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Noise-Induced Global Asymptotic Stability

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We prove analytically that additive and parametric (multiplicative) Gaussian distributed white noise, interpreted in either the Itô or Stratonovich formalism, induces global asymptotic stability in two prototypical dynamical systems designated as supercritical (the Landau equation) and subcritical, respectively. In both systems without noise, variation of a parameter leads to a switching between a single, globally stable steady state and multiple, locally stable steady states. With additive noise this switching is mirrored in the behavior of the extrema of probability densities at the same value of the parameter. However, parametric noise causes a noise-amplitude-dependent shift (postponement) in the parameter value at which the switching occurs. It is shown analytically that the density converges to a Dirac delta function when the solution of the Fokker–Planck equation is no longer normalizable.

KEY WORDS: Stochastic differential equations; Fokker–Planck equation; Hopf bifurcation; Liapunov functions; global stability; noise-induced transitions.

1. INTRODUCTION

The effects of additive and parametric (multiplicative) noise in nonlinear dynamical systems have been the object of intense study.[(1)] Systems that display bifurcations in dynamics in the absence of noise have received the most attention, in part because noise effects in these systems qualitatively mimic first- and second-order phase transitions.[(1)]

The presence of noise in combination with dynamics leads to a situation in which one may describe the global behavior of the system by the evolution of densities. That evolution is described by the Fokker–Planck (parabolic)

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partial differential equation. The steady-state solutions to the Fokker–Planck equation are known as stationary densities.

Most studies[2–4] indicate that additive noise, a term usually taken to imply that noise amplitude is independent of the state variable(s), leads to a bifurcation in the qualitative form of the stationary density at precisely the same parameter value at which the bifurcation occurred in the noise-free system. However, parametric noise, in which the noise amplitude depends on the state variable(s), induces different behaviors in the stationary density. Usually[5–7] parametric noise induces a noise-amplitude-dependent postponement in the parameter value at which these qualitative changes in the stationary density take place relative to the noise-free system, though one study[8] indicates the possibility of an advancement in the bifurcation parameter depending on the relative values of the noise correlation time and the system response time.

In spite of the intense interest in the changes that additive and parametric noise give rise to in stationary densities, there has not–to our knowledge–been any proof of the global convergence of the time-dependent solutions of the Fokker–Planck equation to the (generally unique) stationary density. In this paper we consider two prototypical systems in the presence of additive and parametric noise, and use a recent result to prove the global asymptotic stability of the solutions of the Fokker–Planck equation.

2. PRELIMINARIES

2.1. The Model Systems

In our investigation of the effects of additive and parametric noise, we will consider two specific systems.

Supercritical System. The two-dimensional oscillator system

$$\frac{dr}{dt} = r(c - r^2)$$
$$\frac{d\theta}{dt} = 2\pi \tag{1}$$

in (r, θ) space is an example of a system with a supercritical Hopf bifurcation. For $c < 0$ the origin $r_* = 0$ is the globally stable steady state, while for $c > 0$ all solutions are attracted to the limit cycle defined by $r = \sqrt{c}$.

Here we consider the effects of noise in the analogous one-dimensional system

$$\frac{dx}{dt} = x(c - x^2) \tag{2}$$

obtained by ignoring the angular coordinate θ in Eq. (1), and designate this the *supercritical system.* This equation appears, for example, as the reduced amplitude equation for systems undergoing a supercritical Hopf bifurcation.[2-4,6,9,10] For Eq. (2), it is simple to show that when $c<0$, all solutions are attracted to the single steady state $x_*=0$. Further, when $c>0$, the steady state $x_*=0$ is unstable and $x(t)\to\sqrt{c}$ if $x(0)=x_0>0$, while $x(t)\to-\sqrt{c}$ for $x_0<0$.

Subcritical System. A second simple oscillator system

$$\frac{dr}{dt}=r(c+2r^2-r^4)$$
$$\frac{d\theta}{dt}=2\pi \tag{3}$$

has a subcritical Hopf bifurcation at $c=-1$, as have other systems studied in the presence of noise.[2,5]

In analogy with the previous case, we treat the effects of noise in the one-dimensional system

$$\frac{dx}{dt}=x(c+2x^2-x^4) \tag{4}$$

which we call the *subcritical system.* The solutions of Eq. (4) have the following behavior. For $c<-1$ all solutions $x(t)\to 0$ regardless of the initial condition x_0. However, for $-1<c<0$ there is tristability in that

$$x(t)\to\begin{cases}-[1+(1+c)^{1/2}]^{1/2} & \text{for}\quad x_0<-x_*^+,\quad -x_*^+<x_0<-x_*^-\\ 0 & \text{for}\quad -x_*^-<x_0<x_*^-\\ [1+(1+c)^{1/2}]^{1/2} & \text{for}\quad x_*^+<x_0,\quad x_*^-<x_0<x_*^+\end{cases} \tag{5}$$

where $x_*^+=[1+(1+c)^{1/2}]^{1/2}$ and $x_*^-=[1-(1+c)^{1/2}]^{1/2}$. For $c>0$, the steady state $x_*=0$ becomes unstable and this tristable behavior gives way to a bistability such that

$$x(t)\to\begin{cases}-[1+(1+c)^{1/2}]^{1/2} & \text{for}\quad x_0<0\\ [1+(1+c)^{1/2}]^{1/2} & \text{for}\quad x_0>0\end{cases} \tag{6}$$

