Noise Prevents Collapse of Vlasov-Poisson Point Charges

FRANÇOIS DELARUE
Université de Nice

FRANCO FLANDOLI
Università di Pisa

AND

DARIO VINCENZI
Université de Nice
CNRS

Abstract

We elucidate the effect of noise on the dynamics of N point charges in a Vlasov-Poisson model with a singular bounded interaction force. A too simple noise does not affect the structure inherited from the deterministic system and, in particular, cannot prevent coalescence of point charges. Inspired by the theory of random transport of passive scalars, we identify a class of random fields generating random pulses that are chaotic enough to disorganize the structure of the deterministic system and prevent any collapse of particles. We obtain the strong unique solvability of the stochastic model for any initial configuration of distinct point charges. In the case where there are exactly two particles, we implement the "vanishing noise method" for determining the continuation of the deterministic model after collapse. © 2014 Wiley Periodicals, Inc.

1 Introduction

It is a well-known fact that white noise perturbations improve the well-posedness properties of ordinary differential equations (ODEs), and in particular the uniqueness of the solutions; see, for instance, Krylov and Röckner [20]. The influence of noise on pathologies of partial differential equations (PDEs) is not as well understood. A review of recent results in the direction of uniqueness can be found in [13, 14]. By contrast, whether noise can prevent the emergence of singularities in PDEs is still quite obscure. A further challenging question is whether noise can select a natural candidate for the continuation of solutions after the singularities.

A well-known system in which the form of the singularities is known explicitly is the Vlasov-Poisson equation on the line. We refer the reader to [26] for several examples and for an extensive discussion of related issues, including the connection with the two-dimensional Euler equations (see also [5, 7, 27, 32, 35]). The motivation for the present study is to understand the influence of noise on such singularities.

Communications on Pure and Applied Mathematics, Vol. LXVII, 1700–1736 (2014)

1.1 Vlasov-Poisson Equation on the Line

Consider the following system in the unknown $f : [0,\infty)\times\mathbb{R}\times\mathbb{R} \ni (t,x,v) \mapsto f(t,x,v) \in \mathbb{R}$:

$$\begin{gathered}\frac{\partial f}{\partial t}(t,x,v) + v\frac{\partial f}{\partial x}(t,x,v) + E(t,x)\frac{\partial f}{\partial v}(t,x,v) = 0, \quad f(0,x,v) = f_0(x,v),\\ \rho(t,x) = \int_{\mathbb{R}} f(t,x,v)\,dv, \quad E(t,x) = \int_{\mathbb{R}} F(x-y)\,\rho(t,y)dy,\end{gathered} \tag{1.1}$$

where $F(x)$ is a bounded function that is continuous everywhere except at $x = 0$ and has sided limits in 0^+ and 0^-. If $F(x) = \text{sign}(x)$ (with $\text{sign}(0) = 0$), equation (1.1) is the one-dimensional Vlasov-Poisson model describing the evolution of the phase space density f of a system of electrons (in natural units, in which the elementary charge and the mass of the electron are set equal to 1). Such an equation is known to develop singularities in the case of measure-valued solutions; see [26] and the works discussed therein; see also [18, 24] for references on equations of this form and related particle approximation. For instance, it is possible to design examples of so-called electron sheet structures that collapse into one point in phase space in finite time (f_0 is an electron sheet if it is concentrated on lines, i.e., $f_0(x,v) = f_0(x)\cdot\delta(v - v_0(x))$). A simplified version of this phenomenon is the coalescence[1] of N point charges: as shown below, there are examples of initial conditions of the form $f_0(x,v) = \sum_{i=1}^N a_i\delta(x-x_i)\delta(v-v_i)$, with distinct pairs $(x_1,v_1),\ldots,(x_N,v_N) \in \mathbb{R}^2$ and with $a_1,\ldots,a_N \geq 0$, for which f remains of the same form on some interval $[0,t_0)$ but at some time t_0 degenerates into $f(t_0,x,v) = \delta(x-x_0)\delta(v-v_0)$ with $(x_0,v_0) \in \mathbb{R}^2$.

The main question motivating our work is the following one: does the presence of noise modify the coalescence phenomenon described above? In this framework, the following picture appears as natural to conceive a noisy version of (1.1): when the electric charge is not totally isolated but lives in a medium (a sort of electric bath), a random external force adds to the force generated by the electric field. Under the assumption that the electric charge does not affect the external random field, the simplest structure modeling this situation is a stochastic PDE (SPDE) of the form

$$\frac{\partial f}{\partial t}(t,x,v) + v\frac{\partial f}{\partial x}(t,x,v) + \left(E(t,x) + \varepsilon\circ\frac{dW_t}{dt}\right)\frac{\partial f}{\partial v}(t,x,v) = 0, \tag{1.2}$$

where W is Brownian motion and the Stratonovich integral is used (this is the natural choice when passing to the limit along regular noises).

Unfortunately, the noise in equation (1.2) is too simple to avoid the emergence of singularities such as those described above. Indeed, the random field $\widetilde{f}(t,x,v) =$

[1] Throughout this paper, the terms *coalescence* and *collapse* are used without distinction.

$f(t, x + \varepsilon \int_0^t W_s ds, v + \varepsilon W_t)$ formally satisfies

$$\left[\frac{\partial \widetilde{f}}{\partial t} + v\frac{\partial \widetilde{f}}{\partial x} + \widetilde{E}\frac{\partial \widetilde{f}}{\partial v}\right](t, x, v) = 0 \quad \text{with } \widetilde{E}(t, x) = \int_{\mathbb{R}^2} F(x-y)\widetilde{f}(t, y, v)dy\, dv,$$

so that any concentration point $z_0 = (x_0, v_0)$ of $\widetilde{f}$ at some time t_0 translates into the random concentration point $(x_0 + \varepsilon \int_0^{t_0} W_s ds, v_0 + \varepsilon W_{t_0})$ of f at the same time.

To expect a nontrivial effect of the noise, we must use noises having a refined spatial structure. Specifically, by considering a noisy equation of the form

$$\left[d_t f + v\frac{\partial f}{\partial x} dt + \left(E(t, x)dt + \varepsilon \sum_{k=1}^{\infty} \sigma_k(x) \circ dW_t^k\right)\frac{\partial f}{\partial v}\right](t, x, v) = 0, \tag{1.3}$$

where $((W_t^k)_{t\geq 0})_{k\geq 1}$ is a family of independent Brownian motions, we prove that, under very general conditions on the covariance function

$$Q(x, y) = \sum_{k=1}^{\infty} \sigma_k(x)\sigma_k(y),$$

the following result holds:

THEOREM 1.1. *Given the initial condition $f_0(x, v) = \sum_{i=1}^N a_i \delta(z - z_i)$ with the generic notation $z = (x, v)$ and with distinct initial points $z_i \in \mathbb{R}^2$ and non-negative coefficients a_i, $i = 1, \dots, N$, there is a unique global solution to system* (1.3) *of the form $f(t, x, v) = \sum_{i=1}^N a_i \delta(z - z_i(t))$, where $((z_i(t))_{t\geq 0})_{1\leq i\leq N}$ is a continuous adapted stochastic process with values in $\mathbb{R}^{2N}$ without coalescence in $\mathbb{R}^2$; i.e., with probability* 1, *$z_i(t) \neq z_j(t)$ for all $t \geq 0$ and $1 \leq i, j \leq N$, $i \neq j$.*

The precise assumptions of Theorem 1.1 and the definitions used therein will be specified later. See in particular Section 1.4 and Theorem 4.1. Here it is worth remarking that our study does not cover the case of an *electron sheet*. We nonetheless expect our result to be a first step forward in this direction, since here the number N of particles is arbitrary[2] for a given covariance function $Q(x, y)$. Indeed, the assumption that we shall impose on $Q(x, y)$ guarantees that, for any N, distinct points $(z_1, \dots, z_N)$ in the (x, v)-space are subject to highly uncorrelated impulses. Such a propagation may be seen as a sort of mild spatial chaos produced by the noise. We notice that Example (1.2) discussed above does not enjoy a similar property since the noise $(\varepsilon W_t)_{t\geq 0}$ plugged therein produces the same impulse at every space point, thus acting as a random Galilean transformation.

[2] Note that N is arbitrary but finite and that nothing is said in the paper about the behavior of the system as N tends to infinity.

1.2 Non-Markovian Continuation after a Singularity

As mentioned above, the random perturbation introduced in equation (1.3) may provide some indications about the natural continuation of the solutions after the coalescence of two point charges. (More difficult or generic cases are not clear at this stage of our understanding of the problem.)

Consider the simple example in which $F(x) = \text{sign}(x)$ and

$$f_0(x, v) = \frac{1}{2}\delta(z - z_1) + \frac{1}{2}\delta(z - z_2), \quad z_1 = (-1, v_0),\ z_2 = (1, -v_0),$$

with $v_0 > 0$. As we shall discuss below, the Lagrangian dynamics corresponding to (1.1) is given by the system

$$\frac{dx_i}{dt}(t) = v_i(t), \quad \frac{dv_i}{dt}(t) = F(x_i(t) - x_{\bar{\imath}}(t)), \quad t \geq 0, \tag{1.4}$$

for $i = 1, 2$ and $\bar{\imath} = 2$ if $i = 1$ and vice versa. The initial condition for the above system is $(x_i\,(0)\,, v_i\,(0)) = (\varsigma_i, -\varsigma_i v_0)$ with $\varsigma_i = 1$ if $i = 2$ and $\varsigma_i = -1$ if $i = 1$. When $v_0 = \sqrt{2}$, the functions

$$v_i^*(t) = \varsigma_i(t - v_0), \quad x_i^*(t) = \varsigma_i\left(1 - v_0 t + \frac{t^2}{2}\right), \quad i = 1, 2, \tag{1.5}$$

solve the system for $t \in [0, v_0)$, and the limits of $x_1^*(t)$ and $x_2^*(t)$ coincide as $t \uparrow v_0$. This means that, with the choice $v_0 = \sqrt{2}$, the solutions $(x_i^*(t), v_i^*(t))_{t\in[0,v_0)}$, $i = 1, 2$, converge to the same point $(0, 0)$ as $t \uparrow v_0$, so that the origin $(0, 0)$ is a singular point of the Lagrangian dynamics.

By contrast, Theorem 1.1 states that, for any positive level of noise ε in the noisy formulation (1.3), the random solutions $((x_i^\varepsilon(t), v_i^\varepsilon(t))_{t\geq0})_{i=1,2}$ never meet, with probability 1. It is then natural to investigate the behavior of the stochastic solution as $\varepsilon \to 0$. In Section 2, we shall prove the following theorem under general conditions on the covariance function Q (see Conditions 2.1 and 2.2 together with Proposition 3.7):

THEOREM 1.2. *Let $v_0 = \sqrt{2}$. Then, as $\varepsilon \to 0$, the pair process $((z_i^\varepsilon(t))_{t\geq0})_{i=1,2}$ converges in distribution on the space $\mathcal{C}([0,\infty), \mathbb{R}^2 \times \mathbb{R}^2)$ toward*

$$\frac{1}{2}\delta\big((z_1^*(t), z_2^*(t))_{t\geq0}\big) + \frac{1}{2}\delta\big((z_1^{**}(t), z_2^{**}(t))_{t\geq0}\big), \tag{1.6}$$

where $z_i^*(t) = (x_i^*(t), v_i^*(t))$ for $t \geq 0$ and $i = 1, 2$, and $(z_1^{**}(t), z_2^{**}(t))$ is equal to $(z_1^*(t), z_2^*(t))$ for $t \leq \sqrt{2}$ and $(z_2^*(t), z_1^*(t))$ for $t > \sqrt{2}$.

Theorem 1.2 must be seen as a rule for the continuation of the solutions of the deterministic system (1.1) after a singularity. When the particles meet, they split instantaneously, but they can do it in two different ways: (i) with probability $\frac{1}{2}$, the trajectories meet at coalescence time and then split without crossing each other (i.e., each of the two trajectories keeps the same sign before and after coalescence); (ii) with probability $\frac{1}{2}$, the trajectories meet, cross each other, and then split forever (i.e., the sign of each of them changes exactly at coalescence time). This represents a mathematical description of the repulsive effect of the interaction force F: there is no way for the particles to merge and then form a single particle with a double charge.

This situation can be interpreted as a physical loss of the Markov property: just after coalescence, splitting occurs because the system keeps memory of what its state was before. More precisely, if we model the dynamics of a static single particle with double charge by a pair $(z_1^{00}(t), z_2^{00}(t))_{t\geq 0}$ in phase space, with $z_1^{00}(t) = z_2^{00}(t)$ and $\dot{v}_1^{00}(t) = \dot{v}_2^{00}(t) = 0$ for $t \geq 0$, we get a non-Markovian family of solutions to (1.4). When the trajectories z_1^0 and z_2^0 meet, they do not restart with the same dynamics as z_1^{00} and z_2^{00}. We refer the reader to [10] for other mathematical examples of non-Markovian continuations.

1.3 Vlasov-Poisson-Type System of N Particles in $\mathbb{R}^d$

The problem described in Section 1.1 will be treated as a particular case of the following generalization in $\mathbb{R}^d$ subject to similar constraints as in (1.1):

$$\left[d_t f + (v \cdot \nabla_x f + E(t,x) \cdot \nabla_v f)\, dt + \sum_{k=1}^{\infty} \sigma_k(x) \cdot \nabla_v f \circ dW_t^k\right](t,x,v) = 0,$$

where $\sigma_k : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz-continuous fields that are subject to additional assumptions, which will be specified later (see (A.2–3) in Section 1.4), and $((W_t^k)_{t\geq 0})_{k\in\mathbb{N}\setminus\{0\}}$ are independent one-dimensional Brownian motions. In the following, F will be assumed to be bounded and locally Lipschitz-continuous on any compact subset of $\mathbb{R}^d \setminus \{0\}$, the Lipschitz constant on any ring of the form $\{x \in \mathbb{R}^d : r \leq |x| \leq 1\}$ growing at most as $1/r$ as r tends to 0. In particular, F may be discontinuous at 0. A relevant case is when $F = \nabla U$, where U is a potential with a *Lipschitz point at* 0; i.e., U is Lipschitz-continuous on $\mathbb{R}^d$ and smooth on $\mathbb{R}^d \setminus \{0\}$.

This framework includes the example $F(x) = x/|x|$, $x \in \mathbb{R}^d$, and, as a particular case, the one-dimensional model discussed above, i.e., $F(x) = \mathrm{sign}(x)$, $x \in \mathbb{R}$. By contrast, the d-dimensional Poisson case, where

$$F(x) = \pm x|x|^{-d}, \quad x \in \mathbb{R}^d, \ d \geq 2, \tag{1.7}$$

does not satisfy the aforementioned assumptions, and therefore falls outside this study. The signs "+" and "−" describe repulsive and attractive interactions, respectively; the corresponding models are referred to as *electrostatic* and *gravitational*. For the electrostatic potential, our analysis turns out to be irrelevant in dimension $d \geq 2$ since the deterministic system itself is free of coalescence.

When $F(0) = 0$, the Lagrangian motion associated with the SPDE is

$$\frac{dX_t^i}{dt} = V_t^i, \quad dV_t^i = \sum_{j\neq i} a_j F\big(X_t^i - X_t^j\big)dt + \sum_{k=1}^{\infty} \sigma_k\big(X_t^i\big) \circ dW_t^k, \tag{1.8}$$

for $t \geq 0$ and $1 \leq i \leq N$. Indeed, by applying Itô's formula in the Stratonovich form to the process $(\sum_{1\leq i\leq N} a_i\varphi(X_t^i, V_t^i))_{t\geq 0}$ for a test function $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, it can be shown that the measure-valued process

$$f(t,x,v) = \sum_{i=1}^{N} a_i \delta\bigl(x - X_t^i\bigr)\delta\bigl(v - V_t^i\bigr), \quad t \geq 0, \tag{1.9}$$

solves the SPDE in a weak form.

This paper is devoted to the analysis of system (1.8). Note that the problem would be much easier to handle if each particle were to be forced by an independent Brownian motion. This choice of the noise, however, would break the relation between the Lagrangian dynamics and the SPDE introduced above.

The paper is organized as follows: In Section 2 we start with the proof of Theorem 1.2 in view of its physical interpretation. The vanishing noise method for selecting solutions of singular differential equations goes back to [3], but the examples investigated therein are one-dimensional only. (See also [13], as well as [2, 16].) In [9, 33], the vanishing noise method is also used to investigate the motion of two touching circles moving by mean curvature: this is an example where the motion after the singularity—the configuration in which the two circles touch each other—is not unique and the so-called fattening phenomenon may happen; it is then proven that the zero-noise limit selects a unique continuation after a singularity. The analysis therein reduces to a one-dimensional problem as well. Here Theorem 1.2 applies to a four-dimensional system, which is actually reduced to a two-dimensional one in the proof. In [3], the method for investigating the vanishing-noise behavior of the stochastic differential equation under consideration is mainly of an analytical essence. The proof of Theorem 1.2 below relies on a stochastic expansion of the solutions similar to the one used in [9, 33]: pathwise, the dynamics of $(z_i^\varepsilon)_{i=1,2}$ are expanded with respect to the parameter ε until coalescence occurs; the limit distribution is then given by the distribution of the random coefficients of the expansion. The zero-noise solution in [9, 33] is a Bernoulli distribution on the path space concentrated on two special solutions with equal probabilities. In our case, this is true as well, but only at the level of ODE (1.4). At the level of PDE (1.1), the two different solutions of (1.4) define the same measure-valued solution (it is just an exchange of particles), and consequently there is a unique deterministic continuation for the PDE.

The remainder of the paper is devoted to the proof of Theorem 1.1. We first discuss the structure of the noise that will prevent coalescence from emerging. It should be emphasized that the effect of the noise on the $2N$-system (1.8) is highly nontrivial. Indeed, although it is doubly singular, the noise makes the system fluctuate enough to avoid pathological phenomena such as those observed in the deterministic case. The first singularity is due to the fact that the same Brownian motions act on all the particles (contrary to the classical case when each particle is subject to an independent noise; see, for instance, [19]). If the $(\sigma_k)_{k\in\mathbb{N}\setminus\{0\}}$ were constant, the particles would feel the same impulses and the noise would not have any real effect; in other words, the noise would just act as a random translation of the system, as in (1.2). Thus the point is to design a noise allowing displacements of distinct particles in distinct directions.

To reach the desired effect, the covariance matrix $(Q(x^i, x^j))_{1 \le i,j \le N}$ must be strictly positive for any vector $(x^1, \dots, x^N) \in (\mathbb{R}^d)^N$ with pairwise distinct entries. We give some examples in Section 3. These examples are inspired by the Kraichnan noise used in the theory of random transport of passive scalars (see, for example, [12, 22, 29]). Because of the regularity properties we assume on Q, they are just related with the Batchelor regime of the Kraichnan model and not with the so-called "turbulent regime," the structure of which is too singular for our analysis (see Section 3.2). Nevertheless, the model considered here does not have the same interpretation as the Kraichnan model, since in (1.8) the noise acts on the velocities. A possible way to relate equation (1.8) to turbulence theory would consist in penalizing the drift of the velocity of the i^{th} particle by $-V_t^i$. This model would describe the motion of interacting heavy particles in a random velocity; see [4].

