol. 1, Springer, Berlin, 1931.

2. K. Kreith, *A generalized Picone identity*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **45** (1968), 217-220. MR **40** #4601.

3. C. L. Dolph, *Nonlinear integral equations of the Hammerstein type*, Trans. Amer. Math. Soc. **66** (1949), 289-307. MR **11**, 367.

4. C. A. Swanson, *A comparison theorem for elliptic differential equations*, Proc. Amer. Math. Soc. **17** (1966), 611-616. MR 34 #1663.

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