2.2. Densities and the Fokker–Planck Equation

In considering the effects of noise in systems like (2) or (4), we may think of the general one-dimensional differential equation

$$\frac{dx}{dt}=g(x)$$

and the corresponding stochastic differential equation

$$\frac{dx}{dt} = g(x) + \sigma(x)\xi \tag{7}$$

where ξ is a (Gaussian-distributed) white-noise perturbation with zero mean and unit variance, and $\sigma(x)$ is the amplitude of the perturbation.

Under some standard regularity conditions, the process $x(t)$, which is the solution of the stochastic differential equation (7), has a density function $u(t, x)$ defined by

$$\text{prob}\{a < x(t) < b\} = \int_a^b u(t, z)\, dz, \qquad a, b \in R$$

It is well known that the density $u(t, x)$ satisfies the parabolic differential equation (*Fokker–Planck equation*)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2[\sigma^2(x)u]}{\partial x^2} - \frac{\partial[G(x)u]}{\partial x} \tag{8}$$

where the function G is given by

$$G = g \tag{9a}$$

when the Itô calculus is used to interpret (7), or

$$G = g + \frac{1}{4}\frac{\partial[\sigma^2(x)]}{\partial x} \tag{9b}$$

when the Stratonovich calculus is used.[1] The Fokker–Planck equation can also be written in the equivalent form

$$\frac{\partial u}{\partial t} = -\frac{\partial S}{\partial x} \tag{10}$$

where

$$S = -\frac{1}{2}\frac{\partial[\sigma^2(x)u]}{\partial x} + Gu \tag{11}$$

is called the *probability current.*

As usual, we say that an L^1 function f is a density if f is nonnegative and its integral over its domain is identically equal to 1, i.e., it is *normalized.* Given an initial density $f(x) = u(0, x)$ and the solution $u(t, x)$ of (8), we may write this solution formally as

$$u(t, x) = P_t f(x)$$

where P_t is a *Markov operator,* i.e., P_t is a linear operator and for every density f, $P_t f$ is also a density. Thus, the Fokker–Planck equation governs the evolution of the flow of densities $\{P_t f\}$.

When stationary solutions of (8), denoted by $f_*(x)$ and defined by $P_t f_* = f_*$ for all t, exist, they are given as the generally unique (up to a multiplicative constant) solution of

$$\frac{1}{2}\frac{\partial^2[\sigma^2 f_*]}{\partial x^2} - \frac{\partial[G f_*]}{\partial x} = 0 \tag{12}$$

Integration of Eq. (12) by parts with the assumption that the probability current S vanishes at the integration limits, followed by a second integration, yields the solution

$$f_*(x) = \frac{K}{\sigma^2(x)} \exp\left[\int^x \frac{2G(z)}{\sigma^2(z)}\, dz\right] \tag{13}$$

This stationary solution f_* will be a density if and only if there exists a positive constant $K > 0$ such that f_* can be normalized.

3. ADDITIVE NOISE

For the supercritical system (2) and the subcritical system (4) in the presence of additive noise, the corresponding stochastic differential equations are of the form

$$\frac{dx}{dt} = g(x) + \sigma\xi \tag{14}$$

where σ is a positive constant and

$$g(x) = \begin{cases} x(c - x^2), & \text{supercritical} \\ x(c + 2x^2 - x^4), & \text{subcritical} \end{cases} \tag{15}$$

Thus, in the additive noise case, reference to Eqs. (14) and (15) makes it clear that there is always a positive probability that $x(t)$ may take on negative values starting from a positive position and *vice versa.* Therefore it is natural to consider this problem for $-\infty < x < \infty$.

Furthermore, since the noise amplitude σ is constant with additive noise, Eqs. (9a) and (9b) make it clear that the corresponding Fokker–Planck equations are identical in the Itô and Stratonovich interpretations. Specifically, they take the forms

$$\frac{\partial u}{\partial t}=\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial x}[x(c-x^2)u] \tag{16}$$

and

$$\frac{\partial u}{\partial t}=\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial x}[x(c+2x^2-x^4)u] \tag{17}$$

in the super- and subcritical cases, respectively.