The second singularity of the model is inherited from the kinetic structure of the deterministic counterpart: the noise only acts as an additional random force; i.e., it is only plugged into the equation of the velocity. In other words, the coupled system for $(X_t^i, V_t^i)_{1 \le i \le N}$ is degenerate. We shall show in Section 3.3 that the ellipticity properties of the noise in $\mathbb{R}^{Nd}$ actually lift up to hypoellipticity properties in $\mathbb{R}^{2Nd}$.

Once the structure of the noise is defined, we are ready to tackle the problem of noncoalescence. We first establish that the Lagrangian dynamics is well posed for Lebesgue a.e. initial configuration of distinct particles. This does not require any special feature of the noise. By specifying the form of the noise according to the requirements discussed in Section 3, we then prove the well-posedness and the absence of collapse for *all* initial conditions of the particle system with pairwise distinct entries. To prove these results, we exploit the hypoellipticity of the whole system; see Section 4.3. The main lines of Section 4 are connected with the strategy already developed in [15] (see [28] for a deterministic counterpart) in order to prove that noise may prevent N-point vortices driven by two-dimensional Euler equations from collapsing. However, here both the framework and the results are quite different. In [15], the noise is finite dimensional, the dimension depending upon the number of particles; the noise is only given implicitly from a generic existence result; moreover, the dynamics of the particles is nondegenerate. Here the structure of the noise is explicit and is independent of the number of particles; moreover, the dynamics of the particles is degenerate.

1.4 Assumptions

For simplicity, in (1.8) we choose $a_i = 1/N$ for $1 \le i \le N$. We also assume that

(A.1) F is bounded everywhere on $\mathbb{R}^d$ and locally Lipschitz-continuous on any compact subset of $\mathbb{R}^d \setminus \{0\}$. Moreover,

$$\sup_{0<r\le 1} \sup_{\substack{r\le|x|,|y|\le 1\\ x\ne y}} \left[r\frac{|F(x)-F(y)|}{|x-y|} \right] < +\infty.$$

Possible examples are: for $d=1$, $F(x)=\operatorname{sgn}(x)$, $x\in\mathbb{R}\setminus\{0\}$; for $d\ge 2$, $F(x)=x/|x|$, $x\in\mathbb{R}^d\setminus\{0\}$.

(A.2) For each $k\in\mathbb{N}\setminus\{0\}$, $\sigma_k:\mathbb{R}^d\to\mathbb{R}^d$ is Lipschitz-continuous, and the series $\sum_{k=1}^{\infty}\sigma_k^{\alpha}(\tilde{x})\sigma_k^{\beta}(\tilde{y})$ converges uniformly with respect to $(\tilde{x},\tilde{y})$ in any compact subset of $\mathbb{R}^d\times\mathbb{R}^d$ for each $\alpha,\beta=1,\dots,d$. The covariance function of the random field $\mathbb{R}^d\ni\tilde{x}\mapsto\sum_{k=1}^{\infty}\sigma_k(\tilde{x})W_1^k$ is defined as follows:

$$Q(\tilde{x},\tilde{y})=\sum_{k=1}^{\infty}\sigma_k(\tilde{x})\otimes\sigma_k(\tilde{y})\in\mathbb{R}^{d\times d}. \tag{1.10}$$

It is of positive type, that is, $\sum_{i,j=1}^{n}\langle Q(\tilde{x}^j,\tilde{x}^i)\tilde{v}^i,\tilde{v}^j\rangle_{\mathbb{R}^d}\ge 0$ for any $n\ge 1,\tilde{x}^1,\dots,\tilde{x}^n,\tilde{v}^1,\dots,\tilde{v}^n\in\mathbb{R}^d$.

(A.3) $Q(\tilde{x},\tilde{y})$ is bounded on the diagonal, that is, $\sup_{\tilde{x}\in\mathbb{R}^d}|Q(\tilde{x},\tilde{x})|<+\infty$. Furthermore, it satisfies the Lipschitz-type regularity property:

$$\sup_{\substack{\tilde{x},\tilde{y}\in\mathbb{R}^d\\ \tilde{x}\ne\tilde{y}}}\frac{|Q(\tilde{x},\tilde{x})+Q(\tilde{y},\tilde{y})-Q(\tilde{x},\tilde{y})-Q(\tilde{y},\tilde{x})|}{|\tilde{x}-\tilde{y}|^2}<+\infty. \tag{1.11}$$

(A.4) $Q(\tilde{x},\tilde{y})$ is *strictly positive on* $\Gamma_{x,N}=\{(x^1,\dots,x^N)\in\mathbb{R}^{Nd}:x^i\ne x^j$ whenever $i\ne j\}$; that is, for all $(x^1,\dots,x^N)\in\Gamma_{x,N}$ and $v=(v^1,\dots,v^N)\in\mathbb{R}^{Nd}\setminus\{0\}$,

$$\sum_{i,j=1}^{N}\langle Q(x^j,x^i)v^i,v^j\rangle_{\mathbb{R}^d}>0.$$

The regularity assumptions on σ_k and Q in (A.2) and (A.3) are strongly related to each other. Specifically, the Lipschitz condition (1.11) implies a strong Lipschitz property of the fields $(\sigma_k)_{k\in\mathbb{N}\setminus\{0\}}$:

$$\begin{aligned}&\sum_{k=1}^{\infty}\big(\sigma_k^{\alpha}(\tilde{x})-\sigma_k^{\alpha}(\tilde{y})\big)\big(\sigma_k^{\beta}(\tilde{x})-\sigma_k^{\beta}(\tilde{y})\big)=\\ &\quad Q_{\alpha\beta}(\tilde{x},\tilde{x})-Q_{\alpha\beta}(\tilde{x},\tilde{y})-Q_{\alpha\beta}(\tilde{y},\tilde{x})+Q_{\alpha\beta}(\tilde{y},\tilde{y})\le C|\tilde{x}-\tilde{y}|^2.\end{aligned} \tag{1.12}$$

Conversely, equation (1.11) holds if the Lipschitz constants of the $(\sigma_k)_{k\in\mathbb{N}\setminus\{0\}}$ are square-summable.

In practice, the covariance function Q is given first, i.e., given a function $Q : \mathbb{R}^{2d} \ni (\tilde{x}, \tilde{y}) \mapsto Q(\tilde{x}, \tilde{y})$ with values in the set of symmetric matrices of size $d \times d$ satisfying (A.3) and of positive type, Q may be expressed as a covariance function of the form (1.10) for some fields $(\sigma_k)_{k \in \mathbb{N} \setminus \{0\}}$ satisfying (A.2). We refer the reader to theorem 4.2.5 in [21] for more details. In this framework, a sufficient condition to guarantee (1.11) is: Q is of class $\mathcal{C}^2$ with bounded mixed derivatives, that is, $\sup_{(\tilde{x}, \tilde{y}) \in \mathbb{R}^d \times \mathbb{R}^d} |\partial^2_{\tilde{x}, \tilde{y}} Q(\tilde{x}, \tilde{y})| < +\infty$. Indeed, Lipschitz property (1.11) then follows from a straightforward Taylor expansion.

As a consequence of (A.2), the Stratonovich integrals in SDE (1.8) are (formally) equal to Itô integrals, and hence (1.8) will be interpreted in the usual Itô form

$$\frac{dX_t^i}{dt} = V_t^i, \quad dV_t^i = \frac{1}{N} \sum_{j \neq i} F\big(X_t^i - X_t^j\big) dt + \sum_{k=1}^{\infty} \sigma_k\big(X_t^i\big) dW_t^k, \tag{1.13}$$

$t \geq 0$, $i \in \{1, \dots, N\}$. Indeed, the local martingale part of $(\sigma_k(X_t^i))_{t \geq 0}$ is zero, since $(\sigma_k(X_t^i))_{t \geq 0}$ is of bounded variation, so that

$$\big[\sigma_k(X_\cdot^i), W^k\big]_t = 0, \quad t \geq 0, \; i \in \{1, \dots, N\}, \; k \in \mathbb{N} \setminus \{0\}.$$

We shall not treat this equivalence more rigorously, and from now on we shall adopt the Itô formulation.

1.5 Useful Notation

Throughout this paper, the number N of particles is fixed, and thus the dependence of the constants upon N is not investigated. For any $n \in \mathbb{N} \setminus \{0\}$, $z \in \mathbb{R}^n$, and $r > 0$, $B_n(z, r)$ is the closed ball of dimension n, center z, and radius r; Leb_n is the Lebesgue measure on $\mathbb{R}^n$. The volume of $B_n(z, r)$ is denoted by $\mathcal{V}_n(r)$. The configurations of the N-particle system in the phase space are generally denoted by z or Z. Positions are denoted by x or X and velocities by v or V. Similarly, the typical notation for a single particle in the phase space is $\tilde{z} = (\tilde{x}, \tilde{v})$, $\tilde{x}$ standing for its position and $\tilde{v}$ for its velocity. The set of pairs of different indices in the particle system is denoted by $\Delta_N = \{(i, j) \in \{1, \dots, N\}^2 : i \neq j\}$. Moreover, we introduce $\Gamma_N = \{(z^1, \dots, z^N) \in \mathbb{R}^{2Nd} : \forall (i, j) \in \Delta_N, z_i \neq z_j\}$ and $\Gamma_{x,N} = \{(x^1, \dots, x^N) \in \mathbb{R}^{Nd} : \forall (i, j) \in \Delta_N, x_i \neq x_j\}$. We also define the following projection mappings:

$$\begin{aligned} \Pi_x : \mathbb{R}^{2Nd} \ni z = (z^j)_{1 \leq j \leq N} = ((x^j, v^j))_{1 \leq j \leq N} &\mapsto \Pi_x(z) \\ &= (x^j)_{1 \leq j \leq N} \in \mathbb{R}^{Nd}, \end{aligned}$$

$$\tilde{\pi}_x : \mathbb{R}^{2d} \ni \tilde{z} = (\tilde{x}, \tilde{v}) \mapsto \tilde{x} \in \mathbb{R}^d,$$

$$\begin{aligned} \pi_{i,x} : \mathbb{R}^{2Nd} \ni z = (z^j)_{1 \leq j \leq N} = ((x^j, v^j))_{1 \leq j \leq N} &\mapsto \pi_{i,x}(z) \\ &= x^i \in \mathbb{R}^d. \end{aligned}$$

Π_v, $\widetilde{\pi}_v$, and $\pi_{i,v}$ are defined analogously. We then denote $\pi_i = (\pi_{i,x}, \pi_{i,v})$.

In the following, equation (1.13) will also be written in the compact form

$$dZ_t = \mathbb{F}(Z_t)dt + \sum_{k=1}^{\infty} \mathbb{A}_k(Z_t)dW_t^k, \quad t \geq 0, \tag{1.14}$$

where $Z_t = (X_t, V_t)$, with $X_t = (X_t^1, \dots, X_t^N)$ and $V_t = (V_t^1, \dots, V_t^N)$, and

$$\begin{aligned} &\mathbb{F} : \mathbb{R}^{Nd} \times \mathbb{R}^{Nd} \ni (x = (x^1, \dots, x^N), v) \\ &\quad \mapsto \left(v, \left(\frac{1}{N} \sum_{j \neq i} F(x^i - x^j) \right)_{1 \leq i \leq N} \right) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}, \\ &\mathbb{A}_k : \mathbb{R}^{Nd} \times \mathbb{R}^{Nd} \ni (x, v) \mapsto (0, A_k(x)) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}, \\ &A_k : (\mathbb{R}^d)^N \ni (x^1, \dots, x^N) \mapsto \big(\sigma_k(x^1), \dots, \sigma_k(x^N)\big) \in (\mathbb{R}^d)^N. \end{aligned} \tag{1.15}$$

For any $t \geq 0$, the $2d$-coordinates of Z_t will be denoted by $Z_t^i = \pi_i(Z_t) = (X_t^i, V_t^i)$, $i \in \{1, \dots, N\}$. Similarly, we shall denote by $(\mathbb{F}^i \equiv \pi_i(\mathbb{F}))_{1 \leq i \leq N}$ and $(\mathbb{A}_k^i \equiv \pi_i(\mathbb{A}_k))_{1 \leq i \leq N}$ the $2d$-components of $\mathbb{F}$ and $\mathbb{A}_k$+ for $k \in \mathbb{N} \setminus \{0\}$.

2 Continuation: Proof of Theorem 1.2

In this section, we identify general conditions on the structure of the noise in (1.3) under which Theorem 1.2 holds. Typical examples are given in Proposition 3.7. Throughout this section, we thus consider the four-dimensional system:

$$dX_t^{i,\varepsilon} = V_t^{i,\varepsilon}dt, \quad dV_t^{i,\varepsilon} = \operatorname{sign}\big(X_t^{i,\varepsilon} - X_t^{\bar{\imath},\varepsilon}\big) + \varepsilon \sum_{k=1}^{\infty} \sigma_k\big(X_t^{i,\varepsilon}\big)dW_t^k, \tag{2.1}$$

for $t \geq 0$, $i = 1, 2$, and $\bar{\imath} = 2$ if $i = 1$ and vice versa. We assume below that $(X_0^{i,\varepsilon}, V_0^{i,\varepsilon}) = (\varsigma_i, -\varsigma_i\sqrt{2})$, with $\varsigma_i = 1$ if $i = 2$ and $\varsigma_i = -1$ if $i = 1$. As a first general condition (once more, we refer the reader to Proposition 3.7 for examples), we set:

Condition 2.1. For any $\varepsilon > 0$, Theorem 1.1 applies and thus (2.1) has a unique strong solution that satisfies $\mathbb{P}\{\forall t \geq 0, (X_t^{1,\varepsilon}, V_t^{1,\varepsilon}) \neq (X_t^{2,\varepsilon}, V_t^{2,\varepsilon})\} = 1$.

We shall define $Z_t^{i,\varepsilon} = (X_t^{i,\varepsilon}, V_t^{i,\varepsilon})$, $\varepsilon > 0$, $i = 1, 2$. When $\varepsilon = 0$, the curves

$$X_t^{i,0} = \varsigma_i \left(1 - \frac{t}{\sqrt{2}}\right)^2, \quad V_t^{i,0} = \varsigma_i\big(-\sqrt{2} + t\big), \quad t \geq 0,$$

solve the system (2.1) but merge at time $t_0 = \sqrt{2}$. We shall once more define $Z_t^{i,0} = (X_t^{i,0}, V_t^{i,0})$, $i = 1, 2$. Note that $Z_t^{2,0} = -Z_t^{1,0}$ for all $t \geq 0$.

We are to prove that $(Z_t^{1,\varepsilon}, Z_t^{2,\varepsilon})_{t\geq 0}$ converges in distribution on the space $\mathcal{C}([0,+\infty),\mathbb{R}^4)$ toward $(1/2)\delta_{(Z_t^{1,+},Z_t^{2,+})_{t\geq 0}} + (1/2)\delta_{(Z_t^{1,-},Z_t^{2,-})_{t\geq 0}}$, where

$$(Z_t^{1,+}, Z_t^{2,+}) = (Z_t^{1,0}, Z_t^{2,0}), \quad t \geq 0,$$

$$(Z_t^{1,-}, Z_t^{2,-}) = \begin{cases} (Z_t^{1,0}, Z_t^{2,0}), & t \in [0, t_0], \\ (Z_t^{2,0}, Z_t^{1,0}), & t > t_0. \end{cases}$$

The whole point is to investigate the differences:

$$\mathbb{X}_t^\varepsilon = \frac{X_t^{2,\varepsilon} - X_t^{1,\varepsilon}}{2}, \quad \mathbb{V}_t^\varepsilon = \frac{V_t^{2,\varepsilon} - V_t^{1,\varepsilon}}{2}, \quad t \geq 0, \ \ \varepsilon > 0. \tag{2.2}$$

We shall use the second condition (see Proposition 3.7 as an example):

Condition 2.2. Assume $(Z_t^{1,\varepsilon}, Z_t^{2,\varepsilon})_{t\geq 0}$ satisfies (2.1) but with an arbitrary random initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon}) \in \Gamma_2$ that is independent of the noise $((W_t^k)_{t\geq 0})_{k\geq 1}$. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the augmented filtration that is generated by the initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon})$ and by the noise $((W_t^k)_{t\geq 0})_{k\geq 1}$. Then there exists an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(B_t^\varepsilon)_{t\geq 0}$ such that, for all $t \geq 0$,

$$d\mathbb{X}_t^\varepsilon = \mathbb{V}_t^\varepsilon dt, \quad d\mathbb{V}_t^\varepsilon = \operatorname{sign}(\mathbb{X}_t^\varepsilon)dt + \varepsilon\sigma(\mathbb{X}_t^\varepsilon)dB_t^\varepsilon, \tag{2.3}$$

where σ is a $\mathcal{C}^2$ function from $\mathbb{R}$ to $\mathbb{R}$, depending on the $(\sigma_k)_{k\geq 1}$ only (in particular, σ is independent of the initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon})$ and of ε), with bounded derivatives of order 1 and 2, and such that $\sigma(0) = 0$ and $\sigma(1) > 0$.

Defining $\mathbb{Z}_t^\varepsilon = (\mathbb{X}_t^\varepsilon, \mathbb{V}_t^\varepsilon)$ for any $t \geq 0$, we first investigate the solutions of (2.3) when $\varepsilon = 0$. We have the obvious lemma:

LEMMA 2.3. *For $\varepsilon = 0$, all the solutions of* (2.3) *with $\varepsilon = 0$ and $(\mathbb{X}_0^0, \mathbb{V}_0^0) = (1, -\sqrt{2})$ have the form*

$$\mathbb{Z}_t^0 = (\mathbb{X}_t^0, \mathbb{V}_t^0) = \left(\frac{(t_0 - t)^2}{2}, t - t_0\right) \quad \text{for } 0 \leq t \leq t_0 = \sqrt{2}. \tag{2.4}$$

We emphasize that uniqueness fails after coalescence time t_0. Indeed, any $(\mathbb{Z}_t^0)_{t\geq 0}$ with $(\mathbb{Z}_t^0)_{0\leq t\leq t_0}$ as in (2.4), $\mathbb{Z}_t^0 = (0,0)$ for $t_0 \leq t \leq t_1$, and $\mathbb{Z}_t^0 = \pm((t-t_1)^2/2, t-t_1)$ for $t \geq t_1$, where $t_1 \geq t_0$ may be real or infinite, is a solution of (2.3) when $\varepsilon = 0$ therein. We claim the following:

PROPOSITION 2.4. *Given $\tau^\varepsilon = \inf\{t \geq 0 : \mathbb{X}_t^\varepsilon \leq 0\}$, for any $\delta > 0$ and $M > t_0$,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon \in (t_0 - \delta, t_0 + \delta)\} = \frac{1}{2}, \quad \lim_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon \geq M\} = \frac{1}{2}. \tag{2.5}$$

Moreover, defining $\tau_2^\varepsilon = \inf\{t > \tau^\varepsilon : \mathbb{X}_t^\varepsilon \geq 0\}$, we have, for any $M > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\tau_2^\varepsilon \geq M\} = 1. \tag{2.6}$$

Proposition 2.4 suggests that, in the limiting regime $\varepsilon \to 0$, the trajectories of the two particles cross with probability equal to $\frac{1}{2}$, and, if so, they just cross once, at coalescence time. This is one step forward in the proof of Theorem 1.2. Specifically, we prove below that Proposition 2.4 implies Theorem 1.2.

