Noise-Sustained Convective Structures in Nonlinear Optics

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Evidence of noise-sustained patterns in nonlinear optical systems is given. They are found in passive optical cavities, filled by Kerr type nonlinear media, when the angle of incidence of the pump beam is not zero, in a regime of convective instability. These patterns arise as a macroscopic manifestation of dynamically amplified noise, with amplification factors of up to 10^5. We characterize the difference between noise-sustained and deterministic patterns in terms of statistical properties of the field spectral intensity. [S0031-9007(97)04420-7]

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Processes in which intrinsic microscopic noise of a system is amplified so that it manifests macroscopically provide a way to probe and characterize noise. Three important examples are the noise-triggered decay of an unstable state, the critical fluctuations close to an instability (which give rise to noisy precursors [1] of the state beyond the instability) and convectively unstable states in which fluctuations are dynamically amplified while convected away through the boundaries of the system [2]. In this paper we address this third situation in the context of nonlinear optics. We show the existence of noise-sustained patterns which appear as a spatially structured macroscopic manifestation of quantum noise.

The other two examples of noise amplification have been previously addressed in optical systems. The switch-on of a laser is an example of decay of an unstable state in which laser radiation is built up from quantum noise. In fact, the idea of a "statistical microscope" was put forward to use this process, with amplification factors of 10^8, as a sensitive test of quantum fluctuations [3]. However, this gives no information about spatial correlations, a topic which has been considered recently through looking at amplified critical fluctuations in transverse pattern formation in nonlinear passive cavities [4–6]. This pattern formation problem has been the object of an intense investigation [7,8], partly because of the foreseen applications in all-optical processing and storage of information [9] and the possibility of studying the interface between classical and macroscopic quantum patterns [10]. The quantum noise reduction associated with a pattern forming instability and a Heisenberg-type relation between near- and far-field patterns appears when looking at the noisy precursor of the pattern that will emerge beyond the instability [4–6,10]. From the classical viewpoint, this situation is conceptually equivalent to experiments in fluid dynamics where patterns induced by thermal fluctuations can be observed just below the onset of convection [11]. The strength of thermal noise is determined from the power spectrum of the fluctuating pattern (rolls or hexagons) which identifies a preferred wave number below threshold. The alternative situation considered in fluid dynamics to characterize thermal fluctuations is that of a convectively unstable state in an open flow. Macroscopic noise-sustained patterns arise here by noise being amplified by factors of up to 10^5. Thus, we find two different situations: just below threshold the noisy precursors, which are weakly damped fluctuations, and just above threshold the convective noise-sustained patterns, which result from the amplification of fluctuations by the deterministic dynamics. Quantitative detailed experiments in Taylor-Couette flow [12], in which the noise strength is the only free parameter, have been used to determine thermal noise characteristics, including the noise power spectrum. What we explore in this paper is the counterpart of this situation in nonlinear optics, in which quantum noise might sustain a macroscopic pattern.

The system we consider is a cavity [ring or Fabry-Perot (FP)] filled with a nonlinear, Kerr-type media [7] and pumped by an external laser beam. This kind of device presents transversally uniform steady states and bistability [13]. If the amplitude of the cavity field exceeds a certain threshold the uniform steady state becomes unstable and a spatial structure might form [14,15]. It is also in this system where the question of the manifestation of quantum fluctuations in pattern formation was first addressed [4,5]. For this system we predict that *optical noise-sustained structures* can be observed in the regime of *convective instability*. A necessary condition for this regime is the existence of a drift (or group-velocity) term in the governing equation. Such a contribution arises naturally if the incident pump beam is slightly tilted; then, the pattern which forms drifts always in time and, therefore, at a fixed spatial position, the field pulses (drift instability) as theoretically predicted [15] and experimentally observed [16,17]. Three regimes can actually be found: absolute instability, absolute stability (i.e., when a perturbation of the steady state may or may not grow at a fixed point in space), and the *convectively unstable* regime [18]. The latter refers to the case when a perturbation is unstable in a reference frame moving at the drift velocity but it does not grow locally. In other words the growing perturbation drifts from its original position, and therefore it finally moves outside the system. However, under this condition, a pattern may arise if noise is present; in fact, noise can excite the convectively unstable modes and the pattern, though drifting, is regenerated in a continuous fashion at each position and at all times. The role of noise in optical systems in the convectively unstable regime has not been previously addressed.