3.1. Stationary Solutions

It is straightforward to show that the stationary solution (13) to the Fokker–Planck equation (12) is given by

$$f_*(x)=K_1 e^{\beta x^2(2c-x^2)/4c} \tag{18}$$

for the supercritical system, where $\beta=2c/\sigma^2$, and by

$$f_*(x)=K_2 e^{\beta x^2(3c+3x^2-x^4)/6c} \tag{19}$$

for the subcritical system. It is easy to show that the normalization constants K_1 and K_2 always exist and thus the $f_*(x)$ defined by (18) and (19) are stationary densities.

In Fig. 1a we show the stationary density given in Eq. (18) for the supercritical system as a function of the parameter c. As might be expected on intuitive grounds, for $c<0$, the stationary density $f_*(x)$ has a single maximum centered at $x=0$, the location of the globally stable steady state of the unperturbed system (2). Once $c\geq 0$, the stationary density $f_*(x)$ shows two maxima centered at $x=\pm\sqrt{c}$, the locally stable steady states of (2), and a local minimum at the unstable steady state $x=0$.

Figure 1b shows the stationary density for the subcritical system, again as a function of c, given in Eq. (19). For $c<-1$, the stationary density $f_*(x)$ has a single maximum located at $x=0$, the globally stable steady of the unperturbed system (4). For $-1<c<0$, where the tristable behavior of (4) occurs, the stationary densities still have an absolute maximum at $x=0$, but also display maxima at $x=\pm[1+(1+c)^{1/2}]^{1/2}$ that become progressively more prominent as c increases. Finally, for $c>0$ the stationary density has absolute maxima located at $x=\pm[1+(1+c)^{1/2}]^{1/2}$ and a local minimum at $x=0$.