PROPOSITION 2.4 $\Rightarrow$ THEOREM 1.2. As a consequence of (A.3), the family $((Z_t^{1,\varepsilon}, Z_t^{2,\varepsilon})_{t\geq 0})_{0<\varepsilon\leq 1}$ is tight. We denote by μ a weak limit of the family of measures $(\mathbb{P}_{(Z_\cdot^{1,\varepsilon}, Z_\cdot^{2,\varepsilon})})_{0<\varepsilon\leq 1}$ as $\varepsilon \to 0$ on the space of continuous functions $\mathcal{C}([0,+\infty), \mathbb{R}^4)$, the canonical process on $\mathcal{C}([0,+\infty), \mathbb{R}^4)$ being denoted by $(\xi_\cdot^1 = (\chi_\cdot^1, \nu_\cdot^1), \xi_\cdot^2 = (\chi_\cdot^2, \nu_\cdot^2))$. We shall also denote $\chi_t = (\chi_t^2 - \chi_t^1)/2$ and $\nu_t = (\nu_t^2 - \nu_t^1)/2$, $t \geq 0$.

Under the measure μ, $\xi_\cdot^i = (\chi_\cdot^i, \nu_\cdot^i)$, $i = 1, 2$, satisfies $\dot{\chi}_t^i = \nu_t^i$, $|\dot{\nu}_t^i| \leq 1$, $t \geq 0$, $i = 1, 2$. We now make use of Proposition 2.4. Given $M > 0$, we have, on the set $\{\tau^\varepsilon \geq M\}$,

$$V_t^{i,\varepsilon} = V_0^{i,\varepsilon} + \varsigma_i t + \varepsilon \sum_{k\geq 1} \int_0^t \sigma_k(X_s^{i,\varepsilon}) dW_s^k, \quad 0 \leq t \leq M,$$

where ς_i is equal to 1 if $i = 2$ and -1 if $i = 1$, so that, for any $\eta > 0$,

$$\liminf_{\varepsilon\to 0} \mathbb{P}\Big\{ \sup_{0\leq t\leq M} \big| V_t^{i,\varepsilon} - V_0^{i,\varepsilon} - \varsigma_i t \big| \leq \eta,\ i = 1, 2 \Big\} \geq \lim_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon \geq M\}.$$

By using the portmanteau theorem, we deduce that

$$\mu\{\nu_t^i = \nu_0^i + \varsigma_i t,\ 0 \leq t \leq M,\ i = 1, 2\} \begin{cases} = 1 & \text{if } M < t_0, \\ \geq \frac{1}{2} & \text{if } M > t_0. \end{cases} \tag{2.7}$$

Therefore, under μ, $(\xi_t^1)_{0\leq t\leq t_0}$ coincides with $(Z_t^{1,0})_{0\leq t\leq t_0}$ and $(\xi_t^2)_{0\leq t\leq t_0}$ coincides with $(Z_t^{2,0})_{0\leq t\leq t_0}$. In particular, under μ, $\xi_{t_0}^1 = \xi_{t_0}^2 = (0, 0)$. Similarly, we also infer from (2.7) that, with probability greater than $\frac{1}{2}$ under μ, $(\xi_t^1, \xi_t^2) = (Z_t^{1,+}, Z_t^{2,+})$ for any $t \geq 0$.

By the same argument, for $\delta > 0$ small and $M > t_0 + \delta$, we deduce from Proposition 2.4 that

$$\mu\big\{\nu_t^i = \nu_{t_0+\delta}^i - \varsigma_i[t - (t_0 + \delta)],\ t_0 + \delta \leq t \leq M,\ i = 1, 2\big\} \geq \lim_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon \leq t_0 + \delta, \tau_2^\varepsilon \geq M\} = \frac{1}{2}.$$

By letting δ tend to 0 and M to $+\infty$, we obtain that, with probability greater than $\frac{1}{2}$ under μ, $(\xi_t^1, \xi_t^2) = (Z_t^{1,-}, Z_t^{2,-})$ for any $t \geq 0$. □

2.1 Key Lemmas by Integration by Parts

The proof of Proposition 2.4 relies on two key lemmas, each proven by using integration by parts.

LEMMA 2.5. *Let* $\mathcal{N}^+ = [0,+\infty)^2 \setminus \{(0,0)\}$ *and* $\mathcal{N}^- = (-\infty,0]^2 \setminus \{(0,0)\}$. *Consider also the following sets of initial conditions for* (2.1): $\Gamma^\pm = \{(z^1 = (x^1,v^1), z^2 = (x^2,v^2)) : (x^2-x^1, v^2-v^1) \in \mathcal{N}^\pm\}$. *Then there exists a constant* $c > 0$ *such that, for any* $M > 0$ *and any compact subset* $K \subset \mathbb{R}^4$,

$$\lim_{\varepsilon\to 0} \inf_{(z^1,z^2)\in K\cap\Gamma^\pm} \mathbb{P}\{\forall t \in [0,M] \ \pm \mathbb{X}_t^\varepsilon \geq ct^2 \mid (Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon}) = (z^1,z^2)\} = 1.$$

PROOF. In the whole proof, the initial condition $(z^1,z^2) \in K \cap \Gamma^+$ is given, i.e., $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon}) = (z^1,z^2) \in K \cap \Gamma^+$. Writing $z^i = (x^i,v^i)$, $i = 1,2$, we define $x = (x^2-x^1)/2$ and $v = (v^2-v^1)/2$. Without loss of generality, we assume that $x > 0$. Indeed, when $x = 0$, v must be positive, so that, in very short time, both $\mathbb{X}^\varepsilon$ and $\mathbb{V}^\varepsilon$ are positive. By the Markov property (which holds for the four-dimensional system because of strong uniqueness), we are then led back to the case when x and v are positive. By Condition 2.2, we can write

$$d\mathbb{V}_t^\varepsilon = \operatorname{sign}(\mathbb{X}_t^\varepsilon)dt + \varepsilon\sigma(\mathbb{X}_t^\varepsilon)dB_t^\varepsilon, \quad t \geq 0,$$

where $(B_t^\varepsilon)_{t\geq 0}$ is a one-dimensional Brownian motion. Using the smoothness of σ, we perform the following integration by parts:

$$d(\mathbb{V}_t^\varepsilon - \varepsilon\sigma(\mathbb{X}_t^\varepsilon)B_t^\varepsilon) = (\operatorname{sign}(\mathbb{X}_t^\varepsilon) - \varepsilon\sigma'(\mathbb{X}_t^\varepsilon)\mathbb{V}_t^\varepsilon B_t^\varepsilon)dt.$$

Recalling that $\tau^\varepsilon = \inf\{t \geq 0 : \mathbb{X}_t^\varepsilon \leq 0\}$, we have

$$\mathbb{V}_t^\varepsilon - \varepsilon\sigma(\mathbb{X}_t^\varepsilon)B_t^\varepsilon \geq t - \varepsilon\int_0^t \sigma'(\mathbb{X}_s^\varepsilon)\mathbb{V}_s^\varepsilon B_s^\varepsilon\, ds, \quad 0 \leq t \leq \tau^\varepsilon.$$

In the event $A_1^\varepsilon = \{\sup_{0\leq t\leq M} |\sigma'(\mathbb{X}_t^\varepsilon)\mathbb{V}_t^\varepsilon B_t^\varepsilon| \leq \frac{1}{2\varepsilon}\}$, we have

$$d\mathbb{X}_t^\varepsilon \geq \left(\frac{t}{2} + \varepsilon\sigma(\mathbb{X}_t^\varepsilon)B_t^\varepsilon\right)dt \geq \left(\frac{t}{2} - C\varepsilon\mathbb{X}_t^\varepsilon|B_t^\varepsilon|\right)dt, \quad 0 \leq t \leq \tau^\varepsilon \wedge M,$$

where here C is the Lipschitz constant of σ. We conclude that

$$d\bar{\mathbb{X}}_t^\varepsilon \geq \frac{t}{2}\exp\left(C\varepsilon\int_0^t |B_s^\varepsilon| ds\right)dt \quad \text{with } \bar{\mathbb{X}}_t^\varepsilon = \mathbb{X}_t^\varepsilon \exp\left(C\varepsilon\int_0^t |B_s^\varepsilon| ds\right)$$

for $0 \leq t \leq \tau^\varepsilon \wedge M$. Therefore, on A_1^ε, τ^ε must be greater than M so that the above expression holds up to time M (at least). We deduce that $d\bar{\mathbb{X}}_t^\varepsilon \geq (t/2)dt$ for $0 \leq t \leq M$, and hence $\bar{\mathbb{X}}_t^\varepsilon \geq t^2/4$ for $0 \leq t \leq M$.

Intersect now A_1^ε with $A_2^\varepsilon = \{\sup_{0\leq t\leq M} |B_t^\varepsilon| \leq \frac{1}{\varepsilon M}\}$. Then, on $A_1^\varepsilon \cap A_2^\varepsilon$,

$$\mathbb{X}_t^\varepsilon \geq \frac{t^2}{4}\exp(-C), \quad 0 \leq t \leq M.$$

To complete the proof, it remains to note (from a standard tightness argument) that $\mathbb{P}(A_1^\varepsilon \cap A_2^\varepsilon) \to 1$ as $\varepsilon \to 0$, uniformly in $(z^1,z^2) \in K$. (The proof when (z^1,z^2) is in Γ^- is similar.) $\square$

We now come back to the case when the initial condition of the four-dimensional system is $((1,-\sqrt{2}),(-1,\sqrt{2}))$. The second key lemma consists in expanding the difference process $(\mathbb{X}^\varepsilon,\mathbb{V}^\varepsilon)$ with respect to ε, up to $\tau^\varepsilon=\inf\{t\geq 0:\mathbb{X}_t^\varepsilon\leq 0\}$.

LEMMA 2.6. *There exist a family of Brownian motions* $((B_t^\varepsilon)_{t\geq 0})_{\varepsilon>0}$ *and a family of random continuous processes* $(g^\varepsilon:\mathbb{R}_+\to\mathbb{R})_{\varepsilon>0}$ *such that*

$$\forall T>0\qquad \lim_{R\to+\infty}\sup_{0<\varepsilon\leq 1}\mathbb{P}\Big\{\sup_{0\leq t\leq T}|g_t^\varepsilon|>R\Big\}=0, \tag{2.8}$$

and the processes

$$d\mathbb{V}_t^{(0)}=dt,\quad d\mathbb{X}_t^{(0)}=\mathbb{V}_t^{(0)}dt,\quad \big(\mathbb{X}_0^{(0)},\mathbb{V}_0^{(0)}\big)=(1,-\sqrt{2}),$$
$$d\mathbb{V}_t^{(1,\varepsilon)}=\sigma\big(\mathbb{X}_t^{(0)}\big)dB_t^\varepsilon,\quad d\mathbb{X}_t^{(1,\varepsilon)}=\mathbb{V}_t^{(1,\varepsilon)}dt,\quad \big(\mathbb{X}_0^{(1,\varepsilon)},\mathbb{V}_0^{(1,\varepsilon)}\big)=(0,0),$$

satisfy

$$\big|\mathbb{X}_t^\varepsilon-\big(\mathbb{X}_t^{(0)}+\varepsilon\mathbb{X}_t^{(1,\varepsilon)}\big)\big|+\big|\mathbb{V}_t^\varepsilon-\big(\mathbb{V}_t^{(0)}+\varepsilon\mathbb{V}_t^{(1,\varepsilon)}\big)\big|\leq\varepsilon^2\big|g_t^\varepsilon\big|,\quad 0\leq t\leq\tau^\varepsilon. \tag{2.9}$$

PROOF. From Condition 2.2, we can write

$$d\mathbb{V}_t^\varepsilon=dt+\varepsilon\sigma(\mathbb{X}_t^\varepsilon)dB_t^\varepsilon,\quad 0\leq t\leq\tau^\varepsilon,$$

for some one-dimensional Brownian motion $(B_t^\varepsilon)_{t\geq 0}$, whence

$$d\big[\delta\mathbb{X}_t^\varepsilon\big]=\delta\mathbb{V}_t^\varepsilon dt,\quad d\big[\delta\mathbb{V}_t^\varepsilon\big]=\varepsilon\big[\sigma\big(\mathbb{X}_t^\varepsilon\big)-\sigma\big(\mathbb{X}_t^{(0)}\big)\big]dB_t^\varepsilon,\quad 0\leq t\leq\tau^\varepsilon,$$

with $\delta\mathbb{X}_t^\varepsilon=\mathbb{X}_t^\varepsilon-(\mathbb{X}_t^{(0)}+\varepsilon\mathbb{X}_t^{(1,\varepsilon)})$ and $\delta\mathbb{V}_t^\varepsilon=\mathbb{V}_t^\varepsilon-(\mathbb{V}_t^{(0)}+\varepsilon\mathbb{V}_t^{(1,\varepsilon)})$. We perform the same integration by parts as above, with

$$\delta\overline{\mathbb{V}}_t^\varepsilon=\delta\mathbb{V}_t^\varepsilon-\varepsilon\big(\sigma(\mathbb{X}_t^\varepsilon)-\sigma\big(\mathbb{X}_t^{(0)}\big)\big)B_t^\varepsilon.$$

Then we can find a family of random continuous functions $((v_t^{0,\varepsilon})_{t\geq 0})_{\varepsilon>0}$ satisfying (2.8) such that

$$\begin{aligned}
d\big[\delta\overline{\mathbb{V}}_t^\varepsilon\big]&=-\varepsilon\big(\sigma'\big(\mathbb{X}_t^\varepsilon\big)\mathbb{V}_t^\varepsilon-\sigma'\big(\mathbb{X}_t^{(0)}\big)\mathbb{V}_t^{(0)}\big)B_t^\varepsilon\,dt\\
&=-\varepsilon\big(\sigma'\big(\mathbb{X}_t^\varepsilon\big)\mathbb{V}_t^\varepsilon-\sigma'\big(\mathbb{X}_t^{(0)}+\varepsilon\mathbb{X}_t^{(1,\varepsilon)}\big)\big(\mathbb{V}_t^{(0)}+\varepsilon\mathbb{V}_t^{(1,\varepsilon)}\big)\big)B_t^\varepsilon\,dt\\
&\quad+\varepsilon^2v_t^{0,\varepsilon}\,dt\\
&=-\varepsilon\sigma'\big(\mathbb{X}_t^\varepsilon\big)B_t^\varepsilon\delta\mathbb{V}_t^\varepsilon\,dt\\
&\quad-\varepsilon\big(\sigma'\big(\mathbb{X}_t^\varepsilon\big)-\sigma'\big(\mathbb{X}_t^{(0)}+\varepsilon\mathbb{X}_t^{(1,\varepsilon)}\big)\big)\big(\mathbb{V}_t^{(0)}+\varepsilon\mathbb{V}_t^{(1,\varepsilon)}\big)B_t^\varepsilon\,dt+\varepsilon^2v_t^{0,\varepsilon}\,dt.
\end{aligned}$$

Since σ' is Lipschitz-continuous, we can find two families of random functions $((v_t^{1,\varepsilon})_{t\geq 0})_{\varepsilon>0}$ and $((v_t^{2,\varepsilon})_{t\geq 0})_{\varepsilon>0}$ satisfying (2.8) such that

$$d\big[\delta\overline{\mathbb{V}}_t^\varepsilon\big]=\varepsilon v_t^{1,\varepsilon}\delta\mathbb{X}_t^\varepsilon\,dt+\varepsilon v_t^{2,\varepsilon}\delta\overline{\mathbb{V}}_t^\varepsilon\,dt+\varepsilon^2v_t^{0,\varepsilon}\,dt. \tag{2.10}$$

In a similar way, we can find two families of random functions $((x_t^{0,\varepsilon})_{t\geq 0})_{\varepsilon>0}$ and $((x_t^{1,\varepsilon})_{t\geq 0})_{\varepsilon>0}$ satisfying (2.8) such that

$$d\big[\delta\mathbb{X}_t^\varepsilon\big]=\varepsilon x_t^{1,\varepsilon}\delta\mathbb{X}_t^\varepsilon\,dt+\delta\overline{\mathbb{V}}_t^\varepsilon\,dt+\varepsilon^2x_t^{0,\varepsilon}\,dt. \tag{2.11}$$

The result can be easily obtained by bounding the resolvent of the linear system (2.10)–2.11 in terms of the bounds for $x^{1,\varepsilon}$, $v^{1,\varepsilon}$, and $v^{2,\varepsilon}$. □

2.2 Proof of Proposition 2.4

We emphasize that $\mathbb{V}_t^{(0)} = -\sqrt{2} + t$ and $\mathbb{X}_t^{(0)} = (1 - t/\sqrt{2})^2$. Moreover,

$$
\begin{aligned}
\mathbb{V}_t^{(1,\varepsilon)} &= \int_0^t \sigma\big(\mathbb{X}_s^{(0)}\big) dB_s^\varepsilon = \int_0^t \sigma\left[\left(1 - \frac{s}{\sqrt{2}}\right)^2\right] dB_s^\varepsilon,\\
\mathbb{X}_t^{(1,\varepsilon)} &= \int_0^t \int_0^s \sigma\left[\left(1 - \frac{r}{\sqrt{2}}\right)^2\right] dB_r^\varepsilon \\
&= \int_0^t (t-r)\sigma\left[\left(1 - \frac{r}{\sqrt{2}}\right)^2\right] dB_r^\varepsilon.
\end{aligned}
\tag{2.12}
$$

By choosing $t_0 = \sqrt{2}$ and recalling that $\sigma(1) \neq 0$, we deduce the following:

LEMMA 2.7. *The r.v.'s $(\mathbb{X}_{t_0}^{(1,\varepsilon)})_{\varepsilon>0}$ have the same Gaussian law with zero mean and nonzero variance. In particular,* $\mathbb{P}\{\mathbb{X}_{t_0}^{(1,\varepsilon)} > 0\} = \mathbb{P}\{\mathbb{X}_{t_0}^{(1,\varepsilon)} \geq 0\} = \frac{1}{2}$.

We claim the following:

LEMMA 2.8. *For any real $M > t_0 = \sqrt{2}$ and any $\delta > 0$,*

(i) $\lim_{\varepsilon\to 0} \mathbb{P}\big(\{\tau^\varepsilon \leq M\} \cap \{\mathbb{X}_{t_0}^{(1,\varepsilon)} > \delta\}\big) = 0$,

(ii) $\lim_{\varepsilon\to 0} \mathbb{P}\big(\{\tau^\varepsilon > M\} \cap \{\mathbb{X}_{t_0}^{(1,\varepsilon)} < -\delta\}\big) = 0$,

(iii) $\lim_{\varepsilon\to 0} \mathbb{P}\big(\{\tau^\varepsilon \leq M\} \cap \{\tau^\varepsilon \notin (t_0 - \delta, t_0 + \delta)\}\big) = 0$.