The equation governing the slowly varying electric field amplitude A in a FP cavity filled with a Kerr medium and excited by an external laser pump E_0 is [7,15,19]

$$\partial_t A - 2\alpha_0 \partial_x A = i\partial_x^2 A - [1 + i\eta(\Delta - |A|^2)]A + E_0 + \sqrt{\epsilon}\,\xi(x,t)\,, \quad (1)$$

where Δ represents the cavity detuning, η gives the sign of the Kerr nonlinearity (1 for self-focusing, -1 for self-defocusing), and α_0 depends on the angle of incidence of the pump into the cavity. The complex stochastic variable $\xi(x,t)$ is Gaussian with zero mean and correlation $\langle \xi(x,t), \xi^*(x',t')\rangle = 2\delta(x - x')\delta(t - t')$, and it gives a standard semiclassical model of noise. In the linearized version of the Langevin equation of (1) it may describe quantum noise in the Wigner representation, as considered in [6] for the optical parametric oscillator. It can also account for thermal and input field fluctuations. A stochastic dynamics description of Eq. (1) when no drift term is present was given in [19].

Equation (1) has homogeneous steady-state solutions A_0 given by $A_0[1 + i\eta(\Delta - |A_0|^2)] = E_0$ and for $\Delta > \sqrt{3}$ bistability occurs [13]. On writing $A = A_0(1 + \sigma)$ the eigenvalues of the linear evolution matrix for a weak perturbation σ in the Fourier space (at wave number q) are, for $\eta = 1$ [20],

$$\omega_{\pm}(q) = 2i\alpha_0 q - 1 \pm \sqrt{|A_0|^4 - (2|A_0|^2 - \Delta - q^2)^2}\,. \quad (2)$$

The homogeneous steady state is unstable for $\mathrm{Re}(\omega_+) > 0$, and the marginal stability curve is given by $|A_0|^2 = [2(\Delta + q^2) - \sqrt{(\Delta + q^2)^2 - 3}\,]/3$. Its minimum corresponds to $q_c^2 = 2 - \Delta$ when $\Delta < 2$, and $q_c^2 = 0$ when $\Delta > 2$. Hence, for $\Delta < 2$, the instability threshold is given by $|A_0|^2_< = 1$, $q_c^2 = 2 - \Delta$. On the other hand, for $\Delta > 2$, the instability threshold is given by $|A_0|^2_> = [2\Delta - \sqrt{\Delta^2 - 3}\,]/3$, $q_c^2 = 0$. This instability threshold is shown as a solid line in Fig. 1. Above threshold, when $\Delta < 2$, $\mathrm{Re}(\omega_+)$ is maximum for $q^2 = q_c^2 = 2|A_0|^2 - \Delta$ and when $\Delta > 2$ for $q^2 = q_c^2 = 0$.

Because of the presence of an advective-type term in the case of oblique input fields, the instabilities described in the preceding section are convective up to the absolute instability threshold. The nature of the instability may be determined, as usual, through the evaluation of the maximum growth rate of a perturbation of the form $\exp(kx + \omega t)$, where $k = k' + ik''$ is complex, at a fixed location [18,21]. The growth rate is given by $\mathrm{Re}[\omega_+(-ik)]$ where ω_+ is defined in Eq. (2). If $\mathrm{Re}[\omega_+(-ik)]$ is negative, perturbations decay locally, since they are advected away by the drift and the instability is convective. If $\mathrm{Re}[\omega_+(-ik)]$ is positive, the instability is absolute. The threshold for absolute instability (Fig. 1) is then given by $\mathrm{Re}[\omega_+(-ik_s)] = 0$ where k_s is the complex vector k at which the velocity of the fastest growing perturbation front is zero [21]. This is determined by

