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NOMALIZERS OF NONNORMAL SUBGROUPS OF FINITE p-GROUPS

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Abstract. Assume G is a finite p-group and i is a fixed positive integer. In this paper, finite p-groups G with $|N_G(H):H| = p^i$ for all nonnormal subgroups H are classified up to isomorphism. As a corollary, this answers Problem 116(i) proposed by Y. Berkovich in his book "Groups of Prime Power Order Vol. I" in 2008.

1. Introduction

Assume G is a group and H is a subgroup of G. A simple fact is that $H \lhd G$ if and only if $N_G(H) = G$. H is called *self-normalizing* if $N_G(H) = H$; H is called *an abnormal subgroup* if $g \in \langle H, H^g\rangle$ for all $g \in G$. R. W. Carter [3] proved an abnormal subgroup must be a self-normalizing. Obviously, the concept of abnormal subgroups (self-normalizing) is an extreme case of normal subgroups. A. Fattahi [4] determined finite groups with normal and abnormal subgroups (self-normalizing). Since then, Zhang [11, 12, 13, 14] replaced the condition "normal" in [4] by quasinormal, s-quasinormal, seminormal and s-seminormal, respectively, and determined finite groups with quasinormal (s-quasinormal, seminormal and s-seminormal, respectively) and abnormal subgroups (self-normalizing).

It is natural to ask that if the condition "self-normalizing" in [4] is replaced by "$|N_G(H):H| = p_1p_2\cdots p_s$", where p_i is a prime and s is a positive integer, then what can be said about finite groups G with $|N_G(H):H| = p_1p_2\cdots p_s$ for nonnormal subgroups H? It turned out that such groups must be groups of prime power order, i.e., finite p-groups. In this paper, we classified finite p-groups G with $|N_G(H):H| = p^i$ for nonnormal subgroups H, where i is a fixed positive integer. As a corollary, this answers Problem 116(i) proposed by Y. Berkovich in his book "Groups of Prime Power Order Vol. I" in 2008.

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Problem 116(i). Classify the p-groups such that $|N_G(H) : H| = p$ for all nonnormal subgroups $H < G$.

For convenience, we introduce the following symbols.

$\mathcal{S}_1 = \{G \mid G$ with $|N_G(H) : H| = p$ for nonnormal subgroups H of $G\}$;

$\mathcal{S}_2 = \{G \mid G$ with $|N_G(H) : H| = p^2$ for nonnormal subgroups H of $G\}$;

$\mathcal{S}_3 = \{G \mid G$ with $|N_G(H) : H| = p^i$ for nonnormal subgroups H of G, $i \geq 3\}$.

G_n denotes the nth term of the lower central series of a groups G. $M \lessdot G$ denotes M is a maximal subgroup of a group G. In this paper G denotes a finite p-group.

Let G be a finite p-group. For a positive integer i, we define $\Omega_i(G) = \langle a \in G \mid a^{p^i} = 1\rangle$, and $\mho_i(G) = \langle a^{p^i} \mid a \in G\rangle$.

2. Preliminaries

Definition. Assume G is a finite nonabelian group. G is called minimal nonabelian if every proper subgroup of G is abelian; G is said to be a meta-Hamilton group if every proper subgroup of G is abelian or normal. A subgroup H of a group G is called fully-normal if $K \trianglelefteq G$ provided $K \leq H$.

Definition. Assume A and B are subgroups of a group G. If $G = AB$ and $[A, B] = 1$, then G is called a central product of A and B, denoted by $A * B$.

Definition. Assume that $\mathcal{P}$ is a group theoretic property. $\mathcal{P}$ is called inheritable by subgroups if a group G is a $\mathcal{P}$-group, then every subgroup H of G is also a $\mathcal{P}$-group; $\mathcal{P}$ is called inheritable by quotient groups if a group G is a $\mathcal{P}$-group, then every quotient group G/N is also a $\mathcal{P}$-group.

Definition. Assume G is a group of order p^n, $n \geq 2$. G is called a group of maximal class if $c(G) = n - 1$; G is called metaabelian if $G'' = 1$; G is called metacyclic if G has a cyclic normal subgroup N such that G/N is cyclic; G is called p^s-abelian if $(ab)^{p^s} = a^{p^s} b^{p^s}$ for any $a, b \in G$, where s is a positive integer.

Lemma 2.1 ([5, p. 361, 14.2 Hilfssatz])**.** *Assume G is a group of order p^n of maximal class. Then*

(1) $|G/G'| = p^2$, $G' = \Phi(G)$ *and* $d(G) = 2$;
(2) $|G_i/G_{i+1}| = p$, $i = 2, 3, \ldots, n-1$;
(3) *for* $i \geq 2$, G_i *is the unique normal subgroup of order* p^{n-i} *of* G;
(4) *if* $N \trianglelefteq G$, $|G/N| \geq p^2$, *then* G/N *is also a* p*-group of maximal class*;
(5) *for* $0 \leq i \leq n-1$, $Z_i(G) = G_{n-i}$;
(6) *assume* $p > 2$. *If* $n > 3$, *then there does not exist any cyclic normal subgroup of order* p^2.

Lemma 2.2 ([8])**.** *Assume G is a minimal nonabelian p-group. Then G is one of the following groups*:

(1) Q_8;

(2) $M_p(n,m) = \langle a,b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle, n \geq 2, m \geq 1$; (*metacyclic*)

(3) $M_p(n,m,1) = \langle a,b,c \mid a^{p^n} = b^{p^m} = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1 \rangle, n \geq m$. *If* $p = 2$, $m + n \geq 3$ (*non-metacyclic*).

Lemma 2.3 ([6])**.** *Assume G is a finite p-group. If $G/N \cong M_p(n,m)$, where $N \leq Z(G)$ and $|N| = p$, then G is one of the following mutually non-isomorphic groups*:

I. $|G'| = p$

(1) *minimal nonabelian p-groups*;

(2) *direct product of a minimal nonabelian p-group and C_p*;

II. $|G'| = p^2$

$c(G) = 2$

(1) $\langle a,b \mid a^{p^{n+1}} = b^{p^m} = 1, [a,b] = a^{p^{n-1}} \rangle, n \geq 3, m \geq 2$;

(2) $\langle a,b \mid a^{p^{n+1}} = 1, b^{p^m} = a^{p^n}, [a,b] = a^{p^{n-1}} \rangle, m > n \geq 3$;

$c(G) = 3$

(3) $\langle a,b \mid a^8 = b^{2^m} = 1, [a,b] = a^2 \rangle$;

(4) $\langle a,b \mid a^8 = b^{2^m} = 1, [a,b] = a^{-2} \rangle$;

(5) $\langle a,b \mid a^8 = 1, b^{2^m} = a^4, [a,b] = a^{-2} \rangle$;

(6) $\langle a,b \mid a^{p^3} = b^{p^m} = 1, [a,b] = a^p \rangle, p \geq 3, m \geq 2$;

(7) $\langle a,b \mid a^{p^3} = 1, b^{p^m} = a^{p^2}, [a,b] = a^p \rangle, p \geq 3, m \geq 3$.

Lemma 2.4 ([10])**.** $Q_8 * Q_8 \cong D_8 * D_8$.

Lemma 2.5 ([10, p. 51, 2.5.5])**.** *Assume G is a nonabelian 2-group. If $|G : G'| = 4$, then G is of maximal class.*

Lemma 2.6 ([9])**.** *Assume G is a metaabelian group, $a, b \in G$ and m, n are positive integers. Then*

$$[a^m, b^n] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]^{\binom{m}{i}\binom{n}{j}},$$

$$(ab^{-1})^m = a^m \left(\prod_{i+j \leq m} [ia, jb]^{\binom{m}{i+j}} \right) b^{-m},$$

where i, j are integers and satisfy $i + j \leq m$.

Lemma 2.7 ([2, p. 73, Lemma 4.2])**.** *Assume G is a p-group and $|G'| = p$. Then $G = (A_1 * A_2 * \cdots * A_s)Z(G)$, where $A_1, A_2, \ldots, A_s$ are minimal nonabelian p-groups.*

Lemma 2.8 ([7, p. 370, Lemma 3.2])**.** *Assume G is a finite p-group of order $\geq p^3$, $m > 1$ and $p > 2$. Then all nonnormal subgroups of G are of order p^m if and only if $G \cong M_p(n,m)$, where $m \leq n$.*

Lemma 2.9 ([7])**.** *Assume G is a finite non-Dedekind p-group. Then all non-normal subgroups of G are of order p if and only if G is one of the following groups*:

(1) $M_p(m, 1)$;
(2) $M_p(1, 1, 1) * C_{p^n}$;
(3) $D_8 * Q_8$.

