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# Node polynomials for families: methods and applications

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We continue the development of methods for enumerating nodal curves on smooth complex surfaces, extending the range of validity. We apply the new methods in three important cases. First, for up to eight nodes, we prove Göttsche's conjecture about plane curves of low degree. Second, we prove Vainsencher's conjectural enumeration of irreducible six-nodal plane curves on a general quintic threefold in four-space, which is important for Clemens' conjecture and mirror symmetry. Third, we supplement Bryan and Leung's enumeration of nodal curves in a given homology class on an Abelian surface of Picard number 1.



## 1 Introduction

This paper is the second in a series devoted to the enumeration of nodal curves on smooth complex surfaces. The first paper [19] focuses on curves in a "suitably" ample linear system on a fixed ambient surface. This second paper treats more general systems and variable surfaces, and it extends the range of validity. (This paper was once entitled, "Node polynomials for families: results and examples"; however, the words "results" and "examples" mislead some readers, and so were changed.) Here we develop some general methods. However, the importance of methods is in their applications, and we apply the new methods in three important cases: curves of low degree in the plane, plane curves on a threefold in four-space, and homologous curves on an Abelian surface.

Nodal plane curves were enumerated, for up to three nodes, in the third quarter of the nineteenth century, and the general problem has recently been revived; the history is reviewed in Remark 3.7. In particular, Göttsche conjectured in [11, Conj. 4.1, p. 530], that, for each $r$, if $N_r(m)$ denotes the number of curves of degree $m$ with $r$ nodes through $m(m+3)/2 - r$ general points, then $N_r(m)$ is given by a certain "node" polynomial of degree $2r$ in $m$ for $m \geq r/2 + 1$, which is just the range of $m$ where the locus of nonreduced curves is too small to interfere. Our first main result, Theorem 3.1, establishes Göttsche's conjecture for $r \leq 8$.

Theorem 3.1 could be derived from Theorem (1.1) of [19] and the recursive enumerative formula of Caporaso and Harris [4, p. 353]; see the end of Remark 3.7. However, we proceed differently for three reasons. First, our lemmas are also needed to prove our second main result, Theorem 4.1. Second, our enumeration is independent of those of other authors, including Caporaso and Harris. Third, our approach may eventually lead to a proof of Göttsche's conjecture for all $r$, whereas the alternative approach requires evaluating Caporaso and Harris's formula at least once for each $r$, an absurd project.

Theorem 4.1 enumerates, for $m \geq 4$, the 6-nodal plane curves of degree $m$ on a general threefold of degree $m$ in 4-space, or what is the same, its 6-tangent 2-planes. This enumeration provides a nice application of our machinery in the case of a nontrivial family of ambient surfaces. The family consists of all the planes in 4-space, parameterized by the Grassmann variety; so each surface is the same, but the family is nonconstant. The curves are those cut out on the planes by the threefold. The number of curves is given by a certain "node" polynomial of degree 18 in $m$.

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This enumeration was originally done by Vainsencher [34]. Indeed, his paper inspired this one and its companions [19] and [20]; our work just refines and extends his. Here, notably, we develop some new ways of extending the range of validity of the enumerations. For example, for plane curves on threefolds, Vainsencher's Propositions 3.5 and 4.1 imply only that there exists some undetermined integer $m_0$ such that the enumeration is valid for $m \geq m_0$, whereas we prove validity for $m \geq 4$.

The case $m = 5$ is particularly important because of Clemens' conjecture and mirror symmetry. Clemens' conjecture [7, p. 639] asserts notably that, on the general quintic threefold, there are only finitely many rational curves of each degree, and all are smooth. Their number was predicted in 1991 in a dramatic application of mirror symmetry, its first application to enumerative geometry. This enumeration is revisited several times in Cox and Katz's lovely text [9].

These irreducible 6-nodal plane quintics are rational, but singular! So this part of Clemens' original conjecture is false, and is not made part of the conjecture's modern formulation [9, p. 202]. Furthermore, mirror symmetry includes these 6-nodal curves in its count. However, Pandharipande [9, (7.54), p. 206] found something worse: each 6-nodal curve has six previously unconsidered double covers. So, in degree 10, mirror symmetry simply produced the wrong number. It cannot be the number of all rational curves, smooth and singular! It is too large by six times the number of irreducible 6-nodal curves.

The irreducible 6-nodal curves too were originally enumerated by Vainsencher in [34, pp. 513–514], and we recover his number in our third main result, Theorem 4.3, basically pursuing his approach, but following our own improved way through the computations. Namely, we use Theorem 4.1 to obtain the number of all 6-nodal curves, and from it, we subtract the number of reducible ones. Far more importantly, we again advance Vainsencher's work by establishing, for the first time, the validity of the numbers involved.

Our fourth and last main result, Theorem 5.2, enumerates the irreducible curves having $r$ nodes and lying in a given homology class $\gamma$ on an Abelian surface $A$ with Picard number 1. Say $\gamma$ has self-intersection number $d$, and is $m$ times the positive primitive class. Set $g := d/2 - r + 1$, and let $N_{g,r}$ be the number of curves through $g$ general points. Theorem 5.2 asserts that, if $r \leq 8$, then $N_{g,r}$ is given by a certain polynomial of degree $r + 1$ in $g$ for $g > g_0$, where $g_0$ is a certain number depending on $m$ and $r$, but not on $A$. The nine polynomials are listed in Table 5.1.

The first theorem of this sort was proved by Bryan and Leung [3, Thm. 1.1, p. 312], using symplectic methods. Their theorem is valid for any $r$ and $g$, provided $A$ is generic in the following sense: given the underlying topological space, the complex structure of $A$ is generic among those for which the given class $\gamma$ is algebraic. It follows (as stated in the proof of [3, Lem. 5]) that $A$ has Picard number 1 and that $\gamma$ is primitive, that is, $m = 1$. By contrast, we fix $A$, not $\gamma$; moreover, our methods are algebraic-geometric and rather different. Thus, for $r \leq 8$ and $g > g_0$, our work recovers and extends theirs by different means.

Bryan and Leung expressed the $N_{g,r}$ essentially as follows:

$$\sum_{r\geq 0} (N_{g,r}/g)q^r \;=\; \Bigl(\sum_{k\geq 1} k\sigma_1(k)q^{k-1}\Bigr)^{g-1} \quad \text{where} \quad \sigma_1(k) \;:=\; \sum_{d|k} d\,.$$

Say the logarithms of the left and right sides are $\sum_{r\geq 1} a_r q^r/r!$ and $(g-1)\sum_{r\geq 1} b_r q^r/r!$. Then $a_r = (g-1)b_r$ for $r \geq 1$. Moreover, $b_1, \ldots, b_8$ are these integers:

$$6,\ -12,\ 168,\ -2448,\ 46944,\ -1071360,\ 29064960,\ -921110400\,.$$

Furthermore, there is a weighted homogeneous polynomial $P_r$ of degree $r$ such that

$$N_{g,r} \;=\; gP_r(a_1, \ldots, a_r)/r!\,.$$

The $P_r$ are defined by the formal identity (2.2), and are known as the *Bell polynomials*. They appear in all our enumerations, although in the case at hand they enter our work somewhat differently.

Closely related is the enumeration of the $r$-nodal curves lying in a given linear-equivalence subclass and passing through $g - 2$ general points. Various cases have been discussed by various authors, and their work is surveyed in Remark 5.4. In particular, Göttsche conjectured a generating function similar to the one above for $N_{g,r}$, and Bryan and Leung proved it when $A$ is generic in the above sense. Supplementing their work, we can modify the proof of our Theorem 5.1 to prove Göttsche's conjecture when $r \leq 8$ and $m > (3r + 5)/2$.

All our enumerations are carried out on the basis of Theorem 2.5. Its statement is implicit in Section 4 of [19]; its proof is outlined there, and is completed in [20]. Here, notably we refine our treatment of the key cycles. In [19], they are placed under unnecessarily stringent genericity hypotheses, which likely are not satisfied in the present circumstances. So we must adopt a more liberal definition of these cycles, and develop suitable conditions that imply the cycles have the right support and are reduced.

More precisely, Theorem 2.5 concerns a smooth, projective family of surfaces, $\pi\colon F \to Y$, where $Y$ is equidimensional and Cohen–Macaulay. In Section 4 of [19], mistakenly, $Y$ is not assumed to be Cohen–Macaulay; on the other hand, unnecessarily, $Y$ is assumed to be reduced, and the surfaces $\pi^{-1}(y)$, to be irreducible.

Let $D \subset F$ be a $Y$-flat closed family of curves. Denote its rational equivalence class by $v$, and the Chern class $c_i\big(\Omega^1_{F/Y}\big)$ by $w_i$. Partition $Y$ into locally closed subsets: one $Y(\infty)$ where the fibers $D_y$ have a multiple component, and each other where the $D_y$ have a given equisingularity type. Given $r$, assume that, if nonempty, $Y(\infty)$ has codimension at least $r+1$ and that each remaining nonempty subset has codimension at least $\min(r+1,\, c)$ where $c$ is its expected codimension.

Consider the set of $y \in Y$ where the $D_y$ are $r$-nodal. Theorem 2.5 asserts that this set is either empty or exactly of codimension $r$; either way, its closure is the support of a natural nonnegative cycle $U(r)$. Furthermore, if $r \le 8$, then the class $[U(r)]$ is equal to $P_r(a_1, \ldots, a_r)/r!$ where $P_r$ is the Bell polynomial, where $a_q := \pi_* b_q$, and where $b_q$ is a certain polynomial in the classes $v,\ w_1,\ w_2$.

In order to apply Theorem 2.5, we must check that the relevant subsets of $Y$ have appropriate codimensions. To do so, we modify several arguments in [19], and thereby obtain better results. In the case of an ambient Abelian surface, basically we replace the Gotzmann regularity theorem and Bertini's theorem by the Beltrametti–Sommese $k$-very ampleness theorem. In the case of curves in the plane, we take a different tack: we work directly on $Y$ using some of Greuel and Lossen's results about equisingular families of curves. Finally, in the case of 6-nodal plane curves on a threefold, we derive what we need from our work with curves in the plane.

In each case, therefore, Theorem 2.5 provides us with an enumerating cycle $U(r)$ and an effective expression for its class $[U(r)]$. To complete the enumeration, we must show that $U(r)$ is reduced so that we know that each $r$-nodal curve is counted with multiplicity 1. We do so by carrying a bit further our analysis of the relevant subsets of $Y$. Finally, we need to work out the cycles $b_q,\ a_q$, and $P_r$. This work is done in Section 4 of [19] for any linear system on any fixed surface, and so it applies in particular to the case of curves in the plane. In the remaining two cases, the details are explained, but the more mechanical calculations are omitted.

In short, in Section 2, we state the general enumeration theorem, Theorem 2.5, and explain its ingredients: the Bell polynomials, the polynomials giving the $b_q$ in terms of $v,\ w_1,\ w_2$, the key subsets of $Y$, and the enumerating cycle $U(r)$. In Sections 3, 4, and 5, we work out in detail the three cases: the plane, a threefold in four-space, and an Abelian surface.

## 2 The general theorem

In this section, we discuss the general enumeration theorem, Theorem 2.5, that we use in the following sections. In Remark 2.7, we conjecture a possible generalization.

Let $Y$ be an equidimensional Cohen–Macaulay scheme of finite type over the complex numbers. Let $\pi\colon F \to Y$ be a smooth projective family of surfaces, and $D$ a relative effective divisor on $F/Y$. Fix $r \ge 0$, and consider the points $y \in Y$ parameterizing the curves $D_y$ with precisely $r$ nodes.

Theorem 2.5 says that these $y$ are enumerated by a cycle $U(r)$, and that if $r \le 8$, then the rational equivalence class $[U(r)]$ is given by a universal polynomial in the classes $y(a,b,c)$ that are defined as follows:

$$y(a,b,c) \;:=\; \pi_* v^a w_1^b w_2^c \;\text{ where }\; v \;:=\; [D] \;\text{ and }\; w_i \;:=\; c_i\big(\Omega^1_{F/Y}\big)\,. \tag{2.1}$$

The only hypotheses are that certain key subsets of $Y$ have appropriate dimensions.

The universal polynomial has a special shape, which makes it much easier to find and evaluate. Namely, define auxiliary polynomials $P_i(a_1, \ldots, a_i)$ via this formal identity in $t$:

$$\sum_{i\ge 0} P_i t^i/i! \;:=\; \exp\left(\sum_{j\ge 1} a_j t^j/j!\right). \tag{2.2}$$

For example, $P_0 = 1$, and $P_1 = a_1$, and $P_2 = a_1^2 + a_2$, and $P_3 = a_1^3 + 3a_1a_2 + a_3$. If we assign $a_j$ weight $j$, then $P_i(a_1, \dots, a_i)$ is weighted homogeneous of degree $i$. These polynomials are known as the (*complete*) *Bell polynomials*, and have been studied by a number of authors; see Comtet's book [8, pp. 144–148].

The universal polynomial can be obtained from $P_r(a_1, \dots, a_r)/r!$ by replacing each $a_q$ by a certain linear combination of the $y(a, b, c)$ with $a + b + 2c = q + 2$. Equivalently, we can set $a_q := \pi_* b_q$ where $b_q$ is a certain weighted homogeneous polynomial of degree $q+2$ in $v,\ w_1,\ w_2$ if we assign $v$ and $w_1$ weight 1 and $w_2$ weight 2. The $b_q$ are given by a simple algorithm; it was stated informally in Section 4 of [19], and is stated in pseudo-code in Algorithm 2.3 below.

**Algorithm 2.3. Pseudocode for the $b_q(v, w_1, w_2)$**

INPUT: indeterminates $v,\ w_1,\ w_2$.
OUTPUT: polynomials $b_q(v, w_1, w_2)$ for $q = 1, \dots, 8$.

FUNCTION: $Q(i, R)$.

INPUT: an integer $i$ and a polynomial $R(v, w_1, w_2)$.
LOCAL: an indeterminate $e$.

$R' := R\big(v - ie,\ w_1 + e,\ w_2 - e^2\big)$.
$R''$ := the remainder in $e$ of $R'$ on division by $\big(e^3 + w_1e^2 + w_2e\big)$.

RETURN: $Q(i, R) := -\mathrm{Coeff}\big(R'',\ e^2\big)$.