PROOF. Given $M > \sqrt{2}$, we introduce $E_M^\varepsilon = \{\tau^\varepsilon \leq M\}$. We know from Lemma 2.6 that there exists a tight family of random variables $(\zeta_M^\varepsilon)_{0<\varepsilon\leq 1}$ such that

$$
\big|\mathbb{X}_t^\varepsilon - \big(\mathbb{X}_t^{(0)} + \varepsilon \mathbb{X}_t^{(1,\varepsilon)}\big)\big| \leq \varepsilon^2 \zeta_M^\varepsilon, \quad 0 \leq t \leq \tau^\varepsilon \wedge M. \tag{2.13}
$$

Therefore, on E_M^ε, we can choose $t = \tau^\varepsilon$ in the equation above. Since $\mathbb{X}_{\tau^\varepsilon}^\varepsilon = 0$, we deduce that

$$
\left|\left(1 - \frac{\tau^\varepsilon}{\sqrt{2}}\right)^2 + \varepsilon \mathbb{X}_{\tau^\varepsilon}^{(1,\varepsilon)}\right| \leq \varepsilon^2 \zeta_M^\varepsilon. \tag{2.14}
$$

Up to a modification of ζ_M^ε, we deduce (which is (iii))

$$
|\tau^\varepsilon - \sqrt{2}|^2 \leq \varepsilon \zeta_M^\varepsilon. \tag{2.15}
$$

We now prove (i). From (2.14), we deduce that $\mathbb{X}_{\tau^\varepsilon}^{(1,\varepsilon)} \leq \varepsilon \zeta_M^\varepsilon$ on E_M^ε. Since $\mathbb{X}^{(1,\varepsilon)}$ is Lipschitz-continuous on the interval $[0, M]$, it follows from (2.15) that there exists a tight family of random variables $(C_M^\varepsilon)_{0<\varepsilon\leq 1}$ such that

$$
\begin{aligned}
\mathbb{X}_{t_0}^{(1,\varepsilon)} &= \mathbb{X}_{\tau^\varepsilon}^{(1,\varepsilon)} + \mathbb{X}_{t_0}^{(1,\varepsilon)} - \mathbb{X}_{\tau^\varepsilon}^{(1,\varepsilon)} \\
&\leq \varepsilon \zeta_M^\varepsilon + C_M^\varepsilon |\tau^\varepsilon - t_0| \leq \varepsilon \zeta_M^\varepsilon + C_M^\varepsilon \varepsilon^{\frac{1}{2}} (\zeta_M^\varepsilon)^{\frac{1}{2}};
\end{aligned}
$$

that is, for every $\delta > 0$, $\lim_{\varepsilon\to 0} \mathbb{P}(E^\varepsilon_M \cap \{\mathbb{X}^{(1,\varepsilon)}_{t_0} > \delta\}) = 0$.

We finally prove (ii). From (2.13), we know that $\varepsilon \mathbb{X}^{(1,\varepsilon)}_{t_0} \geq \mathbb{X}^\varepsilon_{t_0} - \varepsilon^2 \zeta^\varepsilon_{t_0}$. Therefore, on $(E^\varepsilon_M)^{\mathsf{c}}$, $\varepsilon \mathbb{X}^{(1,\varepsilon)}_{t_0} \geq -\varepsilon^2 \zeta^\varepsilon_{t_0}$. This proves that, for every $\delta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\big((E^\varepsilon_M)^{\mathsf{c}} \cap \{\mathbb{X}^{(1,\varepsilon)}_{t_0} < -\delta\}\big) = 0.$$

□

LEMMA 2.9. *It holds that*

(i) $\forall M > \sqrt{2}$, $\lim_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon > M\} = \frac{1}{2}$,

(ii) $\forall \delta > 0$, $\lim_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon \in (t_0 - \delta, t_0 + \delta)\} = \frac{1}{2}$.

In particular, τ^ε *converges in law to* $\frac{1}{2}\delta_{t_0} + \frac{1}{2}\delta_{+\infty}$.

PROOF. From Lemmas 2.7 and 2.8, for any $M > \sqrt{2}$ and any $\delta > 0$,

$$\limsup_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon > M\} \leq \limsup_{\varepsilon\to 0} \mathbb{P}\{\mathbb{X}^{(1,\varepsilon)}_{t_0} \geq -\delta\} = \mathbb{P}\{\mathbb{X}^{(1,1)}_{t_0} \geq -\delta\}.$$

By letting δ tend to 0, we obtain $\limsup_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon > M\} \leq \frac{1}{2}$. Similarly,

$$\limsup_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon \leq M\} \leq \mathbb{P}\{\mathbb{X}^{(1,1)}_{t_0} \leq 0\} = \frac{1}{2}.$$

From this limit, we deduce (i). Then (ii) follows from (iii) in Lemma 2.8. □

We finally claim the following:

LEMMA 2.10. *Let* $\sigma^\varepsilon = \inf\{t \geq 0 : \mathbb{V}^\varepsilon_t \geq 0\}$. *Then, for all* $M > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon \leq M, \sigma^\varepsilon < \tau^\varepsilon\} = 0, \tag{2.16}$$

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\tau^\varepsilon_2 \geq M\} = 1. \tag{2.17}$$

PROOF. From Lemma 2.9, we can assume $M > \sqrt{2}$. We then begin with the proof of (2.16). By the Markov property, we note that

$$\mathbb{P}\{\tau^\varepsilon \leq M, \sigma^\varepsilon < \tau^\varepsilon\} \leq \int_{\Gamma_2} \mathbb{1}_{\{\tilde{\pi}_x(z^2 - z^1) > 0, \tilde{\pi}_v(z^2 - z^1) = 0\}} \times \mathbb{P}\{\tau^\varepsilon \leq M \mid (Z^{1,\varepsilon}_0, Z^{2,\varepsilon}_0) = (z^1, z^2)\} d\eta^\varepsilon(z^1, z^2),$$

where η^ε is the conditional law of $(Z^{1,\varepsilon}_{\rho^\varepsilon}, Z^{2,\varepsilon}_{\rho^\varepsilon})$ given that $\rho^\varepsilon \leq M$, with $\rho^\varepsilon = \inf(\sigma^\varepsilon, \tau^\varepsilon)$, under the initial condition $((1, -\sqrt{2}), (-1, \sqrt{2}))$. By using (i) in Lemma 2.9, it is easy to see that the distributions $(\eta^\varepsilon)_{0<\varepsilon\leq 1}$ are tight. According to Lemma 2.5, this property implies (2.16). Similarly, we have

$$\mathbb{P}\{\tau^\varepsilon_2 \leq M, \tau^\varepsilon < \sigma^\varepsilon\} \leq \int_{\Gamma_2} \mathbb{1}_{\{\tilde{\pi}_x(z^2 - z^1) = 0, \tilde{\pi}_v(z^2 - z^1) < 0\}} \times \mathbb{P}\{\tau^\varepsilon \leq M \mid (Z^{1,\varepsilon}_0, Z^{2,\varepsilon}_0) = (z^1, z^2)\} d\eta^\varepsilon(z^1, z^2),$$

which tends to 0 by the same argument as above. Since

$$\lim_{\varepsilon\to 0}\mathbb{P}\{\tau_2^\varepsilon \le M, \sigma^\varepsilon < \tau^\varepsilon\} \le \lim_{\varepsilon\to 0}\mathbb{P}\{\tau^\varepsilon \le M, \sigma^\varepsilon < \tau^\varepsilon\} = 0,$$

we deduce (2.17). (Recall that $\mathbb{P}\{\tau^\varepsilon = \sigma^\varepsilon\} = 0$ from Condition 2.1.) □

3 Structure of the Noise

In this section, we investigate the meaning of Assumption (A.4). First, we translate it into an ellipticity property of the noise. Second, we discuss some general examples inspired by turbulence theory. Finally, we prove that ellipticity of the noise lifts up to hypoellipticity of any mollified version of equation (1.13). Throughout this section, we shall use the notation introduced in (1.14).

3.1 Ellipticity of the Noise

When $F = 0$ and $x \in \Gamma^{\mathrm{c}}_{x,N}$, $\mathrm{Span}\{A_k(x)\}_{k\in\mathbb{N}\setminus\{0\}} \subsetneq \mathbb{R}^{Nd}$ and therefore the velocity component in (1.13) only moves along a restricted number of directions. By contrast, when $x \in \Gamma_{x,N}$, the noise generated at x is nondegenerate because of the *strict positivity* of $Q(\tilde{x}, \tilde{y})$ on $\Gamma_{x,N}$ in (A.4):

LEMMA 3.1. *Q satisfies* (A.4) if and only if

$$\mathrm{Span}\{A_k(x)\}_{k\in\mathbb{N}\setminus\{0\}} = \mathbb{R}^{Nd} \quad \forall x = (x^1, \dots, x^N) \in \Gamma_{x,N}. \tag{3.1}$$

PROOF. For any $v = (v^1, \dots, v^N) \in \mathbb{R}^{Nd}\setminus\{0\}$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \langle A_k(x), v\rangle^2_{\mathbb{R}^{Nd}} &= \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{N}\langle \sigma_k(x^i), v^i\rangle_{\mathbb{R}^d}\Big)^2 \\ &= \sum_{i,j=1}^{N}\langle Q(x^j, x^i)v^i, v^j\rangle_{\mathbb{R}^d} > 0.\end{aligned}$$ □

Below we exhibit interesting examples of strictly positive covariance functions $Q(\tilde{x}, \tilde{y})$ that are space-homogeneous. We shall assume that there is a symmetric $d \times d$ matrix-valued function $Q(\tilde{x})$ such that $Q(\tilde{x}, \tilde{y}) = Q(\tilde{x} - \tilde{y}) = Q(\tilde{y} - \tilde{x})$, with the following spectral representation:

$$Q(\tilde{x}) = \int_{\mathbb{R}^d} e^{ik\cdot\tilde{x}}\mathcal{Q}(k)\,dk, \quad \tilde{x} \in \mathbb{R}^d, \tag{3.2}$$

where the spectral density $\mathcal{Q}$ takes values in the space of nonnegative real symmetric $d \times d$ matrices with coordinates in $L^1(\mathbb{R}^d)$ and satisfies $\mathcal{Q}(-k) = \mathcal{Q}(k)$, $k \in \mathbb{R}^d$. (Above, $k \cdot \tilde{x}$ is a shortened notation for $\langle k, \tilde{x}\rangle_{\mathbb{R}^d}$.) In this framework, we have the general criterion:

LEMMA 3.2. *Assume that* $\mathcal{Q}$ *has the property that for any* $\mathbb{R}^d$*-valued trigonometric polynomial* $v(k)$ *of the form* $v(k) = \sum_{j=1}^N v^j e^{ik\cdot x^j}$, *for some* $(x^1,\dots,x^N)$, $(v^1,\dots,v^N) \in \mathbb{R}^{Nd}$ *and for* $i^2=-1$, *the a.e. equality*

$$\langle \mathcal{Q}(k)v(k), v(k)\rangle_{\mathbb{C}^d} = 0 \quad \textit{for a.e. } k \in \mathbb{R}^d$$

implies $v(k)=0$ *for any* $k\in\mathbb{R}^d$, *where* $\langle\cdot,\cdot\rangle_{\mathbb{C}^d}$ *denotes the Hermitian product in* $\mathbb{C}^d$. *Then* $Q(\tilde{x},\tilde{y})$ *is* strictly positive on $\Gamma_{x,N}$. (*Recall that, for any* $u,u'\in\mathbb{C}^d$, $\langle u,u'\rangle_{\mathbb{C}^d} = \sum_{j=1}^d \overline{u}^j(u')^j$, $\overline{u}$ *denoting the conjugate of* u. *We shall also write* $\langle u,u'\rangle_{\mathbb{C}^d} = \langle \overline{u},u'\rangle_{\mathbb{R}^d}$ *with an abuse of notation.*)

PROOF. The proof follows from the identity

$$\sum_{j,\ell=1}^N \langle Q(x^j,x^\ell)v^\ell, v^j\rangle_{\mathbb{R}^d} = \int_{\mathbb{R}^d} \langle \mathcal{Q}(k)v(k), v(k)\rangle_{\mathbb{C}^d}\, dk, \tag{3.3}$$

where $v(k) = \sum_{j=1}^N v^j e^{ik\cdot x^j}$. Indeed, $v(k)=0$ for any $k\in\mathbb{R}^d$ implies $(v^1,\dots,v^N)=0$ since $v(k)$ is a (vector-valued) trigonometric polynomial driven by pairwise different vectors $x^1,\dots,x^N$ (see Remark 3.3 below). □

Remark 3.3. Let $f:\mathbb{R}^d\to\mathbb{C}$ be of the form

$$f(k) = \sum_{j=1}^N (a^j + \langle k, v^j\rangle_{\mathbb{C}^d})e^{ix^j\cdot k}, \quad k\in\mathbb{R}^d,$$

where $a^j\in\mathbb{C}$, $v^j\in\mathbb{C}^d$, and $(x^1,\dots,x^N)\in\Gamma_{x,N}$. If there is a Borel set $A\subset\mathbb{R}^d$ of positive Lebesgue measure such that $f=0$ on A, then $a^j=0$ and $v^j=0$ for any $j=1,\dots,N$. Indeed, by a standard extension of the principle of analytic continuation, $f(k)=0$ for any $k\in\mathbb{R}^d$. Given a smooth function $\varphi:\mathbb{R}^d\to\mathbb{C}$ with compact support, we denote by $\widehat{\varphi}$ its Fourier transform. We have $\int_{\mathbb{R}^d} f(k)\widehat{\varphi}(k)dk = 0$ and thus $\sum_{j=1}^N [a^j\varphi(x^j) - i\langle v^j, \nabla\varphi(x^j)\rangle_{\mathbb{R}^d}] = 0$. Since the points x^i are all distinct, we may construct a function φ such that $\varphi(x^j) = \overline{a}^j$ and $\nabla\varphi(x^j) = i\overline{v}^j$.

From Lemma 3.2 and Remark 3.3, we get as a first example:

PROPOSITION 3.4. *If* $\mathcal{Q}(k)$ *is strictly positive definite on a Borel subset of* $\mathbb{R}^d$ *of positive Lebesgue measure, then* $Q(\tilde{x},\tilde{y})$ *is* strictly positive on $\Gamma_{x,N}$.

3.2 Isotropic Random Fields

Proposition 3.4 does not cover important examples, such as the following one, which appears in the literature on turbulent dispersion of passive scalars:

Example 3.5. We say that Q is *isotropic* if $Q(U\tilde{x}) = UQ(\tilde{x})U^\top$ for any $\tilde{x}\in\mathbb{R}^d$ and $U\in\mathcal{O}(\mathbb{R}^d)$, where $\mathcal{O}(\mathbb{R}^d)$ is the set of orthogonal matrices of dimension d

($U^\top$ denotes the transpose of U). This is the case when $\mathcal{Q}(k)$ in (3.2) has the form

$$\mathcal{Q}(k) = \pi_k f(|k|), \quad \text{that is,} \quad Q(\tilde{x}) = \int_{\mathbb{R}^d} e^{ik\cdot\tilde{x}} \pi_k f(|k|) dk,$$

where $\pi_k = 1$ if $d = 1$ and $\pi_k = (1-\mathsf{p})I_d + |k|^{-2}(\mathsf{p}d-1)k \otimes k$ for some $\mathsf{p} \in [0,1]$ if $d \geq 2$, and $f : [0,+\infty) \to \mathbb{R}$ is in $L^1([0,+\infty))$ and satisfies

$$f(r) \geq 0 \quad \text{for a.e. } r > 0. \tag{3.4}$$

The matrix $\mathcal{Q}(k)$ is symmetric, it satisfies $\mathcal{Q}(-k) = \mathcal{Q}(k)$, and it is almost everywhere nonnegative because (we restrict the proof to $d \geq 2$)

$$\begin{aligned} |k|^2 \langle \pi_k w, w\rangle_{\mathbb{C}^d} &= (1-\mathsf{p})|k|^2|w|^2 + (\mathsf{p}d-1)|\langle k, w\rangle_{\mathbb{C}^d}|^2 \\ &\geq (1-\mathsf{p})|\langle k, w\rangle_{\mathbb{C}^d}|^2 + (\mathsf{p}d-1)|\langle k, w\rangle_{\mathbb{C}^d}|^2 \\ &= \mathsf{p}(d-1)|\langle k, w\rangle_{\mathbb{C}^d}|^2 \geq 0. \end{aligned} \tag{3.5}$$

(Here the inequality is given in $\mathbb{C}^d$ but only the $\mathbb{R}^d$ part is useful to prove the nonnegativity of $\mathcal{Q}(k)$. The full inequality in $\mathbb{C}^d$ will be used later.)

We refer to [22, 23, 25, 29, 34] for references where this form (for particular choices of f) is used or investigated. This class of covariances is related to the Batchelor regime of the Kraichnan model, where $f(r) = (r_0^2 + r^2)^{-(d+\varpi)/2}$ with $\varpi = 2$ (see [6, 12]). In the limit $r_0 \to 0$, the covariance of the increments of the noise is scale invariant with scaling exponent equal to 2. The "turbulent regime" of the Kraichnan model ($0 \leq \varpi < 2$) is in contrast not included in our main final result because of the regularity properties we require on Q.

PROPOSITION 3.6. *If there exists a Borel set $A \subset [0,\infty)$ such that* $\mathrm{Leb}_1(A) > 0$ *and $f(r) > 0$ for $r \in A$, then $Q(\tilde{x}, \tilde{y})$ is* strictly positive on $\Gamma_{x,N}$.

PROOF. From Lemma 3.2 it is sufficient to prove that the condition

$$f(|k|)\langle \pi_k v(k), v(k)\rangle_{\mathbb{C}^d} = 0 \quad \text{for a.e. } k \in \mathbb{R}^d$$

implies $v(k) = 0$ for any $k \in \mathbb{R}^d$ when $v(k)$ has the form $v(k) = \sum_{j=1}^N v^j e^{ik\cdot x^j}$ for some $(v^1, \dots, v^N) \in \mathbb{R}^{Nd}$ and $(x^1, \dots, x^N) \in \Gamma_{x,N}$. Since $f \neq 0$ on A, it holds $\langle \pi_k v(k), v(k)\rangle_{\mathbb{C}^d} = 0$ for k in a Borel subset $A^* \subset \mathbb{R}^d$ of positive measure. We now prove that this implies $v(k) \equiv 0$.

We focus on the condition $\langle \pi_k w, w\rangle_{\mathbb{C}^d} = 0$ for some $w \in \mathbb{C}^d$. When $d = 1$, this condition implies $w = 0$. For $\mathsf{p} \in (0,1]$, $d > 1$, inequality (3.5) implies $\mathsf{p}(d-1)|\langle k, w\rangle_{\mathbb{C}^d}|^2 = 0$, and thus $\langle k, w\rangle_{\mathbb{C}^d} = 0$. Finally, in the case $\mathsf{p} = 0$, $d > 1$, for all $w \in \mathbb{C}^d$ we have

$$|k|^2\langle \pi_k w, w\rangle_{\mathbb{C}^d} = |k|^2|w|^2 - |\langle k, w\rangle_{\mathbb{C}^d}|^2,$$

and thus $\langle \pi_k w, w\rangle_{\mathbb{C}^d} = 0$ implies that $w = \lambda k$ for some $\lambda \in \mathbb{R}$ if $k \neq 0$.