Lemma 2.10 ([15, Lemma 2.4])**.** *Assume E is a minimal nonabelian subgroup of a finite p-group. If $[G, E] = E'$, then $G = E * C_G(E)$.*

Lemma 2.11 ([15, Proposition 2.5])**.** *Assume G is a finite p-group and $|G'| = p$. If $H \leq G$ and $H \nleq Z(G)$, then $H \trianglelefteq G$ if and only if $G' \leq H$.*

Lemma 2.12. *Assume G is a finite non-Dedekind group. If every nonnormal subgroup H of G satisfies $|N_G(H) : H| = p_1 p_2 \cdots p_s$, where p_i is a prime and s is a positive integer, then $p_1 = p_2 = \cdots = p_s = p$. That is, G is a p-group.*

Proof. By hypothesis we have G is nilpotent. Since G is non-Dedekind, there exists $P_i \in \mathrm{Syl}_{p_i}(G)$ such that P_i is non-Dedekind. Assume $G = P_i \times K$. Since P_i is non-Dedekind, there exists $H < P_i$ and $H \ntrianglelefteq P_i$. Moreover, $H \ntrianglelefteq G$. Since $(|H|, |K|) = 1$, $N_G(H) = N_{P_i}(H) \times K$. It follows that $|N_G(H) : H| = |N_{P_i}(H) : H||K| = {p_i}^{x_i}|K|$. Since $H \times K \ntrianglelefteq G$, $N_G(H \times K) = N_G(H)$. It follows that $|N_G(H \times K) : H \times K| = |N_{P_i}(H) : H| = {p_i}^{x_i}$. By hypothesis again, we have $K = 1$. Thus $G = P_i$. Assume $p = p_i$. Then G is a p-group. □

Lemma 2.13. (1) *If $G \in \mathcal{S}_1$, then $H \in \mathcal{S}_1$ for $H \leq G$.*
(2) *If $G \in \mathcal{S}_i$, then $G/N \in \mathcal{S}_i$ for $N \trianglelefteq G$, where $i \geq 1$*

Proof. (1) Assume $H \leq G$ and $K \leq H$. If $K \ntrianglelefteq H$, then $K \ntrianglelefteq G$. By hypothesis, $|N_G(K) : K| = p$. Since $|N_H(K) : K| \leq |N_G(K) : K|$ and $K < N_H(K)$, $|N_H(K) : K| = p$. So $H \in \mathcal{S}_1$.

(2) Assume $N \trianglelefteq G$ and $H/N \ntrianglelefteq G/N$. Then $H \ntrianglelefteq G$. It follows that $|N_{G/N}(H/N) : H/N| = |N_G(H)/N : H/N| = |N_G(H) : H| = p^i$. Thus $G/N \in \mathcal{S}_i$. □

3. Classifying $\mathcal{S}_1$

Lemma 3.1. *A finite group G is a Dedekind group if and only if all cyclic subgroups of G are normal.*

Lemma 3.2. *A subgroup N of a finite group G is fully-normal if and only if all cyclic subgroups of N are normal in G.*

Lemma 3.3. *Assume G is a finite p-group. If $|G| \leq p^3$, then $G \in \mathcal{S}_1$.*

Lemma 3.4. *Assume G is a minimal nonabelian p-group and $|G| \geq p^4$. If $G \in \mathcal{S}_1$, then $G \cong M_p(2, 2)$.*

Proof. By Lemma 2.2, $G \cong M_p(n, m, 1)$ or $G \cong M_p(n, m)$. If $G \cong M_p(n, m, 1)$, then by hypothesis we have $n + m + 1 \geq 4$. From $n \geq m$ we get $n \geq 2$. Let $G = \langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle$. Obviously, $\langle b\rangle \ntrianglelefteq G$. On the other hand, $N_G(\langle b\rangle) \geq \langle a^p\rangle \times \langle c\rangle \times \langle b\rangle$. Thus $|N_G(\langle b\rangle) : \langle b\rangle| \geq p^n \geq p^2$, a contradiction. Assume $G \cong M_p(n, m)$. If $n \geq 3$, then let $G = \langle a, b \mid a^{p^n} = b^{p^m} = 1, [a, b] = a^{p^{n-1}}\rangle$. Obviously, $\langle b\rangle \ntrianglelefteq G$. On the other hand, $N_G(\langle b\rangle) \geq \langle a^p\rangle \times \langle b\rangle$. Thus $|N_G(\langle b\rangle) : \langle b\rangle| \geq p^{n-1} \geq p^2$, a contradiction. So $n = 2$. If $m \geq 3$, then let $G = \langle a, b \mid a^{p^2} = b^{p^m} = 1, [a, b] = a^p\rangle$, where $\langle ab^{p^{m-2}}\rangle \ntrianglelefteq G$, $o(\langle ab^{p^{m-2}}\rangle) = p^2$. But $N_G(\langle ab^{p^{m-2}}\rangle) = \langle a\rangle \times \langle b^p\rangle$. So $|N_G(\langle ab^{p^{m-2}}\rangle) : \langle ab^{p^{m-2}}\rangle| \geq p^{m-1} \geq p^2$, a contradiction. So $m = 2$. It follows that $G \cong M_p(2, 2)$. □

Theorem 3.5. *Assume G is a finite p-group, $p > 2$ and $|G| \geq p^4$. Then $G \in \mathcal{S}_1$ if and only if G is abelian or $G \cong M_p(2, 2)$.*

Proof. $\Longleftarrow$: If G is abelian, the conclusion is true. Assume $G \cong M_p(2, 2)$. Since $\Omega_1(G) = Z(G)$, all subgroups of order p are normal. Obviously, all subgroups of order p^3 are normal. So if $H \ntrianglelefteq G$, then $|H| = p^2$. It follows that $H < N_G(H) < G$. Thus $|N_G(H) : H| = p$. That is, $G \in \mathcal{S}_1$.

$\Longrightarrow$: We use induction on $|G|$. If $|G| = p^4$ and $G \in \mathcal{S}_1$, then we can prove G is abelian or $G \cong M_p(2, 2)$. The conclusion is true. Assume the conclusion is true for groups of order $< |G|$. Since G is a p-group, there exists $N \leq G' \cap Z(G)$ and $|N| = p$. By Lemma 2.13 and $|G/N| < |G|$, we have, by induction hypothesis, G/N is abelian or $G/N \cong M_p(2, 2)$.

If $G/N \cong M_p(2, 2)$, then $|(G/N)'| = |G'N/N| = |G'/G' \cap N| = |G'/N| = p$. Thus $|G'| = p^2$. By Lemma 2.3, $G \cong \langle a, b \mid a^{p^3} = b^{p^2} = 1, [a, b] = a^p\rangle$. Thus $\langle b^p\rangle \ntrianglelefteq G$, $|\langle b^p\rangle| = p$. But $N_G(\langle b^p\rangle) = \langle a^p, b\rangle \cong M_p(2, 2)$. Thus $|N_G(\langle b^p\rangle) : \langle b^p\rangle| = p^3$, a contradiction.

If G/N is abelian, then G is abelian or nonabelian. If G is nonabelian, then $|(G/N)'| = |G'N/N| = |G'/G' \cap N| = |G'/N| = 1$. So $|G'| = p$. By Lemma 2.7, $G \cong A_1 * A_2 * \cdots * A_s Z(G)$, where A_i is minimal nonabelian. Assume $G = A_1 * K$ without loss of generality. If $K \nleq A_1$, then there exists $g \in K \backslash A_1$ such that $N_G(H) \geq \langle N_{A_1}(H), g\rangle$ for any $H \ntrianglelefteq A_1$. Thus $|N_G(H) : H| \geq p^2$, a contradiction. It follows that $K \leq A_1$. That is, $G = A_1$. By Lemma 3.4 we get $G \cong M_p(2, 2)$. □

Lemma 3.6. *Assume G is a 2-group of maximal class. Then*
(i) $G/Z(G) \cong D_{2^{n-1}}$;
(ii) *every maximal subgroup of G is cyclic or of maximal class.*

Proof. By [5, Chapter III, 11.9b Satz], G is isomorphic to one of the following: D_{2^n}, Q_{2^n} or SD_{2^n}. It is straightforward by a simple calculation. □

Theorem 3.7. *Assume G is a group of order 2^n. Then $G \in \mathcal{S}_1$ if and only if G is one of the following mutually non-isomorphism groups*

I. *Dedekind 2-groups*;
II. 2*-groups of maximal class*;
III. $\langle a, b \mid a^{2^{n-2}} = b^4 = 1, [a, b] = a^{-2}\rangle$;
IV. $\langle a, b \mid a^{2^{n-2}} = b^4 = 1, [a, b] = a^{-2+2^{n-3}}\rangle$.

Proof. $\Longleftarrow$: If G is a 2-group of maximal class, we prove $G \in \mathcal{S}_1$.

Assume $H \ntrianglelefteq G$. If $|G : H| = 2^2$, then $|N_G(H) : H| = 2$. Assume $|G : H| \geq 2^3$ without loss of generality.

Assume G is a counterexample of the smallest order. Then there exists $H \ntrianglelefteq G$ such that $|N_G(H) : H| > 2$. Since $N_G(H) < G$, there exists $M \lessdot G$ such that $N_G(H) \leq M$. By Lemma 3.6 we get M is cyclic or a 2-group of maximal class. If M is cyclic, then $H \trianglelefteq G$, a contradiction. So M is a 2-group of maximal class. If $H \trianglelefteq M$, then $|M : H| \geq 2^2$ by $|G : H| \geq 2^3$. By Lemma 2.1 there exists i such that $H = M_i$. Thus $H \mathrm{char} M \trianglelefteq G$. It follows that $H \trianglelefteq G$, a contradiction. Thus $H \ntrianglelefteq M$. Since G is a counterexample of the smallest order, $|N_M(H) : H| = 2$. On the other hand, $N_M(H) = N_G(H) \cap M = N_G(H)$. So $|N_G(H) : H| = 2$, a contradiction again. So the counterexample does not exist.