$x_2 := v^3 + v^2w_1 + vw_2$.
FOR $s$ FROM 0 TO 2 DO
$b_{s+1} := P_s(Q(2, b_1), \dots, Q(2, b_s))x_2$.

$x_3 := v^6 + 4v^5w_1 + 5v^4\big(w_1^2 + w_2\big) + v^3\big(2w_1^3 + 11w_1w_2\big) + v^2\big(6w_1^2w_2 + 4w_2^2\big) + 4vw_1w_2^2$.
FOR $s$ FROM 3 TO 6 DO
$b_{s+1} := P_s(Q(2, b_1), \dots, Q(2, b_s))x_2 - s(s-1)(s-2)P_{s-3}(Q(3, b_1), \dots, Q(3, b_{s-3}))x_3$.

$$x_4 := v^{10} + 10v^9w_1 + v^8\big(40w_1^2 + 15w_2\big) + v^7\big(82w_1^3 + 111w_1w_2\big) + v^6\big(91w_1^4 + 315w_1^2w_2 + 63w_2^2\big) + v^5\big(52w_1^5 + 29w_1^3w_2 + 324w_1w_2^2\big) + v^4\big(12w_1^6 + 282w_1^4w_2 + 593w_1^2w_2^2 + 85w_2^3\big) + v^3\big(72w_1^5w_2 + 464w_1^3w_2^2 + 259w_1w_2^3\big) + v^2\big(132w_1^4w_2^2 + 246w_1^2w_2^3 + 36w_2^4\big) + v\big(72w_1^3w_2^3 + 36w_1w_2^4\big).$$

$b_8 := P_7(Q(2, b_1), \dots, Q(2, b_7))x_2 - 7 \cdot 6 \cdot 5\, P_4(Q(3, b_1), \dots, Q(3, b_4))x_3 + 3281 \cdot 7!\, x_4$.

Hypothesis (i) of Theorem 2.5 concerns the set $Y(\infty)$ of $y \in Y$ such that the curve $D_y$ has a multiple component, or equivalently, is nonreduced. Now, let $\mathbf{D}$ be a "minimal Enriques diagram" as defined in Section 2 of [19]. Hypothesis (ii) concerns the (locally closed) set $Y(\mathbf{D})$ of $y \in (Y - Y(\infty))$ such that $D_y$ has $\mathbf{D}$ as its associated diagram.

Briefly put, these $\mathbf{D}$ are abstract combinatorial structures that represent the equisingularity types of reduced curves on smooth surfaces. The $\mathbf{D}$ associated to a curve $C$ is made from its directed resolution graph $\Gamma$. Weight $\Gamma$ with the multiplicities of the strict transforms of $C$, and equip $\Gamma$ with the binary relation of "proximity"; by definition, one infinitely near point of $C$ is *proximate* to a second if the first lies on the strict transform of the exceptional divisor of the blowup centered at the second. By the theorem of embedded resolution, almost all infinitely near points have multiplicity 1, and are proximate solely to their immediate predecessors. Form all the infinite unbroken successions of these points, and consider the corresponding vertices in the weighted and equipped $\Gamma$; remove these vertices to get $\mathbf{D}$.

From $\mathbf{D}$, we can, in principle, determine all the numerical invariants of the equisingularity class of $C$. Six such invariants were studied in Sections 2 and 3 of [19], and they will be used here; so we recall them now. Each is given by a formula in these basic numbers:

$$\begin{aligned} m_V &:= \text{the multiplicity, or weight, of the vertex } V \in \mathbf{D}\,, \\ \operatorname{frs}(\mathbf{D}) &:= \text{the number of free vertices in } \mathbf{D}\,, \\ \operatorname{rts}(\mathbf{D}) &:= \text{the number of roots in } \mathbf{D}\,, \end{aligned}$$

where a *root* is an initial vertex and a *free vertex* is one that is not proximate to a remote predecessor (so a root is free). Each remaining vertex is proximate to two vertices, and is said to be a *satellite* of the more distant of the two.

The six numerical invariants are the following:

$$\begin{aligned} \dim(\mathbf{D}) &:= \operatorname{rts}(\mathbf{D}) + \operatorname{frs}(\mathbf{D})\,, & \delta(\mathbf{D}) &:= \sum_{V\in\mathbf{D}} \binom{m_V}{2}\,, \\ \deg(\mathbf{D}) &:= \sum_{V\in\mathbf{D}} \binom{m_V+1}{2}\,, & r(\mathbf{D}) &:= \sum_{V} \Bigl(m_V - \sum_{W\succ V} m_W\Bigr)\,, \\ \operatorname{cod}(\mathbf{D}) &:= \deg(\mathbf{D}) - \dim(\mathbf{D})\,, & \mu(\mathbf{D}) &:= 2\delta(\mathbf{D}) - r(\mathbf{D}) + \operatorname{rts}(\mathbf{D})\,, \end{aligned}$$

where $W \succ V$ means that $W$ is proximate to $V$. The numbers in the right column are, respectively, equal to the $\delta$-invariant, the number of branches, and the Milnor number of $C$. The numbers in the left column have geometric meanings, which were discussed in Section 3 of [19], and will become clear when we use them.

For example, if $C$ has precisely $r$ nodes, then its diagram consists simply of $r$ roots of multiplicity 2; this diagram is denoted $r\mathbf{A}_1$. If $C$ has a simple cusp, then its diagram consists of three vertices: a root of multiplicity 2, followed by a free vertex of multiplicity 1, followed by a final vertex of multiplicity 1 and proximate to the root. This diagram is denoted $\mathbf{A}_2$. Many more examples are discussed in Section 2 of [19]; in fact, there is there a classification of all the $\mathbf{D}$ with a single root $R$ and with $\operatorname{cod}(\mathbf{D}) \le 10$ (whence $m_R \le 4$) and also of all those $\mathbf{D}$ with $m_R \le 3$.

The following lemma will be used to prove the first assertion of Theorem 2.5.

**Lemma 2.4** *Only finitely many distinct minimal Enriques diagrams arise from the fibers of $D/Y$.*

P r o o f. As $C$ ranges over the fibers, the numbers $\dim H^1(\mathcal{O}_C)$ are bounded, say by $p$. Fix a relatively ample sheaf on $F/Y$; then the numbers $\deg C$ are defined, and they too are bounded, say by $m$. Fix an arbitrary reduced $C$, and let $f\colon C' \to C$ be the normalization map. Then the number of connected components of $C$ is equal to $\dim H^0(\mathcal{O}_C)$, and the number of irreducible components of $C$ is equal to $\dim H^0(\mathcal{O}_{C'})$. Hence

$$\dim H^0(\mathcal{O}_C) \;\ge\; 1 \quad\text{and}\quad \dim H^0(\mathcal{O}_{C'}) \;\le\; m\,.$$

Consider the standard short exact sequence,

$$0 \longrightarrow \mathcal{O}_C \longrightarrow f_*\mathcal{O}_{C'} \longrightarrow f_*\mathcal{O}_{C'}/\mathcal{O}_C \longrightarrow 0\,.$$

In view of the preceding paragraph, this sequence yields the bound,

$$\dim f_*\mathcal{O}_{C'}/\mathcal{O}_C \;\le\; p+m-1\,.$$

Let $\mathbf{D}$ be the diagram of $C$. Then $\dim f_*\mathcal{O}_{C'}/\mathcal{O}_C = \delta(\mathbf{D})$ by the Noether–Enriques theorem; see [19, Prop. (3.1), p. 220]. Hence, by the definition of $\delta(\mathbf{D})$,

$$\sum_{V\in\mathbf{D},\; m_V\ge 2} m_V/2 \;\le\; \delta(\mathbf{D}) \;\le\; p+m-1\,.$$

Thus the number of vertices $V$ with $m_V \ge 2$ is bounded, and the weights $m_V$ themselves are bounded.

It remains to bound the number of vertices $V$ with $m_V = 1$. Each free $V$ determines a distinct branch by [19, Lem. (2.1), p. 214]. So the number of free $V$ is bounded by $r(\mathbf{D})$, which is equal to the total number branches of $C$ through all of its singular points [19, Prop. (3.1), p. 220]. Hence, by Part (i) of Lemma 3.5 in the next section, the number of $V$ with $m_V = 1$ is bounded by $\sum m_R$ where $R$ ranges over the roots of $\mathbf{D}$. However, $\sum m_R$ is bounded by virtue of the last display in the preceding paragraph. The proof is now complete. □

We can now state our general theorem, and prove its first assertion. The rest of the proof is found in [20]. (See also Section 4 of [19].)

**Theorem 2.5** *In the above setup, assume*
(i) *if* $Y(\infty) \neq \emptyset$*, we have* $\operatorname{cod} Y(\infty) \geq r+1$*, and*
(ii) *for each* $\mathbf{D}$ *such that* $Y(\mathbf{D}) \neq \emptyset$*, we have* $\operatorname{cod} Y(\mathbf{D}) \geq \min(r+1, \operatorname{cod} \mathbf{D})$.
*Then either* $Y(r\mathbf{A}_1)$ *is empty, or it has pure codimension* $r$*; in either case, its closure* $\overline{Y(r\mathbf{A}_1)}$ *is the support of a natural nonnegative cycle* $U(r)$*. Furthermore, if* $r \leq 8$*, then the rational equivalence class* $[U(r)]$ *is given by the formula*

$$[U(r)] \;=\; P_r(a_1,\ldots,a_r)/r! \quad \textit{where} \quad a_q \;:=\; \pi_* b_q$$

*and* $b_q$ *is a certain polynomial in* $v,\, w_1,\, w_2$*, namely, that output by* Algorithm 2.3.

Proof. In the relative Hilbert scheme $\operatorname{Hilb}^r_{F/Y}$, form the open subscheme $H(r)$ parameterizing the sets $G$ of $r$ distinct points in the fibers of $F/Y$. Re-embed $H(r)$ in $\operatorname{Hilb}^{3r}_{F/Y}$ by sending a $G$ to the subscheme defined by the square of its ideal; this embedding is well defined because $F/Y$ is smooth. Next, form the intersection

$$Z(r) \;:=\; H(r) \cap \operatorname{Hilb}^{3r}_{D/Y}\,,$$

its closure $\overline{Z(r)}$, and the fundamental cycle $\big[\,\overline{Z(r)}\,\big]$. Push $\big[\,\overline{Z(r)}\,\big]$ down to $Y$; the result is, by definition, $U(r)$.

On $Y$, the image of $\overline{Z(r)}$ contains, as a dense subset, the image of $Z(r)$. The latter image consists of $Y(\infty)$ plus the set of all $y \in (Y - Y(\infty))$ such that $D_y$ has $r$ or more distinct singular points. The latter condition implies that the minimal Enriques diagram $\mathbf{D}$ of $D_y$ has $r$ roots or more. So, $\operatorname{cod}(\mathbf{D}) \geq r$, and $\operatorname{cod}(\mathbf{D}) = r$ if and only if $\mathbf{D} = r\mathbf{A}_1$ (because no final vertex, or leaf, of $\mathbf{D}$ can be a free vertex of multiplicity 1). Hence, either $y \in Y(\mathbf{D})$ with $\operatorname{cod}(\mathbf{D}) > r$ or else $y \in Y(r\mathbf{A}_1)$. Also, the fiber of $Z(r)$ over $y$ is finite, and it has cardinality 1 if $y \in Y(r\mathbf{A}_1)$; moreover, the image of $Z(r)$ contains $Y(r\mathbf{A}_1)$.

Each component of $Z(r)$ is of dimension at least $\dim\,(Y) - r$, because $H(r)$ is of dimension $\dim\,(Y) + 2r$ and because $\operatorname{Hilb}^{3r}_{D/Y}$ is the zero scheme of a regular section of a bundle of rank $3r$ on $\operatorname{Hilb}^{3r}_{F/Y}$. Now, by the preceding paragraph, the fibers of $Z(r)/Y$ are finite off $Y(\infty)$, and have cardinality precisely 1 over $Y(r\mathbf{A}_1)$. Moreover, the image of $Z(r)$ is contained in $Y(r\mathbf{A}_1)$ plus the union of $Y(\infty)$ and certain $Y(\mathbf{D})$ with $\operatorname{cod}(\mathbf{D}) > r$; these $\mathbf{D}$ are finite in number by Lemma 2.4 above. Hence the hypotheses of the theorem imply that $U(r)$ is a cycle of pure codimension $r$, and its support is $\overline{Y(r\mathbf{A}_1)}$. The first assertion is now proved. □

Since the characteristic is 0, a map between integral schemes has degree 1 if its fibers have cardinality one. Hence the above considerations also yield the following lemma, which we use in conjunction with Theorem 2.5.

**Lemma 2.6** *The enumerating cycle* $U(r)$ *is reduced if and only if the scheme* $Z(r)$ *is reduced on an open set that dominates* $Y(r\mathbf{A}_1)$.

**Remark 2.7** It is natural to conjecture that the theorem generalizes to any $r$. More precisely, for any $r$, the hypotheses of the theorem should imply that the class $[U(r)]$ is given by a universal polynomial in the classes $y(a,b,c)$. Moreover, this polynomial should be of the form $P_r(a_1,\ldots,a_r)/r!$ where $a_q := \pi_* b_q$ and $b_q$ is the weighted homogeneous polynomial output by a suitable extension of Algorithm 2.3 and evaluated at $v,\, w_1,\, w_2$.

It is also natural to conjecture that the theorem generalizes so as to enumerate the $y \in Y$ such that $D_y$ has a given equisingularity type, say that represented by a minimal Enriques diagram $\mathbf{D}$. More precisely, set $r := \operatorname{cod}(\mathbf{D})$, and let $\rho$ be the number of roots of $\mathbf{D}$. Then the hypotheses should imply that the closure of $Y(\mathbf{D})$ is the support of a natural positive cycle $U(\mathbf{D})$, and its class $[U(\mathbf{D})]$ is given by a universal polynomial of degree $\rho$ in the $y(a,b,c)$ with $a+b+2c \leq r+2$.

Evidence for this conjecture is provided by the case of a suitably general linear system on a fixed smooth irreducible projective surface. First, Theorem (1.2) of [19, p. 210], enumerates the curves with a triple point of a given type and additionally up to three nodes. Second, this conjecture implies Göttsche's conjecture [11, Rmk. 5.4, p. 532], which enumerates the curves with several ordinary multiple points.