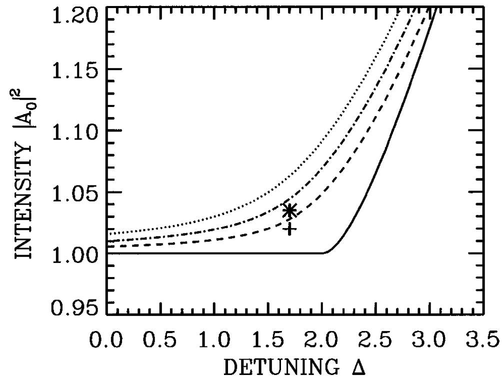

FIG. 1. Stability diagram as a function of the detuning Δ: solid curve is the threshold of convective instability. The other curves, obtained solving numerically $\mathrm{Re}[\omega_+(-ik_s)] = 0$ and Eq. (3), are the absolute instability thresholds for different α_0 (dashed, $\alpha_0 = 0.15$; dash-dotted, $\alpha_0 = 0.2$; and dotted, $\alpha_0 = 0.25$). The quantities plotted in all the figures of this paper are dimensionless.

$$\mathrm{Re}\left(\frac{d\omega_+(-ik)}{dk}\Big|_{k_s}\right) = \mathrm{Im}\left(\frac{d\omega_+(-ik)}{dk}\Big|_{k_s}\right) = 0\,, \qquad \mathrm{Re}\left(\frac{d^2\omega_+(-ik)}{dk^2}\Big|_{k_s}\right) > 0\,. \quad (3)$$

Note that, unlike the result reported in Ref. [15], here the absolute instability threshold depends on the incidence angle; this fact might be an explanation for the numerical observation in Ref. [22]. The convective instability appears also for $\Delta > 2$, for the homogeneous perturbations ($q_c^2 = 0$). For a given $\alpha_0 \neq 0$ convectively unstable solutions are those in the region between the solid curve and the calculated threshold. There, a pattern can still be observed if noise is present as we demonstrate through the numerical solutions of Eq. (1). The detuning is fixed to $\Delta = 1.7$, a value set in order to observe the phenomenon easily without additional complications. In fact, according to Fig. 1 as $\Delta \to 2$ the convectively unstable region gets larger but, as said, for $\Delta > \sqrt{3}$ the system shows bistability and the $q = 0$ mode is unstable, becoming the most unstable for $\Delta > 2$. Note that the criterion given for the transition from convective to absolute instability is of linear nature. Nonlinear terms can shift this threshold if the bifurcation is strongly subcritical [23]. For the chosen value of Δ we have checked that the bifurcation is weakly subcritical and our numerical results indicate that the linear criterion is valid. The pump used was a super-Gaussian beam, $E_0(x,t) = E_m \exp[-(x/x_0)^{2m}/2]$, with

$m = 5$ and $x_0 = 250$. We integrated Eq. (1) in Fourier space with periodic boundary conditions; the spatial window (640 units) was large enough to yield an almost zero pump (and thus field) on both sides.

The spatiotemporal evolution of the intensity $|A|^2$ is shown in Fig. 2 for three cases. In 2(a) the pump intensity is above the threshold of absolute instability (star in Fig. 1) and a spatial structure forms and tends to spread to the whole beam. Figure 2(b) corresponds to the regime of the convective instability (cross in Fig. 1) without noise ($\epsilon = 0$). The structure is formed from an initial perturbation, but it drifts away from the pump region and finally disappears. Figure 2(c) has been obtained with the same parameters of Fig. 2(b) but with nonzero noise. A pattern is continuously formed and drifts outside the pump beam. By comparing 2(a) with 2(c) note that the noise-sustained pattern rises appreciably at a random spatial position for different times. The phenomenon described occurs for any nonzero value of the noise intensity; the average time delay and the "jitter" of the formation of the edge of the structure can be used, as done in the experiments reported in Ref. [12], to determine the noise level.