If G is the group of type III, then $G/\langle b^2\rangle = \langle \overline{a}, \overline{b} \mid \overline{a}^{2^{n-2}} = \overline{b}^2 = 1, [\overline{a}, \overline{b}] = \overline{a}^{(-2)}\rangle \cong D_{2^{n-1}}$. Let $H \ntrianglelefteq G$. Then $\langle b^2\rangle \leq H$. If not, let $K = \langle a, b^2\rangle$. Then $H \leq K$. In fact, $H \leq K \Longleftrightarrow G - H \supseteq G - K \Longleftrightarrow \forall g \in G - K \Longrightarrow g \in G - H \Longleftrightarrow \forall g \in G - K \Longrightarrow g \notin H$. Since every element of G has the form $b^j a^i$, $j = 0, 1, 2, 3$, we need to prove $ba^i, b^{-1}a^i \notin H$. By calculation, we have $(ba^i)^2 = ba^iba^i = b^2(a^b)^i a^i = b^2$ for any i. So $ba^i \notin H$. In the same way, $(b^{-1}a^i)^2 = b^{-1}a^ib^{-1}a^i = b^3a^ib^3a^i = ba^iba^i = b^2$. So $b^{-1}a^i \notin H$. Thus $H \leq K = \langle a, b^2\rangle$. Moreover, $\langle a, b^2\rangle$ is a fully-normal subgroup of G. In fact, let $g = a^ib^{2j}$ for any $g \in \langle a, b^2\rangle$. Then $g^a = (a^ib^{2j})^a = g$, $g^b = (a^ib^{2j})^b = (a^b)^ib^{2j} = a^{-i}b^{2j} = g^{-1}$. So $\langle g\rangle \trianglelefteq G$. By Lemma 3.2, $\langle a, b^2\rangle$ is a fully-normal subgroup of G. It follows that $H \trianglelefteq G$, a contradiction. It follows that $\langle b^2\rangle \leq H$.

Since $H \ntrianglelefteq G$, $\overline{H} \ntrianglelefteq \overline{G}$. Since $\overline{G}$ is of maximal class, $|N_{\overline{G}}(\overline{H}) : \overline{H}| = 2$. Thus $|N_G(H) : H| = |N_{\overline{G}}(\overline{H}) : \overline{H}| = 2$. That is, the group of type III is in $\mathcal{S}_1$.

If G is the group of type IV, considering $G/\langle b^2\rangle$ and $G/\langle a^{2^{n-3}}b^2\rangle$, then

$$\begin{aligned} G/\langle b^2\rangle &= \langle \overline{a}, \overline{b} \mid \overline{a}^{2^{n-2}} = \overline{b}^2 = 1, [\overline{a}, \overline{b}] = \overline{a}^{(-2+2^{n-3})}\rangle \\ &\cong SD_{2^{n-1}}, \\ G/\langle a^{2^{n-3}}b^2\rangle &= \langle \overline{a}, \overline{b} \mid \overline{a}^{2^{n-2}} = 1, \overline{b}^2 = \overline{a}^{2^{n-3}}, [\overline{a}, \overline{b}] = \overline{a}^{(-2+2^{n-3})}\rangle \\ &\cong SD_{2^{n-1}}. \end{aligned}$$

Let $H \ntrianglelefteq G$. We prove $\langle b^2\rangle \leq H$ or $\langle a^{2^{n-3}}b^2\rangle \leq H$ as follows. If not, then, letting $K = \langle a, b^2\rangle$, we can prove $H \leq K$. In fact, $H \leq K \Longleftrightarrow G - H \supseteq G - K \Longleftrightarrow \forall g \in G - K \Longrightarrow g \in G - H \Longleftrightarrow \forall g \in G - K \Longrightarrow g \notin H$. Since every element of G has the form b^ja^i, $j = 0, 1, 2, 3$, we need to prove $ba^i, b^{-1}a^i \notin H$. By calculation, we have $(ba^i)^2 = ba^iba^i = b^2(a^b)^ia^i = b^2a^{(-1+2^{n-3})i}a^i =$

$b^2a^{2^{n-3}i}$ for any i. If $(i,2)=1$, then $(ba^i)^2 = b^2a^{2^{n-3}}$; if $2 \mid i$, then $(ba^i)^2 = b^2$. So $ba^i \notin H$. In the same way, $(b^{-1}a^i)^2 = b^{-1}a^ib^{-1}a^i = b^3a^ib^3a^i = ba^iba^i$. So $b^{-1}a^i \notin H$. Thus $H \le K = \langle a, b^2\rangle$. Moreover, $\langle a, b^2\rangle$ is a fully-normal subgroup of G. In fact, let $g = a^ib^{2j}$ for any $g \in \langle a, b^2\rangle$. Then $g^a = (a^ib^{2j})^a = g$, $g^b = (a^ib^{2j})^b = (a^b)^ib^{2j} = a^{-i+2^{n-3}i}b^{2j}$. If $(i,2)=1$, $g^b = a^{-i+2^{n-3}}b^{2j} = g^{-1+2^{n-3}}$; if $2 \mid i$, $g^b = a^{-i}b^{2j} = g^{-1}$. Thus $\langle g\rangle \trianglelefteq G$. By Theorem 3.2, we get $\langle a, b^2\rangle$ is a fully-normal subgroup of G. Thus $H \trianglelefteq G$, a contradiction. It follows that $\langle b^2\rangle \le H$ or $\langle a^{2^{n-3}}b^2\rangle \le H$.

Since $H \ntrianglelefteq G$, $\overline{H} \ntrianglelefteq \overline{G}$. Since $\overline{G}$ is of maximal class, $|N_{\overline{G}}(\overline{H}) : \overline{H}| = 2$. Thus $|N_G(H) : H| = |N_{\overline{G}}(\overline{H}) : \overline{H}| = 2$. That is, the group of type IV is in $\mathcal{S}_1$.

$\Longrightarrow$: Case 1: $d(G) \ge 3$.

Assume $G \in \mathcal{S}_1$ and G is nonabelian. We prove G is a nonabelian Dedekind 2-group as follows.

Assume G is a counterexample of the smallest order. Since $\Phi(G) \ne 1$, we can take $N \le \Phi(G)$ such that $|N| = 2$ and $N \trianglelefteq G$. Thus $d(G/N) = d(G) \ge 3$. By Lemma 2.13, G/N is a nonabelian Dedekind 2-group. Since G is a counterexample of the smallest order, by Lemma 3.1 there exists $a \in G$ such that $\langle a\rangle \ntrianglelefteq G$. So $N \cap \langle a\rangle = 1$. Since $N_G(\langle a\rangle) \ge \langle N, a\rangle = N \times \langle a\rangle$, $|N_G(\langle a\rangle) : \langle a\rangle| = 2$ by hypothesis. So $N_G(\langle a\rangle) = N \times \langle a\rangle$. Since G/N is a Dedekind 2-group, $(N \times \langle a\rangle)/\langle a\rangle \trianglelefteq G/N$. Thus $N \times \langle a\rangle \trianglelefteq G$.

We calculate $|\{\langle a\rangle^g \mid g \in G\}|$ as follows. First, we prove $|\{\langle a\rangle^g \mid g \in G\}| = |G : N_G(\langle a\rangle)| \ge 2^2$. Since $d(G) \ge 3$, $|G/\langle a, \Phi(G)\rangle| \ge 2^2$. Let $\overline{G} = G/N_G(\langle a\rangle)$. Then $\Phi(\overline{G}) = \Phi(G/N_G(\langle a\rangle)) = (\Phi(G)N_G(\langle a\rangle))/N_G(\langle a\rangle)$, and $\Phi(G)N_G(\langle a\rangle) = \Phi(G)(N \times \langle a\rangle) = \Phi(G)\langle a\rangle = \langle a, \Phi(G)\rangle$. So $\overline{G}/\Phi(\overline{G}) \cong G/(\langle a, \Phi(G)\rangle)$. Since $|G : (\langle a, \Phi(G)\rangle)| \ge 2^2$, $|\overline{G}/\Phi(\overline{G})| \ge 2^2$. That is, $d(\overline{G}) \ge 2$. Therefore $|\overline{G}| \ge 2^2$, that is, $|G : N_G(\langle a\rangle)| \ge 2^2$.

On the other hand, since $\langle a\rangle \le N \times \langle a\rangle$, $\langle a\rangle^g \le (N \times \langle a\rangle)^g = N \times \langle a\rangle$. So $\langle a\rangle^g \lessdot N \times \langle a\rangle$. Since $d(N \times \langle a\rangle) = 2$, $N \times \langle a\rangle$ has three maximal subgroups. It follows that $|\{\langle a\rangle^g \mid g \in G\}| \le 2$, a contradiction. So the counterexample does not exist.

Case 2: $d(G) \le 2$.

We use induction on $|G|$. If $|G| = 2^4$ and $G \in \mathcal{S}_1$, then we can prove G is a Dedekind group, a group of order 2^4 of maximal class 2^4 or $G \cong M_2(2,2)$. The conclusion is true. Assume the conclusion is true for groups of order $< |G|$. Since G is a 2-group, there exists $N \le G' \cap Z(G)$ and $|N| = 2$. Since the condition is inheritable by quotient groups and $|G/N| < |G|$, G/N is one of the groups listed in theorem by induction hypothesis.

If G/N is abelian, then, in the same way as that in the case $p > 2$, we have G is abelian or $G \cong Q_8, D_8$ or $M_2(2,2)$.

If G/N is a 2-groups of maximal class, then $|(G/N)'| = 2^{n-2}$ by Lemma 2.1. Since $(G/N)' = G'N/N \cong G'/G' \cap N = G'/N$, $|G'/N| = 2^{n-2}$. Thus $|G'| = 2^{n-1}$. It follows that $|G/G'| = 4$. By Lemma 2.5 we get G is a 2-group of maximal class.