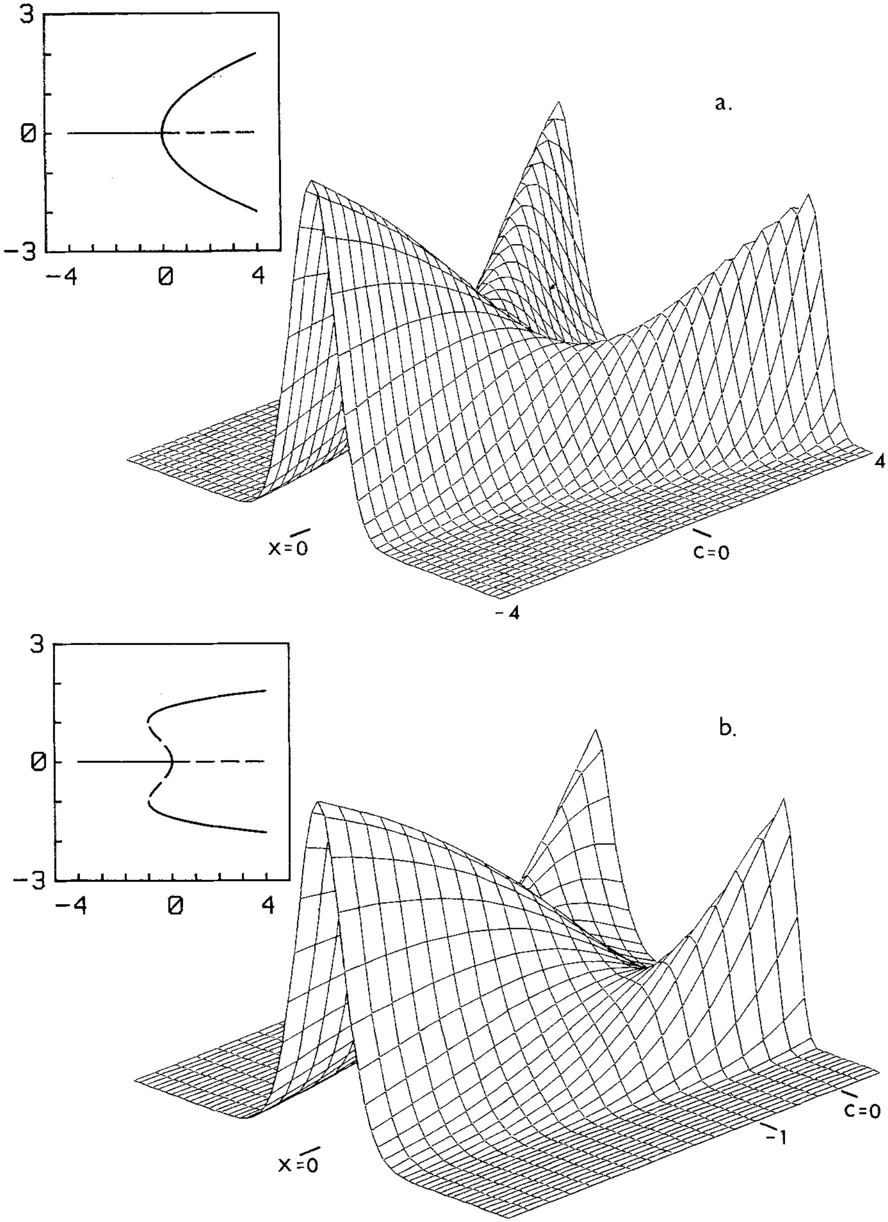

Fig. 1. Globally stable stationary densities in the presence of additive noise, as functions of x and the parameter c, for (a) the supercritical system (2) and (b) the subcritical system (4). To aid in visualization, in each part the insert shows the location of the maxima in the stationary density as a solid line in the (c, x) plane. The dashed line in the insert of (b) corresponds to the minimum in the density.

3.2. Asymptotic Stability of the Stationary Solutions

We now turn to a consideration of the stability of the stationary densities determined in the previous section.

We first define the property of stability by saying that Eq. (8) is *globally asymptotically stable* if

$$\lim_{t \to \infty} u(t, x) = \lim_{t \to \infty} P_t f(x) = f_*(x)$$

for *all* initial densities $f(x)$, i.e., $P_t f$ converges to f_* in L^1 norm.[11] We will alternately say that f_* is *globally asymptotically stable* under this circumstance.

For parabolic equations whose solutions are given by an integral operator with a sufficiently smooth kernel, it is possible to prove their global asymptotic stability via a Liapunov function approach. Both Fokker–Planck equations (16) and (17) are quite regular from this point of view, since they are uniformly parabolic (σ^2 is a positive constant) and $xg(x) < 0$ for sufficiently large x. These properties ensure that the solutions of Eqs. (16) and (17) will decay at least exponentially as $x \to \pm\infty$.

We define a *Liapunov function* $V: R \to R$ as a C^2 function with the following properties:

1. $V(x) \geqslant 0$ for all x.
2. $\lim_{x \to \pm\infty} V(x) = \infty$.
3. $V(x) \leqslant \rho e^{\delta |x|}$ and $|dV/dx| \leqslant \rho e^{\delta |x|}$ for some positive constants ρ and δ.

It has been shown[11,12] that the existence of a Liapunov function V satisfying

$$\sigma^2 \frac{\partial^2 V}{\partial x^2} + g(x) \frac{\partial V}{\partial x} \leqslant -\alpha V(x) + \beta \tag{20}$$

where α and β are positive constants, implies that the Fokker–Planck equation (8) is globally asymptotically stable.

Let $V(x) = x^2$, so V is a Liapunov function, and consider the supercritical system with additive noise. Inequality (20) becomes, in this case,

$$2\sigma^2 + (2c + \alpha) x^2 - 2x^4 \leqslant \beta \tag{21}$$

This is clearly satisfied for arbitrary fixed $\alpha > 0$ and sufficiently large $\beta > 0$, thus proving the global asymptotic stability of the Fokker–Planck equation (16) for additive noise applied to the supercritical system (2).

Retain $V(x) = x^2$ for the subcritical system (4) with additive noise. An entirely analogous argument suffices to show that positive constants α and β can be found such that inequality (20) is satisfied, thus establishing the global asymptotic stability of Eq. (17).

Hence, the entrance of white noise perturbations to either the supercritical or subcritical systems (2) and (4) in an additive fashion always leads to globally asymptotically stable behavior.

4. PARAMETRIC NOISE

Both the supercritical and subcritical systems contain a single parameter c, and in this section we investigate the effects of noise in this parameter by replacing c with

$$c + \sigma\xi$$

where $\sigma > 0$ is a constant. As a result of this assumption, the stochastic differential equation (7) takes the form

$$\frac{dx}{dt} = g(x) + \sigma x\xi \tag{22}$$

where $g(x)$ is given by Eq. (15). From Eq. (22) in conjunction with (15) it is clear that $x(t) = 0$ is always a solution. Therefore, for any $x_0 > 0$ the solution $x(t)$ will always be positive. For $x_0 < 0$ we will have $x(t) < 0$. Thus, in contrast to the situation with additive noise, in the presence of parametric noise we need only consider $-\infty < x \leqslant 0$ or $0 \leqslant x < \infty$. As the results are symmetric, we take $0 \leqslant x < \infty$.

With parametric noise, it is no longer the case that the Fokker–Planck equation corresponding to (22) will be the same for the Itô and Stratonovich interpretations.[1] Hence, assume first that we are using the Itô calculus, and replace c by c_I to denote this distinction. Then, using (8) and (9a), we obtain the corresponding Fokker–Planck equations

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2[x^2 u]}{\partial x^2} - \frac{\partial}{\partial x}[x(c_I - x^2)u], \qquad \text{supercritical} \tag{23}$$

and

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2[x^2 u]}{\partial x^2} - \frac{\partial}{\partial x}[x(c_I + 2x^2 - x^4)u], \qquad \text{subcritical} \tag{24}$$

4.1. Stationary Solutions

Supercritical System. For parametric noise in the supercritical system it is a straightforward application of Eq. (13) to show that the stationary solution $f_*(x)$ of the Fokker–Planck equation (23) is given by

$$f_*(x) = Kx^{\gamma} e^{-x^2/\sigma^2} \tag{25}$$

where $\gamma = (2c_{\mathrm{I}}/\sigma^2) - 2$.