Coming back to the main line of the proof, we have $\langle \pi_k v(k), v(k)\rangle_{\mathbb{C}^d} = 0$ for all $k \in A^*$. Depending on the values of p and d, this implies at least one of the three following conditions: $v(k) = 0$ for all $k \in A^*$, or $\langle k, v(k)\rangle_{\mathbb{C}^d} = 0$ for all $k \in A^*$, or $v(k) \parallel k$ for all $k \in A^*$ (except maybe at $k = 0$). From Remark 3.3, $v^j = 0$ for $j = 1, \dots, N$ in the two first cases. In the third case, we note that $v(k) \parallel k$ may be written as $\sum_{j=1}^N \langle v_\ell^j e_{\ell'} - v_{\ell'}^j e_\ell, k\rangle_{\mathbb{C}^d} e^{ix^j\cdot k} = 0$ for $1 \le \ell, \ell' \le d$, where $(e_\ell)_{1\le\ell\le d}$ is the canonical basis of $\mathbb{C}^d$. Again from Remark 3.3, this also implies $v^j = 0$ for $j = 1, \dots, N$ (see also [17, theorem 4.7] and [11]). □

We are now able to give examples for which Theorem 1.2 applies:

PROPOSITION 3.7. *In the case when* $d = 1$, *consider* Q *as in Example* 3.5, *with* $f \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ *satisfying the assumption of Proposition* 3.6 *together with* $\int_0^{+\infty} k^4 f(k)dk < +\infty$. *Then Conditions* 2.1 *and* 2.2 *in Section* 2 *are satisfied with*

$$\sigma(\widetilde{x}) = \operatorname{sign}(\widetilde{x})\sqrt{(Q(0) - Q(\widetilde{x}))/2} \quad \textit{for } \widetilde{x} \in \mathbb{R}.$$

PROOF. Consider the framework introduced in Section 2 and recall (2.1) and (2.2). Existence and uniqueness in Condition 2.1 follow from Theorem 4.1 below. In order to prove (2.3), we consider an arbitrary random initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon}) \in \Gamma_2$, independent of the noise $((W_t^k)_{t\ge0})_{k\ge1}$. For any $t \ge 0$,

$$\begin{aligned} d\mathbb{X}_t^\varepsilon &= \operatorname{sign}(\mathbb{X}_t^\varepsilon)dt + \varepsilon \sum_{k\ge1} \frac{\sigma_k(X_t^{2,\varepsilon}) - \sigma_k(X_t^{1,\varepsilon})}{2}\, dW_t^k \\ &= \operatorname{sign}(\mathbb{X}_t^\varepsilon)\left(dt + \varepsilon\sqrt{\frac{\rho_t^\varepsilon}{2}}\, dB_t^\varepsilon\right), \end{aligned}$$

with $\rho_t^\varepsilon = Q(0) - Q(\mathbb{X}_t^\varepsilon) \ge 0$ and

$$\begin{aligned} dB_t^\varepsilon &= \left(\mathbb{1}_{\{\mathbb{X}_t^\varepsilon \ge 0\}} - \mathbb{1}_{\{\mathbb{X}_t^\varepsilon < 0\}}\right) \\ &\quad \times \left(\sum_{k\ge1} \mathbb{1}_{\{\rho_t^\varepsilon > 0\}} \frac{\sigma_k(X_t^{2,\varepsilon}) - \sigma_k(X_t^{1,\varepsilon})}{\sqrt{2\rho_t^\varepsilon}}\, dW_t^k + \mathbb{1}_{\{\rho_t^\varepsilon = 0\}} dW_t^1\right). \end{aligned}$$

It is easily checked that $d\langle B^\varepsilon\rangle_t = dt$. By Lévy's theorem, $(B_t^\varepsilon)_{t\ge0}$ is a Brownian motion with respect to the augmented filtration generated by the initial condition $(Z_0^{1,\varepsilon}, Z_0^{2,\varepsilon})$ and by the noise $((W_t^k)_{t\ge0})_{k\ge1}$.

We now investigate the properties of σ. Clearly, $\sigma(0) = 0$. We prove below that σ is $\mathcal{C}^2$ with bounded derivatives and that $\sigma(1) > 0$. We have

$$Q(0) - Q(\widetilde{x}) = \widetilde{x}^2 \int_{\mathbb{R}} \frac{1 - \cos(k\widetilde{x})}{\widetilde{x}^2} f(|k|)dk = \widetilde{x}^2 \int_{\mathbb{R}} k^2 \varphi(k\widetilde{x}) f(|k|)dk,$$

with $\varphi(u) = u^{-2}(1 - \cos(u))$, $\varphi(0) = \frac{1}{2}$. Clearly, φ is infinitely differentiable with bounded derivatives. Therefore, the function $\Phi : \mathbb{R} \ni \widetilde{x} \mapsto \int_{\mathbb{R}} k^2 \varphi(k\widetilde{x}) f(|k|)dk$ is twice continuously differentiable with bounded derivatives. At $\widetilde{x} = 0$, $\Phi(0) >$

0, and hence $\sqrt{\Phi}$ is twice continuously differentiable in the neighborhood of 0. Then the function σ, which can be written $\sigma(\tilde{x}) = \tilde{x}\sqrt{\Phi(\tilde{x})}$ for $\tilde{x} \in \mathbb{R}$, is twice continuously differentiable in the neighborhood of 0. Away from 0, the function $\mathbb{R} \ni \tilde{x} \mapsto Q(0) - Q(\tilde{x})$ has positive values, and therefore its square root and σ are also both twice continuously differentiable. The derivatives of order 1 and 2 of σ are bounded since the derivatives of order 1 and 2 of Q are bounded and $Q(0) - Q(\tilde{x}) \to Q(0) > 0$ as $|\tilde{x}| \to +\infty$. Moreover, $\sigma(1)$ is clearly positive. □

3.3 Hypoellipticity of the N-Point Motion

The ellipticity of the noise turns into hypoellipticity of the system in the following sense (the proof is standard and is thus left to the reader):

PROPOSITION 3.8. *Assume that F and σ_k, for any $k \in \mathbb{N} \setminus \{0\}$, are of class $\mathcal{C}^1$ on $\mathbb{R}^d$ and that, for every $x = (x^1, \dots, x^N) \in \Gamma_{x,N}$,* $\mathrm{Span}\{A_k(x)\}_{k\in\mathbb{N}\setminus\{0\}} = \mathbb{R}^{Nd}$. *Then, for every $z \in \mathbb{R}^{2Nd}$ of the form $z = (x, v)$ with $x = (x^1, \dots, x^N) \in \Gamma_{x,N}$, $v \in \mathbb{R}^{Nd}$, we have* $\mathrm{Span}\{\mathbb{A}_k(z), [\mathbb{A}_k, \mathbb{F}](z)\}_{k\in\mathbb{N}\setminus\{0\}} = \mathbb{R}^{2Nd}$, *with $\mathbb{A}_k$ and $\mathbb{F}$ as in* (1.15). (*Here, $[\cdot,\cdot]$ stands for the Lie bracket of vector fields.*)

The precise formulation of hypoellipticity in our framework is given below:

PROPOSITION 3.9. *In addition to* (A.1)–(A.4)*, assume that F is Lipschitz-continuous on the whole $\mathbb{R}^d$. Then, for every initial condition $Z^0 = z \in \mathbb{R}^{2Nd}$, equation* (1.13) *admits a unique strong solution. Moreover, the mappings $\varphi_t : \mathbb{R}^{2Nd} \ni z \mapsto Z_t$ subject to $Z_0 = z$, $t \geq 0$, form a stochastic flow of homeomorphisms on $\mathbb{R}^{2Nd}$. Finally, for any $t > 0$, the marginal law of the $2Nd$-dimensional vector Z_t is absolutely continuous with respect to the Lebesgue measure when $z = (x, v)$ satisfies $x \in \Gamma_{x,N}$.*

PROOF. The unique strong solvability and the homeomorphism property of the flow can be found in [30] and [21, chap. 4, sec. 5]. When both $\mathbb{F}$ and the coefficients $(\sigma_k)_{k\in\mathbb{N}\setminus\{0\}}$ are smooth, with derivatives of any order in $\ell^2(\mathbb{N} \setminus \{0\})$, absolute continuity then follows from Proposition 3.8 and a suitable version of Hörmander's theorem for systems driven by an infinite-dimensional noise. See, for example, [30, theorem 4.3].

Here the coefficients are not smooth. However, absolute continuity directly follows from the Bouleau and Hirsch criterion. By proposition 2.2 in [30], $(Z_t)_{t\geq 0}$ is differentiable in the sense of Malliavin with $\sum_{k=1}^{+\infty} \mathbb{E} \int_0^t |D_s^k Z_t|^2 \, ds < +\infty$ for any $t \geq 0$. We also know that

$$D_r^k Z_t = Y_t (Y_r)^{-1} \mathbb{A}_k(Z_r), \quad 0 \leq r \leq t, \tag{3.6}$$

the equality holding true in $\mathbb{R}^{2Nd}$, where $(Y_t)_{t\geq 0}$ is an $\mathbb{R}^{2Nd\times 2Nd}$-valued process and solves a linear SDE of the form

$$Y_t = I_{2Nd} + \sum_{k=1}^{+\infty} \int_0^t \alpha_k(s) Y_s \, dW_s^k + \int_0^t \alpha_0(s) Y_s \, ds, \quad t \geq 0. \tag{3.7}$$

In the above SDE, the processes $(\alpha_k(s))_{s\geq 0}$, $k \in \mathbb{N}$, are bounded and progressively measurable, and the infinite-dimensional process $((|\alpha_k(s)|)_{s\geq 0})_{k\in\mathbb{N}\setminus\{0\}}$ is bounded in $\ell^2(\mathbb{N} \setminus \{0\})$. When the coefficients $\mathbb{F}$ and $\mathbb{A}_k$, $k \in \mathbb{N} \setminus \{0\}$, in the compact formulation (1.14) are smooth, it holds that $\alpha_0(s) = \nabla\mathbb{F}(Z_s)$ and $\alpha_k(s) = \nabla\mathbb{A}_k(Z_s)$, $k \in \mathbb{N} \setminus \{0\}$. We then use the following notation: given a square matrix M of size $2Nd \times 2Nd$, we denote by $[M]_{x,x}$, $[M]_{x,v}$, $[M]_{v,x}$, and $[M]_{v,v}$ the blocks of size $Nd \times Nd$ corresponding to the decomposition of a vector $z \in \mathbb{R}^{2Nd}$ into coordinates $x = \Pi_x(z)$ and $v = \Pi_v(z)$ in $\mathbb{R}^{Nd}$. With this notation, $[\alpha_k(s)]_{x,v}$ and $[\alpha_k(s)]_{v,v}$ are 0 since $\mathbb{A}_k$ is independent of v. Similarly, $[\alpha_k(s)]_{x,x}$ is 0 since $\Pi_x(\mathbb{A}_k) \equiv 0$, and $[\alpha_0(s)]_{x,x} = [\alpha_0(s)]_{v,v} = 0$ since $\Pi_x(\mathbb{F}) \equiv v$ and $\Pi_v(\mathbb{F})$ is independent of v. Moreover, $[\alpha_0(s)]_{x,v} = I_{Nd}$. By using a mollification argument, it can be shown that these relations remain true in the Lipschitz setting. Finally, as in the finite-dimensional framework, we can check that Y_t is invertible a.s. for any $t > 0$, the inverse having finite polynomial moments of any order.

For small r, $Y_r = I_{2Nd} + o_r(1)$, where $o_r(1)$ stands for the Landau notation and converges to 0 with r a.s. Therefore, from the equalities $[\alpha_0(s)]_{x,x} = [\alpha_k(s)]_{x,x} = [\alpha_k(s)]_{x,v} = [\alpha_0(s)]_{v,v} = [\alpha_k(s)]_{v,v} = 0$ and $[\alpha_0(s)]_{x,v} = I_{Nd}$, we deduce that $[Y_r]_{x,x} = I_{Nd} + ro_r(1)$ and $[Y_r]_{v,v} = I_{Nd} + o_r(1)$ and that $[Y_r]_{x,v}$ can be expanded as follows:

$$[Y_r]_{x,v} = rI_{Nd} + ro_r(1).$$

Defining $\mathcal{Z}_r = (Y_r)^{-1}\mathbb{A}_k(Z_r) \in \mathbb{R}^{2Nd}$ and writing $\mathcal{Z}_r$ under the form $\mathcal{Z}_r = ((\mathcal{X}_r^i, \mathcal{V}_r^i))_{1\leq i\leq N}$, we have $Y_r\mathcal{Z}_r = \mathbb{A}_k(Z_r)$, whence

$$[Y_r]_{x,x}\mathcal{X}_r + [Y_r]_{x,v}\mathcal{V}_r = 0, \quad [Y_r]_{v,x}\mathcal{X}_r + [Y_r]_{v,v}\mathcal{V}_r = A_k(X_r),$$

that is, $\mathcal{X}_r + r\mathcal{V}_r = ro_r(1)$ and $o_r(1)\mathcal{X}_r + \mathcal{V}_r = A_k(X_r) + o_r(1)$. We deduce $\mathcal{V}_r = A_k(X_r) + o_r(1)$ and $-\mathcal{X}_r = -rA_k(X_r) + ro_r(1)$, so that, from (3.6),

$$(Y_t)^{-1}D_r^k Z_t = (-rA_k(x) + ro_r(1), A_k(x) + o_r(1)). \tag{3.8}$$

(The above equality holds a.s., $o_r(1)$ being random itself.) For a given $\omega \in \Omega$ for which (3.8) holds true, consider $\zeta = ((\chi^i, \nu^i))_{1\leq i\leq N} \in \mathbb{R}^{2Nd}$ such that $\langle D_r^k Z_t, \zeta\rangle_{\mathbb{R}^d} = 0$ for any $0 \leq r \leq t$ and $k \in \mathbb{N} \setminus \{0\}$. By changing ζ into $((Y_t)^{-1})^\top\zeta$, we deduce from (3.8) that

$$-r\sum_{i=1}^N \langle\sigma_k(x^i), \chi^i\rangle_{\mathbb{R}^d} + \sum_{i=1}^N \langle\sigma_k(x^i), \nu^i\rangle_{\mathbb{R}^d} = ro_r(1)|\chi| + o_r(1)|\nu|.$$

By letting $r \to 0$, we get $\nu \perp A_k(x)$ for any $k \in \mathbb{N} \setminus \{0\}$. From (A.4), $\nu = 0$. By dividing the above equality by r and letting $r \to 0$, it is possible to show that $\chi = 0$. We complete the proof by using the Bouleau and Hirsch criterion; see [31, theorem 2.1.2]. □

4 Noncoalescence of the Stochastic Dynamics

We now prove the main result of the paper:

THEOREM 4.1. *Under* (A.1)–(A.4), *for any* $z \in \Gamma_N$ *there exists a unique solution* $(Z_t(z))_{t\geq 0}$ *of* (1.13) *with initial condition* z. *The solution satisfies* $\mathbb{P}\{\forall t \geq 0,\ Z_t(z) \in \Gamma_N\} = 1$ *and* $\mathbb{P}\{\mathrm{Leb}_1\{t \geq 0 : \Pi_x(Z_t(z)) \in \Gamma^{\mathrm{c}}_{x,N}\} = 0\} = 1$.

The proof is split into three parts: we first establish a priori estimates for a regularized version of (1.13); by using a compactness argument, we deduce that strong unique solvability holds for Lebesgue almost every starting point; by taking advantage of the absolute continuity of the marginal laws of the regularized system, we establish strong unique solvability for any $z \in \Gamma_N$.

4.1 Smoothed System of Equations

For every $\varepsilon > 0$, let $F_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ be equal to F outside $B_d(0,\varepsilon)$, but be smooth inside $B_d(0,\varepsilon)$, with $\sup_{x\in\mathbb{R}^d} |F_\varepsilon(x)| \leq \sup_{x\in\mathbb{R}^d} |F(x)| + 1$. Given such an F_ε, we consider equation (1.13), but with F_ε instead of F therein (or, equivalently, the compact writing (1.14) when driven by $\mathbb{F}_\varepsilon$, with an appropriate definition of $\mathbb{F}_\varepsilon$ in (1.15)). From Proposition 3.9, the smoothed system is uniquely solvable for every initial condition in $\mathbb{R}^{2Nd}$, the solution being generically denoted by $(Z^\varepsilon_t = (X^\varepsilon_t, V^\varepsilon_t))_{t\geq 0}$, with $X^\varepsilon_t = (X^{i,\varepsilon}_t)_{1\leq i\leq N}$ and $V^\varepsilon_t = (V^{i,\varepsilon}_t)_{1\leq i\leq N}$, and the associated flow by $\varphi^\varepsilon_t : \mathbb{R}^{2Nd} \ni z \mapsto Z^\varepsilon_t$ with $Z^\varepsilon_0 = z$, $t \geq 0$.

From the a.e. equality $\mathrm{div}_{(x,v)}\,\mathbb{F}_\varepsilon = 0$ and $\nabla_v A_k = 0$ for all $k \in \mathbb{N}$ (the divergence being computed in the phase space), we directly obtain the following:

LEMMA 4.2. *For any* $t \geq 0$, $\varphi^\varepsilon_t(\cdot)$ *preserves the Lebesgue measure, that is, for all measurable and nonnegative* g,

$$\mathbb{E}\int_{\mathbb{R}^{2Nd}} g\big(\varphi^\varepsilon_t(z)\big)dz = \int_{\mathbb{R}^{2Nd}} g(z')dz'.$$

PROPOSITION 4.3. *Let* $\log^+ : (0,+\infty) \ni r \mapsto \log^+(r)$ *be the function equal to* 0 *for* $r \geq 1$ *and to* $-\log r$ *for* $r \in (0,1)$. *For every* $R \geq 1$, *let*

$$h_R(z) = \mathbb{1}_{\{|z|\leq R\}} \sum_{(i,j)\in\Delta_N} \log^+ |z^i - z^j|, \quad z \in \mathbb{R}^{2Nd}.$$

Then, for any $R_0, R, T > 0$ *there exists a constant* C *such that, for any* $\varepsilon > 0$,

$$\int_{B_{2Nd}(0,R_0)} \mathbb{E}\Big[\sup_{t\in[0,T]} h_R\big(\varphi^\varepsilon_t(z)\big)\Big]dz \leq C. \tag{4.1}$$

PROOF.

Step 1. For a smooth function $\phi : \mathbb{R} \to [0,1]$, with support included in $(0,1)$ and with $\int_0^1 |\phi'(r)|dr \leq 2$ (which is the case if, for some $r_0 \in (0,1)$, ϕ is nondecreasing on $[0,r_0]$ and nonincreasing on $[r_0,1]$), let $\log^+_\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be the smooth function:

$$\log_\phi^+(r) = \int_r^1 \frac{\phi(s)}{s}\, ds \ \text{ for } r \geq 0 \quad \text{whence} \quad \left|\frac{d}{dr}\log_\phi^+(r)\right| \leq \frac{1}{r} \quad \text{for } r > 0.$$

As ϕ increases towards the indicator function of the interval $(0,1)$, $\log_\phi^+(r)$ increases towards $\log^+(r)$. Given the function $\mathbb{R}^{2d} \ni \tilde{z} \mapsto \log_\phi^+(|\tilde{z}|)$, we have

$$|\nabla[\log_\phi^+(|\tilde{z}|)]| \leq C\frac{\mathbb{1}_{\{|\tilde{z}|\leq 1\}}}{|\tilde{z}|}, \tag{4.2}$$

$$|\nabla^2[\log_\phi^+(|\tilde{z}|)]| \leq \frac{C}{|\tilde{z}|^2}(1 + |\phi'|(|\tilde{z}|)) \cdot \mathbb{1}_{\{|\tilde{z}|\leq 1\}}, \tag{4.3}$$

for a constant C that is independent of the details of ϕ.

Given $R > 0$, let $\theta_R : \mathbb{R}^{2Nd} \to [0,1]$ be a smooth function equal to 1 on $B_{2Nd}(0,R)$, equal to 0 outside $B_{2Nd}(0,R+2)$, with values in $[0,1]$ and with $\sup_{z\in\mathbb{R}^{2Nd}} |\nabla\theta_R(z)| \leq 1$ and $\sup_{z\in\mathbb{R}^{2Nd}} |\nabla^2\theta_R(z)| \leq 1$. Define

$$h_\phi^{\theta_R}(z) = \theta_R(z) \sum_{(i,j)\in\Delta_N} \log_\phi^+(|z^i - z^j|), \quad z \in \mathbb{R}^{2Nd}.$$

We prove below that, given $R_0, R > 0$, there exists a constant C, independent of ε and of the details of ϕ and θ_R in $B_{2Nd}(0,R+2)\backslash B_{2Nd}(0,R)$, such that

$$\mathbb{E}\int_{B(0,R_0)} \sup_{t\in[0,T]} h_\phi^{\theta_R}(\varphi_t^\varepsilon(z))\,dz \leq C. \tag{4.4}$$

Then, inequality (4.1) can be obtained by letting ϕ increase towards $\mathbb{1}_{(0,1)}$ and by using the monotone convergence theorem.