If G/N is the group of type III. That is, $\overline{G} \cong \langle \overline{a}, \overline{b} \mid \overline{a}^{2^{n-2}} = \overline{b}^4 = 1, [\overline{a}, \overline{b}] = \overline{a}^{(-2)} \rangle$. Assume $N = \langle x \rangle$. Then $G = \langle a, b \mid a^{2^{n-2}} = x^i, b^4 = x^j, [a, b] = a^{-2}x^k, x^2 = 1, [x, a] = [x, b] = 1 \rangle$. Let $K = \langle a^{-2}x^k \rangle$. It is easy to prove G/K is abelian. Thus $G' \le K$. But $K \le G'$. So $G' = K$. That is, G' is cyclic. Since $|G'| = 2^{n-2}$, it follows by $[a, b]^{2^{n-2}} = 1$ that $o(a) = 2^{n-1}$ and $N = \langle a^{2^{n-2}} \rangle$. Thus we get the following groups:

(a1) $\langle a, b \mid a^{2^{n-1}} = 1, b^4 = 1, [a, b] = a^{-2} \rangle$;
(a2) $\langle a, b \mid a^{2^{n-1}} = 1, b^4 = 1, [a, b] = a^{-2+2^{n-2}} \rangle$;
(a3) $\langle a, b \mid a^{2^{n-1}} = 1, b^4 = a^{2^{n-2}}, [a, b] = a^{-2} \rangle$;
(a4) $\langle a, b \mid a^{2^{n-1}} = 1, b^4 = a^{2^{n-2}}, [a, b] = a^{-2+2^{n-2}} \rangle$;

Obviously, (a1) $\cong$ the group of type III; (a2) $\cong$ the group of type IV.

For (a3), let $H = \langle a^{2^{n-3}} b^2 \rangle$. Then $|H| = 2$ and $H \ntrianglelefteq G$, $N_G(H) = \langle a, b^2 \rangle \lessdot G$. Thus $|N_G(H) : H| \ge 2^3$, a contradiction. For (a4), let $a' = ab^2, b' = b$. Then (a4) $\cong$ (a3).

If $G/N \cong$ the group of type IV, then, by a similar argument as that case of above paragraph, no new groups occur. The theorem is proved. □

4. Classifying $\mathcal{S}_2$

Lemma 4.1. *Assume G is a p-group. If $G \in \mathcal{S}_2$ and $|G : H| \le p^2$ for $H \le G$, then $H \trianglelefteq G$. In particular, if G is nonabelian, then $G \cong Q_8$ or $|G| \ge p^4$.*

Proof. Assume $|G| = p^n$ and $|G : H| = p^2$. If $H \ntrianglelefteq G$, then $H < N_G(H) < G$. Thus $|N_G(H)| = p^{n-1}$. It follows that $|N_G(H) : H| = p$, a contradiction. So $H \trianglelefteq G$. In particular, if G is nonabelian and $|G| = p^3$, then $G \cong Q_8$. □

Lemma 4.2. *Assume G is a minimal nonabelian p-group and $|G| \ge p^4$. Then $G \in \mathcal{S}_2$ if and only if $G \cong M_p(3, m)$, where $m \le 3$.*

Proof. $\Longrightarrow$: By Lemma 2.2, $G \cong M_p(n, m, 1)$ or $G \cong M_p(n, m)$.

If $G \cong M_p(n, m) = \langle a, b \mid a^{p^n} = 1, b^{p^m} = 1, [a, b] = a^{p^{n-1}} \rangle$, then, if $n \ge 4$, letting $H = \langle b \rangle$, we get $H \ntrianglelefteq G$. But $N_G(H) \ge \langle a^p \rangle \times \langle b \rangle$. Thus $|N_G(H) : H| \ge p^{n-1} \ge p^3$, a contradiction. If $n = 2$, letting $H = \langle b \rangle$, we get $H \ntrianglelefteq G$. But $|G : H| = p^2$. This contradicts Lemma 4.1. If $G \cong M_p(3, m)$ and $m \ge 4$, letting $H = \langle ab^{p^{m-3}} \rangle$, we get $H \ntrianglelefteq G$. But $N_G(H) \ge \langle a \rangle \times \langle b^p \rangle$. Thus $|N_G(H) : H| \ge p^{m-1} \ge p^3$, a contradiction. Thus $G \cong M_p(3, m)$, where $m \le 3$.

If $G \cong M_p(n, m, 1)$, then, by a similar argument as that case of above paragraph, the case does not occur.

$\Longleftarrow$: We check case by case.

If $G \cong M_p(3, 3)$, then $Z(G) = \langle a^p \rangle \times \langle b^p \rangle, G' = \langle a^{p^2} \rangle, \Omega_1(G) = \langle a^{p^2} \rangle \times \langle b^{p^2} \rangle, \Omega_2(G) = \langle a^p \rangle \times \langle b^p \rangle$. If $|H| = p^5$, then $H \lessdot G$. Thus $H \trianglelefteq G$. If $|H| = p^4$, then $|H \cap \langle a \rangle| \ne 1$. If not, $|H\langle a \rangle| > |G|$, a contradiction. It follows that $G' \le H$, and $H \trianglelefteq G$. If $|H| = p^3$ and $H \ntrianglelefteq G$, then $|H \cap Z(G)| \le p^2$, and $N_G(H) \ge \langle H, Z(G) \rangle$. Thus $|N_G(H)| \ge \frac{|H||Z(G)|}{|H \cap Z(G)|} \ge \frac{p^3 \cdot p^4}{p^2} = p^5$. It follows that

$|N_G(H)| = p^5$, and $|N_G(H) : H| = p^2$. If $|H| = p^2$, then $H \leq \Omega_2(G) \leq Z(G)$. Thus $H \trianglelefteq G$. If $|H| = p$, then $H \leq \Omega_1(G) \leq Z(G)$. Thus $H \trianglelefteq G$. So $G \in \mathcal{S}_2$.

If $G \cong M_p(3, 2)$ or $G \cong M_p(3, 1)$, then $G \in \mathcal{S}_2$ by a similar argument as that case of above paragraph. □

Lemma 4.3. *Assume G is a non-Dedekind p-group and $|G| \geq p^4$, K is a minimal nonabelian p-group. If $G \cong K \times C_p$, then $G \notin \mathcal{S}_2$.*

Proof. By Lemma 2.2, $K \cong M_p(n, m)$ or $M_p(n, m, 1)$. If $K \cong M_p(n, m, 1)$, then $G \cong K \times N$, where $N \cong C_p$, and $G/N \cong M_p(n, m, 1)$. By Lemma 2.13 and Lemma 4.2, $G \notin \mathcal{S}_2$. If $K \cong M_p(n, m)$, then $G = \langle a, b, c \mid a^{p^n} = 1, b^{p^m} = 1, c^p = 1, [a, b] = a^{p^{n-1}}, [a, c] = [b, c] = 1\rangle$. If $n \geq 3$, then, by letting $H = \langle b\rangle$, we get $H \ntrianglelefteq G$. But $N_G(H) = \langle a^p\rangle \times \langle b\rangle \times \langle c\rangle$. Thus $|N_G(H) : H| \geq p^n \geq p^3$, a contradiction. If $n = 2$, let $H \cong \langle b, c\rangle$. Then $H \ntrianglelefteq G$. But $|G : H| = p^2$. This contradicts Lemma 4.1. So $G \notin \mathcal{S}_2$. □

Theorem 4.4. *Assume G is a finite p-groups. Then $G \in \mathcal{S}_2$ if and only if G is one of the following mutually non-isomorphic groups*

(1) $M_p(3, m)$, *where* $m \leq 3$;
(2) $M_p(1, 1, 1) * C_{p^2}$;
(3) $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = 1, [a, c] = b^{kp}, [b, c] = a^p b^{hp}\rangle$, *if* $p > 2$, $k + 4^{-1}h^2$ *is a fixed quadratic non-residue* (mod p), *where* $k = 1$ *or* ν, ν *is a fixed quadratic non-residue* (mod p), $h = 0, 1, \ldots, \frac{p-1}{2}$. *If* $p = 2$, *then* $k = 1, h = 1$;
(4) $Q_8 \times C_4$;
(5) $\langle a, b, c \mid a^4 = b^4 = 1, c^2 = b^2, [a, b] = 1, [a, c] = b^2, [b, c] = a^2\rangle$;
(6) $\langle a, b, c \mid a^4 = b^4 = c^4 = 1, [a, b] = c^2, [a, c] = b^2c^2, [b, c] = a^2b^2, [c^2, a] = [c^2, b] = 1\rangle$;
(7) *Dedekind groups.*

Proof. $\Longrightarrow$: We use induction on $|G|$. If $|G| = p^4$ and $G \in \mathcal{S}_2$, then we can prove G is a Dedekind group, or $G \cong M_p(3, 1)$ or $G \cong M_p(1, 1, 1) * C_{p^2}$. The conclusion is true. Assume the conclusion is true for groups of order $< |G|$. Since G is a p-group, there exists $N \leq G' \cap Z(G)$ and $|N| = p$. Since the condition is inheritable by quotient groups and $|G/N| < |G|$, G/N is the group listed in Theorem by induction hypothesis.

Case 1: If $G/N \cong M_p(3, m) = \langle \bar{a}, \bar{b} \mid \bar{a}^{p^3} = 1, \bar{b}^{p^m} = 1, [\bar{a}, \bar{b}] = \bar{a}^{p^2}\rangle$, then, by $|(G/N)'| = |G'N/N| = |G'/G' \cap N| = |G'/N| = p$, we have $|G'| = p^2$. By Lemma 2.3, $G \cong \langle a, b \mid a^{p^4} = 1, b^{p^m} = 1, [a, b] = a^{p^2}\rangle$, where $m = 2, 3$. Let $H = \langle b^p\rangle$. Then $H \ntrianglelefteq G$. But $N_G(H) \geq \langle a^{p^2}, b\rangle$. So $|N_G(H) : H| \geq p^3$, a contradiction.