It is easy to construct a natural candidate for the cycle $U(\mathbf{D})$ by generalizing the construction of $U(r)$. Namely, set $d := \deg(\mathbf{D})$, and in $\operatorname{Hilb}^d_{F/Y}$ form the set $H_{F/Y}(\mathbf{D})$ of points parameterizing the complete ideals with $\mathbf{D}$

as associated diagram. For example, $H(r\mathbf{A}_1) = H(r)$. (The set $H_{F/Y}(\mathbf{D})$ is studied in Section 4 of [19] and is studied further in [20].) Owing to the work of Nobile and Villamayor [25, Thm. 2.6 and Prop. 3.4], or to that of Lossen [22, Prop. 2.19, p. 35], $H_{F/Y}(\mathbf{D})$ is locally closed; in fact, it is a smooth $Y$-scheme. Form the intersection $Z(\mathbf{D}) := H_{F/Y}(\mathbf{D}) \cap \mathrm{Hilb}^d_{D/Y}$, its closure $\overline{Z(\mathbf{D})}$, and the fundamental cycle $\big[\overline{Z(\mathbf{D})}\big]$. Push $\big[\overline{Z(\mathbf{D})}\big]$ down to $Y$; take the result to be $U(\mathbf{D})$.

The support of $U(\mathbf{D})$ contains, as a dense subset, the set of $y \in Y$ such that the Enriques diagram of $D_y$ contains $\mathbf{D}$. Hence the hypotheses of the theorem imply that the support of $U(\mathbf{D})$ is equal to the closure of $Y(\mathbf{D})$.

## 3 Plane curves

Let $N_r(m)$ be the (unweighted) number of reduced plane curves of degree $m$, that possess exactly $r$ (ordinary) nodes and that contain $m(m+3)/2 - r$ points in general position. In this section, for $r \le 8$, we prove Göttsche's conjecture [11, Conj. 4.1, p. 530], about $N_r(m)$; more precisely, we prove Theorem 3.1, which is our first main result. The proof relies on Theorem 2.5, which is solely responsible for the restriction $r \le 8$: if Theorem 2.5 is proved for more values of $r$, then Theorem 3.1 will follow for these same values. We end the section with a survey of related work and with some instructive examples.

**Theorem 3.1** *Assume $r \le 8$ and $m \ge r/2 + 1$. Then*

$$N_r(m) \;=\; P_r(a_1, \ldots, a_r)/r!$$

*where $P_r$ is the Bell polynomial, defined by Identity* (2.2)*, and the $a_q$ are the quadratic polynomials in $m$ listed in Table* 3.2.

**Table 3.2. The polynomials $a_q(m)$ for plane curves**

$$\begin{aligned}
a_1 &= 3m^2 - 6m + 3 \;=\; 3(m-1)^2\\
a_2 &= -42m^2 + 117m - 75 \;=\; -3(m-1)(14m-25)\\
a_3 &= 1380m^2 - 4728m + 3798\\
a_4 &= -72360m^2 + 287010m - 271242\\
a_5 &= 5225472m^2 - 23175504m + 24763752\\
a_6 &= -481239360m^2 + 2334195360m - 2748951000\\
a_7 &= 53917151040m^2 - 281685755520m + 359332109280\\
a_8 &= -7118400139200m^2 + 39618359640720m - 54066876993360
\end{aligned}$$

Proof. We apply Theorem 2.5. Let $Y$ be the projective space parameterizing the plane curves of degree $m$, so $\dim Y = m(m+3)/2$. Set $S := \mathbf{P}^2$ and $F := S \times Y$, and let $D \subset F$ be the total space of curves.

Consider the set $Y(\infty)$ of $y \in Y$ such that the curve $D_y$ has an $s$-fold component for some $s \ge 2$. If $m = 1$, then $Y(\infty)$ is empty. Suppose $m \ge 2$. Then the $D_y$ with $s = 2$ form a subset of maximal dimension, namely, $(m-2)(m+1)/2 + 2$. Hence $\operatorname{cod} Y(\infty) = 2m - 1$. Since $m \ge r/2 + 1$ by hypothesis, Hypothesis (i) of Theorem 2.5 follows. Furthermore, its Hypothesis (ii) holds owing to Parts (i) and (ii) of Lemma 3.3 below. Hence we may apply Theorem 2.5.

Theorem 2.5 implies that the closure of $Y(r\mathbf{A}_1)$ is the support of a nonnegative cycle $U(r)$, whose class is equal to $P_r(a_1, \ldots, a_r)/r! \cdot h^r$ where the $a_q$ are certain integers and where $h := c_1(\mathcal{O}_Y(1))$. In fact, the argument at the top of p. 232 of [19] shows that the $a_q$ are equal to certain linear combinations of the four basic Chern numbers $d$, $k$, $s$ and $x$. These combinations are listed on p. 210 of [19]. Moreover, since $S := \mathbf{P}^2$, the four numbers are, respectively, $m^2$, $-3m$, 9 and 3. Formal calculations now yield the values in Table 3.2.

Finally, $U(r)$ is reduced by Lemma 3.4 below. Let $M \subset Y$ be the linear space representing the plane curves that contain $m(m+3)/2 - r$ points in general position. Then $M \cap U(r)$ is finite, reduced, and contained in $Y(r\mathbf{A}_1)$ by Lemma (4.7) on p. 232 of [19]. Hence $N_r(m)$ is equal to $P_r(a_1, \ldots, a_r)/r!$, and the proof is complete. □

**Lemma 3.3** *Assume $m \geq r/2 + 1$. Let $Y$ be the projective space of plane curves of degree $m$, and let $\mathbf{D}$ be a (nonempty) minimal Enriques diagram such that $Y(\mathbf{D}) \neq \emptyset$.*

(i) *If $\operatorname{cod}(\mathbf{D}) \leq r$, then $\operatorname{cod}(Y(\mathbf{D}), Y) = \operatorname{cod}(\mathbf{D})$ and $Y(\mathbf{D})$ is smooth. Moreover, then $Y(\mathbf{D})$ represents the functor of $\mathbf{D}$-equisingular families of plane curves of degree $m$ (their parameter spaces need not be reduced).*

(ii) *If $\operatorname{cod}(\mathbf{D}) \geq r + 1$, then $\operatorname{cod}(Y(\mathbf{D}), Y) \geq r + 1$.*

Proof. Let $C$ be a curve corresponding to an arbitrary (closed) point of $Y(\mathbf{D})$. For a moment, suppose that $\mathbf{D}$ consists of one vertex of multiplicity $m$. Then $\operatorname{cod}(\mathbf{D}) = \binom{m+1}{2} - 2$. Furthermore, $C$ has an ordinary $m$-fold point. Hence $Y(\mathbf{D})$ is smooth, it represents the functor, and $\operatorname{cod}(Y(\mathbf{D}), Y) = \operatorname{cod}(\mathbf{D})$ owing to Greuel and Lossen's [12, Cor. 5.1 a), p. 339]. Thus Parts (i) and (ii) hold in this case.

For the rest of the proof, suppose therefore that $\mathbf{D}$ does not consists of one vertex of multiplicity $m$. Then $C$ is not a union of $m$ concurrent lines. Now, $\deg C = m$. So $m \geq 3$.

Let $\tau^{\mathrm{es}}$ be the colength of the global equisingular ideal of $C$ in $\mathbf{P}^2$. If $4m > 4 + \tau^{es}$, then $Y(\mathbf{D})$ is smooth, it represents the functor, and $\operatorname{cod}(Y(\mathbf{D}), Y) = \tau^{es}$ owing to Greuel and Lossen's [12, Cor. 3.9 b) and d), p. 334], which applies since $C$ is not a union of $m$ concurrent lines and since $m \geq 3$.

Consider the multigerm of $C$ along its singular locus, a corresponding miniversal deformation base space $B$, and the subspace of equisingular deformations $B^{\mathrm{es}}$. Then $B^{\mathrm{es}}$ is smooth and $\operatorname{cod}(B^{\mathrm{es}}, B) = \tau^{es}$ owing to Wahl's [35, Thm. 7.4, p. 162]. However, $\operatorname{cod}(B^{es}, B) = \operatorname{cod}(\mathbf{D})$ by [19, Cor. (3.3), p. 222], (closely related formulas were given by Wall [36, Thm. 8.1, p. 505], by Mattei [23, Thm. (4.2.1), p. 323] and by T. de Jong [15, Thm. 3.5]; the present authors are grateful to T. de Jong for pointing out the first two references). Thus $\tau^{\mathrm{es}} = \operatorname{cod}(\mathbf{D})$.

Suppose $\operatorname{cod}(\mathbf{D}) \leq r$. Now, $r \leq 2m - 2$ by hypothesis. Also, $m \geq 2$; in fact, $m \geq 3$. So $2m - 2 < 4m - 4$. Hence $4m - 4 > \operatorname{cod}(\mathbf{D})$. So $4m > 4 + \tau^{es}$ by the preceding paragraph. By the paragraph before it, Part (i) therefore holds.

Suppose that $\operatorname{cod}(\mathbf{D}) \geq r + 1$ instead. If $4m - 4 > \operatorname{cod}(\mathbf{D})$, then as in the preceding case, $\operatorname{cod}(Y(\mathbf{D}), Y) = \operatorname{cod}(\mathbf{D})$, and so Part (ii) holds. So suppose that $4m - 4 \leq \operatorname{cod}(\mathbf{D})$. Now, $\operatorname{cod}(\mathbf{D}) \leq 2\delta(\mathbf{D})$ by Part (v) of Lemma 3.5. Hence $2m - 2 \leq \delta(\mathbf{D})$. Now, $\delta(\mathbf{D})$ is equal to the genus discrepancy by the Noether–Enriques theorem; see [19, Prop. (3.1), p. 220]. Hence $\operatorname{cod} Y(\mathbf{D}) \geq \delta(\mathbf{D})$, and if equality holds, then $\mathbf{D} = \delta(\mathbf{D})\mathbf{A}_1$, owing to Zariski's [37, Thm. 2, p. 220]. Now, for any $s$, we have $\operatorname{cod}(s\mathbf{A}_1) = s$ and $\delta(s\mathbf{A}_1) = s$. Therefore, if $\operatorname{cod} Y(\mathbf{D}) = \delta(\mathbf{D})$, then $\operatorname{cod} Y(\mathbf{D}) = \operatorname{cod}(\mathbf{D})$, and so Part (ii) holds in this case. However, if $\operatorname{cod} Y(\mathbf{D}) > \delta(\mathbf{D})$, then $\operatorname{cod} Y(\mathbf{D}) > 2m - 2$ since $2m - 2 \leq \delta(\mathbf{D})$, and so Part (ii) holds in any case. The proof is now complete. □

**Lemma 3.4** *Consider the cycle $U(r)$ of Theorem 2.5. If $m \geq r/2 + 1$, then $U(r)$ is reduced.*

Proof. By definition, $U(r)$ is the image on $Y$ of the fundamental cycle of the closure $\overline{Z(r)}$ of the intersection $Z(r) := H(r) \cap \operatorname{Hilb}^{3r}_{D/Y}$. By Lemma 2.6, $U(r)$ is reduced if $Z(r)$ is reduced on an open set $Z^0$ that dominates $Y(r\mathbf{A}_1)$. We now construct such a $Z^0$ by taking the inverse image of a suitable dense open subset $Y(r\mathbf{A}_1)^0$ of $Y(r\mathbf{A}_1)$, and then we prove that the map $Z^0 \to Y$ factors through the reduced scheme $Y(r\mathbf{A}_1)^0$ and that the induced map $Z^0 \to Y(r\mathbf{A}_1)^0$ is an isomorphism.

Take any dense open subscheme $Y^0$ of $Y$ such that $Y^0 \cap \overline{Y(r\mathbf{A}_1)} \subset Y(r\mathbf{A}_1)$, and denote the preimage of $Y^0$ in $Z(r)$ by $Z^0$. Taking $Y^0$ smaller if necessary, we may assume that the map $Z^0 \to Y^0$ is finite. Set $D^0 := D \times_Y Z^0$. Via the projection to $H(r)$, view $Z^0$ as the parameter space of a flat family of $r$ distinct points in the fibers of $F/Y$; denote the total space by $W^0$. Then, over a point of $Z^0$, the fiber of $W^0$ is just the set of $r$ nodes of the the fiber of $D^0$. Let $W^0_{(2)}$ be the infinitesimal thickening of $W^0$ defined by the square of its ideal. Since $Z^0 \subset Z(r)$, we have $W^0_{(2)} \subset D^0$.

Let $\beta\colon F^\star \to F \times_Y Z^0$ be the blowup along $W^0$. Set $E^\star := \beta^{-1}W^0$, so $E^\star$ is the exceptional divisor. Set $D^\star := \beta^{-1}D^0 - 2E^\star$. Then $D^\star$ is effective since $W^0_{(2)} \subset D^0$. Moreover, the fibers of $D^\star/Z^0$ are the proper transforms of the fibers of $D^0/Z^0$. Hence $D^\star/Z^0$ is smooth, and $(D^\star \cap E^\star)/Z^0$ is a family of $r$ pairs of distinct points. Thus, after localizing via the étale covering $W^0/Z^0$, we obtain an $r\mathbf{A}_1$-equisingular section of $D^0 \times_{Z^0} W^0/W^0$; in other words, $D^0/Z^0$ is an equisingular family of $r$-nodal curves. Now, $Y(r\mathbf{A}_1)$ represents the functor of such families by Part (i) of Lemma 3.3. Hence, the map $Z^0 \to Y$ factors through the reduced subscheme $Y(r\mathbf{A}_1)$, so through its dense open subscheme $Y^0 \cap Y(r\mathbf{A}_1)$. Set $Y(r\mathbf{A}_1)^0 := Y^0 \cap Y(r\mathbf{A}_1)$.

The map $Z^0 \to Y(r\mathbf{A}_1)^0$ is finite and surjective; moreover, its fibers have cardinality 1 by the analysis in the middle of Section 2. To prove that this map is an isomorphism, it suffices, since the characteristic is 0, to prove that each closed fiber is reduced. Suppose one isn't. Then it contains a copy of Spec(A) where $A := \mathbb{C} + \mathbb{C}\epsilon$ is the ring of dual numbers. Let $C$ be the $r$-nodal curve in question. Then $W^0 \otimes A/A$ is an étale family supported on the set of nodes of $C$. Furthermore, its infinitesimal thickening $W^0_{(2)} \otimes A$ is contained in $C \otimes A$.