With parametric noise, a stationary density will not exist for some parameter values. In order that f_* be a density, it must be integrable on R^+, and from (25) this is only possible if $\gamma > -1$, or

$$c_{\mathrm{I}} > \tfrac{1}{2}\sigma^2 \tag{26}$$

Thus, in sharp contrast to the results for additive noise, for parametric noise a stationary density $f_*(x)$ in the supercritical case exists for only a limited range of values of the parameter c_{I} as defined by inequality (26).

In Fig. 2 we show the graph of the stationary densities $f_*(x)$ given by Eq. (25) for the range of c_{I} values for which it exists. For $\sigma^2/2 < c_{\mathrm{I}} < \sigma^2$ the

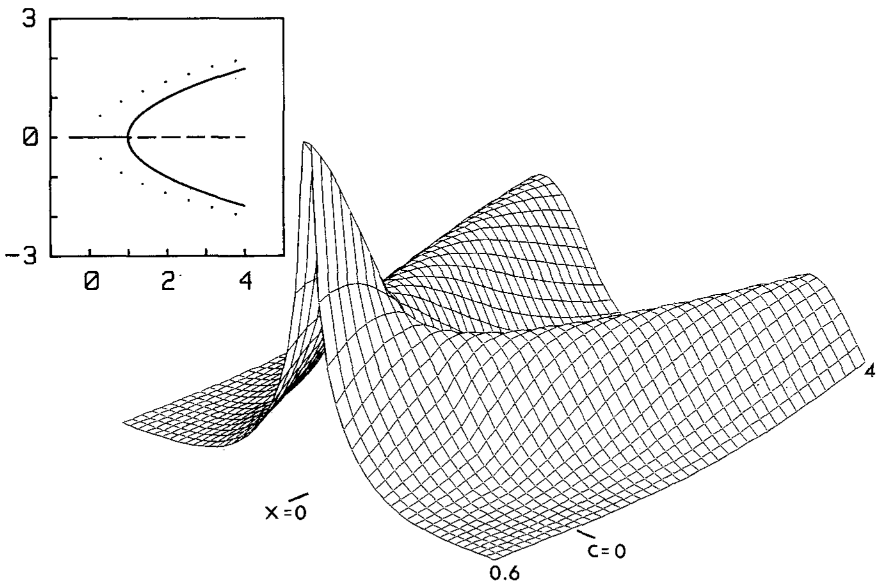

Fig. 2. Globally stable stationary densities for the supercritical system (2) with parametric noise under the Itô interpretation. For clearer viewing, the density for $0 \leqslant x$ is reflected as a mirror image to $x < 0$ and also displayed. The inset shows the location of the maxima (solid line) and minima (dashed line) of the densities in the (c_{I}, x) plane, and the location of the globally stable steady states (dotted line) in the absence of noise.

density has a single maximum at $x = 0$. However, once $c_{\rm I} > \sigma^2$, the stationary density $f_*(x)$ has a local minimum at $x = 0$ and a maximum at $x = (c_{\rm I} - \sigma^2)^{1/2}$. Thus, with parametric noise there is not only a shift in the value of the parameter $c_{\rm I}$ at which there is a transition between the stationary density having a maximum at $x = 0$ and a nonzero value of x, but there is also a shift in the nonzero location of the maximum in the stationary density below that of the globally stable steady state in the absence of noise ($x = \sqrt{c_{\rm I}}$) toward zero. It is only as $c_{\rm I}$ becomes large that the location of the density maximum starts to approximate $\sqrt{c_{\rm I}}$.

All of these calculations and conclusions are precisely the same if the Stratonovich interpretation is used in place of the Itô formulation. One must only replace $c_{\rm I}$ everywhere by $c_{\rm S} = c_{\rm I} + (\sigma^2/2)$ for the formulas and conclusions to be applicable to the Stratonovich case.

Subcritical System. As in the previous section, it is an elementary consequence of Eq. (13) that the stationary solution of the (Itô) Fokker–Planck equation for the subcritical case with parametric noise is given by

$$f_*(x) = Kx^{\gamma} e^{x^2(4 - x^2)/2\sigma^2} \tag{27}$$

where γ is as before. For the $f_*(x)$ defined in (27) to be a stationary density, precisely the same conditions must hold as for the supercritical system of the previous section. Namely, $f_*(x)$ will be stationary density of the Fokker–Planck equation if and only if inequality (26) is satisfied.