Case 2: If $G/N \cong M_p(1, 1, 1) * C_{p^2} = \langle \bar{a}, \bar{b}, \bar{c}\rangle = \langle \bar{a}^{p^2} = 1, \bar{b}^p = 1, \bar{c}^p = 1, [\bar{b}, \bar{c}] = \bar{a}^p, [\bar{a}, \bar{b}] = 1, [\bar{a}, \bar{c}] = 1\rangle$. Let $N = \langle x\rangle$. Then $G = \langle a, b, c \mid a^{p^2} = x^i, b^p = x^j, c^p = x^k, [a, b] = x^l, [a, c] = x^m, [b, c] = a^p x^n, x^p = 1, [x, a] = [x, b] = [x, c] = 1\rangle$.

It is easy to prove the following facts:

1. $o(a) = p^2$;
2. $G' \cong C_p \times C_p$ and $c(G) = 2$;
3. $\mho_1(G) \leq Z(G)$;
4. j, k are not zero in the same time.

We discuss in two cases: (i) $b^p \neq 1$ and $c^p = 1$, (ii) $b^p \neq 1$ and $c^p \neq 1$.

(i) $b^p \neq 1$ and $c^p = 1$.

Then $G = \langle a, b, c \mid a^{p^2} = 1, b^p = x^j, c^p = 1, [a, b] = x^l, [a, c] = x^m, [b, c] = a^p x^n, x^p = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. Let $H = \langle a, c\rangle$. If $[a, c] = 1$, then $H \ntrianglelefteq G$, and $|G : H| = p^2$. This contradicts Lemma 4.1. Thus $m \not\equiv 0 \pmod p$.

(ia) $p > 2$.

If $[a, b] \neq 1$, then, by letting $b_1 = bc^t$ satisfying $l+mt \equiv 0 \pmod p$, it reduces to the case $[a, b] = 1$. Assume $G = \langle a, b, c \mid a^{p^2} = 1, b^p = x, c^p = 1, [a, b] = 1, [a, c] = x^m, [b, c] = a^p x^n, x^p = 1, [x, a] = [x, b] = [x, c] = 1\rangle \cong \langle a, b, c \mid a^{p^2} = 1, b^{p^2} = 1, c^p = 1, [a, b] = 1, [a, c] = b^{mp}, [b, c] = a^p b^{np}\rangle$.

If $m \equiv s^2 \pmod p$, then, replacing a by $a^{s^{-1}}$, and c by $c^{s^{-1}}$, and letting $h = ns^{-1}$, we have $G = \langle a, b, c \mid a^{p^2} = 1, b^{p^2} = 1, c^p = 1, [a, b] = 1, [a, c] = b^p, [b, c] = a^p b^{hp}\rangle$. Replacing a by a^{-1}, and c by c^{-1}, we have $h \leq \frac{p-1}{2}$. Let $H = \langle ab^i, c\rangle$. If $hi + 1 \equiv i^2 \pmod p$, in other words, $1 + \frac{h^2}{4}$ is a quadratic residue (mod p), then $H \cong M_p(2, 1)$, and $|H| = p^3$. Obviously, $H \ntrianglelefteq G$. But $|G : H| = p^2$. This contradicts Lemma 4.1. So $1 + \frac{h^2}{4}$ is a quadratic non-residue (mod p).

If $m \equiv \nu s^2 \pmod p$, where $\nu \neq 0$, then, replacing a by $a^{s^{-1}}$, c by $c^{s^{-1}}$ and letting $h = ns^{-1}$, $G = \langle a, b, c \mid a^{p^2} = 1, b^{p^2} = 1, c^p = 1, [a, b] = 1, [a, c] = b^{\nu p}, [b, c] = a^p b^{hp}\rangle$. Again replacing a by a^{-1} and c by c^{-1}, we have $h \leq \frac{p-1}{2}$. Let $H = \langle ab^i, c\rangle$. If $hi + \nu \equiv i^2 \pmod p$, in other words, $\nu + \frac{h^2}{4}$ is a quadratic residue (mod p), then $H \cong M_p(2, 1)$, and $|H| = p^3$. Obviously, $H \ntrianglelefteq G$, But $|G : H| = p^2$. This contradicts Lemma 4.1. So $\nu + \frac{h^2}{4}$ is a quadratic non-residue (mod p).

So, if $p > 2$ and $G/N \cong M_p(1, 1, 1) * C_{p^2}$, then G is the group of type (3).

(ib) $p = 2$.

Assume $G = \langle a, b, c \mid a^4 = 1, b^2 = x, c^2 = 1, [b, c] = a^2 x^n, [a, b] = x^l, [a, c] = x, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. If $l = n$, then, by letting $H = \langle ab, c\rangle$, we have $H \ntrianglelefteq G$. But $|G : H| = p^2$. This contradicts Lemma 4.1. So $l \neq n$. Thus we get two groups:

$$\begin{aligned}
G_{(11)} &= \langle a, b, c \mid a^4 = 1, b^2 = x, c^2 = 1, [b, c] = a^2, [a, b] = x, [a, c] = x, x^2 = 1, \\
&\qquad [x, a] = [x, b] = [x, c] = 1\rangle \\
&\cong \langle a, b, c | a^4 = 1, b^4 = 1, c^2 = 1, [b, c] = a^2, [a, b] = b^2, [a, c] = b^2\rangle, \\
G_{(12)} &= \langle a_1, b_1, c_1 | {a_1}^4 = 1, {b_1}^2 = x_1, {c_1}^2 = 1, [b_1, c_1] = {a_1}^2 x_1, [a_1, b_1] = 1, \\
&\qquad [a_1, c_1] = x_1, {x_1}^2 = 1, [x_1, a_1] = [x_1, b_1] = [x_1, c_1] = 1\rangle
\end{aligned}$$

$$\cong \langle a_1, b_1, c_1 | {a_1}^4 = 1, {b_1}^4 = 1, {c_1}^2 = 1, [b_1, c_1] = {a_1}^2{b_1}^2, [a_1, b_1] = 1, [a_1, c_1] = {b_1}^2\rangle.$$

Let $\sigma : a_1 \longrightarrow a,\ b_1 \longrightarrow abc,\ c_1 \longrightarrow c$. Then $G_{(11)} \cong G_{(12)} \cong$ the group of type (3).

(ii) $b^p \neq 1$ and $c^p \neq 1$.

If $p > 2$, then, by letting $c_1 = cb^t$ satisfying $jt + k \equiv 0 \pmod p$, it reduces to the case (ia).

If $p = 2$, assume $G = \langle a, b, c \mid a^4 = 1, b^2 = x, c^2 = x, [b, c] = a^2x^n, [a, b] = x^l, [a, c] = x^m, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. Since $G' \cong C_2 \times C_2$, l, m are not zero in the same time.

(ii-1) $m = 0, l = 1$.

If $n = 0$, then $G \cong \langle a, b, c \mid a^4 = 1, b^4 = 1, c^2 = b^2, [b, c] = a^2, [a, b] = b^2, [a, c] = 1\rangle$. Let $a_1 = a, b_1 = c, c_1 = b$. Then $G \cong$ the group of type (5). If $n = 1$, then $G \cong \langle a, b, c \mid a^4 = 1, b^4 = 1, c^2 = b^2, [b, c] = a^2b^2, [a, b] = b^2, [a, c] = 1\rangle$. Let $a_1 = a, b_1 = b, c_1 = abc$. Then $G \cong G_{(11)} \cong$ the group of type (3).

(ii-2) $m = 1, l = 0$.

Then $G = \langle a, b, c \mid a^4 = 1, b^2 = x, c^2 = x, [b, c] = a^2x^n, [a, b] = 1, [a, c] = x, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. Let $a_1 = a, b_1 = c, c_1 = b$. Then it reduces to the case $(\mathrm{ii} - 1)$.

(ii-3): $m = 1, l = 1$.

Then $G = \langle a, b, c \mid a^4 = 1, b^2 = x, c^2 = x, [b, c] = a^2x^n, [a, b] = x, [a, c] = x, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. Let $a_1 = a, b_1 = b, c_1 = abc$. Then it reduces to the case (i) or $(\mathrm{ii} - 1)$.

Case 3: If $G/N \cong \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{p^2} = 1, \overline{b}^{p^2} = 1, \overline{c}^p = 1, [\overline{a}, \overline{b}] = \overline{1}, [\overline{a}, \overline{c}] = \overline{b}^{kp}, [\overline{b}, \overline{c}] = \overline{a}^p\overline{b}^{hp}\rangle$. Assume $N = \langle x\rangle$, then $G = \langle a, b, c \mid a^{p^2} = x^i, b^{p^2} = x^j, c^p = x^k, [a, b] = x^l, [a, c] = b^{kp}x^m, [b, c] = a^pb^{hp}x^n, x^p = 1, [x, a] = [x, b] = [x, c] = 1\rangle$.

First, we prove the following facts:

1. $o(a) = p^2$ and $o(b) = p^2$;
2. $G' \cong C_p^3$ and $c(G) = 2$;
3. $c^p \neq 1$.

In fact, since $c(G/N) = 2$, $(G/N)_3 = G_3N/N = 1$. Thus $G_3 \leq N \leq Z(G)$. That is, $G_4 = 1$. Since $G'' \leq G_4$, G' is abelian. Since $|(G/N)'| = p^2$, $|G'| = p^3$. It follows by $c^p \in Z(G)$ that $[a, c^p] = 1$, $[b, c^p] = 1$.