This situation is untenable. Indeed, work locally analytically at one of the nodes of $C$. Choose coordinates $X$, $Y$ so that $C : XY = 0$. Say $W^0 \otimes A$ is defined by $X - a\epsilon = 0$ and $Y - b\epsilon = 0$. Then the ideal of $W^0_{(2)} \otimes A$ is generated by the three polynomials,

$$X^2 - 2a\epsilon X\,, \quad XY - \epsilon(aY + bX)\,, \quad Y^2 - 2b\epsilon Y\,.$$

However, this ideal does not contain $XY$. Thus the lemma is proved. □

**Lemma 3.5** *Let $\mathbf{D}$ be a minimal Enriques diagram with one root $R$. Let $\mathbf{S}$ be the set of satellites of $\mathbf{D}$, and set $e(\mathbf{D}) := \mu(\mathbf{D}) + m_R - 1$. Then*

(i) $m_R = r(\mathbf{D}) + \sum_{V\in\mathbf{S}} m_V$;

(ii) $\delta(\mathbf{D}) \le \operatorname{cod}(\mathbf{D})$, *with equality if and only if* $\mathbf{D} = \mathbf{A}_1$;

(iii) $\operatorname{cod}(\mathbf{D}) \le \mu(\mathbf{D})$, *with equality if and only if* $\mathbf{D}$ *is* $\mathbf{A}_k$, $\mathbf{D}_k$, $\mathbf{E}_6$, $\mathbf{E}_7$, *or* $\mathbf{E}_8$;

(iv) $\mu(\mathbf{D}) \le 2\delta(\mathbf{D})$, *with equality if and only if* $r(\mathbf{D}) = 1$;

(v) $\operatorname{cod}(\mathbf{D}) \le 2\delta(\mathbf{D})$, *with equality if and only if* $\mathbf{D}$ *is either* $\mathbf{A}_{2i}$, $\mathbf{E}_6$, *or* $\mathbf{E}_8$;

(vi) $2\delta(\mathbf{D}) \le e(\mathbf{D})$, *with equality if and only if* $m_R = r(\mathbf{D})$;

(vii) $e(\mathbf{D}) \le \operatorname{cod}(\mathbf{D}) + \delta(\mathbf{D})$, *with equality if and only if* $\mathbf{D} = \mathbf{A}_1$ *or* $\mathbf{D} = \mathbf{A}_2$;

(viii) $e(\mathbf{D}) \le 2\operatorname{cod}(\mathbf{D})$, *with equality if and only if* $\mathbf{D} = \mathbf{A}_1$.

Proof. Consider Part (i). Let $V$ and $W$ be vertices, and write $W \succ_{\mathrm{imm}} V$ if $W$ is an immediate successor of $V$. If not, but $W$ is proximate to $V$, then $W$ is a satellite of $V$. By the "law of proximity," $W$ cannot also be a satellite of a second vertex. Now, by definition, $r(\mathbf{D}) := \sum_V \bigl(m_V - \sum_{W\succ V} m_W\bigr)$. Hence

$$r(\mathbf{D}) = \sum_V \left( m_V - \sum_{W\succ_{\mathrm{imm}}V} m_W \right) - \sum_{V\in\mathbf{S}} m_V\,.$$

In the first sum, all the terms cancel except $m_R$. Thus Part (i) holds.

Consider Part (ii). Denote the set of free vertices other than $R$ by $\mathbf{F}$. Since $R$ is the only root, the definitions yield

$$\begin{aligned} \operatorname{cod}(\mathbf{D}) - \delta(\mathbf{D}) &= \sum_V \left( \binom{m_V+1}{2} - \binom{m_V}{2} \right) - 1 - \operatorname{frs}(\mathbf{D}) \\ &= \sum_V m_V - 1 - \operatorname{frs}(\mathbf{D}) \\ &= (m_R - 2) + \sum_{V\in\mathbf{F}} (m_V - 1) + \sum_{V\in\mathbf{S}} m_V\,. \end{aligned} \tag{3.1}$$

Since $m_R \ge 2$, the last term is nonnegative. Moreover, it vanishes if and only if $m_R = 2$, and $m_V = 1$ for all $V \in \mathbf{F}$, and there are no satellites. However, the latter three conditions hold if and only if $\mathbf{D} = A_1$; see [19, Section 2]. Thus Part (ii) holds.

Consider Part (iii). The definitions yield

$$\mu(\mathbf{D}) - \operatorname{cod}(\mathbf{D}) = 2\delta(\mathbf{D}) - \deg(\mathbf{D}) + 2 + \operatorname{frs}(\mathbf{D}) - r(\mathbf{D})\,.$$

Now, $\operatorname{frs}(\mathbf{D})$ is simply the total number of vertices less the number of satellites; so

$$\operatorname{frs}(\mathbf{D}) = \sum_V 1 - \sum_{V\in\mathbf{S}} 1\,.$$

Hence Part (i) yields

$$\begin{aligned}\mu(\mathbf{D})-\operatorname{cod}(\mathbf{D}) &= \sum_V\left(2\binom{m_V}{2}-\binom{m_V+1}{2}+1\right)+(2-m_R)+\sum_{V\in\mathbf{S}}(m_V-1)\\ &= \binom{m_R-2}{2}+\sum_{V\in\mathbf{F}}\binom{m_V-1}{2}+\sum_{V\in\mathbf{S}}\binom{m_V}{2}.\end{aligned}$$

The last term is nonnegative. Moreover, it vanishes if and only if (1) $m_R$ is 2 or 3, and (2) $m_V$ is 1 or 2 for all $V\in\mathbf{F}$, and (3) $m_V$ is 1 for all satellites $V$. However, the latter three conditions hold if and only if $\mathbf{D}$ is either $\mathbf{A}_k$, $\mathbf{D}_k$, $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$; see [19, Section 2]. Thus Part (iii) holds.

Part (iv) follows immediately from the definition of $\mu(\mathbf{D})$ since $r(\mathbf{D})\geq 1$.

Part (v) follows immediately from Parts (iii) and (iv) and from Table 2-1 of [19, p. 219], which lists the values of $r(\mathbf{D})$ for all the $\mathbf{D}$ in question.

Consider Part (vi). The definitions of $e(\mathbf{D})$ and $\mu(\mathbf{D})$ yield

$$e(\mathbf{D})-2\delta(\mathbf{D}) = m_R-r(\mathbf{D}). \tag{3.2}$$

Now, $m_R\geq r(\mathbf{D})$ by Part (i), and Part (vi) follows.

Consider Part (vii). Together (3.1) and (3.2) yield

$$\operatorname{cod}(\mathbf{D})+\delta(\mathbf{D})-e(\mathbf{D}) = r(\mathbf{D})-2+\sum_{V\in\mathbf{F}}(m_V-1)+\sum_{V\in\mathbf{S}}m_V.$$

Suppose $r(\mathbf{D})\geq 2$. Then the right side is nonnegative. Since every free vertex of multiplicity 1 must be followed by a satellite, the right side vanishes if and only if $r(\mathbf{D})=2$ and there are no other vertices than the root, hence, if and only if $\mathbf{D}=\mathbf{A}_1$.

Suppose $r(\mathbf{D})=1$. Then there is at least one satellite by Part (i). Hence, the right side is nonnegative. It vanishes precisely when there is just one free vertex other than $R$ and just one satellite, and both have weight 1. The latter condition implies $m_R=2$ by Part (i) since $r(\mathbf{D})=1$. So the condition holds if and only if $\mathbf{D}=\mathbf{A}_2$. Thus Part (vii) holds.

Finally, Part (viii) follows immediately from Parts (vii) and (ii). Thus the lemma is proved. □

**Remark 3.6** Lemma 3.5 is purely combinatorial. However, it can be interpreted geometrically, as asserting properties of an arbitrary curve $C$ belonging to $Y(\mathbf{D})$. Indeed, $r(\mathbf{D})$, $\delta(\mathbf{D})$, and the other numerical characters of $\mathbf{D}$ are equal to corresponding characters of $C$. All but $e(\mathbf{D})$ were treated in [19, Section 3]; however, $e(\mathbf{D})$ is equal to the multiplicity of the Jacobian ideal, owing directly to Teissier's work [33, 1.6, p. 300].

Correspondingly, Lemma 3.5 can be proved via alternative geometric arguments. For example, the inequality $\operatorname{cod}(\mathbf{D})\leq\mu(\mathbf{D})$ of Part (iii) just says that the modality $\operatorname{mod}(C)$ is nonnegative. Indeed, $\operatorname{mod}(C)=\mu(C)-\tau^{\mathrm{es}}$ by Greuel and Lossen's [12, Lem. 1.3, p. 326], and $\tau^{\mathrm{es}}=\operatorname{cod}(\mathbf{D})$ by the proof of Lemma 3.3 above. Alternatively, $\mu(C)\geq\tau^{\mathrm{es}}$ because the equisingular ideal contains the Jacobian ideal by Wahl's work [35]; see the proof of Prop. 6.1, top of p. 159.

Similarly, the inequality $e(\mathbf{D})\leq 2\operatorname{cod}(\mathbf{D})$ of Part (viii) holds because the equisingular, equiclassical, and equigeneric ideals form an ascending chain. Indeed, the inequality was proved this way by Greuel, Lossen, and Shustin in [13, Lem. 2.2, p. 601]. (However, there's a typo in the proof: the colengths of the equiclassical and equigeneric ideals are transposed.) These authors and others write "$\kappa$" instead of "$e$".

**Remark 3.7** The formula $N_1(m)=3m^2-6m+3$ was given by Steiner [31, p. 499] in 1848, but it was probably known earlier. After all, $N_1(m)$ is simply the number of singular members of a general pencil of plane curves of degree $m$. So $N_1(m)$ is just the degree of the discriminant of a general ternary form of degree $m$, viewed as a polynomial in its coefficients, because the discriminant is the resultant (or "eliminant") of the three partials.

The formula $N_2(m)=3/2(m-1)(m-2)(3m^2-3m-11)$ was given by Cayley [5, Art. 33, p. 306] in 1863. He considered a varying pencil, and formed its "double discriminant." One factor has $N_2(m)$ as its degree. Cayley determined the degrees of the other two factors and the degree of the double discriminant, then he divided.

The same formula was found a little differently by Salmon [29, Appendix IV, p. 506] (there is a typo: a "1" instead of an "11"). Salmon acknowledged Cayley's work, saying: "Mr. Cayley had arrived at these numbers by a different process in a Memoir communicated to the Cambridge Philosophical Society, but not yet published." First, Salmon computed the number of curves with either two nodes or one cusp, $9/2(m-1)(m-2)\big(m^2-m-1\big)$; then he subtracted the number of curves with one cusp, $12(m-1)(m-2)$. (See also the bottom of p. 361 in Salmon's [30].)

The formula for $N_2(m)$ was recovered implicitly via a third method by S. Roberts [27, p. 276] in 1867. At the bottom of p. 275, he said his work agrees with Salmon's.

The formula $N_3(m) = 9/2m^6 - 27m^5 + 9/2m^4 + 423/2m^3 - 229m^2 - 829/2m + 525$ was given implicitly by Roberts [28, pp. 111–112] in 1875. His primary interest lay in his new method for obtaining the degree of the polynomial condition that three ternary equations have three common solutions. As an application, he discussed the theory of the reciprocal, or dual, surface of a surface of degree $m$ in $\mathbf{P}^3$. In effect, he determined the numbers $\beta$ of curves with one tacnode and $\gamma$ of curves with one node and one cusp; he explicitly gave the formulas,

$$\begin{aligned}\beta &= 50m^2 - 192m + 168\,;\\ \gamma &= 12(m-3)\big(3m^3 - 6m^2 - 11m + 18\big)\,.\end{aligned}$$

He also explicitly gave the formula,

$$\beta + \gamma + N_3(m) \;=\; 1/2\big(9m^6 - 54m^5 + 81m^4 + 63m^3 - 190m^2 + 11m + 90\big)\,.$$

However, he did not solve for $N_3(m)$, which he denoted by $t$.

The formulas for $N_1(m)$, $N_2(m)$, and $N_3(m)$ were recovered implicitly, and analogous formulas for $N_4(m)$, $N_5(m)$, and $N_6(m)$ were obtained explicitly, by Vainsencher [34, p. 515] in 1995. Vainsencher did not discuss the validity of these particular formulas, but his general results, Propositions 3.5 and 4.1, do imply that there exists some undetermined $m_0$ such that, for $m > m_0$, the formulas do hold.

The formulas for $N_1(m)$, $N_2(m)$, and $N_3(m)$ were recovered explicitly by Harris and Pandharipande [14] later in 1995. They used an interesting new method, involving the geometry of the Hilbert scheme of points in $\mathbf{P}^2$ and the Bott residue formula. In 1997, in the paragraph before Definition 3.4 and in Prop. 3.6 in [6], Choi derived from Ran's Theorem 5 in [26] that $N_r(m)$ is, for every $r$ and $m > r$, given by a polynomial in $m$ of degree $2r$.

A recursive formula was obtained by Caporaso and Harris [4, p. 353] in 1998, which makes it possible to compute $N_r(m)$ for every $r$ and $m$. From this formula, though, it is not at all clear that, when $r$ is fixed, $N_r(m)$ is given by a polynomial of degree $2r$ in $m$ for $m > m_0$ for some $m_0$. Nevertheless, as Göttsche observed in [11, Rmk. 4.2, p. 530], and Choi observed in [6, Rmk. 3.5.2], if it is assumed that $N_r(m)$ is given by such a "node" polynomial for a known $m_0$, then it is possible to work out the coefficients.

Given a specific value for $m_0$, such as Choi's value $m_0 = r$ mentioned above or the value $m_0 = 3r - 1$ for $r \le 8$ given in Thm. (1.1) of [19], it would be possible to use Caporaso and Harris's formula to check the validity of the values given by the polynomial for $m_0 \ge m \ge r/2 + 1$. Thus, given an $r$ and an $m_0$, it would be possible to prove Göttsche's conjecture [11, Conj. 4.1], and so, for $r \le 8$, to obtain another proof of Theorem 3.1.