Figure 3 graphically presents the stationary density given by (27) for the range of $c_{\rm I}$ for which inequality (26) is satisfied. The density for the subcritical system in the presence of parametric noise has two qualitatively different behaviors as the parameter $c_{\rm I}$ is varied. The appearance of either of these behaviors depends on the noise amplitude σ.

For noise amplitudes satisfying $0 < \sigma^2 < 2$, a new feature unobserved in the supercritical system appears as shown in Fig. 3a. Namely, for f_* defined by (27) and this range of σ, as $c_{\rm I}$ is increased past $\sigma^2/2$, f_* may be normalized, and the resulting stationary density has a maximum located at

$$x = [1 + (1 + c_{\rm I} - \sigma^2)^{1/2}]^{1/2} \tag{28}$$

and a singularity at $x = 0$ which only exists for $\sigma^2/2 < c_{\rm I} < \sigma^2$. [The condition $0 < \sigma^2 < 2$ may seem dimensionally incorrect at first glance. However, it is simply a consequence of the choice of parameters in Eq. (3).]

However, as illustrated in Fig. 3b, for higher noise amplitudes such that $\sigma^2 > 2$, for $\sigma^2/2 < c_{\rm I} < \sigma^2 - 1$ the density f_* has a single maximum located at $x = 0$. As $c_{\rm I}$ is increased, once $\sigma^2 - 1 < c_{\rm I}$, then f_* has a relative maximum at $x = 0$ and a second maximum located at the same location

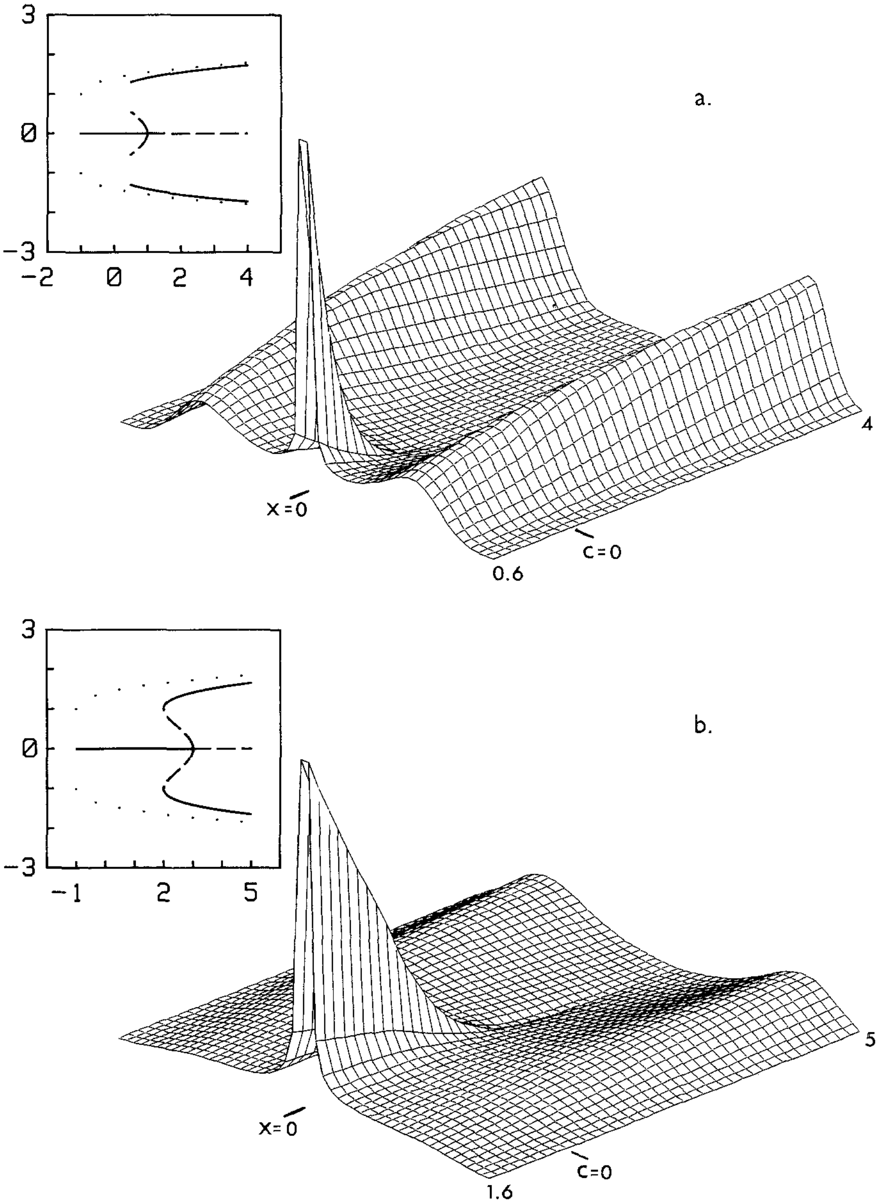

Fig. 3. Globally stable stationary densities for the subcritical system (4) in the presence of parametric noise (Itô interpretation), reflected across the x axis as in Fig. 2. In (a) the qualitative situation for $0<\sigma^2<2$ is depicted using $\sigma^2=1$, while (b), with $\sigma^2=3$, illustrates the qualitative features found when $2<\sigma^2$. In both (a) and (b), the types of lines in the inserts have the same meaning as in Fig. 2.