By the formula of Lemma 2.6, we have

$$\begin{aligned}[b^{kp}, c] &= [b, c]^{kp}[b, c, b]^{\binom{kp}{2}} = (a^pb^{hp}x^n)^{kp}[a^pb^{hp}x^n, b] \\ &= a^{kp^2}[a^p, b^{-hp}]b^{hkp^2}[a^pb^{hp}, b] \\ &= a^{kp^2}b^{hkp^2}[a^p, b]^{b^{hp}} = a^{kp^2}b^{hkp^2}.\end{aligned}$$

It follows that

$$1 = [a, c^p] = [a, c]^p[a, c, c]^{\binom{p}{2}} = [a, c]^p[b^{kp}, c]^{\binom{p}{2}} = b^{kp^2}(a^{kp^2}b^{hkp^2})^{\binom{p}{2}}. \tag{1}$$

$$(2)\qquad \begin{aligned} 1 = [b, c^p] &= [b, c]^p [b, c, c]^{\binom{p}{2}} = (a^p b^{hp} x^n)^p [a^p b^{hp} x^n, c]^{\binom{p}{2}} \\ &= a^{p^2} b^{hp^2} ([a^p, c]^{b^{hp}} [b^{hp}, c])^{\binom{p}{2}} = a^{p^2} b^{hp^2} (b^{kp^2} a^{hp^2 b^{h^2 p^2}})^{\binom{p}{2}}. \end{aligned}$$

If $p > 2$, then it follows from (1) and (2) that $[a, c^p] = b^{kp^2} = 1$, $[b, c^p] = a^{p^2} b^{hp^2} = 1$. Thus $o(a) = p^2$, $o(b) = p^2$.

If $p = 2$, then $h = k = 1$. It follows from (1) and (2) that $[a, c^2] = a^4 = 1$, $[b, c^2] = b^4 = 1$. Thus $o(a) = 4$, $o(b) = 4$.

Since $\overline{G}' = \overline{G'} = \langle \overline{a^p}, \overline{b^p} \rangle$, $G' = \langle a^p, b^p, x\rangle \cong {C_p}^3$. So $l \neq 0$. Since $G' \leq Z(G)$, $c(G) = 2$. Moreover, assume $c^p \neq 1$. If not, let $H = \langle c \rangle$. Then $H \ntrianglelefteq G$. But $N_G(H) \geq \langle a^p, b^p, c, x\rangle$. So $|N_G(H) : H| \geq p^3$, a contradiction.

If $p > 2$, assume $G = \langle a, b, c \mid a^{p^2} = 1, b^{p^2} = 1, c^p = x, [a, b] = x^l, [a, c] = b^{kp} x^m, [b, c] = a^p b^{hp} x^n, x^p = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. Obviously, $|G| = p^6$. Since $c(G) = 2$, G is p-abelian. It follows that $\Omega_1(G) = \langle a^p\rangle \times \langle b^p\rangle \times \langle x\rangle \leq Z(G)$. We will prove G does not satisfy the condition of theorem.

Assume $H \leq G$. If $|H| = p^3$, then H is abelian. In fact, if $\exp(H) = p$, then $H = \Omega_1(G)$. Thus H is abelian. If $\exp(H) = p^2$ and H is not abelian, then $H \cong M_p(2, 1)$. This contradicts $\Omega_1(G) \leq Z(G)$. If $|H| = p^5$, then $H \lhd G$. If $|H| = p^4$ and $H \trianglelefteq G$, then G is meta-Hamilton p-group. But by the classification of meta-Hamilton p-group [1], we know G is not a meta-Hamilton p-group. Thus there exists a nonnormal subgroup H of order p^4. It follows that $|G : H| = p^2$. But this contradicts Lemma 4.1. So G does not satisfy the condition of theorem.

If $p = 2$, assume $G = \langle a, b, c \mid a^4 = 1, b^4 = 1, c^2 = x, [a, b] = x, [a, c] = b^2 x^m, [b, c] = a^2 b^2 x^n, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$, where $m, n = 0, 1$. If $m = n$, then, by letting $H = \langle ac, b\rangle \cong M_2(2, 2)$, we have $|H| = 2^4$. Obviously, $H \ntrianglelefteq G$. But $|G : H| = 2^2$. This contradicts Lemma 4.1. If $m = 0, n = 1$, then, by letting $H = \langle ab, bc\rangle \cong M_2(2, 2)$, we have $|H| = 2^4$. Obviously, $H \ntrianglelefteq G$. But $|G : H| = p^2$. This contradicts Lemma 4.1 again. If $m = 1, n = 0$, Then $G \cong \langle a, b, c \mid a^4 = 1, b^4 = 1, c^4 = 1, [a, b] = c^2, [a, c] = b^2 c^2, [b, c] = a^2 b^2, [c^2, a] = [c^2, b] = 1\rangle \cong$ the group of type (6).

Case 4: If $G/N \cong \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^4 = 1, \overline{b}^2 = \overline{a}^2, \overline{c}^4 = 1, [\overline{a}, \overline{b}] = \overline{a}^2, [\overline{a}, \overline{c}] = \overline{1}, [\overline{b}, \overline{c}] = \overline{1}\rangle$. Assume $N = \langle x\rangle$. Then $G = \langle a, b, c \mid a^4 = x^i, b^2 = a^2 x^j, c^4 = x^k, [a, b] = a^2 x^l, [a, c] = x^m, [b, c] = x^n, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$.

It is easy to prove the following facts:

1. $o(a) = 4$;
2. $G' \cong C_2 \times C_2$ and m, n are not 0 in the same time.

If $k = 0$, then $G = \langle a, b, c \mid a^4 = 1, b^2 = a^2 x^j, c^4 = 1, [a, b] = a^2 x^l, [a, c] = x^m, [b, c] = x^n, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. Let $H = \langle a^n b^m, c\rangle \cong C_4 \times C_4$. Then $|H| = 2^4$. Obviously, $H \ntrianglelefteq G$. But $|G : H| = p^2$. This contradicts Lemma 4.1.

If $k = 1$, then $G = \langle a, b, c \mid a^4 = 1, b^2 = a^2 x^j, c^4 = x, [a, b] = a^2 x^l, [a, c] = x^m, [b, c] = x^n, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. Let $H = \langle a^n b^m\rangle \cong C_4$. Then

$|H| = 2^2$. Obviously, $H \ntrianglelefteq G$. But $N_G(H) \geq \langle a^n b^m, c\rangle \cong C_4 \times C_8$. It follows that $N_G(H) \cong C_4 \times C_8$. Thus $|N_G(H) : H| = 2^3$, a contradiction.

Case 5: If $G/N = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^4 = \bar{b}^4 = 1, \bar{c}^2 = \bar{b}^2, [\bar{a}, \bar{b}] = \bar{1}, [\bar{a}, \bar{c}] = \bar{b}^2, [\bar{b}, \bar{c}] = \bar{a}^2\rangle$. Assume $N = \langle x\rangle$. Then $G = \langle a, b, c \mid a^4 = x^i, b^4 = x^j, c^2 = b^2x^k, [a, b] = x^m, [a, c] = b^2x^n, [b, c] = a^2x^l, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$.

It is easy to prove the following facts:

1. $o(a) = 4$ and $o(b) = 4$;
2. $G' \cong C_2^3$ and $m = 1$;
3. $c(G) = 2$;
4. $\Omega_1(G) = \langle a^2, b^2, c^2\rangle \leq Z(G)$.

By the above facts we can assume $G = \langle a, b, c \mid a^4 = 1, b^4 = 1, c^2 = b^2x^k, [a, b] = x, [a, c] = b^2x^n, [b, c] = a^2x^l, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. By discussing the possible values for k, n, l, we know there exists $H \cong M_2(2, 2)$, and $H \ntrianglelefteq G$. But $|G : H| = 2^2$. This contradicts Lemma 4.1.

Case 6: If $G/N \cong \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^4 = \bar{b}^4 = \bar{c}^4 = 1, [\bar{b}, \bar{c}] = \bar{a}^2\bar{b}^2, [\bar{a}, \bar{b}] = \bar{c}^2, [\bar{a}, \bar{c}] = \bar{b}^2\bar{c}^2, [\bar{c}^2, \bar{a}] = 1, [\bar{c}^2, \bar{b}] = 1\rangle$. Assume $N = \langle x\rangle$. Then $G = \langle a, b, c \mid a^4 = x^i, b^4 = x^j, c^4 = x^k, [b, c] = a^2b^2x^l, [a, b] = c^2x^m, [a, c] = b^2c^2x^n, [c^2, a] = x^s, [c^2, b] = x^t, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$.

It is easy to prove the following facts:

1. $o(a) = 4$;
2. $o(b) = o(c) = 4$;
3. $G' = \langle a^2\rangle \times \langle b^2\rangle \times \langle c^2\rangle \times \langle x\rangle \cong C_2^4$.

By above facts we can assume $G = \langle a, b, c \mid a^4 = 1, b^4 = 1, c^4 = 1, [b, c] = a^2b^2x^l, [a, b] = c^2x^m, [a, c] = b^2c^2x^n, [c^2, a] = x^s, [c^2, b] = x^t, x^2 = 1, [x, a] = [x, b] = [x, c] = 1\rangle$. If $s = t = 0$, then, letting $H = \langle a\rangle$, we have $H \ntrianglelefteq G$. But $N_G(H) \geq \langle a, b^2, c^2, x\rangle$. It follows that $|N_G(H) : H| \geq 2^3$, a contradiction. If s and t are not zero in the same time, then, by letting $H = \langle c^2\rangle$, we have $H \ntrianglelefteq G$. But $N_G(H) \geq \langle a^2, b^2, c, x\rangle$. It follows that $|N_G(H) : H| \geq 2^3$, a contradiction.