**Example 3.8** It is useful to look at some basic examples. First, note that, for any given $m$, there are several special ranges for $r$. For $r \le \min(2m-2,\, 8)$, the formula $N_r(m) = P_r(m)/r!$ holds by Theorem 3.1. For $r = 2m-1$, the sets $Y(r\mathbf{A}_1)$ and $Y(\infty)$ have the same dimension when both are nonempty; see the first part of the proof of that theorem. Both sets are empty when $m = 1$, but $Y(\infty)$ is nonempty for $m \ge 2$. For $r > (m-1)(m-2)/2$, if $y \in Y(r\mathbf{A}_1)$, then $D_y$ is reducible; otherwise, $D_y$ would have strictly negative geometric genus since $(m-1)(m-2)/2$ is equal to its arithmetic genus $p$. For $r = m(m-1)/2$, if $y \in Y(r\mathbf{A}_1)$, then $D_y$ is the union of $m$ lines. Finally, for $r > m(m-1)/2$, the set $Y(r\mathbf{A}_1)$ is empty; indeed, applying the argument in the middle of Lemma 2.4 with $C := D_y$, we see that $r \le p + m - 1$, with equality if and only if $D_y$ has $m$ irreducible components.

Suppose $m = 1$. Then $D_y$ is a line for every $y \in Y$. So $N_0(1) = 1$, and $N_r(1) = 0$ for $r \ge 1$. On the other hand, direct computation yields $P_0(1) = 1$, and $P_r(1) = 0$ for $r = 1, 2$, but $P_3(1)/3! = 75$. Thus $N_r(1) = P_r(1)/r!$ holds for $r = 0, 1, 2$ but not for $r = 3$. The hypothesis $m \ge r/2 + 1$ of Theorem 3.1 fails for $r \ge 1$. However, the hypotheses of Theorem 2.5 are satisfied for every $r$. Hence, the proof of Theorem 3.1

shows that the formula $N_r(1) = P_r(1)/r!$ must hold for $r = 0, 1, 2$. For $r \geq 3$, the value of $P_r(1)$ is irrelevant since $U(r)$ vanishes by reason of dimension.

Suppose $m = 2$. Then $N_r(2) = P_r(2)/r!$ holds for $r = 0, 1, 2$ by Theorem 3.1. If $y \in Y(\mathbf{A}_1)$, then $D_y$ is a line-pair. So $N_1(2)$ is the number of line-pairs through four points in general position. This number is $\binom{4}{2}/2$, or 3, since two of the four points will determine one of the lines, and the remaining two points will determine the other. Now, $Y(\mathbf{D})$ is empty for any $\mathbf{D}$ other than $\mathbf{A}_1$. So $N_r(2) = 0$ for $r \geq 2$. On the other hand, direct computation yields $P_1(2) = 3$ and $P_2(2) = 0$, but $P_3(2)/3! = -32$. Thus $N_r(2) = P_r(2)/r!$ checks for $r = 1, 2$.

The equation $N_r(2) = P_r(2)/r!$ fails, however, for $r = 3$, although Theorem 2.5 nearly applies. Indeed, all the relevant $Y(\mathbf{D})$ are empty, and $Y(\infty)$ has its expected codimension, namely 3, but Hypothesis (i) requires $\operatorname{cod} Y(\infty) > 3$. In fact, $Y(\infty)$ is the Veronese surface in $Y = \mathbf{P}^5$. So two general hyperplanes intersect each other and $Y(\infty)$ in four distinct points. If each hyperplane represents the conics that contain a given point in $\mathbf{P}^2$, then the four points of intersection coalesce in the point that represents the double-line through the two points in $\mathbf{P}^2$. Since $P_3(2)/3! = -32$, this double-line may be interpreted as four coincident double-lines, each the equivalent of $-8$ three-nodal conics.

Finally, consider the case $m = 5$ and $r = 8$. Here, $N_8(5)$ is the number of 8-nodal quintics through 12 points in general position. Since $8 > (5-1)(5-2)/2$, these quintics are reducible. So each is either the union of two smooth conics and a line, or the union of a nodal cubic and a line-pair. Hence

$$N_8(5) \;=\; \frac{\binom{12}{5}\binom{7}{5}}{2} + N_1(3)\,\frac{\binom{12}{8}\binom{4}{2}}{2} \;=\; 8316 + 17820 \;=\; 26136\,.$$

Therefore, Theorem 3.1 implies that $P_8(5) = 26136 \times 8!$.

The value of $P_8(5)$ can be used to determine the multiplier $3281 \cdot 7!$ of $x_4$ in $b_8$ in Algorithm 2.3. Indeed, as indicated in [19, Section 4], residual intersection theory shows that $x_4$ appears with some multiplier, whose value does not vary with $F/Y$ and $D$. Hence this value can be determined from any particular example that can be worked out by other means.

## 4 Threefolds in four-space

In this section, we enumerate the 6-nodal plane curves on a general threefold in 4-space, recovering Vainsencher's formula. In fact, we correct a typo: the multiplier 5 appearing in Theorem 4.1 is lacking in [34, p. 522]. More importantly, we establish the formula's validity, which was left unaddressed in [34]. Then, in Theorem 4.3, we rederive the number of irreducible 6-nodal plane quintic curves on a general quintic threefold; again we follow Vainsencher's approach [34, pp. 523–524] (recovering his number), but also establish the number's validity. Its significance is recalled in the introduction.

**Theorem 4.1** *In $\mathbf{P}^4$, consider a general threefold $Q$ of degree $m$. If $m \geq 4$, then $Q$ contains precisely the following number of* 6*-nodal plane curves of degree $m$:*

$$\begin{aligned}&5(m^{18} - 12m^{17} + 24m^{16} + 155m^{15} - 405m^{14} + 1082m^{13} - 18469m^{12} + 66446m^{11}\\ &\quad - 192307m^{10} + 1242535m^9 - 4049006m^8 + 11129818m^7 - 53664614m^6 + 166756120m^5\\ &\quad - 415820104m^4 + 1293514896m^3 - 2517392160m^2 + 1781049600m)/6!\,.\end{aligned}$$

Proof. Let's apply Theorem 2.5 again. Fix $r \leq 6$. Let $Y$ be the Grassmann variety of 2-planes $H$ in $\mathbf{P}^4$, and $\mathcal{Q}$ the tautological rank-3 quotient of $\mathcal{O}_Y^5$. Let

$$F \;:=\; \mathbf{P}(\mathcal{Q}) \subset Y \times \mathbf{P}^4 \tag{4.1}$$

be the total space of planes, $\pi\colon F \to Y$ and $p\colon F \to \mathbf{P}^4$ the projections. Set

$$D \;:=\; F \cap (Y \times Q)\,.$$

Since $Q$ is smooth, $Q$ contains no 2-plane $H$; otherwise, there'd be a normal-bundle surjection, $\mathcal{N}_{H/\mathbf{P}^4} \twoheadrightarrow \mathcal{N}_{Q/\mathbf{P}^4}|H$, in other words, $\mathcal{O}_H(1)^2 \twoheadrightarrow \mathcal{O}_H(m)$; so $3 \times 2 \geq (m+2)(m+1)/2$, contradicting $m \geq 3$. Since $Q$ contains no $H$, each $D_y$ is a curve of degree $m$. Hence $D$ is a relative effective divisor on $F/Y$. Moreover,

$$\mathcal{O}_F(D) \;=\; p^*\mathcal{O}_{\mathbf{P}^4}(m)\,. \tag{4.2}$$

Consider the set $Y(\infty)$ of $y \in Y$ such that the curve $D_y$ has a multiple component. To check Hypothesis (i) of Theorem 2.5, it suffices to prove $\operatorname{cod} Y(\infty) = 2m-1$ if $Y(\infty) \neq \emptyset$, because $2m-1 > r$ as $m \geq 4$ and $r \leq 6$. Let's count constants.

Let $U$ be the scheme of all smooth quintic threefolds, and $P$ the scheme of all quintic curves in the fibers of $F/Y$ $\big($so $P$ is $\mathbf{P}\big(Sym^5(\mathcal{Q}^*)\big)\big)$. Form the natural map,

$$\lambda : U \times Y \longrightarrow P ;$$

it's given by $\lambda(Q, H) = Q \cap H$. Now, $\lambda$ is the restriction to $U \times Y$ of a family over $Y$ of linear projections; each has, as domain, the projective space of all quintic threefolds, and as center at $H \in Y$, the subspace of threefolds containing the plane $H$. Hence, $\lambda$ is smooth, and has irreducible fibers (conceivably $\lambda$ is not surjective).

Let $P_\infty \subset P$ be the subset of quintic curves with a multiple component. The fibers of $P_\infty/Y$ have codimension $2m-1$ by the analysis at the beginning of the proof of Theorem 3.1. So, $\lambda^{-1}P_\infty$ has codimension $2m-1$ if it's nonempty. Now, $Y(\infty)$ is just the fiber of $\lambda^{-1}P_\infty$ over $Q \in U$, and $Q$ is general. Hence $\operatorname{cod} Y(\infty) = 2m-1$ if $Y(\infty) \neq \emptyset$, as desired. Thus Hypothesis (i) of Theorem 2.5 is satisfied.

Hypothesis (ii) is also satisfied, as the same argument shows. Indeed, in the plane, the corresponding condition is satisfied by Lemma 3.3, which applies since $m \geq 6/2 + 1$.

Therefore, we may apply Theorem 2.5. We conclude that the closure $\overline{Y(r\mathbf{A}_1)}$ is the support of a nonnegative cycle $U(r)$, whose class $[U(r)]$ is given by the formula

$$[U(r)] \;=\; P_r(a_1, \ldots, a_r)/r! \quad \text{where} \quad a_q \;:=\; \pi_* b_q$$

and $b_q$ is the polynomial in $v$, $w_1$, $w_2$ output by Algorithm 2.3.

In the present case, $U(r)$ is reduced, as the same argument shows when developed a little further. Indeed, in the plane, the corresponding cycle is reduced by Lemma 3.4. Moreover, by Lemma 2.6, $U(r)$ is reduced if and only if $Z(r)$ is reduced on an open set that dominates $Y(r\mathbf{A}_1)$. So consider the open subset on which $Z(r)$ is reduced. This subset can be shown to dominate $Y(r\mathbf{A}_1)$ by developing the same argument as the formation of $Z(r)$ and $Y(r\mathbf{A}_1)$ is compatible with the constructions involved. Therefore, the desired number of curves is equal to the degree $\int [U(6)]$.

To compute the degree $\int [U(6)]$, set

$$h \;:=\; c_1\big(\mathcal{O}_{\mathbf{P}^4}(1)\big) \quad \text{and} \quad q_i \;:=\; c_i(\mathcal{Q}) .$$

Recall the Euler exact sequence:

$$0 \longrightarrow \Omega^1_{F/Y}(1) \longrightarrow \pi^*\mathcal{Q} \longrightarrow \mathcal{O}_F(1) \longrightarrow 0 .$$

It and Formula (4.2) yield the following formulas:

$$v \;=\; mp^*h \quad \text{and} \quad w_1 \;=\; \pi^*q_1 - 3p^*h \quad \text{and} \quad w_2 \;=\; \pi^*q_2 - 2p^*h\pi^*q_1 + 3p^*h^2 . \tag{4.3}$$

By reason of dimension, $h^j = 0$ for $j > 4$, and $\pi_*p^*h^j = 0$ for $j = 0, 1$; moreover, it is well-known and easy to see that

$$\pi_*p^*h^2 \;=\; [Y] \quad \text{and} \quad \pi_*p^*h^3 \;=\; q_1 \quad \text{and} \quad \pi_*p^*h^4 \;=\; q_1^2 - q_2 . \tag{4.4}$$

It is now a mechanical matter to compute the $a_q$, and then $[U(6)]$, as polynomials in $m$, $q_1$, and $q_2$. Finally, standard Schubert calculus on $Y$ yields the following degrees:

$$\int q_1^6 \;=\; 5 , \quad \int q_1^4 q_2 \;=\; 3 , \quad \int q_1^2 q_2^2 \;=\; 2 , \quad \int q_2^3 \;=\; 1 . \tag{4.5}$$

It is now a mechanical matter to complete the proof. □

**Remark 4.2** Other enumerations fall out of the proof of Theorem 4.1. Indeed, the proof shows that the cycle $U(r)$ is reduced for every $r$, and that its class can be computed mechanically as a polynomial in $m$, $q_1$, and $q_2$. It follows, for example, that in $\mathbf{P}^4$ a general threefold of degree $m \geq 4$ contains precisely

$$5m^9/6 - 5m^8 + 11m^7/2 + 23m^6/6 + 17m^5 + 359m^4/6 - 1165m^3/3 + 1024m^2/3 + 40m$$

3-nodal plane curves whose plane meets three lines in general position.

Indeed, these curves are enumerated by the intersection of $U(3)$ and the three special Schubert cycles defined by the three lines. The intersection is reduced by the theorem of transversality of the general translate [18, Cor. 4, p. 291] (which applies because the characteristic is 0). The class of each Schubert cycle is $q_1$. Hence the number of curves is just $\int [U(3)]q_1^3$, and its value can be computed mechanically.

**Theorem 4.3** *In* $\mathbf{P}^4$*, a general quintic threefold contains precisely* $17,601,000$ *irreducible* $6$*-nodal plane curves of degree* $5$*.*

Proof. (Compare with Vainsencher [34, pp. 523–524].) For $m = 5$, the formula in Theorem 4.1 yields the number 21617125. From it, we must subtract the number of reducible curves. Each is, plainly, one of two types: (1) the transversal union of a smooth conic and a smooth cubic, or (2) the transversal union of a line and a binodal quartic. Each reducible curve is counted with multiplicity 1; indeed, all 21617125 curves are by the theorem. So we may count set-theoretically.

A general quintic $Q$ contains precisely 609250 smooth conics, thanks to the work of S. Katz (see [17, (2), p. 175]). Each conic determines a plane, and it meets $Q$ partially in the conic and residually in a cubic. The cubic is smooth and meets the conic transversally because $Q$ is generic. Indeed, form the space of all triples $(H, A, B)$ where $H$ is a plane, $A$ is a conic in $H$, and $B$ is a cubic in $H$. Form the subset $U$ of $(H, A, B)$ such that $A$ and $B$ are smooth and meet transversally in $H$. Clearly $U$ is open and dense. So its complement $R$ has smaller dimension. Hence, counting constants as in the proof of Theorem 4.1, we conclude that there is no $(H, A, B) \in R$ such that $A \cup B \subset Q$. Thus the first subtrahend is 609250.