Step 2. We now prove (4.4). In the whole argument, we use the compact formulation (1.14). With the notation $g(\tilde{z}) = \log_\phi^+(|\tilde{z}|)$ and for a generic solution $(Z_t^\varepsilon)_{t\geq 0}$ of the smoothed system, we have

$$\begin{gathered} d(\theta_R(Z_t^\varepsilon)g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})) = dI_t^1 + dI_t^2 + dI_t^3, \\ \begin{cases} dI_t^1 = dI_t^{11} + dI_t^{12} \\ dI_t^2 = dI_t^{21} + dI_t^{22}, \end{cases} \end{gathered} \tag{4.5}$$

where

$$dI_t^{11} = \theta_R(Z_t^\varepsilon)\langle\nabla g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}), d(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})\rangle_{\mathbb{R}^{2d}},$$

$$dI_t^{12} = \frac{\theta_R(Z_t^\varepsilon)}{2}\sum_{\alpha,\beta=1}^{2d}\frac{\partial^2 g}{\partial\tilde{z}_\alpha\partial\tilde{z}_\beta}(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})d[(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})_\alpha, (Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})_\beta]_t,$$

$$dI_t^{21} = g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})\langle\nabla\theta_R(Z_t^\varepsilon), dZ_t^\varepsilon\rangle_{\mathbb{R}^{2Nd}},$$

$$dI_t^{22} = \frac{g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})}{2} \sum_{i',j'=1}^{N} \sum_{\alpha,\beta=1}^{2d} \frac{\partial^2 \theta_R}{\partial (z^{i'})_\alpha \partial (z^{j'})_\beta}(Z_t^\varepsilon) d\big[(Z^{i',\varepsilon})_\alpha, (Z^{j',\varepsilon})_\beta\big]_t,$$

$$dI_t^3 = \sum_{k=1}^{\infty} \langle \nabla g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}), [\mathbb{A}_k^i - \mathbb{A}_k^j](Z_t^\varepsilon)\rangle_{\mathbb{R}^{2d}} \langle \nabla \theta_R(Z_t^\varepsilon), \mathbb{A}_k(Z_t^\varepsilon)\rangle_{\mathbb{R}^{2Nd}} dt.$$

We first tackle the mutual variations. From the identity $[\mathbb{A}_k^i - \mathbb{A}_k^j](Z_t^\varepsilon) = \sigma_k(X_t^{i,\varepsilon}) - \sigma_k(X_t^{j,\varepsilon})$ and from (4.2), (1.12), and (A.3),

$$|dI_t^3| \le C|\nabla \theta_R(Z_t^\varepsilon)| dt. \tag{4.6}$$

In order to deal with the term I_t^{12}, we need to analyze the mutual variation $[(Z^{i,\varepsilon} - Z^{j,\varepsilon})_\alpha, (Z^{i,\varepsilon} - Z^{j,\varepsilon})_\beta]_t$. Obviously, we have $[(X^{i,\varepsilon} - X^{j,\varepsilon})_p, (Z^{i,\varepsilon} - Z^{j,\varepsilon})_\beta]_t = 0$ for all $p = 1, \dots, d$ and $\beta = 1, \dots, 2d$, since $X^{i,\varepsilon} - X^{j,\varepsilon}$ is of bounded variation. Moreover, from (1.12) we have

$$\big|d\big[(V_t^{i,\varepsilon} - V_t^{j,\varepsilon})_p, (V_t^{i,\varepsilon} - V_t^{j,\varepsilon})_q\big]_t\big| \le C|X_t^{i,\varepsilon} - X_t^{j,\varepsilon}|^2 dt$$

for $p, q = 1, \dots, d$. From inequality (4.3), we obtain (renaming the constant C)

$$\begin{aligned} |dI_t^{12}| &\le \frac{C\theta_R(Z_t^\varepsilon)(1 + |\phi'|(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|))}{|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|^2} |X_t^{i,\varepsilon} - X_t^{j,\varepsilon}|^2 dt \\ &\le C\theta_R(Z_t^\varepsilon)\big(1 + |\phi'|(|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|)\big) dt. \end{aligned} \tag{4.7}$$

Finally, let us deal with I_t^{22}. As above, only the terms $d[(V_t^{i',\varepsilon})_p, (V_t^{j',\varepsilon})_q]_t$ are nonzero in the variation $d[(Z^{i',\varepsilon})_\alpha, (Z^{j',\varepsilon})_\beta]_t$. From the boundedness of $|Q(\tilde{x}, \tilde{x})|$, we deduce that $|d[(V_t^{i',\varepsilon})_p, (V_t^{j',\varepsilon})_q]_t| \le C\, dt$ and thus

$$|dI_t^{22}| \le Cg(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})|\nabla^2 \theta_R(Z_t^\varepsilon)| dt. \tag{4.8}$$

Step 3. Split now dI_t^{11} and dI_t^{21} into $dI_t^{111} + dI_t^{112}$ and $dI_t^{211} + dI_t^{212}$, where

$$dI_t^{111} = \theta_R(Z_t^\varepsilon)\langle \nabla g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}), \mathbb{F}_\varepsilon^i(Z_t^\varepsilon) - \mathbb{F}_\varepsilon^j(Z_t^\varepsilon)\rangle_{\mathbb{R}^{2d}} dt,$$

$$dI_t^{112} = \theta_R(Z_t^\varepsilon) \sum_{k=1}^{\infty} \langle \nabla g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}), \mathbb{A}_k^i(Z_t^\varepsilon) - \mathbb{A}_k^j(Z_t^\varepsilon)\rangle_{\mathbb{R}^{2d}} dW_t^k,$$

$$dI_t^{211} = g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})\langle \nabla \theta_R(Z_t^\varepsilon), \mathbb{F}_\varepsilon(Z_t^\varepsilon)\rangle_{\mathbb{R}^{2Nd}} dt,$$

$$dI_t^{212} = g(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}) \sum_{k=1}^{\infty} \langle \nabla \theta_R(Z_t^\varepsilon), \mathbb{A}_k(Z_t^\varepsilon)\rangle_{\mathbb{R}^{2Nd}} dW_t^k.$$

From (4.2) and the boundedness of $\mathbb{F}_\varepsilon$ on $B_{2Nd}(0, R)$, we have

$$dI_t^{111} \le C \frac{\theta_R(Z_t^\varepsilon)}{|Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon}|} dt, \quad dI_t^{211} \le Cg(Z_t^{i,\varepsilon} - Z_t^{j,\varepsilon})|\nabla \theta_R(Z_t^\varepsilon)| dt. \tag{4.9}$$

Step 4. We now deal with the martingale terms I^{112} and I^{212}. From (4.2), (1.12), the boundedness of $|Q(\tilde{x},\tilde{x})|$, and Doob's inequality,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|I_T^{112}\big|^2\Big] \le C\,\mathbb{E}\bigg[\int_0^T \theta_R^2\big(Z_s^\varepsilon\big)\,ds\bigg],$$

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|I_T^{212}\big|^2\Big] \le C\,\mathbb{E}\bigg[\int_0^T g^2\big(Z_s^{i,\varepsilon}-Z_s^{j,\varepsilon}\big)\big|\nabla\theta_R\big(Z_s^\varepsilon\big)\big|^2\,ds\bigg].$$

From the above bounds, together with (4.5), (4.6), (4.7), (4.8), and (4.9), and by making use of the estimates

$$\max(\theta_R(z),|\nabla_R\theta(z)|,|\nabla_R^2\theta(z)|) \le \mathbb{1}_{\{|z|\le R+2\}}, \quad z\in\mathbb{R}^{2Nd},$$

$$\max(g(\tilde{z}),g^2(\tilde{z})) \le \frac{C}{|\tilde{z}|}, \quad \tilde{z}\in\mathbb{R}^{2d},$$

we deduce (with $Z_0^\varepsilon = z$)

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\big(\theta_R\big(Z_t^\varepsilon\big)\log_\phi^+\big(\big|Z_t^{i,\varepsilon}-Z_t^{j,\varepsilon}\big|\big)\big)\Big]\\
&\quad\le \theta_R(z)\log_\phi^+(|z^i-z^j|)\\
&\qquad + C\bigg(1+\mathbb{E}\int_0^T \mathbb{1}_{\{|Z_s^\varepsilon|\le R+2\}}\bigg(\frac{1}{|Z_s^{i,\varepsilon}-Z_s^{j,\varepsilon}|}+|\phi'|\big(\big|Z_s^{i,\varepsilon}-Z_s^{j,\varepsilon}\big|\big)\bigg)ds\bigg).
\end{aligned}$$

Step 5. We now integrate on a ball $B_{2Nd}(0,R_0)$ of $\mathbb{R}^{2Nd}$ with respect to the initial conditions. Applying Lemma 4.2, we get

$$\begin{aligned}
&\int_{B_{2Nd}(0,R_0)} \mathbb{E}\Big[\sup_{t\in[0,T]}\big(\theta_R\big(\varphi_t^\varepsilon(z)\big)\log_\phi^+\big(\big|\varphi_t^{i,\varepsilon}(z)-\varphi_t^{j,\varepsilon}(z)\big|\big)\big)\Big]dz\\
&\quad\le \int_{B_{2Nd}(0,R_0)} \theta_R(z)\log_\phi^+(|z^i-z^j|)\,dz\\
&\qquad + C\bigg[R_0^{2Nd}+\mathbb{E}\int_0^T\int_{\{|z|\le R+2\}}\bigg(\frac{1}{|z^i-z^j|}+|\phi'|(|z^i-z^j|)\bigg)dz\,ds\bigg].
\end{aligned}$$

A spherical change of variable shows that the integral of $|\phi'|(|z^i-z^j|)$ is bounded by $C\int_0^1|\phi'(r)|\,dr$, which is less than $2C$. This completes the proof. □

LEMMA 4.4. *Given $R_0, T>0$, define $\mathfrak{m}$ as the normalized product measure*

$$\mathfrak{m} = \mathcal{V}_{2Nd}^{-1}(R_0)\cdot \mathrm{Leb}_{2Nd}\otimes\mathbb{P}$$

on $\mathfrak{O} = B_{2Nd}(0,R_0)\times\Omega$. Then,

$$\lim_{\varsigma\to 0}\sup_{\varepsilon>0}\mathfrak{m}\Big\{\inf_{t\in[0,T]}\inf_{(i,j)\in\Delta_N}\big|\varphi_t^{i,\varepsilon}(z,\omega)-\varphi_t^{j,\varepsilon}(z,\omega)\big| < \varsigma\Big\} = 0.$$

(*For a measurable function* $\phi : \mathfrak{O} \to \mathbb{R}$ *with respect to the product* σ*-field on* $\mathfrak{O}$ *and a Borel set* $A \subset \mathbb{R}$, $\mathfrak{m}\{\phi(z,\omega) \in A\}$ *stands for* $\mathfrak{m}\{(z,\omega) \in \mathfrak{O} : \phi(z,\omega) \in A\}$.)

PROOF. From the boundedness of F_ε and $Q(\tilde{x},\tilde{x})$ and from the Markov inequality, it is easily seen that, for any $R_0 > 0$, there exists a constant C depending only on R_0 and T and such that, for any $R > 0$, $\mathfrak{m}\{\sup_{t\in[0,T]} |\varphi_t^\varepsilon(z,\omega)| > R\} \le C/R$. Moreover, by Proposition 4.3,

$$\mathfrak{m}\Big\{\sup_{t\in[0,T]}\Big[\mathbb{1}_{\{|\varphi_t^\varepsilon(z,\omega)|\le R\}} \sum_{(i,j)\in\Delta_N} \log^+ \big|\varphi_t^{i,\varepsilon}(z,\omega) - \varphi_t^{j,\varepsilon}(z,\omega)\big|\Big] > K\Big\} \le \frac{C'}{K},$$

where C' only depends upon R_0, R, and T. The proof is easily completed. $\square$

LEMMA 4.5. *Given* $R_0, T > 0$, *use the same definition for* $\mathfrak{m}$ *as above. Then,*

$$\lim_{(\varsigma,A)\to(0,+\infty)} \sup_{\varepsilon>0} \sup_{0<\delta_0<1} \mathfrak{m}\Big\{(z,\omega) \in \mathfrak{O} : \\ \mathrm{Leb}_1\Big(t \in [0,T] : \inf_{(i,j)\in\Delta_N} \big|\tilde{\pi}_x\big[\varphi_t^{i,\varepsilon}(z,\omega) - \varphi_t^{j,\varepsilon}(z,\omega)\big]\big| < \frac{\delta_0\varsigma}{A}\Big) > A\delta_0\Big\} = 0.$$

PROOF. The proof follows from Lemma 4.4 and Proposition 4.6 below. Indeed, as a consequence of the boundedness of F_ε and $Q(\tilde{x},\tilde{x})$, the probability that the v-coordinate of $(\varphi_t^\varepsilon(z))_{0\le t\le T}$ is $\frac14$-Hölder-continuous with A' as Hölder constant converges to 1 as A' tends to $+\infty$ uniformly in $\varepsilon > 0$ and in $z \in B_{2Nd}(0, R_0)$. $\square$

PROPOSITION 4.6. *Given* $A, R_0, T > 0$, *let* $(\zeta_t = (\chi_t, \nu_t))_{0\le t\le T}$ *be a continuous path with values in* $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *such that* $\zeta_0 = z \in B_{2Nd}(0,R_0)$, $(\nu_t^i)_{t\ge 0}$ *is a* $\frac14$*-Hölder-continuous* $\mathbb{R}^d$*-valued path with Hölder constant A for* $1 \le i \le N$ *and* $[d\chi_t^i/dt] = \nu_t^i$ *for* $t \in [0,T]$ *and* $i \in \{1,\dots,N\}$. *Then there exists a constant* C *depending only on* d, A, N, R_0, *and* T *and such that, for any* $\varsigma, \delta_0 \in (0,1)$, $\inf_{t\in[0,T]} \inf_{(i,j)\in\Delta_N} |\zeta_t^i - \zeta_t^j| \ge \varsigma$ *implies*

$$\mathrm{Leb}_1\Big(t \in [0,T] : \inf_{(i,j)\in\Delta_N} \big|\chi_t^i - \chi_t^j\big| \le \delta_0 \frac{\varsigma^5}{C}\Big) \le C\delta_0.$$

PROOF. Assume that there exist $\delta \in (0,\varsigma)$, $t_0 \in [0,T)$, and $(i,j) \in \Delta_N$ such that $|\chi_{t_0}^i - \chi_{t_0}^j| \le \delta$. Since $\inf_{t\in[0,T]} \inf_{(i,j)\in\Delta_N} |\zeta_t^i - \zeta_t^j| \ge \varsigma$, we deduce that $|\nu_{t_0}^i - \nu_{t_0}^j| \ge \sqrt{\varsigma^2 - \delta^2}$. From the Hölder property of $(\nu_t)_{0\le t\le T}$, there exists a constant C, independent of ς, t_0, and δ, such that

$$\big|\nu_t^i - \nu_t^j\big| \ge \sqrt{\varsigma^2 - \delta^2} - C(t-t_0)^{\frac14}, \quad t_0 \le t \le T.$$

Therefore, there exists one coordinate $\ell \in \{1,\dots,d\}$ such that

$$\big|\big(\nu_t^i - \nu_t^j\big)_\ell\big| \ge \frac{\sqrt{\varsigma^2-\delta^2} - C(t-t_0)^{1/4}}{\sqrt{d}}, \quad t_0 \le t \le T.$$

For $C(t-t_0)^{1/4} < \sqrt{\varsigma^2-\delta^2}$, the right-hand side is always positive and hence $(\nu_t^i-\nu_t^j)_\ell$ cannot vanish. By continuity, it is of constant sign. Therefore,

$$\begin{aligned}\left|\chi_t^i-\chi_t^j\right| \geq \left|\left(\chi_t^i-\chi_t^j\right)_\ell\right| &\geq \left|\int_{t_0}^t \left(\nu_s^i-\nu_s^j\right)_\ell\, ds\right| - \left|\left(\chi_{t_0}^i-\chi_{t_0}^j\right)_\ell\right| \\ &\geq (t-t_0)\frac{\sqrt{\varsigma^2-\delta^2}-C(t-t_0)^{1/4}}{\sqrt{d}} - \delta\end{aligned}$$

for $C(t-t_0)^{1/4} \leq \sqrt{\varsigma^2-\delta^2}$. For $\delta \leq \varsigma/2$ and $C(t-t_0)^{1/4} \leq \varsigma/4$, we deduce that $|\chi_t^i-\chi_t^j| \geq \varsigma(t-t_0)/(4\sqrt{d})-\delta$. Finally, for $8\sqrt{d}\delta/\varsigma \leq t-t_0 \leq \varsigma^4/(4C)^4$, $|\chi_t^i-\chi_t^j| \geq \delta$. By modifying C if necessary, we deduce that

$$\left|\chi_t^i-\chi_t^j\right| \geq \delta \tag{4.10}$$

for $C\delta/\varsigma \leq t-t_0 \leq \varsigma^4/C$ and $\delta \leq \varsigma/2$. Assume now, without loss of generality, that $C \geq 2$ and choose δ of the form $\delta_0\varsigma^5/C^2$ with $\delta_0 \leq 1$. Define the set

$$\mathcal{I}_x(\delta_0,\varsigma) = \left\{t \in [0,T] : \left|\chi_t^i-\chi_t^j\right| \leq \delta_0\varsigma^5/C^2\right\}.$$

From (4.10), $t_0 \in \mathcal{I}_x(\delta_0,\varsigma) \Rightarrow [t_0+\delta_0\varsigma^4/C, t_0+\varsigma^4/C] \cap \mathcal{I}_x(\delta_0,\varsigma) = \varnothing$. Therefore,

$$\mathrm{Leb}_1(\mathcal{I}_x(\delta_0,\varsigma)) \leq \delta_0\varsigma^4/C\lceil TC/\varsigma^4\rceil \leq \delta_0(T+1). \qquad \square$$

4.2 Noncoalescence for a.e. Initial Configuration

As a consequence of the previous estimates, we prove the following (the result below might be compared with [1, 8], which study the a.e. solvability of ODEs, but therein uniqueness is investigated through the uniqueness of a regular Lagrangian flow):

THEOREM 4.7. *Under Assumptions* (A.1)–(A.3)*, for Lebesgue almost every z, equation* (1.13) *has one and only one global strong solution.*

PROOF.