Case 7: If G/N is abelian and G is not abelian, then $|G'| = p$. By Lemma 2.7, $G \cong A_1 * A_2 * \cdots * A_s Z(G)$. Moreover, assume $G = A_1 * KZ(G)$. If $K \neq 1$, assume $H \ntrianglelefteq A_1$, then $H \ntrianglelefteq G$. $|N_{A_1}(H) : H| \geq p$. We observed $K \leq N_G(H)$, $K \cap A_1 \leq Z(K)$ and $N_G(H) \geq N_{A_1}(H) * K$. Thus $|N_G(H) : H| \geq |N_{A_1}(H)K/H| = \frac{|N_{A_1}(H)||K|}{|N_{A_1}(H)\cap K||H|} \geq \frac{|N_{A_1}(H)||K|}{|A_1\cap K||H|} \geq \frac{|N_{A_1}(H)||K|}{|Z(K)||H|} \geq p^3$, a contradiction. Thus $K = 1$. It follows that $G = A_1Z(G)$.

If $Z(G) \leq A_1$, then $G = A_1$. Thus $G \cong$ the group of type (1) by Lemma 4.2.

If $Z(G) \nleq A_1$, then there exists $g \in Z(G)\backslash A_1$ and $g \in N_G(H)$. If $H \ntrianglelefteq A_1$ and $|N_{A_1}(H) : H| \geq p^2$, then $|N_G(H) : H| \geq p^3$. Thus $G \notin S_2$. It follows that $|N_{A_1}(H) : H| = p$. By Lemma 3.4, $A_1 \cong M_p(2, 1)$, $M_p(1, 1, 1)$ or $M_p(2, 2)$.

If $A_1 \cong M_p(2, 1)$ or $M_p(1, 1, 1)$, then $|Z(G)| \geq p^2$ since $|Z(A_1)| = p$. On the other hand, there exists $H \ntrianglelefteq A_1$ and $H \leq A_1$. Thus $p^2 = |N_G(H) : H| \geq \frac{|N_{A_1}(H)||Z(G)|}{|Z(A_1)||H|} \geq |Z(G)|$. It follows that $|Z(G)| = p^2$. If $Z(G) \cong C_p \times C_p$, then $G \cong M_p(2,1) \times C_p$ or $G \cong M_p(1,1,1) \times C_p$. By Lemma 4.3, $G \notin S_2$. If $Z(G) \cong C_{p^2}$, then $G \cong$ the group of type (2).

If $A_1 \cong M_p(2,2)$, then $|Z(G)| = p^3$ by a similar argument as above paragraph. Since $Z(A_1) \cong C_p \times C_p \le Z(G)$, $Z(G) \cong C_p \times C_p \times C_p$ or $Z(G) \cong C_{p^2} \times C_p$.

If $Z(G) \cong C_p \times C_p \times C_p$, then $G \cong M_p(2,2) \times C_p$. By Lemma 4.2, $G \notin S_2$.

If $Z(G) \cong C_{p^2} \times C_p$, then $G \cong M_p(2,2) * C_{p^2} = \langle a,b,c\rangle = \langle a^{p^2} = 1, b^{p^2} = 1, c^{p^2} = 1, [a,b] = a^p, [a,c] = [b,c] = 1, c^p = a^{ip}b^{jp}\rangle$. If $j \not\equiv 0 \pmod p$, then, by letting $b_1 = a^ib^j, a_1 = a^j$, we have $G = \langle a_1, b_1, c\rangle = \langle {a_1}^{p^2} = 1, {b_1}^{p^2} = 1, c^{p^2} = 1, [a_1,b_1] = {a_1}^p, [a_1,c] = [b_1,c] = 1, c^p = {b_1}^p\rangle$. Let $H = \langle b_1, c\rangle$. Then $H \ntrianglelefteq G$. But $|G : H| = p^2$. This contradicts Lemma 4.1. If $j \equiv 0 \pmod p$, then, letting $H = \langle ca^{-i}\rangle$. Obviously, $H \ntrianglelefteq G$. But $N_G(H) = \langle a, c, b^p\rangle$. Thus $|N_G(H) : H| = p^3$, a contradiction. That means $G \notin S_2$.

If $G/N \cong Q_8 \times C_2$, then $|G'| = 2^2$. Assume $N = \langle x\rangle$. Then $G = \langle a,b,c,x \mid a^4 = x^i, b^2 = a^2x^j, c^2 = x^k, [a,b] = a^2x^l, [a,c] = x^m, [b,c] = x^n, x^2 = 1, [x,a] = [x,b] = [x,c] = 1\rangle$.

Since $a^{-2}b^2 \in Z(G)$, $[a^{-2}b^2, b] = 1$. we get by calculation $[a^{-2}b^2, b] = [a^{-2}, b]^{b^2} = ([a,b]^{-2})^{b^2} = a^{-4} = 1$. Thus $o(a) = 4$. Since $\overline{G}' = \langle\overline{a^2}\rangle$, $G' = \langle a^2, x\rangle = \langle a^2\rangle \times \langle x\rangle$. Moreover, m, n are not zero in the same time. Since $G' \le Z(G)$, $c(G) = 2$. So we can assume $G = \langle a,b,c,x \mid a^4 = 1, b^2 = a^2x^j, c^2 = x^k, [a,b] = a^2x^l, [a,c] = x^m, [b,c] = x^n, x^2 = 1, [x,a] = [x,b] = [x,c] = 1\rangle$.

If $k = 0$, then there exists $H \cong C_4 \times C_2$. It is easy to see that $H \ntrianglelefteq G$. But $|G : H| = p^2$. This contradicts Lemma 4.1. If $k = 1$, then $G \cong$ the group of type (5) by discussing the possible values for j, l, m, n.

$\Longleftarrow$: Case 1: If $G \cong$ the group of type (1), then the conclusion is true by Lemma 4.2.

Case 2: If $G \cong$ the group of type (2), that is, $G = \langle a,b,c \mid a^{p^2} = b^p = c^p = 1, [b,c] = a^p, [a,b] = [a,c] = 1\rangle \cong M_p(1,1,1) * C_{p^2}$, then $Z(G) = \langle a\rangle, G' = \langle a^p\rangle$. If $|H| = p^3$, then $H \lessdot G$. Thus $H \trianglelefteq G$. If $|H| = p^2$, then $|H \cap \langle a\rangle| \neq 1$. It follows that $G' \le H$, so $H \trianglelefteq G$. If $|H| = p$ and $H \ntrianglelefteq G$, then $N_G(H) \ge \langle H, Z(G)\rangle$. Thus $|N_G(H)| \ge p^3$. It follows that $|N_G(H)| = p^3$. So $|N_G(H) : H| = p^2$. That means $G \in \mathcal{S}_2$.

Case 3: If $G \cong$ the group of type (3), where $p > 2$, then $G = \langle a,b,c \mid a^{p^2} = b^{p^2} = c^p = 1, [a,b] = 1, [a,c] = b^{kp}, [b,c] = a^pb^{hp}\rangle$, $k + 4^{-1}h^2$ is a fixed quadratic non-residue (mod p), where $k = 1$ or ν, ν is a fixed quadratic non-residue (mod p), $h = 0, 1, \dots, \frac{p-1}{2}$, then $G' = \langle a^pb^{hp}, b^{kp}\rangle, Z(G) = \Phi(G) = \langle a^p\rangle \times \langle b^p\rangle, G_3 = 1, c(G) = 2$, and G is p-abelian.

It is easy to prove that all quotient groups of order p^4 of G are isomorphic to $M_p(1,1,1) * C_{p^2}$.

For any $H \ntrianglelefteq G$, we prove $|N_G(H) : H| = p^2$ as follows. So $G \in \mathcal{S}_2$.

If $|H| = p^3$, then $|H \cap G'| \le p$. If $H \cap G' = 1$, since $N_G(H) \ge \langle H, G'\rangle$ and $|G| = |\langle H, G'\rangle|$, $H \trianglelefteq G$. This is a contradiction. If $|H \cap G'| = p$, let

$\overline{G} = G/H \cap G'$. Since $|\overline{G}| = p^4$, $\overline{G} \cong M_p(1,1,1) * C_{p^2}$. Since $|\overline{H}| = p^2$, by using the result of Case 2, we have $|\overline{H} \trianglelefteq \overline{G}$. Thus $H \trianglelefteq G$, a contradiction.

If $|H| = p^2$, then $|H \cap G'| \leq p$. If $|H \cap G'| = 1$, then $N_G(H) \geq \langle H, G'\rangle$. Since $H \ntrianglelefteq G$, $|N_G(H)| = p^4$. Thus $|N_G(H) : H| = p^2$. If $|H \cap G'| = p$, let $\overline{G} = G/H \cap G'$. Since $|\overline{G}| = p^4$, $\overline{G} \cong M_p(1,1,1) * C_{p^2}$. If $H \ntrianglelefteq G$, then $\overline{H} \ntrianglelefteq \overline{G}$. By using the result of Case 2, we have $|N_{\overline{G}}(\overline{H}) : \overline{H}| = p^2$. Thus $|N_G(H) : H| = |N_{\overline{G}}(\overline{H}) : \overline{H}| = p^2$.

If $|H| = p$, since $\Omega_1(G) = \langle a^p\rangle \times \langle b^p\rangle \times \langle c\rangle$, assume $H = \langle a^{ip}b^{jp}c^{k'}\rangle$. If $k' \equiv 0 \pmod p$, then $H \trianglelefteq G$, a contradiction. Thus $k' \not\equiv 0 \pmod p, 0 \leq i, j \leq p-1$. $(a^{ip}b^{jp}c^{k'})^{a^s b^t c^u} = a^{ip}b^{jp}c^{k'}a^{-k'tp}b^{-hk'tp-kk'sp} \in \langle a^{ip}b^{jp}c^{k'}\rangle$. We get

$$\begin{cases} -k't \equiv 0 & \pmod p \\ -hk't - kk's \equiv 0 & \pmod p). \end{cases}$$

It follows by $k \not\equiv 0 \pmod p$ that

$$\begin{cases} t \equiv 0 & \pmod p \\ s \equiv 0 & \pmod p). \end{cases}$$

Thus $N_G(H) = \{a^s b^t c^u \mid t \equiv 0 \pmod p, s \equiv 0 \pmod p\}$. So $|N_G(H)| = p^3$, and $|N_G(H) : H| = p^2$.