A general quintic $Q$ contains precisely 2875 lines (see [17, (1), p. 175]). Fix one, $L$ say. We must enumerate those planes $H$ that meet $Q$ partially in $L$ and residually in a binodal quartic that meets $L$ transversally. We do so by building on the proof of Theorem 4.1; in particular, we use its notation.

In the Grassmannian $Y$ of all planes $H$, form the Schubert variety $Y_L$ of all $H$ that contain $L$. Correspondingly, in the total space $F$, form the preimage $F_L := \pi^{-1}Y_L$. Also, in the space $U$ of all smooth quintic threefolds, form the subspace $U_L$ of those that contain $L$. Then $Q \in U_L$. In fact, we may assume that $(Q, L)$ is a general pair in the space of all such pairs, for this space is irreducible by Katz's Lemma 1.4 of [16, p. 153]. In particular, $Q$ is a general point of $U_L$. Finally, set

$$D_L := F_L \cap (Y_L \times Q) \quad \text{and} \quad D' := D_L - (Y_L \times L)\,; \tag{4.6}$$

so the fibers of $D_L/Y_L$ are the quintic curves cut out by $Q$, and the fibers of $D'/Y_L$ are the residual quartic curves.

In the space $P$ of all quintic curves in the fibers of $F/Y$, form the subspace $P_L$ of those with $L$ as a component, and form the natural map,

$$\lambda_L : U_L \times Y_L \longrightarrow P_L\,,$$

the restriction of $\lambda$. So $\lambda_L$ is the restriction to $U_L \times Y_L$ of a family over $Y_L$ of linear projections; each has, as domain, the projective space of all quintic threefolds containing $L$. In the latter projective space, $U_L$ is open and nonempty. Hence, $\lambda_L$ is smooth, and has irreducible fibers.

Let's apply Theorem 2.5 with $r := 2$ to $D'/Y_L$. Hypotheses (i) and (ii) can be checked by counting constants just as in the proof of Theorem 4.1, after replacing $Y$, $U$ and $P$ with $Y_L$, $U_L$ and $P_L$, because $P_L$ may be viewed as the space of all quartics in the fibers of $F_L/Y_L$. By the same token, the cycle of binodal quartics $D'_y$ is reduced. Similarly, each $D'_y$ meets $L$ transversally, because, given a plane $H$ containing $L$, in the space of binodal quartics in $H$, those tangent to $L$ form a closed subset of codimension 1, as yet another count of constants shows. Therefore, we may apply Theorem 2.5, and use it as follows to get the multiplier of 2875.

We must find the classes $v' := [D']$ and $w'_i := c_i(\Omega^1_{F_L/Y_L})$. First, $w'_i = w_i|F_L$ since $F_L := \pi^{-1}Y_L$. Now, $F := \mathbf{P}(\mathcal{Q})$ by (4.1), so $F_L = \mathbf{P}(\mathcal{Q}|Y_L)$. Hence, the inclusion of $Y_L \times L$ in $F_L$ corresponds to a surjection $\mathcal{Q}|Y_L \twoheadrightarrow \mathcal{O}^2_{Y_L}$. Denote its kernel by $\mathcal{K}$. Then there is a natural exact sequence,

$$\mathcal{K} \otimes \mathit{Sym}(\mathcal{Q}|Y_L)[-1] \longrightarrow \mathit{Sym}(\mathcal{Q}|Y_L) \longrightarrow \mathit{Sym}(\mathcal{O}^2_{Y_L}) \longrightarrow 0\,.$$

Passing to associated sheaves, we see that the image of $(\pi_L^*\mathcal{K})(-1)$ in $\mathcal{O}_{F_L}$ is just the ideal of $L$, where $\pi_L\colon F_L \to Y_L$ is the structure map. Hence $[L] = -c_1\big((\pi_L^*\mathcal{K})(-1)\big)$. Now, we have $c_1(\mathcal{K}) = c_1(\mathcal{Q}|Y_L) = q_1|Y_L$. Hence $[L] = (p^*h - \pi^*q_1)|F_L$. Therefore, (4.6) and (4.3) yield $v' = (4p^*h + \pi^*q_1)|F_L$.

Finally, by Theorem 2.5 and the projection formula, the desired multiplier is equal to $(1/2)\deg P_2(a_1, a_2)$ where, by the projection formula,

$$a_q = [Y_L] \cdot \pi_* b_q(4p^*h + \pi^*q_1, w_1, w_2)\,.$$

Now, by standard Schubert calculus, $[Y_L] = (q_1^2 - q_2)^2$. Owing to (4.3), to (4.4) and to (4.5), it is now a mechanical matter to compute the multiplier; its value is 1185. Therefore, the number of 6-nodal degree 5 plane curves on $Q$ is

$$21617125 - 609250 - 2875 \times 1185 = 17601000\,,$$

and the proof is complete. □

## 5 Abelian surfaces

Fix an Abelian surface $A$. In this section, assuming certain genericity conditions, we enumerate the reduced and irreducible curves $C \subset A$ satisfying these three conditions: they lie in a given algebraic homology class with positive self-intersection, $d$ say; they have given geometric genus $g$; and they pass through the appropriate number of general points. The appropriate number is $g$, as we see while proving Theorem 5.2, our main result of this section.

Each such curve $C$ must have a certain number of singular points, $r$ say; we prove that, under our genericity conditions, all $r$ points are ordinary nodes. Since $g + r$ is the arithmetic genus of $C$, and since the canonical class of $A$ is trivial, the numbers $d$, $g$, and $r$ are, owing to the adjunction formula, related by the equation

$$d = 2g + 2r - 2\,.$$

The number of these $C$ turns out to depend only on $r$ and $g$, and not on $A$; here we must also assume $r \le 8$. So, let's denote the number of $C$, as Bryan and Leung [3] do, by $N_{g,r}$. (They impose no bound on $r$, but do require $A$ to be generic among the Abelian surfaces for which the given homology class is algebraic; whence this class must generate $\mathrm{Pic}(A)/\mathrm{Pic}^0(A)$.) For $r$ fixed, $N_{g,r}$ is given by a "node" polynomial in $g$ of degree $r + 1$. The polynomials are presented in Table 5.1.

**Table 5.1. Node polynomials for $N_{g,r}$**

$$\begin{aligned}
N_{g,0} &= g\\
N_{g,1} &= 6g(g-1)\\
N_{g,2} &= 6g(g-1)(3g-4)\\
N_{g,3} &= 4g(g-1)\big(9g^2 - 27g + 25\big)\\
N_{g,4} &= 6g(g-1)\big(9g^3 - 45g^2 + 94g - 75\big)\\
N_{g,5} &= 12g(g-1)\big(27g^4 - 198g^3 + 687g^2 - 1213g + 860\big)/5\\
N_{g,6} &= 4g(g-1)\big(81g^5 - 810g^4 + 4095g^3 - 11835g^2 + 18409g - 11800\big)/5\\
N_{g,7} &= 24g(g-1)\big(81g^6 - 1053g^5 + 7200g^4 - 29970g^3 + 75814g^2 - 106347g + 62685\big)/35\\
N_{g,6} &= 3g(g-1)\big(486g^7 - 7938g^6 + 69930g^5 - 389970g^4 + 1413384g^3 - 3216332g^2\\
&\qquad + 4143290g - 2279375\big)/35
\end{aligned}$$

Our genericity conditions are specified in the following theorem.

**Theorem 5.2** *In the above setup, assume $A$ has Picard number* 1*, and say the given homology class is $m$ times the positive primitive class. Assume either*
(i) *that $m = 1$ and $g > 5r + 7$ or*
(ii) *that $m \geq 2$ and $g > \big(3m^2r + 3m^2 - 2mr + 2m + 2r - 2\big)/(2m - 2)$.*
*Then the formulas in Table* 5.1 *are valid.*

Proof. Yet again, we apply Theorem 2.5. Let $Y$ be the connected component of $C$ in the Hilbert scheme of $A$. Then $Y$ parameterizes the curves algebraically equivalent to $C$; hence, $Y$ parameterizes the curves homologically equivalent to $C$, since an Abelian variety has no torsion ([2, Prop. 7.1, p. 59]). Set $F := A \times Y$, and let $D \subset F$ be the universal divisor. To handle $Y$ and $D$, we need a well-known description of them, which we now recall.

Fix an invertible sheaf $\mathcal{L}$ on $A$ representing the given homology class. Denote by $\widehat{A}$ the dual abelian surface, and by $\mathcal{P}$ the Poincaré bundle on $A \times \widehat{A}$, which is trivial along $0 \times \widehat{A}$ and $A \times 0$. On $\widehat{A}$, form the direct image

$$\mathcal{Q} \;:=\; p_{2*}\big(\mathcal{P} \otimes p_1^*\mathcal{L}\big)\,. \tag{5.1}$$

If $\mathcal{N}$ is a fiber of $\mathcal{P} \otimes p_1^*\mathcal{L}$, then $H^q(\mathcal{N}) = 0$ for $q \geq 1$ by the Kodaira vanishing theorem, because $\mathcal{N}$ is algebraically equivalent to $\mathcal{L}$, so ample, and because the canonical bundle of $A$ is trivial. Hence $\mathcal{Q}$ is locally free, and its formation commutes with base change.

Because of this commutativity, the rank of $\mathcal{Q}$ may be determined on the fibers, where we may use the Riemann–Roch theorem and the vanishing of the higher cohomology groups; thus,

$$\mathrm{rk}(\mathcal{Q}) \;=\; \int c_1(\mathcal{L})^2/2 \;=\; d/2\,. \tag{5.2}$$

Then $Y := \mathbf{P}(\mathcal{Q}^*)$ where $\mathcal{Q}^*$ is the dual. Hence $Y$ is smooth and irreducible, and

$$\dim Y \;=\; \mathrm{rk}\, Q - 1 + 2 \;=\; d/2 + 1 \;=\; g + r\,.$$

Thus $g$ is the appropriate number, as asserted in the first paragraph of the section.

To construct $D \subset F := A \times Y$, form the natural Cartesian diagram:

$$\begin{array}{ccc} F & \xrightarrow{1\times p} & A \times \widehat{A} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle p_2} \\ Y & \xrightarrow{\;\;p\;\;} & \widehat{A} \end{array}$$

On $Y$, the tautological map $p^*\mathcal{Q}^* \to \mathcal{O}_Y(1)$ and the base-change isomorphism induce the composition

$$\mathcal{O}_Y(-1) \;\longrightarrow\; p^*p_{2*}\big(\mathcal{P} \otimes p_1^*\mathcal{L}\big) \;\xrightarrow{\sim}\; \pi_*(1 \times p)^*\big(\mathcal{P} \otimes p_1^*\mathcal{L}\big)\,.$$

Form its adjoint on $F$,

$$\alpha \,:\; \pi^*\mathcal{O}_Y(-1) \;\longrightarrow\; (1 \times p)^*\big(\mathcal{P} \otimes p_1^*\mathcal{L}\big)\,.$$

The zero locus of $\alpha$ is the universal divisor $D/Y$. Therefore,

$$\mathcal{O}_F(D) \;=\; (1 \times p)^*\mathcal{P} \otimes p_1^*\mathcal{L} \otimes \pi^*\mathcal{O}_Y(1) \tag{5.3}$$

where now $p_1 \colon F \to A$ is the projection.

In order to apply Theorem 2.5, we must check its hypotheses, (i) and (ii). Each of them implies, owing to Lemma 5.3 below, that, if $\mathcal{N}$ is any fiber of $\mathcal{P} \otimes p_1^*\mathcal{L}$, then $\mathcal{N}$ is $k$-very ample for $k := 3(r + 1) - 1$. In other words, given any ideal $\mathcal{I} \subset \mathcal{O}_A$ of colength $3(r + 1)$ or less, the natural map

$$H^0(\mathcal{N}) \;\longrightarrow\; H^0(\mathcal{N}/\mathcal{I}\mathcal{N}) \tag{5.4}$$

is surjective.

Let $\mathbf{D}$ be a minimal Enriques diagram, and set $e := \deg \mathbf{D}$ and $s := \operatorname{cod}(\mathbf{D})$. We must bound $\operatorname{cod} Y(\mathbf{D})$ in terms of $s$. Form the subset $H(\mathbf{D})$ of $\operatorname{Hilb}^e_A$ of complete ideals with $\mathbf{D}$ as diagram. Then $H(\mathbf{D})$ is locally closed, smooth, and equidimensional of dimension $e - s$ by Proposition (3.6) of [19]. Furthermore, $Y \times H(\mathbf{D})$ is equal to the subset $H_{F/Y}(\mathbf{D})$ of $\operatorname{Hilb}^e_{F/Y}$, which was discussed in Remark 2.7. As in that remark, set $Z(\mathbf{D}) := H_{F/Y}(\mathbf{D}) \cap \operatorname{Hilb}^e_{D/Y}$.

Suppose $e \le 3(r+1)$. Let $\mathcal{I}$ be an arbitrary complete ideal with diagram $\mathbf{D}$. Then $\mathcal{I}$ has colength $e$ by [19, (3.4)]. So, if $\mathcal{N}$ is any fiber of $\mathcal{P} \otimes p_1^*\mathcal{L}$, then (5.4) is surjective. Hence

$$\dim H^0(\mathcal{I}\mathcal{N}) \;=\; \dim H^0(\mathcal{N}) - e\,.$$

Furthermore, $H^1(\mathcal{N}) = 0$, as noted above. Hence $H^1(\mathcal{I}\mathcal{N}) = 0$.