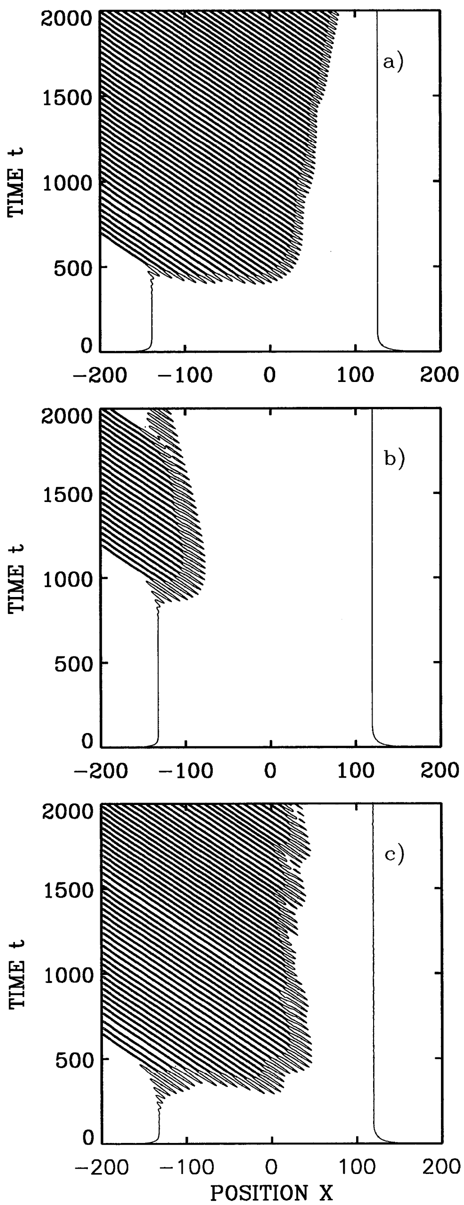

FIG. 2. Spatiotemporal evolution of the field intensity for (a) $|A_0|^2 = 1.035$ ($*$ in Fig. 1) without noise ($\epsilon = 0$), (b) $|A_0|^2 = 1.02$ ($+$ in Fig. 1) without noise, and (c) with noise ($\epsilon = 2.5 \times 10^{-3}$). The contour plot is shown for $|x| < 200$; the integration window was $|x| < 320$; $\alpha_0 = 0.15$.

A quantitative description of the transition from noise- to dynamics-sustained structures can be given in terms of the spectrum obtained by Fourier transforming the time wave form of the amplitude at a fixed spatial position [Fig. 3(a)]. In the $\Omega = 1/t$ frequency domain, the dynamics-sustained pattern spectrum shows a series of well defined lines at frequencies Ω_i which correspond (through the relation $t = x/2\alpha_0$) to the most unstable spatial mode $q_m^2 = 2|A_0|^2 - \Delta$ and its harmonics. By contrast the noise-sustained spectral peaks are sensibly broader. Note that both results have been obtained using the same noise level. Figure 3(b) quantifies [12] this behavior by displaying the variance $\sigma_i^2 = \langle(\Omega - \Omega_i)^2\rangle/\Omega_i^2$ of the first 3 spectral lines as a function of the intensity. The intensity at which the pattern dramatically changes its nature is in very good agreement with that predicted from Fig. 1 ($|A_0|^2 \simeq 1.027$).