[see Eq. (28)] as for $\sigma^2 < 2$. For all values of σ^2, as $c_{\rm I}$ is increased, the location of this maximum tends toward the value of the nonzero steady state $x = [1 + (1 + c_{\rm I})^{1/2}]^{1/2}$ of the unperturbed system (4).

As before, one need only replace $c_{\rm I}$ by $c_{\rm S}$ to obtain the corresponding Stratonovich results.

4.2. Asymptotic Stability with Parametric Noise

In trying to prove that the stationary densities induced by parametric noise are globally asymptotically stable, we no longer have immediately available the Liapunov function technique that we were able to apply so easily in the case of additive noise. This is because with parametric noise, the coefficient $\sigma^2 x^2/2$ vanishes at $x = 0$ and the uniform parabolicity condition is violated at $x = 0$. This fact is crucial.

However, by a straightforward change of variables, we may transform the Fokker–Planck equations (23) and (24) to circumvent this problem, and then again apply the Liapunov function argument.

Define a new variable $y = \ln x$ and a new density $\tilde{u}$ by

$$\tilde{u}(t, y) = e^{2y} u(t, e^y) \tag{29}$$

With these changes, the Fokker–Planck equations (23) and (24) take the form

$$\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2} \sigma^2 \frac{\partial^2 \tilde{u}}{\partial y^2} - \frac{\partial}{\partial y} [\tilde{g}(y) \tilde{u}] \tag{30}$$

where

$$\tilde{g}(y) = \begin{cases} c_{\rm I} - \frac{1}{2}\sigma^2 - e^{2y}, & \text{supercritical} \\ c_{\rm I} - \frac{1}{2}\sigma^2 + 2e^{2y} - e^{4y}, & \text{subcritical} \end{cases} \tag{31}$$

As in the case of additive noise, the uniform parabolicity condition is now satisfied and further $y\tilde{g}(y) < 0$ for sufficiently large y whenever $c_{\rm I} > \sigma^2/2$, which is the range of concern here. Thus, if we are able to find a Liapunov function V which satisfies (20), the asymptotic stability of Eq. (30) will be demonstrated.

Set $q = 2\alpha/(c_{\rm I} - \sigma^2/2)$, where $\alpha > 0$ is the same as in inequality (20). Clearly $c_{\rm I} > \sigma^2/2$ whenever a stationary density of (30) exists, so take $q > 0$. It is evident that

$$V(y) = \cosh(qy)$$

is a Liapunov function. It is easy to show by a straightforward calculation that there are $\alpha > 0$ and $\beta > 0$ such that (20) is satisfied in the new variable

y when $g(x)$ is replaced by $\tilde{g}(y)$ as defined in Eq. (31). Thus, we know the stationary solution of (30) is globally asymptotically stable, which, by the change of variables (29), in turn implies the global asymptotic stability of the stationary solutions of (23) and (24). The same conclusions hold for the Stratonovich interpretation.

4.3. Behavior in the Absence of Asymptotic Stability

The results of the previous section give no insight into the effects of parametric noise for values of the parameter c_{I} when globally stable stationary densities do not exist, i.e., when

$$c_{\mathrm{I}} < \tfrac{1}{2}\sigma^2 \tag{32}$$

This section explores the system behavior for values of the parameter c_{I} satisfying inequality (32) using techniques from the theory of diffusion processes. In this approach, we are returning to the original stochastic differential equation (7) and its specific form (22). We consider these equations for $0 \leqslant x < \infty$. We first show that each trajectory $x(t)$ of (22) converges to zero as $t \to \infty$ with probability 1. We then use this result to show that the densities $u(t, x)$ converge to the Dirac delta function as $t \to \infty$.