Let $\mathcal{I}^\dagger$ be the universal ideal on $A \times H(\mathbf{D})$, and form these two sheaves:

$$\mathcal{F} \;:=\; p_{13}^*\mathcal{I}^\dagger \cdot p_{12}^*(\mathcal{P} \otimes p_1^*\mathcal{L}) \quad \text{and} \quad \mathcal{R} \;:=\; p_{23*}\mathcal{F}$$

where the $p_{ij}$ are the projections from $A \times \widehat{A} \times H(\mathbf{D})$. Every fiber of $\mathcal{F}$ is of the form $\mathcal{I}\mathcal{N}$, and as just noted, $H^1(\mathcal{I}\mathcal{N}) = 0$. Hence $\mathcal{R}$ is locally free on $\widehat{A} \times H(\mathbf{D})$, and the formation of $\mathcal{R}$ commutes with base change. Furthermore, $\operatorname{rk} \mathcal{R} = \dim H^0(\mathcal{I}\mathcal{N})$.

We have $\mathbf{P}(\mathcal{R}^*) = Z(\mathbf{D})$ in $Y \times H(\mathbf{D})$. Indeed, the inclusion $\mathcal{F} \to p_{12}^*\big(\mathcal{P} \otimes p_1^*\mathcal{L}\big)$ and the base-change isomorphism induce a composition on $\widehat{A} \times H(\mathbf{D})$,

$$\mathcal{R} \;\longrightarrow\; p_{23*}p_{12}^*\big(\mathcal{P} \otimes p_1^*\mathcal{L}\big) \;\xrightarrow{\sim}\; p_1^*\mathcal{Q}\,,$$

where the $p_1$'s are different first projections. Dualizing gives a map, $p_1^*\mathcal{Q}^* \to \mathcal{R}^*$. It is surjective because its formation commutes with base change and because its fibers are surjective, since each is the dual of an inclusion of vector spaces of the form $H^0(\mathcal{I}\mathcal{N}) \to H^0(\mathcal{N})$. Thus $\mathbf{P}(\mathcal{R}^*)$ naturally embeds in $\mathbf{P}(p_1^*\mathcal{Q}^*)$, or $Y \times H(\mathbf{D})$.

A nonzero map $\sigma\colon \mathcal{O}_A \to \mathcal{N}$ factors through $\mathcal{I}\mathcal{N}$ if and only if the corresponding map $\sigma \otimes \mathcal{N}^*\colon \mathcal{N}^* \to \mathcal{O}_A$ factors through $\mathcal{I}$, so if and only if the divisor defined by $\sigma$ contains the finite subscheme defined by $\mathcal{I}$. Thus $\mathbf{P}(\mathcal{R}^*)$ and $Z(\mathbf{D})$ have the same sets of closed points. An analogous argument shows that they have the same sets of $T$-points for any $T$; whence, as schemes, $\mathbf{P}(\mathcal{R}^*) = Z(\mathbf{D})$.

Therefore, $Z(\mathbf{D})$ is a smooth and equidimensional scheme since $H(\mathbf{D})$ is so, and

$$\dim Z(\mathbf{D}) \;=\; \operatorname{rk}\mathcal{R} - 1 + \dim\big(H(\mathbf{D}) \times \widehat{A}\,\big) \;=\; \operatorname{rk}\mathcal{Q} - e - 1 + e - s + \dim\widehat{A} =\; \dim(Y) - s\,.$$

So, if the image of $Z(\mathbf{D})$ in $Y$ contains a nonempty set $S$, then $\operatorname{cod} S \ge s$.

To check the hypotheses of Theorem 2.5, we apply the preceding conclusion about $S$ in three cases. First, take $\mathbf{D} := (r+1)\mathbf{A}_1$. Then $e = 3(r+1)$ and $s = r+1$. Furthermore, the image of $Z(\mathbf{D})$ in $Y$ contains $Y(\infty)$. Hence, $\operatorname{cod} Y(\infty) > r$ if $Y(\infty) \neq \emptyset$. Thus Hypothesis (i) of Theorem 2.5 holds.

Second, let $\mathbf{D}'$ be a diagram with $Y(\mathbf{D}') \neq \emptyset$ and $\operatorname{cod}(\mathbf{D}') > r$. Suppose $\mathbf{D}'$ contains a subdiagram $\mathbf{D}$ such that $e \le 3(r+1)$ and $s \ge r+1$. Then the image of $Z(\mathbf{D})$ contains $Y(\mathbf{D}')$. Hence $\operatorname{cod} Y(\mathbf{D}') \ge s > r$.

Such a subdiagram $\mathbf{D}$ exists by [19, (4.4)] if $r \le 7$, and we can easily extend the proof if $r = 8$. (A different proof, valid for any $r$, is given in [20].) We need only check the case where $\mathbf{D}'$ has only one root, say $R$ with weight $m'$. If $m' \ge 5$, then take $\mathbf{D}$ to consist of $R$ with weight 5 so that $e = 15$ and $s = 13$. If $m' = 4$, then $\mathbf{D}'$ cannot have only one vertex since $\operatorname{cod}(\mathbf{D}') > 8$; hence, we may take $\mathbf{D}$ to be $\mathbf{X}_{1,1}$ if $R$ is followed by a vertex of weight 1, and to be $\mathbf{X}_{1,2}$ if $R$ is followed by a vertex of weight 2 or more. If $m' = 3$, then $\mathbf{D}'$ is either $J_{l,j}$ or $E_{6l+j}$ where $l \ge 2$; hence, we may take $\mathbf{D}$ to be $J_{2,0}$ for which $e = 12$ and $s = 10$. Finally, if $m' = 2$, then $\mathbf{D}'$ is $\mathbf{A}_k$ with $k > 8$, and we may take $\mathbf{D}$ to be $\mathbf{A}_9$ so that $e = 15$ and $s = 9$.

Third, let $\mathbf{D}'$ be a diagram with $Y(\mathbf{D}') \neq \emptyset$ and $\operatorname{cod}(\mathbf{D}') \le r$. Then $\deg(\mathbf{D}') \le 3r$ by [19, (4.3)] since $r \le 8$ (a different proof, valid for any $r$, is given in [20]). Hence we may take $\mathbf{D}'$ as $\mathbf{D}$. Then the image of $Z(\mathbf{D})$ contains $Y(\mathbf{D}')$. Hence $\operatorname{cod} Y(\mathbf{D}') \ge \operatorname{cod}(\mathbf{D}')$. Thus Hypothesis (ii) of Theorem 2.5 holds as well.

Therefore, by Theorem 2.5, the closure $\overline{Y(r\mathbf{A}_1)}$ is the support of a nonnegative cycle $U(r)$, whose class $[U(r)]$ is given by a certain expression. We work it out in a moment. (It is here alone that we need the restriction $r \le 8$.) First, however, observe that $U(r)$ is reduced; indeed, $Z(r\mathbf{A}_1)$ is reduced (in fact, smooth) as we proved above, so $U(r)$ is reduced by Lemma 2.6.

Let $M$ be the subscheme of $Y$ parameterizing the curves that pass through $g$ given general points; $M$ is an intersection of divisors, one for each point, see the paragraph after (5.6) below. Hence $M \cap U(r)$ is reduced by the proof of [19, (4.7)], which works virtually without change in the present setting. So we have

$$N_{g,r} \;=\; \int [M] \cdot [U(r)]$$

once we've shown each point of $M \cap U(r)$ represents a curve $C$ that's irreducible.

Suppose some $C$ is reducible, say $C = C_1 + C_2$. Then each $C_i$ has only ordinary nodes, say $r_i$ of them. Moreover, the $C_i$ meet transversally, say in $r_{12}$ points. Then

$$r \;=\; r_1 + r_2 + r_{12}\,.$$

Let $d_i$ be the self-intersection number of $C_i$. Then

$$d \;=\; d_1 + d_2 + 2r_{12}\,.$$

Let $Y_i$ be the component of $C_i$ in the Hilbert scheme of $A$. Then every irreducible component of $Y_i(r_i\mathbf{A}_1)$ is of dimension at least $d_i/2 - r_i + 1$ (with equality if the appropriate $k_i$-ampleness holds). Let $Y_i'$ be the component of $C_i$ in $Y_i(r_i\mathbf{A}_1)$. Summing divisors induces a map $Y_1' \times Y_2' \to \overline{Y(r\mathbf{A}_1)}$. Its fibers are finite. Hence

$$\dim Y(r\mathbf{A}_1) \;\geq\; (d_1/2 - r_1 + 1) + (d_2/2 - r_2 + 1)\,.$$

The right side is equal to $d/2 - r + 2$. The left side is equal to $d/2 - r + 1$. Thus we have a contradiction. So $C$ is irreducible.

Let's now work out the expression for $[U(r)]$. First, note that $w_i = 0$ because $\Omega^1_A = 0$ since $A$ is Abelian. So each $b_q$ reduces to a certain polynomial in $v$. So, to find $a_q := \pi_* b_q$, we must find $\pi_* v^a$ for $a \geq 0$.

By definition, $v := [D]$. So (5.3) yields

$$v \;=\; (1 \times p)^* l + \pi^* h \quad \text{where} \quad l \;:=\; c_1(\mathcal{P} \otimes p_1^* \mathcal{L}) \quad \text{and} \quad h \;:=\; c_1(\mathcal{O}_Y(1))\,.$$

Now, $\pi_*(1 \times p)^* l = p^* p_{2*} l$. Hence the projection formula yields

$$\pi_* v^a \;=\; \sum \binom{a}{i} \big(p^* p_{2*} l^i\big) h^{a-i}\,.$$

So we must compute $p_{2*} l^i$ for $i \geq 0$.

Let $\mu : A \times A \to A$ denote the group law. Given $x \in A$, define $T_x\colon A \to A$ by $T_x y := xy$. Finally, define $\phi\colon A \to \widehat{A}$ by $\phi(x) := T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$. Then

$$(1 \times \phi)^* \mathcal{P} \;=\; \mu^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1}\,,$$

according to [24, p. 151]. Therefore,

$$(1 \times \phi)^* l \;=\; \mu^* c - p_2^* c \quad \text{where} \quad c \;:=\; c_1(\mathcal{L})\,.$$

Consider the Cartesian diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{1 \times \phi} & A \times \widehat{A} \\ \big\downarrow{\scriptstyle p_2} & & \big\downarrow{\scriptstyle p_2} \\ A & \xrightarrow{\phi} & \widehat{A}\,. \end{array}$$

Note that $\phi^* p_{2*} l^i = p_{2*}(1 \times \phi)^* l^i$.

The preceding two equations and the projection formula yield

$$\phi^* p_{2*} l^2 \;=\; p_{2*} \mu^* c^2 - 2c p_{2*} \mu^* c + c^2 p_{2*}[A \times A]\,.$$

Now, $p_{2*}\mu^* c = 0$ and $p_{2*}[A \times A] = 0$ by reason of dimension. Also, $\int c^2 = d$ by definition of $c$, $\mathcal{L}$, and $d$. Furthermore, $p_{2*}\mu^*[x] = [A]$ for any $x \in A$, because

$$\mu^{-1}x \;=\; \bigl\{\bigl(y,\, xy^{-1}\bigr) \mid y \in A\bigr\}\,.$$

Hence $\phi^* p_{2*} l^2 = d[A]$. Since $[A] = \phi^*\bigl[\widehat{A}\,\bigr]$, therefore $p_{2*} l^2 = d\bigl[\widehat{A}\,\bigr]$.

Similarly, we obtain

$$\phi^* p_{2*} l^3 \;=\; -3dc \quad\text{and}\quad \phi^* p_{2*} l^4 \;=\; 6dc^2\,. \tag{5.5}$$

Now, $\phi_*\phi^* z = (\deg\phi)z$ for any $z$ by the projection formula, and $\deg\phi = d^2/4$ by [24, p. 150]. Taking $z := l^3$ and $z := l^4$, we therefore get

$$p_{2*} l^3 \;=\; -(12/d)\phi_* c \quad\text{and}\quad p_{2*} l^4 \;=\; (24/d)\phi_* c^2\,. \tag{5.6}$$

Of course, $p_{2*} l^i = 0$ for $i \neq 2, 3, 4$ by reason of dimension.

We can now mechanically work out an expression for $[U(r)]$ as a linear combination of $(p^*\phi_* c^i)h^{r-i}$ for $i = 0, 1, 2$. However, $N_{g,r} = \int [M]\cdot[U(r)]$. So we must find $[M]$. Well, given a point $x \in A$, define $\iota_x\colon Y \to F$ by $\iota_x(y) := (x, y)$. Then the divisor $\iota_x^{-1} D$ parameterizes the curves that pass through $x$. Owing to (5.3), the class $\bigl[\iota_x^{-1} D\bigr]$ is numerically the same as $h$. Hence $[M]$ is the same as $h^g$. Therefore, owing to the projection formula, we have to find $p_* h^{g+r-i}$ for $i = 0, 1, 2$.

Recall that $Y := \mathbf{P}(\mathcal{Q}^*)$. So $p_* h^{g+r-i} = (-1)^i s_{2-i}(\mathcal{Q})$ for $i = 0, 1, 2$ where the $s_j(\mathcal{Q})$ are the Segre classes. Hence,

$$p_* h^{g+r-2} \;=\; \bigl[\widehat{A}\,\bigr] \quad\text{and}\quad p_* h^{g+r-1} \;=\; -c_1(\mathcal{Q}) \quad\text{and}\quad p_* h^{g+r-1} \;=\; c_1(\mathcal{Q})^2 - c_2(\mathcal{Q})\,.$$

It therefore remains to find $c_1(\mathcal{Q})$ and $c_1(\mathcal{Q})^2$ and $c_2(\mathcal{Q})$.