In general, optical noise-sustained structures should be experimentally observed for pump intensities at the onset of the modulation instability. As a particular example the results of Fig. 2 may correspond, according to the definition of the coefficients and variables of Eq. (1) [15], to a FP cavity, with mirror transmittivity $T = 0.06$ and filled

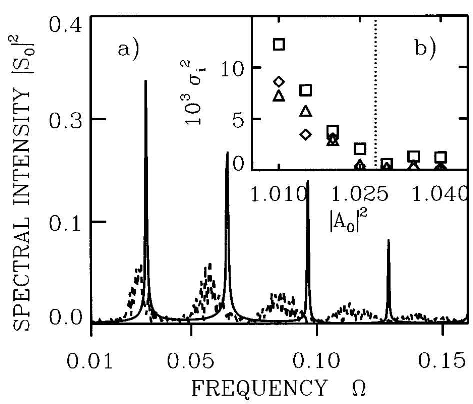

FIG. 3. (a) Spectral intensity of the field amplitude at a fixed spatial position ($x_f = -132.5$) for $|A_0|^2 = 1.035$ (solid curve) and $|A_0|^2 = 1.01$ (dashed curve). The frequencies of the maxima of the first harmonic are $\Omega_{1.035} = 0.032$ (solid curve) and $\Omega_{1.01} = 0.03$ (dashed curve) in good agreement with the predicted values $\Omega_{1.035} = 0.029$ and $\Omega_{1.01} = 0.027$ [$\Omega_I = 2\alpha_0 q_m/(2\pi)$ with $q_m^2 = 2I - \Delta$]. Noise level was the same as for Fig. 2(c). (b) Variance σ_i^2 of the first (squares), second (triangles), and third (diamonds) harmonic. The dotted vertical line indicates the threshold predicted by Fig. 1. As in Fig. 2, $\alpha_0 = 0.15$.

with a $L = 1$ mm thick plate of semiconductor $CdS_{0.5}Se_{0.5}$ (nonlinearity $n_2 = 2.9 \times 10^{-17}$ $m^2 W^{-1}$, refractive index $n_0 = 2.45$). The pump source, lasing at the wavelength of $\lambda = 1.064$ μm ($\lambda_0 = \lambda/n_0 = 0.434$ μm in the medium), should provide a beam with $w = 2x_0\sqrt{L\lambda_0/(2\pi T)} =$ 1.7 cm spot size, an intensity (at threshold) of about $I = |A|^2 = T^2 n_0 \lambda_0/(12\pi L n_2) = 0.35$ MW cm^{-2} at an angle of incidence $\theta = \arcsin[\alpha_0\sqrt{T\lambda_0/(2\pi L)}] = 0.3$ mrad. It is worth stressing that the noise level in the numerical results is 5 orders of magnitude below the intensity level of the noise-sustained pattern. Noise level can be taken, as in fluid dynamics experiments, as the parameter to be determined from comparison of theory and experiment.

In conclusion, we have shown the existence of optical noise-sustained spatial structures in the field intensity. Their growth is induced by the convective instability. A drift, or convective, term arises naturally in the governing equation of a passive optical cavity when there is a tilt of the input pump beam. Though small, this new term causes the pattern to drift outside the pump beam, below the absolute instability threshold, unless noise is present. In this case the structure is locally sustained because, although advected away, it is continuously regenerated by the noise. We have presented the features which distinguish noise-sustained from dynamics-sustained structures.

We finally point out that noise-sustained optical structures should be observable in different systems. Actually, in previous experiments on a drift instability [16,17] the observation of “noisy patterns” has been reported. Despite the fact that the experimental systems are not exactly described by the model presented here, the observed patterns may very well be noise-sustained ones. Moreover, a term proportional to $\partial_x A$ is found in the equations modeling passive cavities filled with nonlinear quadratic media, because of the different group velocities of the fundamental and second harmonic caused by the intrinsic birefringence of the nonlinear crystal. We thus predict that the effect should be observed in optical parametric oscillators too; this subject is now under investigation and will be reported elsewhere.

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