In the qualitative theory of one-dimensional stochastic differential equations like (7), the functions $s(x)$ and $k(x)$, defined by

$$s(x) = \int_1^x e^{-H(y)}\, dy \qquad \text{where} \quad H(y) = 2\int_1^y \frac{g(x)}{\sigma^2(x)}\, dx \tag{33}$$

and

$$k(x) = \int_1^x e^{-H(y)} m(y)\, dy \qquad \text{where} \quad m(y) = 2\int_1^y e^{H(x)} \frac{1}{\sigma^2(x)}\, dx \tag{34}$$

play a crucial role.[13] The function $k(x)$ allows us to evaluate the time interval over which the trajectory is defined and strictly positive. That is, if

$$k(0) = \infty \qquad \text{and} \qquad k(\infty) = \infty \tag{35}$$

then every solution of (7), starting from a finite positive initial value $x(0) > 0$, is defined for all $t \geqslant 0$ and satisfies

$$0 < x(t) < \infty \qquad \text{for} \quad t \geqslant 0$$

with probability 1. Further, by the use of the function $s(x)$, we can describe the behavior of $x(t)$ more precisely. Namely, if $k(x)$ satisfies (35) and

$$s(0) > -\infty, \qquad s(\infty) = \infty \tag{36}$$

then for every trajectory starting from a finite initial value $x(0) > 0$ we have

$$\text{prob}(\lim_{t\to\infty} x(t) = 0) = 1 \tag{37}$$

For Eq. (22) with $g(x)$ given by (15) we have

$$s(x) = \int_1^x y^{-(2c/\sigma^2)} e^{r(y)}\, dy, \qquad m(x) = \frac{2}{\sigma^2} \int_1^x y^{(2c/\sigma^2)-2} e^{r(y)}\, dy$$

and

$$k(x) = \int_1^x y^{-(2c/\sigma^2)} e^{r(y)} m(y)\, dy$$

where

$$r(y) = \begin{cases} \dfrac{1}{\sigma^2}(y^2 - 1), & \text{supercritical} \\ \dfrac{1}{\sigma^2}\left(\dfrac{1}{2} y^4 - 2y^2 + \dfrac{3}{2}\right), & \text{subcritical} \end{cases}$$

With these functions, it is easy to verify that for $2c/\sigma^2 < 1$ both conditions (35) and (36) are satisfied. Thus, we have proved the validity of (37) for c_{I} satisfying (32).

To determine the behavior of the densities $u(t, x)$ from (37), we will use the following standard technique. Let $x(t)$ be an arbitrary solution of (22). Choose an $\varepsilon > 0$ and a sequence of positive numbers $t_n \to \infty$. Consider the sequence of events

$$A_n = \{x(t_n) < \varepsilon\}$$

and define

$$A = \bigcup_{n=1}^{\infty} \bigcap_{k \geqslant n}^{\infty} A_k$$

The event A holds when there is an integer $n \geqslant 1$ such that $x(t_k) < \varepsilon$ for $k \geqslant n$. This is always satisfied when $x(t) \to 0$ as $t \to \infty$. Thus, condition (37) implies $\text{prob}(A) = 1$ and consequently

$$\lim_{n\to\infty} \text{prob}\left(\bigcap_{k \geqslant n}^{\infty} A_k\right) = 1$$

Finally, since $\text{prob}(A_n) \geq \text{prob}(\bigcap_{k \geq n}^{\infty} A_k)$, we obtain

$$\lim_{n \to \infty} \text{prob}(A_n) = 1 \tag{38}$$

On the other hand, from the definition of the density $u(t, x)$ it follows that

$$\text{prob}(A_n) = \int_0^{\varepsilon} u(t_n, y) \, dy \tag{39}$$

Since the sequence t_n was arbitrary, conditions (38) and (39) imply that

$$\lim_{t \to \infty} \int_0^{\varepsilon} u(t, y) \, dy = 1 \qquad \text{for} \quad \varepsilon > 0 \tag{40}$$

Thus, for c_{I} satisfying (32) the densities $u(t, x)$ converge to a Dirac delta function as $t \to \infty$.

These results, and in particular Eq. (37), were obtained using the properties of stochastic differential equations interpreted in the Itô sense. However, the partial differential equations (23) and (24) may be considered as independent objects. Thus, if the solutions satisfy (40) for $c_{\text{I}} < \frac{1}{2}\sigma^2$, then the same behavior is preserved when c_{I} is replaced by c_{S}.

5. SUMMARY AND CONCLUSION

In this paper we have shown analytically that additive and parametric (multiplicative) noise, interpreted in either the Itô or Stratonovich formalism, induces global asymptotic stability in two systems, one of which has received attention as the Landau equation.

In both systems without noise, variation of the parameter c leads to a switching between a single globally stable steady state and multiple locally stable steady states. With additive noise this switching is mirrored in the behavior of the extrema of globally stable probability densities at the same value of c. However, parametric noise causes a noise-amplitude-dependent shift (postponement) in the value of c at which the switching occurs.

Under suitable restrictions these results can be extended to more general polynomial forms $g(x)$ in which there are multiple bifurcations in the absence of noise. Further, it will be interesting to examine the situation where colored noise is used, as opposed to the white noise considered here.

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