Since $A$ is abelian, the Todd class of $p_2$ is trivial. So the Riemann–Roch theorem yields the following relation among the Chern characters:

$$\operatorname{ch}\bigl(p_{2*}\bigl(\mathcal{P} \otimes p_1^*\mathcal{L}\bigr)\bigr) = p_{2*}\bigl(\operatorname{ch}\bigl(\mathcal{P} \otimes p_1^*\mathcal{L}\bigr)\bigr)\,.$$

Now, $\mathcal{Q} := p_{2*}\bigl(\mathcal{P} \otimes p_1^*\mathcal{L}\bigr)$ by (5.1). So, owing to (5.2), the left side is equal to

$$d/2 + c_1(\mathcal{Q}) + \bigl(c_1(\mathcal{Q})^2 - 2c_2(\mathcal{Q})\bigr)/2\,.$$

On the other hand, the right side is equal to $p_{2*}\bigl(\sum l^i/i!\bigr)$. Hence

$$p_{2*} l^3 \;=\; 6c_1(\mathcal{Q}) \quad\text{and}\quad p_{2*} l^4 \;=\; 12\bigl(c_1(\mathcal{Q})^2 - 2c_2(\mathcal{Q})\bigr)\,. \tag{5.7}$$

So (5.6) yields

$$c_1(\mathcal{Q}) \;=\; -(2/d)\phi_* c \quad\text{and}\quad c_2(\mathcal{Q}) \;=\; (1/2)c_1(\mathcal{Q})^2 - (1/d)\phi_*\bigl(c^2\bigr)\,. \tag{5.8}$$

To find $c_1(\mathcal{Q})^2$, use the formulas leading to (5.6). Taking $z := c_1(\mathcal{Q})^2$ gives

$$c_1(\mathcal{Q})^2 \;=\; \bigl(4/d^2\bigr)\phi_*\bigl(\phi^* c_1(\mathcal{Q})\bigr)^2$$

since $\phi^* z^2 = (\phi^* z)^2$. Now, $\phi^* c_1(\mathcal{Q}) = (-d/2)c$ by (5.7) and (5.5). Hence

$$c_1(\mathcal{Q})^2 \;=\; \phi_*\bigl(c^2\bigr). \tag{5.9}$$

By definition, $\int c^2 = d$. So (5.9) and (5.8) yield

$$\int c_1(\mathcal{Q})^2 \;=\; d \quad\text{and}\quad \int c_2(\mathcal{Q}) \;=\; (d/2) - 1\,.$$

It is now a purely mechanical matter to derive the formulas in Table 5.1, and the proof is complete. □

**Lemma 5.3** *Let $S$ be a smooth projective surface with numerically trivial canonical bundle and with Picard number* 1*. Let $\mathcal{N}$ be a line bundle whose homology class is $m$ times the positive primitive class. Set $d := \int c_1(\mathcal{N})^2$, and let $k \geq 0$. Assume either*
(i) *that $m = 1$ and $d > 4(k+1)$ or*
(ii) *that $m \geq 2$ and $(m-1)d > m^2(k+1)$.*
*Then $\mathcal{N}$ is $k$-very ample.*

Proof. If (ii) holds, then $4(m-1)d > 4m^2(k+1)$, and so, since $m^2 \geq 4(m-1)$, then $d > 4(k+1)$. Hence, if either (i) or (ii) holds, then $d > 4(k+1)$.

Let $\mathcal{K}$ be the canonical bundle. Form $\mathcal{K}^{-1} \otimes \mathcal{N}$, and to it, apply Theorem 2.1 of Beltrametti–Sommese's [1, p. 38]. Their theorem implies that, since $d \geq 4k+5$, either $\mathcal{N}$ is $k$-very ample or there exists an effective divisor $D$ such that

$$\int c_1(\mathcal{N})[D] - (k+1) \ \leq \ \int [D]^2 \ \leq \ \int c_1(\mathcal{N})[D]/2\,. \tag{5.10}$$

Suppose such a $D$ exists, and let's derive a contradiction.

Say the class $[D]$ is $t$ times the positive primitive class. Then the second inequality in (5.10) becomes $t \leq m/2$. So $m \geq 2$ since $t \geq 1$. Hence Hypothesis (ii) applies, and so $(m-1)d > m^2(k+1)$. However, the first inequality in (5.10) amounts to $t(m-t)d \leq m^2(k+1)$; whence, $(m-1)d \leq m^2(k+1)$ because $m-1 \leq t(m-t)$ when $1 \leq t \leq m/2$. Thus we have a contradiction, and the proof is complete. □

**Remark 5.4** Similarly, we can enumerate those of the $C$ that lie in a given linear equivalence class and pass through only $g-2$ general points. In fact, modified slightly, the proof of Theorem (5.2) yields the desired enumeration, and shows it is valid when $r \leq 8$ and

$$m \ > \ (3r+5)/2\,. \tag{5.11}$$

Conceivably, some $C$ are reducible, although the corresponding dimension count shows that reducibility is not to be expected; compare [11, Rmk. 3.1, p. 528].

Indeed, the $C$ in the class are parameterized by a fiber of $Y/A$; hence, the enumeration can be accomplished by computing the coefficient of $\big[\widehat{A}\big]$ in $p_*\big(h^{g-2} \cdot [U(r)]\big)$. Furthermore, (5.11) implies $m \geq 2$. Also, (5.11) is equivalent to $2(m-1) > 3(r+1)$. Now, $d/m^2$ is the self-intersection number of the primitive class; so $d/m^2 \geq 2$. Hence Lemma 5.3 implies that, if $\mathcal{N}$ is any fiber of $\mathcal{P} \otimes p_1^*\mathcal{L}$, then $\mathcal{N}$ is $k$-very ample for $k := 3(r+1)-1$. The rest of the proof of validity is virtually the same.

Although Theorem (1.1) of [19] yields the same formula on setting $k$, $s$, and $x$ equal to 0, that theorem only asserts validity when $m \geq 3r$ and $\mathcal{O}_A(C) = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ where $\mathcal{M}$ is very ample and $\mathcal{N}$ is spanned. On the other hand, that theorem does not require the Picard number to be 1 nor the surface to be Abelian. In any event, the formula agrees with Göttsche's Conjecture 2.4 in [11, p. 526].

Some condition like (5.11) is necessary. Indeed, in [10, Rmk. 2.3, p. 581], Debarre considered the case in which $m$ is prime, $d = 2m^2$, and $g = 2$, whence $r = m^2 - 1$. In this case, Göttsche's formula fails as Göttsche [11, Rmk. 3.1, p. 528] expected when $Y(\infty)$ is nonempty.

On the other hand, the case $m = 1$ is rather interesting and has attracted some attention. In this case, the method of Theorem 5.2 shows that Göttsche's formula is valid when $r \leq 8$ and $d > 12(r+1)$, whereas Theorem (1.1) of [19] asserts nothing. Using symplectic methods, Bryan and Leung [3] showed that the formula is valid for all $r$ and $d$ when $A$ is generic among the Abelian surfaces for which the given homology class is algebraic. Using complex analytic methods, in [11, Thm. 3.2, p. 528], Göttsche showed that the formula is valid for all $r$ when $g = 2$ and the homology class is a polarization of type $(1, n)$. Independently and somewhat differently, Debarre [10] proved the same result. Earlier, see [34, Ex. 5.6, p. 521], Schoen treated the case of a polarization of type $(1, 5)$ on a general Horrocks–Mumford Abelian surface.

**Remark 5.5** In Lemma 5.3, if $S$ is a K3 surface, then in Condition (i) we may replace $d > 4(k+1)$ with $d \geq 4k$. Indeed, the proof is the same, except that, instead of applying Theorem 2.1 of Beltrametti–Sommese [1, p. 38], we apply Theorem 1.1 of Knutsen [21], which says that, since $S$ is a K3 surface and $d \geq 4k$, either $\mathcal{N}$ is $k$-very ample or there exists an effective divisor $D$ such that (5.10) holds.

Similarly, if $S$ is an Enriques surface, then we may replace both (i) and (ii) with the single condition that $d \geq 4(k+1)$. Indeed, Theorem 1.2 of Knutsen's [21] asserts that, since $S$ is an Enriques surface and $d \geq 4(k+1)$, either $\mathcal{N}$ is $k$-very ample or there exists an nonzero effective divisor $D$ with nonpositive self-intersection; however, the latter is impossible since $S$ has Picard number 1 by hypothesis.

For these $S$, the method of proof of Theorem 5.2 shows that the formula provided by Theorem (1.1) of [19] is valid for more $m$. Of course, some restriction on $m$ is necessary. Indeed, in [32, Ex. (3.13), p. 252], Tannenbaum gave a simple example of a complete linear system on a K3 surface $S$ such that $\operatorname{cod} Y(4\mathbf{A}_1) < 4$.

In Tannenbaum's example, $S$ is an arbitrary smooth quartic in $\mathbf{P}^3$. The system is the one cut by the quadrics; so it is parameterized by a projective space $Y$ of dimension 9. A general plane section of $S$ is smooth. So a general plane-pair section has two smooth components that meet transversally in four points. Hence $\dim Y(4\mathbf{A}_1) \geq 6$, and so $\operatorname{cod} Y(4\mathbf{A}_1) < 4$.

Furthermore, if $S$ is generic, then its Picard group is generated by $\mathcal{O}_S(1)$ by the Noether–Lefschetz theorem. In particular, the Picard number is 1. Also, if a quadric section is not reduced, then it must be twice a plane section. Since planes and quadrics are determined by their sections, the quadric must be a double plane. Hence $\dim Y(\infty) = 3$, and so $\operatorname{cod} Y(\infty) = 6$. Thus, unexpectedly, there are infinitely many 4-nodal quadric sections through 5 general points, and all are reduced.

## References


[1] M. Beltrametti and A. J. Sommese, Zero cycles and $k$-th order embeddings of smooth projective surfaces, in: Problems in the Theory of Surfaces and their Classification (Cortona, 1988), Sympos. Math. XXXII (Academic Press, London, 1991), pp. 33–48.

[2] C. Birkenhake and H. Lange, Complex tori, Progr. Math. 177 (Birkhäuser, 1999).

[3] J. Bryan and N. C. Leung, Generating functions for the number of curves on abelian surfaces, Duke Math. J. **99**, 311–328 (1999).

[4] L. Caporaso and J. Harris, Counting plane curves of any genus, Invent. Math. **131**, 345–392 (1998).

[5] A. Cayley, On the theory of involution, Trans. Cambridge Phil. Soc. **XI**, Part I, 21–38 (1866); Coll. Math. Papers of A. Cayley **V**, [348], 295–312 (1892).

[6] Y. Choi, On the degree of Severi varieties, preprint 1997, available from http://math.ucr.edu/ychoi/paper.html.

[7] H. Clemens, Curves on higher-dimensional complex projective manifolds, in: Proc. International Cong. Math., Berkeley, 1986 (AMS, 1987), pp. 634–640.

[8] L. Comtet, Analyse Combinatoire I, SUP **4** (Presses Universitaires de France, 1970).

[9] D. Cox and S. Katz, Mirror symmetry and algebraic geometry, Math. Surveys and Monographs Vol. 68 (AMS, 1999).

[10] O. Debarre, On the Euler characteristic of generalized Kummer varieties, Amer. J. Math. **121**, 577–586 (1999).

[11] L. Göttsche, A conjectural generating function for numbers of curves on surfaces, Comm. Math. Phys. **196**, 523–533 (1998).

[12] G.-M. Greuel and C. Lossen, Equianalytic and equisingular families of curves on surfaces, Manuscripta Math. **91**, 323–342 (1996).

[13] G.-M. Greuel, C. Lossen, and E. Shustin, New asymptotics in the geometry of equisingular families of curves, Int. Math. Res. Not. **13**, 595–611 (1997).

[14] J. Harris and R. Pandharipande, Severi degrees in cogenus 3, alg-geom/9504003.

[15] T. de Jong, Equisingular deformations of plane curve and of sandwiched singularities, arXiv: math.AG/0011097.

[16] S. Katz, On the finiteness of rational curves on quintic threefolds, Compositio Math. **60**, 151–162 (1986).

[17] S. Katz, Rational curves on Calabi-Yau threefolds, in: Essays on Mirror Manifolds, edited by S.-T. Yau, Int. Series in Math. Physics (International Press 1992), pp. 168–180.

[18] S. L. Kleiman, The transversality of a general translate, Compositio Math. **28**, 287–297 (1974).

[19] S. Kleiman and R. Piene, Enumerating singular curves on surfaces, in: Algebraic Geometry — Hirzebruch 70, Cont. Math. **241**, 209–238 (1999) (corrections and revision in math.AG/9903192).

[20] S. Kleiman and R. Piene, Node polynomials for curves on surfaces, to appear.

[21] A. L. Knutsen, On $k$th order embeddings of K3 surfaces and Enriques surfaces, Manuscripta Math. **104**, 211–237 (2001).

[22] C. Lossen, The geometry of equisingular and equianalytical families of curves on a surface, Dr. dissertation, Universität Kaiserslautern (1998).

[23] J. F. Mattei, Modules de feuilletages holomorphes singuliers. I. Équisingularité, Invent. Math. **103**, 297–325 (1991).

[24] D. Mumford, Abelian Varieties (Oxford University Press, 1970).

[25] A. Nobile and O. E. Villamayor, Equisingular stratifications associated to families of planar ideals, J. Algebra **193**, 239–259 (1997).

[26] Z. Ran, Enumerative geometry of singular plane curves, Invent. Math. **97**, 447–465 (1989).

[27] S. Roberts, Sur l'ordre des conditions pour la coexistence des équations algébriques à plusieurs variables, J. Reine Angew. Math. **67**, 266–278 (1867).

[28] S. Roberts, On a simplified method of obtaining the order of algebraical conditions, Proc. Lond. Math. Soc. **VI**, 101–113 (1875).

[29] G. Salmon, A Treatise on the Analytic Geometry of Three Dimensions, 2nd edition (Hodges, Smith, and Co., Dublin, 1865).

[30] G. Salmon, Higher Plane Curves, 3rd edition, Chelsea reprint (1879).

[31] J. Steiner, Allgemeine Eigenschaften der algebraischen Curven, Berlin. Ber. 310–315 (1848); J. Reine Angew. Math. **47**, 1–6 (1853); Ges. Werke, herausgegeben von K. Weierstrass, **2**, 495–500 (Berlin, 1882).

[32] A. Tannenbaum, Families of curves with nodes on K3-surfaces, Math. Ann. **260**, 239–253 (1980).

[33] B. Teissier, Cycles évanescents, sections planes et conditions de Whitney, in: Singularités à Cargèse, Astérisque **7–8**, 285–362 (1973).

[34] I. Vainsencher, Enumeration of $n$-fold tangent hyperplanes to a surface, J. Algebraic Geom. **4**, 503–526 (1995).

[35] J. M. Wahl, Equisingular deformations of plane algebroid curves, Trans. Amer. Math. Soc. **193**, 143–170 (1974).

[36] C. T. C. Wall, Notes on the classification of singularities, Proc. Lond. Math. Soc. **48**, 461–513 (1984).

[37] O. Zariski, Dimension-theoretic characterization of maximal irreducible algebraic systems of plane nodal curves of a given order $n$ and with a given number $d$ of nodes, Amer. J. Math. **104**, 209–226 (1982).