Step 1. Here we consider $\Xi = \mathcal{C}([0,+\infty),\mathbb{R}^{2Nd}) \otimes \mathcal{C}([0,+\infty),\mathbb{R})^{\otimes \mathbb{N}\setminus\{0\}}$ endowed with the product σ-field $\mathcal{X}$ of the Borel σ-fields. For $R_0 > 0$ and $\varepsilon > 0$ and with the same notation as in Lemma 4.4, we endow the pair $(\Xi,\mathcal{X})$ with the probability $\mathbb{Q}^\varepsilon$ defined on the cylinders as

$$\begin{aligned}&\mathbb{Q}^\varepsilon(A_0 \times A_1 \times \cdots \times A_k \times \mathcal{C}([0,+\infty),\mathbb{R}) \times \cdots) = \\ &\qquad \mathfrak{m}\left\{\left(\varphi_t^\varepsilon(z)\right)_{t\geq 0} \in A_0,\ \left(W_t^1,\dots,W_t^k\right) \in A_1 \times \cdots \times A_k\right\},\end{aligned}$$

where A_0 is a Borel subset of $\mathcal{C}([0,+\infty),\mathbb{R}^{2Nd})$ and $A_1,\dots,A_N$ are Borel subsets of $\mathcal{C}([0,+\infty),\mathbb{R})$. The σ-field $\mathcal{X}$ coincides with the Borel σ-field generated by the standard product metric on the product space Ξ. In particular, the notion of tightness is relevant for probability measures on the pair $(\Xi,\mathcal{X})$: it is easily checked that the family $(\mathbb{Q}^\varepsilon)_{\varepsilon>0}$ is tight.

Denoting by $\mathbb{Q}$ the limit of some convergent sequence $(\mathbb{Q}^{\varepsilon_n})_{n\in\mathbb{N}}$ for a decreasing sequence of positive reals $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0, we investigate the properties of the canonical process under $\mathbb{Q}$, denoted by $\xi_t = (\xi_t^k)_{k\in\mathbb{N}}$, $(\xi_t^0)_{t\geq 0}$ being $\mathbb{R}^{2Nd}$-valued and each $(\xi_t^k)_{t\geq 0}$, $k \geq 1$, being $\mathbb{R}$-valued. Clearly, the family $((\xi_t^k)_{t\geq 0})_{k\in\mathbb{N}\setminus\{0\}}$ is a family of independent Brownian motions under $\mathbb{Q}$. Moreover, the marginal law of $\Xi \ni \xi \mapsto \xi_0^0$ is the uniform distribution on the ball $B_{2Nd}(0, R_0)$.

For any $\varsigma > 0$, the set $\{\xi \in \Xi : \inf_{t\in[0,T]} \inf_{(i,j)\in\Delta_N} |\pi_i(\xi_t^0) - \pi_j(\xi_t^0)| < \varsigma\}$ is open in Ξ. By using the portmanteau theorem to pass to the limit in Lemma 4.4 and letting ς tend to 0, we deduce that, for any $T > 0$,

$$\mathbb{Q}\Big\{\xi \in \Xi : \inf_{(i,j)\in\Delta_N} \inf_{t\in[0,T]} \big|\pi_i\big(\xi_t^0\big) - \pi_j\big(\xi_t^0\big)\big| = 0\Big\} = 0. \tag{4.11}$$

Similarly, the set $\{\xi \in \Xi : \mathrm{Leb}_1(t \in [0,T] : \inf_{(i,j)\in\Delta_N} |\pi_{i,x}(\xi_t^0) - \pi_{j,x}(\xi_t^0)| < \delta_0\varsigma/A) > A\delta_0\}$ is open in Ξ. By using the portmanteau theorem to pass to the limit in Lemma 4.5 and letting δ_0 tend to 0 first and then (ς, A) to $(0, +\infty)$, we obtain

$$\mathbb{Q}\Big\{\xi \in \Xi : \mathrm{Leb}_1\Big(t \in [0,T] : \inf_{(i,j)\in\Delta_N} \big|\pi_{i,x}\big(\xi_t^0\big) - \pi_{j,x}\big(\xi_t^0\big)\big| = 0\Big) > 0\Big\} = 0. \tag{4.12}$$

Let now $\xi_t^0 = (\chi_t^0, \nu_t^0)$, with $\chi_t^0 = \Pi_x(\xi_t^0)$ and $\nu_t^0 = \Pi_v(\xi_t^0)$, $t \geq 0$. Also let

$$\widetilde{\nu}_t^0 = \nu_t^0 - \int_0^t \Pi_v\big(\mathbb{F}\big(\chi_s^0\big)\big)ds, \quad t \geq 0. \tag{4.13}$$

We claim that $(\widetilde{\nu}_t^0)_{t\geq 0}$ is a square-integrable continuous martingale under $\mathbb{Q}$ with respect to the filtration $(\mathcal{G}_t^0 = \sigma(\xi_s^k, s \leq t, k \in \mathbb{N}))_{t\geq 0}$ with the mutual variations

$$\big[(\widetilde{\nu}^0)^i, (\widetilde{\nu}^0)^j\big]_t = \int_0^t Q\big(\big(\chi_s^0\big)^i, \big(\chi_s^0\big)^j\big)ds, \quad i, j \in \{1, \dots, d\}, \tag{4.14}$$

$$[(\widetilde{\nu}^0)^i, \xi^k]_t = \int_0^t \sigma_k\big(\big(\chi_s^0\big)^i\big)ds, \quad i \in \{1, \dots, d\},\ k \in \mathbb{N} \setminus \{0\}. \tag{4.15}$$

The proof is quite standard and consists in passing to the limit in the martingale properties characterizing the dynamics of $(Z_t^\varepsilon)_{t\geq 0}$. The only difficulty is to pass to the limit along the mollified drifts. For $T > 0$, we thus prove that

$$\left(\int_0^t \mathbb{F}_\varepsilon\big(\xi_s^0\big)ds\right)_{0\leq t\leq T} \cdot \mathbb{Q}^\varepsilon \underset{\varepsilon\to 0}{\Rightarrow} \left(\int_0^t \mathbb{F}\big(\xi_s^0\big)ds\right)_{0\leq t\leq T} \cdot \mathbb{Q}, \tag{4.16}$$

where the left- and right-hand sides denote the distributions of the specified processes under the specified measures on $\mathcal{C}([0,T], \mathbb{R}^{2Nd})$ and $\Rightarrow$ stands for the convergence in distribution. We emphasize that as a consequence of the boundedness of F_ε there exists a constant $C > 0$, independent of ε such that, for any $a > 0$ and any $\varepsilon' > \varepsilon$,

$$\mathbb{Q}^{\varepsilon}\left(\sup_{0\le t\le T}\left|\int_0^t \mathbb{F}_{\varepsilon}(\xi_s^0)ds - \int_0^t \mathbb{F}(\xi_s^0)ds\right| > a\right) \le \mathbb{Q}^{\varepsilon}\Big(C\,\mathrm{Leb}_1\big(t\in[0,T]: \inf_{(i,j)\in\Delta_N}\big|\pi_{i,x}(\xi_t^0) - \pi_{j,x}(\xi_t^0)\big| \le \varepsilon'\big) \ge a\Big).$$

The event in the right-hand side is closed in Ξ, so that

$$\limsup_{\varepsilon\to 0}\mathbb{Q}^{\varepsilon}\left(\sup_{0\le t\le T}\left|\int_0^t \mathbb{F}_{\varepsilon}(\xi_s^0)ds - \int_0^t \mathbb{F}(\xi_s^0)ds\right| > a\right) \le \mathbb{Q}\Big(C\,\mathrm{Leb}_1\big(t\in[0,T]: \inf_{(i,j)\in\Delta_N}\big|\pi_{i,x}(\xi_t^0) - \pi_{j,x}(\xi_t^0)\big| \le \varepsilon'\big) \ge a\Big). \tag{4.17}$$

By letting ε' tend to 0 in (4.17), we deduce from (4.12) that the left-hand side is 0. Therefore, to prove (4.16), it is sufficient to prove

$$\left(\int_0^t \mathbb{F}(\xi_s^0)ds\right)_{0\le t\le T}\cdot\mathbb{Q}^{\varepsilon} \underset{\varepsilon\to 0}{\Rightarrow} \left(\int_0^t \mathbb{F}(\xi_s^0)ds\right)_{0\le t\le T}\cdot\mathbb{Q}.$$

By the dominated convergence theorem, the map

$$\mathcal{C}([0,T],\mathbb{R}^{2Nd}) \ni (\xi_t^0)_{0\le t\le T} \mapsto \left(\int_0^t \mathbb{F}(\xi_s^0)ds\right)_{0\le t\le T} \in \mathcal{C}([0,T],\mathbb{R}^{2Nd})$$

is continuous at any path ξ^0 for which

$$\mathrm{Leb}_1\Big(t\in[0,T]: \inf_{(i,j)\in\Delta_N}\big|\pi_{i,x}(\xi_t^0) - \pi_{j,x}(\xi_t^0)\big| = 0\Big) = 0.$$

Again from (4.12), this is true a.s. under $\mathbb{Q}$: by the continuous mapping theorem, we complete the proof of (4.16). Thus, $(\widetilde{\nu}^0)_{t\ge 0}$ in (4.13) satisfies the announced martingale property.

Step 2. Denote by $(\mathcal{G}_t)_{t\ge 0}$ the right-continuous version of $(\mathcal{G}_t)_{t\ge 0}$ augmented with $\mathbb{Q}$-null sets. Clearly, $(\widetilde{\nu}_t^0)_{t\ge 0}$ is a square-integrable continuous martingale under $\mathbb{Q}$ with respect to $(\mathcal{G}_t)_{t\ge 0}$ and both (4.14) and (4.15) remain true. In particular, we can compute

$$\left[\widetilde{\nu}^0 - \sum_{k\in\mathbb{N}\setminus\{0\}}\int_0^{\cdot} A_k(\chi_s^0)d\xi_s^k\right]_t = 0, \quad t\ge 0,$$

so that, $\mathbb{Q}$-a.s.,

$$\xi_t^0 = \xi_0^0 + \int_0^t \mathbb{F}(\xi_s^0)ds + \sum_{k\in\mathbb{N}\setminus\{0\}}\int_0^t \mathbb{A}_k(\chi_s^0)d\xi_s^k, \quad t\ge 0. \tag{4.18}$$

We denote by $(\mathfrak{Q}(\cdot,z))_{z\in B_{2Nd}(0,R_0)}$ a family of regular conditional probabilities of $\mathbb{Q}$ given the random variable $\Xi \ni \xi \mapsto \xi_0^0$. It is easy to see that, for a.e. $z \in B_{2Nd}(0,R_0)$, $((\xi_t^k)_{t\ge 0})_{k\in\mathbb{N}\setminus\{0\}}$ are independent Brownian motions under

$\mathfrak{Q}(\cdot, z)$, $(\widetilde{v}_t^0)_{t\geq 0}$ is a square-integrable continuous martingale under $\mathfrak{Q}(\cdot, z)$ with respect to $(\mathcal{G}_t)_{t\geq 0}$, and both (4.14) and (4.15) remain true under $\mathfrak{Q}(\cdot, z)$. We deduce that, for a.e. $z \in B_{2Nd}(0, R_0)$, (4.18) holds true $\mathfrak{Q}(\cdot, z)$-a.s. with $\xi_0^0 = z$ therein. (Note that, a.e. on $B_{2Nd}(0, R_0)$, the version of the stochastic integral may be chosen independently of z. Of course, its distribution under $\mathfrak{Q}(\cdot, z)$ depends on z.) From (4.11) and (4.12), we deduce that, for a.e. $z \in B_{2Nd}(0, R_0)$, there is no coalescence in the phase space with probability 1 under $\mathfrak{Q}(\cdot, z)$ and that the set of instants where coalescence occurs in the space of positions is of zero Lebesgue measure with probability 1 under $\mathfrak{Q}(\cdot, z)$.

Step 3. We now prove that pathwise uniqueness holds for solutions that remain in Γ_N a.s. We are thus given two solutions $(\zeta_t)_{t\geq 0}$ and $(\zeta'_t)_{t\geq 0}$ of (1.13) with $\zeta_0 = \zeta'_0 = z \in \Gamma_N$, $(\zeta_t)_{t\geq 0}$ being a.s. free of coalescence in the phase space. (Processes $(\zeta_t)_{t\geq 0}$ and $(\zeta'_t)_{t\geq 0}$ are $\mathbb{R}^{2Nd}$-valued and play the same role as $(Z_t)_{t\geq 0}$.) Denoting by $\tau = \inf\{t \geq 0 : \zeta_t \neq \zeta'_t\}$, we thus have to prove that $\mathbb{P}\{\tau = +\infty\} = 1$.

On the set $\{\tau < +\infty\}$ (if not empty), we have $\zeta_\tau = \zeta'_\tau \in \Gamma_N$ since $(\zeta_t)_{t\geq 0}$ is free of coalescence in the phase space. In other words, we have, $\mathbb{P}$-a.s., $\zeta_{\tau\wedge n} = \zeta'_{\tau\wedge n} \in \Gamma_N$ for all $n \in \mathbb{N}$. Pathwise uniqueness follows from the following:

LEMMA 4.8. *Let Z_0 be a random variable with values in Γ_N, and let $(\zeta_t)_{t\geq 0}$ and $(\zeta'_t)_{t\geq 0}$ stand for two solutions of* (1.13) *with initial condition Z_0, $(\zeta_t)_{t\geq 0}$ being free of coalescence in the phase space. Then there exists a stopping time ρ, $\mathbb{P}\{\rho > 0\} = 1$, such that $(\zeta_t)_{t\geq 0}$ and $(\zeta'_t)_{t\geq 0}$ are equal a.s. on* $[0, \rho]$.

By applying Lemma 4.8 with $Z_0 = \zeta_{\tau\wedge n}$ as initial conditions, we deduce that $\mathbb{P}\{\tau \geq n\} = 1$ for any $n \in \mathbb{N}$, that is, $\mathbb{P}\{\tau = +\infty\} = 1$, as announced above.

We turn to the proof of Lemma 4.8. We write $Z_0 = (X_0, V_0)$ with $X_0 = \Pi_x(Z_0)$ and $V_0 = \Pi_v(Z_0)$, and $\zeta_t = (\chi_t, \nu_t)$ with $\chi_t = \Pi_x(\zeta_t)$ and $\nu_t = \Pi_v(\zeta_t)$ for $t \geq 0$. In a similar fashion, we write $\zeta'_t = (\chi'_t, \nu'_t)$ for $t \geq 0$. Letting $\rho^{1,i,j} = \inf(\rho_\zeta^{1,i,j}, \rho_{\zeta'}^{1,i,j})$ with

$$\rho_\zeta^{1,i,j} = \inf\{t \geq 0 : |\chi_t^i - \chi_t^j| \leq |X_0^i - X_0^j|/2\}$$

(with a similar definition for $\rho_{\zeta'}^{1,i,j}$), and $\rho^{2,i,j} = \inf(\rho_\zeta^{2,i,j}, \rho_{\zeta'}^{2,i,j})$ with

$$\rho_\zeta^{2,i,j} = \inf\{t > 0 : |\chi_t^i - \chi_t^j| \leq t|V_0^i - V_0^j|/2\}$$

(with the convention that $\rho_\zeta^{2,i,j} = 0$ if $V_0^i - V_0^j = 0$ and with a similar definition for $\rho_{\zeta'}^{2,i,j}$), we define $\rho^{i,j} = \max(\rho^{1,i,j}, \rho^{2,i,j})$.

We first prove that $\rho^{i,j}$ is a.s. positive. If $\rho^{1,i,j}(\omega)$ is 0 for a given $\omega \in \Omega$, it holds that $\rho_\zeta^{1,i,j}(\omega) = 0$ or $\rho_{\zeta'}^{1,i,j}(\omega) = 0$, so that $|X_0^i(\omega) - X_0^j(\omega)| = 0$. Since Z_0 has values in Γ_N, we have $|V_0^i(\omega) - V_0^j(\omega)| > 0$. Since the paths of $(\zeta_t)_{t\geq 0}$

and $(\zeta'_t)_{t\geq 0}$ are (a.s.) $\frac{1}{4}$-Hölder-continuous, we also have

$$\left|\nu_t^i(\omega) - \nu_t^j(\omega) - \left(V_0^i(\omega) - V_0^j(\omega)\right)\right| \leq C(\omega)t^{\frac{1}{4}}, \quad t \in [0,1], \tag{4.19}$$

for a finite constant $C(\omega)$ depending on ω. We deduce that, for any $t \in (0,1]$,

$$\left|\chi_t^i(\omega) - \chi_t^j(\omega) - t\left(V_0^i(\omega) - V_0^j(\omega)\right)\right| \leq C(\omega)t^{\frac{5}{4}}.$$

Therefore,

$$\left|\chi_t^i(\omega) - \chi_t^j(\omega)\right| \geq \frac{t}{2}\left|V_0^i(\omega) - V_0^j(\omega)\right| \tag{4.20}$$

for $C(\omega)t^{1/4} \leq (\frac{1}{2})|V_0^i(\omega) - V_0^j(\omega)|$. Therefore, $\rho_\zeta^{2,i,j}(\omega) > 0$. Similarly, $\rho_{\zeta'}^{2,i,j}(\omega) > 0$.

On $[0,\rho]$, the drift $F(\zeta_t^i - \zeta_t^j)$ in (1.13) satisfies

$$\left|F\left(\chi_t^i - \chi_t^j\right) - F\left((\chi'_t)^i - (\chi'_t)^j\right)\right| \leq Ct^{-1}|\chi_t - \chi'_t| \leq C \sup_{0\leq s\leq t}|\nu_s - \nu'_s|, \tag{4.21}$$

where the constant C depends upon the randomness only through Z^0. Indeed, if $\rho^{1,i,j} > 0$, $|X_0^i - X_0^j|$ must be (strictly) positive so that F is locally Lipschitz-continuous; if $\rho^{2,i,j} > 0$, the bound follows from (4.20) and (A.1). Therefore, on $[0,\rho]$, the drift $F(\zeta_t^i - \zeta_t^j)$ coincides with some functional $G((\nu_s^i - \nu_s^j)_{0\leq s\leq t})$ of the whole path $(\nu_s^i - \nu_s^j)_{0\leq s\leq t}$, G being bounded and locally Lipschitz-continuous with respect to the L^∞-norm. We then write

$$\begin{aligned} d\nu_t^i &= \frac{1}{N}\sum_{j\neq i} G\left(\left(\nu_s^i - \nu_s^j\right)_{0\leq s\leq t}\right)dt \\ &\quad + \sum_{k=1}^{+\infty}\sigma_k\left(\chi_t^i\right)dW_t^k, \quad i \in \{1,\dots,d\}, \end{aligned} \tag{4.22}$$

for $t \in [0,\rho]$ with $\rho = \inf_{i\neq j}\rho^{i,j}$. (Obviously, (4.22) is also satisfied by ζ'.) Equation (4.22) can be regarded as a functional equation driven by bounded and locally Lipschitz-continuous coefficients. The Lipschitz constants of the coefficients on any balls are finite random variables depending upon ω only through the initial condition Z_0. Lemma 4.8 follows from $\mathbb{P}\{\sup_{0\leq s\leq \rho}|\nu_s - \nu'_s| = 0 \mid Z_0\} = 1$.

Step 4. We have proven weak existence and strong uniqueness for a.e. initial condition $z \in \Gamma_N$. Following the proof by Yamada and Watanabe in the finite-dimensional case, we deduce that both strong existence and strong uniqueness hold for a.e. initial condition $z \in \Gamma_N$. □

4.3 Noncoalescence for Any Initial Condition in Γ_N

We now complete the proof of Theorem 4.1. To this aim, we first prove the following:

LEMMA 4.9. *For any* $z \in \Gamma_N$, *there exists a unique solution* $(\varphi_t(z))_{0 \le t \le \tau_z}$ *to* (1.13), *with* z *as initial condition, on the interval* $[0, \tau_z]$, *where* $\tau_z = \inf\{t \ge 0 : \varphi_t(z) \in \Gamma_N^{\mathsf{c}}\}$. *Moreover, the mapping*

$$\Gamma_N \ni z \mapsto (\varphi_{t \wedge \tau_z}(z))_{t \ge 0} \in \mathcal{C}([0, +\infty), \mathbb{R}^{2Nd})$$

is measurable.

PROOF. The whole difficulty is to handle the possible coalescence of the particles in the space of positions. By induction, we build a nondecreasing sequence of stopping times $(\tau_z^n)_{n \in \mathbb{N}}$ such that $\tau_z^n \to \tau_z$ a.s. as n tends to $+\infty$ and (1.13) has a unique solution $(\varphi_t(z))_{0 \le t \le \tau_z^n}$ on each $[0, \tau_z^n]$ with z as initial solution for any $n \in \mathbb{N}$. The stopping time τ_0 is set equal to 0. Given $(\varphi_t(z) = (\chi_t(z), \nu_t(z)))_{0 \le t \le \tau_z^n}$ for some $n \in \mathbb{N}$, we can follow the proof of Lemma 4.8 and build a (unique) solution $(\varphi_t(z))_{\tau_z^n \le t \le \tau_z^{n+1}}$ to (1.13) on $[\tau_z^n, \tau_z^{n+1}]$, where $\tau_z^{n+1} = \tau_z \wedge \tau_z^{\prime,n+1}$ with $\tau_z^{\prime,n+1} = \inf_{i \ne j} \rho^{i,j,n+1}$ and $\rho^{i,j,n+1} = \max(\rho^{1,i,j,n+1}, \rho^{2,i,j,n+1})$,

$$\rho^{1,i,j,n+1} = \inf\{t \ge \tau_z^n : |\chi_t^i(z) - \chi_t^j(z)| \le |\chi_{\tau_z^n}^i(z) - \chi_{\tau_z^n}^j(z)|/2\},$$
$$\rho^{2,i,j,n+1} = \inf\{t > \tau_z^n : |\chi_t^i(z) - \chi_t^j(z)| \le (t - \tau_z^n)|\nu_{\tau_z^n}^i(z) - \nu_{\tau_z^n}^j(z)|/2\}.$$

Clearly, the sequence $(\tau_z^n)_{n \in \mathbb{N}}$ is nondecreasing.

On each step, existence and uniqueness hold since equation (1.13) can be written as a functional SDE on the interval $[\tau_z^n, \tau_z^{n+1}]$ with bounded and locally Lipschitz-continuous coefficients. (As already emphasized in the proof of Lemma 4.8, the Lipschitz constants of the coefficients on bounded sets depend on the initial position $\varphi_{\tau_z^n}(z)$. This fact, however, has no consequences.)

Almost surely, the sequence $(\tau_z^n)_{n \in \mathbb{N}}$ cannot have an accumulation point before $\varphi(z)$ hits Γ_N^{c}, as otherwise the modulus of continuity of $\varphi(z)$ would blow up. Once more, the precise argument goes back to the proof of Lemma 4.8: the length $\tau_z^{n+1} - \tau_z^n$ depends: (i) on the modulus of continuity of the path $(\varphi_t(z))_{\tau_z^n \le t \le \tau_z^{n+1}}$ (the length of the interval is controlled from below when the modulus is controlled from above), (ii) on the distance $\mathrm{dist}(\varphi_{\tau_z^n}(z), \Gamma_N^{\mathsf{c}})$ (the length of the interval is controlled from below when the distance is bounded away from 0), and (iii) on the norm $|\varphi_{\tau_z^n}(z)|$ (the length of the interval is controlled from below when the norm is bounded away from ∞). The modulus of continuity is controlled in terms of the bounds of the coefficients by Kolmogorov's criterion, the norm of $\varphi(z)$ is controlled in terms of the bounds of the coefficients as well, and the distance from $\varphi(z)$ to Γ_N^{c} is bounded from below on any $[0, \tilde{\tau}_z^\varsigma]$, with $\tilde{\tau}_z^\varsigma = \inf\{t \ge 0 : \mathrm{dist}(\varphi_t(z), \Gamma_N^{\mathsf{c}}) \le \varsigma\}$ for $\varsigma > 0$. This proves that, a.s., $\sup_{n \ge 1} \tau_z^n \ge \tilde{\tau}_z^\varsigma$ for any $\varsigma > 0$, that is $\sup_{n \ge 1} \tau_z^n \ge \lim_{\varsigma \to 0} \tilde{\tau}_z^\varsigma = \tau_z$. □

PROOF OF THEOREM 4.1. We now complete the proof. We add a point Δ to $\mathbb{R}^{2Nd}$ and set $\varphi_t(z) = \Delta$ for $t \ge \tau_z$, $z \in \Gamma_N$, when $\tau_z < \infty$. The resulting family of processes $(\varphi_t(z))_{t \ge 0}$, $z \in \Gamma_N$, has $\Gamma_N \cup \Delta$ as state space. It is a homogeneous

Markov process. By Theorem 4.7, $\mathbb{P}\{\tau_z = +\infty\} = 1$ for a.e. $z \in \mathbb{R}^{2Nd}$. In particular, for any $0 < \varepsilon < T$ we can write

$$\mathbb{P}\{\varphi_{[\varepsilon,T]}(z) \in \Gamma_N\} = \int_{\Gamma_N \cup \{\Delta\}} \mathbb{P}\{\varphi_{[0,T-\varepsilon]}(z') \in \Gamma_N\} \mu_{\varphi_\varepsilon(z)}(dz'),$$

where

$$\{\varphi_{[\varepsilon,T]}(z) \in \Gamma_N\} = \{\omega \in \Omega : \varphi_t(z)(\omega) \in \Gamma_N \text{ for any } t \in [\varepsilon, T]\}$$

and $\mu_{\varphi_\varepsilon(z)}$ is the law of $\varphi_\varepsilon(z)$. By Theorem 4.7, we can find a Borel subset $\mathcal{N} \subset \mathbb{R}^{2Nd}$ of zero Lebesgue measure such that $\mathbb{P}\{\varphi_{[0,T-\varepsilon]}(z) \in \Gamma_N\} = 1$ for all $z \in \mathcal{N}^c$. Then

$$\begin{aligned} \mathbb{P}\{\varphi_{[\varepsilon,T]}(z) \in \Gamma_N\} &\geq \int_{\mathcal{N}^c} \mathbb{P}\{\varphi_{[0,T-\varepsilon]}(z') \in \Gamma_N\} \mu_{\varphi_\varepsilon(z)}(dz') \\ &= 1 - \mu_{\varphi_\varepsilon(z)}(\mathcal{N}). \end{aligned} \tag{4.23}$$

Now, assume that $z = (x, v)$ is such that $x \in \Gamma_{x,N}$ (so that $z \in \Gamma_N$). Then there exists $\varepsilon^* > 0$ such that $\inf_{i \neq j} |x^i - x^j| > \varepsilon^*$. Defining $\tau_z^* = \inf\{t \geq 0 : \inf_{i \neq j} |\pi_{i,x}(\varphi_t(z)) - \pi_{j,x}(\varphi_t(z))| \leq \varepsilon^*\}$, we have

$$\begin{aligned} \mu_{\varphi_\varepsilon(z)}(\mathcal{N}) &\leq \mathbb{P}\{\varphi_\varepsilon^*(z) \in \mathcal{N}, \tau_z^* > \varepsilon\} + \mathbb{P}\{\tau_z^* \leq \varepsilon\} \\ &\leq \mathbb{P}\big\{\varphi_\varepsilon^*(z) \in \mathcal{N}\big\} + \mathbb{P}\big\{\tau_z^* \leq \varepsilon\big\}, \end{aligned} \tag{4.24}$$

where $(\varphi_t^*(z))_{t \geq 0}$ stands for the solution of (1.13) when the system is driven by a Lipschitz drift that coincides with the original one on $\{(x^1, \dots, x^N) \in \mathbb{R}^{2Nd} : \inf_{i \neq j} |x^i - x^j| > \varepsilon^*\}$. As $\mathcal{N}$ is Lebesgue-negligible and the law of $\varphi_\varepsilon^*(z)$ on $\mathbb{R}^{2Nd}$ is absolutely continuous with respect to the Lebesgue measure when $x \in \Gamma_{x,N}$, we deduce that $\mathbb{P}\{\varphi_\varepsilon^*(z) \in \mathcal{N}\} = 0$ (see Proposition 3.9). By letting ε tend to 0, we deduce that $\mathbb{P}\{\varphi_{[0,T]} \in \Gamma_N\} = 1$. Indeed,

$$\bigcap_{\varepsilon > 0} \{\varphi_{[\varepsilon,T]}(z) \in \Gamma_N\} = \{\varphi_{(0,T]}(z) \in \Gamma_N\} = \{\varphi_{[0,T]}(z) \in \Gamma_N\}$$

since $\varphi_0(z) \in \Gamma_N^c$ implies $\varphi_t(z) = \Delta$ for any $t > 0$.

Assume now that $z \in \Gamma_N$ but $x \notin \Gamma_{x,N}$. From the proof of Lemma 4.8, we know that, $\mathbb{P}$ a.s., there exists a nonempty interval $(0, \rho(\omega))$ such that $\varphi_t(z) \in \Gamma_{x,N}$, where ρ is a stopping time. (When $x^i = x^j$, $|\pi_{i,x}(\varphi_t(z)) - \pi_{j,x}(\varphi_t(z))| \geq (t/2)|v^i - v^j|$ for $t > 0$ small and is thus nonzero for $t > 0$ small.) In particular, $\tau_z(\omega) \geq \rho(\omega)$. For any $\delta \in (0, \varepsilon)$, we have

$$\begin{aligned} \mu_{\varphi_\varepsilon(z)}(\mathcal{N}) &\leq \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \delta < \rho\} + \mathbb{P}\{\rho \leq \delta\} \\ &\leq \mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \Pi_x(\varphi_\delta(z)) \in \Gamma_{x,N}\} + \mathbb{P}\{\rho \leq \delta\}. \end{aligned} \tag{4.25}$$

By the Markov property, we then obtain

$$\mathbb{P}\{\varphi_\varepsilon(z) \in \mathcal{N}, \Pi_x(\varphi_\delta(z)) \in \Gamma_{x,N}\} = \int_{\mathbb{R}^{2Nd}} \mathbb{P}\{\varphi_{\varepsilon-\delta}(z') \in \mathcal{N}\} \mathbb{1}_{\{\Pi_x(z') \in \Gamma_{x,N}\}} \mu_{\varphi_\delta(z)}(dz'). \tag{4.26}$$

Returning to (4.23), we write ε as $\varepsilon = \varepsilon_1 + \varepsilon_2$ with $\varepsilon_1, \varepsilon_2 > 0$. Choosing $\delta = \varepsilon_1$ in (4.25), we have $\varepsilon - \delta = \varepsilon_2$ in (4.26). From (4.24) and from Lebesgue's dominated convergence theorem, we know that the right-hand side of (4.26) tends to 0 as $\varepsilon_2 = \varepsilon - \delta$ tends to 0. By passing to the limit in (4.23) and using (4.25), we obtain that $\mathbb{P}\{\varphi_{[\varepsilon_1,T]}(z) \in \Gamma_N\} \geq 1 - \mathbb{P}\{\rho \leq \varepsilon_1\}$ for any $\varepsilon_1 > 0$. Finally, by letting ε_1 tend to 0, we conclude that $\mathbb{P}\{\varphi_{[0,T]}(z) \in \Gamma_N\} = 1$. □

Bibliography

[1] Ambrosio, L. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.* **158** (2004), no. 2, 227–260. doi:10.1007/s00222-004-0367-2

[2] Attanasio, S.; Flandoli, F. Zero-noise solutions of linear transport equations without uniqueness: an example. *C. R. Math. Acad. Sci. Paris* **347** (2009), no. 13–14, 753–756. doi:10.1016/j.crma.2009.04.027

[3] Bafico, R.; Baldi, P. Small random perturbations of Peano phenomena. *Stochastics* **6** (1981/82), no. 3–4, 279–292. doi:10.1080/17442508208833208

[4] Bec, J.; Cencini, M.; Hillerbrand, R. Heavy particles in incompressible flows: the large Stokes number asymptotics. *Phys. D* **226** (2007), no. 1, 11–22. doi:10.1016/j.physd.2006.10.007

[5] Carmack, L.; Majda, G. Concentrations in the one-dimensional Vlasov-Poisson equations: additional regularizations. *Phys. D* **121** (1998), no. 1–2, 127–162. doi:10.1016/S0167-2789(98)00029-3

[6] Celani, A.; Rubenthaler, S.; Vincenzi, D. Dispersion and collapse in stochastic velocity fields on a cylinder. *J. Stat. Phys.* **138** (2010), no. 4–5, 579–597. doi:10.1007/s10955-009-9875-1

[7] Delort, J.-M. Existence de nappes de tourbillon en dimension deux. *J. Amer. Math. Soc.* **4** (1991), no. 3, 553–586. doi:10.2307/2939269

[8] DiPerna, R. J.; Lions, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** (1989), no. 3, 511–547. doi:10.1007/BF01393835

[9] Dirr, N.; Luckhaus, S.; Novaga, M. A stochastic selection principle in case of fattening for curvature flow. *Calc. Var. Partial Differential Equations* **13** (2001), no. 4, 405–425. doi:10.1007/s005260100080

[10] E, W.; Vanden-Eijnden, E. A note on generalized flows. *Phys. D* **183** (2003), no. 3–4, 159–174. doi:10.1016/S0167-2789(03)00183-0

[11] Eyink, G. L.; Xin, J. Existence and uniqueness of L^2-solutions at zero-diffusivity in the Kraichnan model of a passive scalar. Preprint, 1996. arXiv:chao-dyn/9605008

[12] Falkovich, G.; Gawędzki, K.; Vergassola, M. Particles and fields in turbulence. *Rev. Mod. Phys.* **73** (2001), no. 4, 913–975. doi:10.1103/RevModPhys.73.913

[13] Flandoli, F. *Random perturbation of PDEs and fluid dynamic models.* Lectures from the 40th Probability Summer School Held in Saint-Flour, 2010. Lecture Notes in Mathematics, 2015. Springer, Heidelberg, 2011. doi:10.1007/978-3-642-18231-0

[14] Flandoli, F.; Gubinelli, M.; Priola, E. Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.* **180** (2010), no. 1, 1–53. doi:10.1007/s00222-009-0224-4

[15] Flandoli, F.; Gubinelli, M.; Priola, E. Full well-posedness of point vortex dynamics corresponding to stochastic 2D Euler equations. *Stochastic Process. Appl.* **121** (2011), no. 7, 1445–1463. doi:10.1016/j.spa.2011.03.004

[16] Gradinaru, M.; Herrmann, S.; Roynette, B. A singular large deviations phenomenon. *Ann. Inst. H. Poincaré Probab. Statist.* **37** (2001), no. 5, 555–580. doi:10.1016/S0246-0203(01)01075-5

[17] Hakulinen, V. Passive advection and the degenerate elliptic operators M_n. *Comm. Math. Phys.* **235** (2003), no. 1, 1–45. doi:10.1007/s00220-002-0778-0

[18] Hauray, M.; Jabin, P.-E. N-particles approximation of the Vlasov equations with singular potential. *Arch. Ration. Mech. Anal.* **183** (2007), no. 3, 489–524. doi:10.1007/s00205-006-0021-9

[19] Kotelenez, P. *Stochastic ordinary and stochastic partial differential equations: transition from microscopic to macroscopic equations.* Stochastic modelling and applied probability, 58. Springer, New York, 2008.

[20] Krylov, N. V.; Röckner, M. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131** (2005), no. 2, 154–196. doi:10.1007/s00440-004-0361-z

[21] Kunita, H. Stochastic differential equations and stochastic flows of diffeomorphisms. *École d'été de probabilités de Saint-Flour, XII—1982*, 143–303. Lecture Notes in Mathematics, 1097. Springer, Berlin, 1984. doi:10.1007/BFb0099433

[22] Le Jan, Y. On isotropic Brownian motions. *Z. Wahrsch. Verw. Gebiete* **70** (1985), no. 4, 609–620. doi:10.1007/BF00531870

[23] Le Jan, Y.; Raimond, O. Integration of Brownian vector fields. *Ann. Probab.* **30** (2002), no. 2, 826–873. doi:10.1214/aop/1023481009

[24] Lions, P.-L.; Perthame, B. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.* **105** (1991), no. 2, 415–430. doi:10.1007/BF01232273

[25] Lototskii, S. V.; Rozovskii, B. L. The passive scalar equation in a turbulent incompressible Gaussian velocity field. *Uspekhi Mat. Nauk* **59** (2004), no. 2(356), 105–120; translation in *Russian Math. Surveys* **59** (2004), no. 2, 297–312. doi:10.1070/RM2004v059n02ABEH000719

[26] Majda, A. J.; Bertozzi, A. L. *Vorticity and incompressible flow.* Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.

[27] Majda, A. J.; Majda, G.; Zheng, Y. X. Concentrations in the one-dimensional Vlasov-Poisson equations. I. Temporal development and non-unique weak solutions in the single component case. *Phys. D* **74** (1994), no. 3–4, 268–300. doi:10.1016/0167-2789(94)90198-8

[28] Marchioro, C.; Pulvirenti, M. *Mathematical theory of incompressible nonviscous fluids.* Applied Mathematical Sciences, 96. Springer, New York, 1994. doi:10.1007/978-1-4612-4284-0

[29] Monin, A. S.; Yaglom, A. M. *Statistical fluid mechanics. Vol. II.* MIT Press, Cambridge, Mass., 1975.

[30] Nguyen, M. D.; Nualart, D.; Sanz, M. Application of Malliavin calculus to a class of stochastic differential equations. *Probab. Theory Related Fields* **84** (1990), no. 4, 549–571. doi:10.1007/BF01198319

[31] Nualart, D. *The Malliavin calculus and related topics.* Probability and Its Applications (New York). Springer, New York, 1995.

[32] Poupaud, F. Diagonal defect measures, adhesion dynamics and Euler equation. *Methods Appl. Anal.* **9** (2002), no. 4, 533–561.

[33] Souganidis, P. E.; Yip, N. K. Uniqueness of motion by mean curvature perturbed by stochastic noise. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** (2004), no. 1, 1–23. doi:10.1016/S0294-1449(03)00029-5

[34] Yaglom, A. M. Certain types of random fields in n-dimensional space similar to stationary stochastic processes. *Teor. Veroyatnost. i Primenen* **2** (1957), 292–338.

[35] Zheng, Y. X.; Majda, A. J. Existence of global weak solutions to one-component Vlasov-Poisson and Fokker-Planck-Poisson systems in one space dimension with measures as initial data. *Comm. Pure Appl. Math.* **47** (1994), no. 10, 1365–1401. doi:10.1002/cpa.3160471004

François Delarue
Laboratoire J.-A. Dieudonné
UMR CNRS 7351
Université de Nice Sophia-Antipolis
Parc Valrose
06108 NICE Cedex 02
FRANCE
E-mail: delarue@unice.fr

Franco Flandoli
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 1
56127 PISA
ITALY
E-mail: flandoli@dma.unipi.it

Dario Vincenzi
Laboratoire J.-A. Dieudonné
UMR CNRS 7351
Université de Nice Sophia-Antipolis
Parc Valrose
06108 NICE Cedex 02
FRANCE
E-mail: vincenzi@unice.fr

Received November 2012.