If $p = 2$, then $G \cong \langle a, b, c \mid a^4 = 1, b^4 = 1, c^2 = 1, [a, b] = 1, [a, c] = b^2, [b, c] = a^2b^2\rangle$. $G' = \langle a^2\rangle \times \langle b^2\rangle = Z(G)$.

It is easy to prove that all quotient groups of order 2^4 of G are isomorphic to $Q_8 * C_4$.

For any $H \ntrianglelefteq G$, we prove $|N_G(H) : H| = 2^2$ as follows. So $G \in \mathcal{S}_2$.

If $|H| = 2^3$, then $|H \cap G'| \leq 2$. If $|H \cap G'| = 1$, then $H \trianglelefteq G$ by $N_G(H) \geq HG' = G$, a contradiction. If $|H \cap G'| = 2$, let $\overline{G} = G/H \cap G'$. Since $|\overline{G}| = 2^4$, $\overline{G} \cong Q_8 * C_4$. Since $|\overline{H}| = 2^2$, in the same way as that of Case 2, we have $\overline{H} \trianglelefteq \overline{G}$. Thus $H \trianglelefteq G$, a contradiction.

If $|H| = 2^2$, then $|H \cap G'| \leq 2$. If $|H \cap G'| = 1$, then $|N_G(H)| \geq |HG'| = 2^4$. That means $|N_G(H)| = 2^4$. Thus $|N_G(H) : H| = 2^2$. If $|H \cap G'| = 2$, let $\overline{G} = G/H \cap G'$. Since $|\overline{G}| = 2^4$, $\overline{G} \cong Q_8 * C_4$. But $|\overline{H}| = 2$, and $\overline{H} \ntrianglelefteq \overline{G}$. In the same way as that of Case 2, we have $|N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$. Thus $|N_G(H) : H| = |N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$.

If $|H| = 2$, then we determine H and $N_G(H)$ as follows.

For any $g \in G$, we have $g = a^i b^j c^k$. If $o(g) = 2$, then

$$\begin{aligned} (a^i b^j c^k)^2 &= (a^i b^j)^2 [a^i b^j, c^{-k}] c^{2k} \\ &= a^{2i}[a^i, b^{-j}] b^{2j} [a^i, c^{-k}][b^j, c^{-k}] c^{2k} \\ &= a^{2i} b^{2j} b^{-2ik} a^{-2jk} b^{-2jk} c^{2k} \\ &= a^{2(i-jk)} b^{2(j-ik-jk)} c^{2k} = 1. \end{aligned}$$

It follows that

$$\begin{cases} i - jk \equiv 0 & \pmod 2 \\ j - ik - jk \equiv 0 & \pmod 2. \end{cases}$$

Moreover,

$$\begin{cases} i \equiv 0 & \pmod 2 \\ j \equiv 0 & \pmod 2. \end{cases}$$

So we can assume $H = \langle a^{2i}b^{2j}c^k\rangle$. If $k \equiv 0 \pmod 2$, then $H \trianglelefteq G$, a contradiction. Thus $k \not\equiv 0 \pmod 2, i, j = 0, 1$.

Assume $a^sb^tc^u \in N_G(H)$. Then $(a^{2i}b^{2j}c^k)^{a^sb^tc^u} = a^{2i}b^{2j}(c^kb^{-2ks})^{b^tc^u} = a^{2i}b^{2j}c^ka^{-2tk}b^{-2tk}b^{-2ks} = a^{2i}b^{2j}c^ka^{-2tk}b^{-2tk-2ks} \in \langle a^{2i}b^{2j}c^k\rangle$. It follows that

$$\begin{cases} tk \equiv 0 & \pmod 2 \\ ks \equiv 0 & \pmod 2. \end{cases}$$

Moreover,

$$\begin{cases} t \equiv 0 & \pmod 2 \\ s \equiv 0 & \pmod 2. \end{cases}$$

It follows that $N_G(H) = \{a^sb^tc^u \mid t \equiv 0 \pmod 2,\ s \equiv 0 \pmod 2\}$. Thus $|N_G(H)| = 2^3$, and $|N_G(H) : H| = 2^2$.

Case 4: If $G \cong$ the group of type (4), that is, $G = \langle a, b, c \mid a^4 = 1, b^2 = a^2, c^4 = 1, [a, b] = a^2, [a, c] = [b, c] = 1\rangle$, then $G' = \langle a^2\rangle, Z(G) = \langle a^2\rangle \times \langle c\rangle$.

For any $H \ntrianglelefteq G$, we prove $|N_G(H) : H| = 2^2$ as follows. That means $G \in \mathcal{S}_2$.

If $|H| = 2^3$, then $|H \cap \langle a\rangle| \neq 1$. If not, since $|H \cap \langle a, b\rangle| \geq 2$, $H \cap \langle a, b\rangle$ must contain an element of order 2. But $\langle a, b\rangle \cong Q_8$ has unique element a^2 of order 2, so $a^2 \in H$, a contradiction. Thus $H > G'$, that means $H \trianglelefteq G$, a contradiction. If $|H| = 2^2$, then $|H \cap Z(G)| \leq 2$. But $|N_G(H)| \geq |HZ(G)| = \frac{|H||Z(G)|}{|H\cap Z(G)|} \geq 2^4$, so $|N_G(H)| = 2^4$. Thus $|N_G(H) : H| = 2^2$. If $|H| = 2$, since $\Omega_1(G) = \langle a^2\rangle \times \langle c^2\rangle \leq Z(G)$, $H \trianglelefteq G$, a contradiction.

Case 5: If $G \cong$ the group of type (5), i.e., $G = \langle a, b, c \mid a^4 = 1, b^4 = 1, c^2 = b^2, [a, b] = 1, [a, c] = b^2, [b, c] = a^2\rangle$, then $G' = \langle a^2\rangle \times \langle b^2\rangle = Z(G) = \Omega_1(G)$, and so $H \cap G' \neq 1$ for any $H \leq G$.

It is easy to prove that all quotient groups of order p^4 of G are isomorphic to $Q_8 * C_4$ or $Q_8 \times C_2$.

For any $H \ntrianglelefteq G$, we prove $|N_G(H) : H| = 2^2$ as follows, That means $G \in \mathcal{S}_2$.

If $|H| = 2^3$, then $|H \cap G'| = 2$. Let $\overline{G} = G/H \cap G'$. Since $|\overline{G}| = 2^4$, $\overline{G} \cong Q_8 * C_4$ or $Q_8 \times C_2$. If $\overline{G} \cong Q_8 * C_4$, since $|\overline{H}| = 2^2$, $\overline{H} \trianglelefteq \overline{G}$ by the same argument as that of Case 2. So $H \trianglelefteq G$, a contradiction. If $\overline{G} \cong Q_8 \times C_2$, then $\overline{H} \trianglelefteq \overline{G}$. That means $H \trianglelefteq G$, a contradiction.

If $|H| = 2^2$, then $|H \cap G'| = 2$. Let $\overline{G} = G/H \cap G'$. Since $|\overline{G}| = 2^4$, $\overline{G} \cong Q_8 * C_4$ or $Q_8 \times C_2$. If $\overline{G} \cong Q_8 * C_4$, since $|\overline{H}| = 2$, $|N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$ by the same argument as that of Case 2. Thus $|N_G(H) : H| = |N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$. If $\overline{G} \cong Q_8 \times C_2$, then $\overline{H} \trianglelefteq \overline{G}$. So $H \trianglelefteq G$, a contradiction.

If $|H| = 2$, since $\Omega_1(G) = \langle a^2\rangle \times \langle b^2\rangle \leq Z(G)$, $H \trianglelefteq G$, a contradiction.

Case 6: If $G \cong$ the group of (6), i.e., $G = \langle a, b, c \mid a^4 = 1, b^4 = 1, c^4 = 1, [a, b] = c^2, [a, c] = b^2c^2, [b, c] = a^2b^2, [c^2, a] = [c^2, b] = 1\rangle$, then $G' = \langle a^2\rangle \times \langle b^2\rangle \times \langle c^2\rangle = Z(G) = \Omega_1(G)$, and so $H \cap G' \neq 1$ for any $H \leq G$.

It is easy to prove that all quotient groups of order p^5 of G are isomorphic to the group of type (3).

For any $H \ntrianglelefteq G$, we prove $|N_G(H) : H| = 2^2$ as follows, That means $G \in \mathcal{S}_2$.

If $|H| = 2^4$, then $|H \cap G'| \leq 2^2$. If $|H \cap G'| = 2$, let $\overline{G} = G/H \cap G'$. Since $|\overline{G}| = 2^5$, $\overline{G} \cong$ the group of type (3), Since $|\overline{H}| = 2^3$, $\overline{H} \trianglelefteq \overline{G}$ by the same argument as that of Case 3, a contradiction. If $|H \cap G'| = 2^2$, then there exists $N \leq H \cap G'$ and $|N| = 2$ such that $G/H \cap G' \cong G/N/H \cap G'/N$. Since $G/N \cong$ the group of type (3), $G/H \cap G' \cong$ the group of type (2) by the same argument as that of Case 3, Since $|\overline{H}| = 2^2$, $\overline{H} \trianglelefteq \overline{G}$ by the same argument as that of Case 2. So $H \trianglelefteq G$, a contradiction.

If $|H| = 2^3$, then $|H \cap G'| \leq 2^2$. If $|H \cap G'| = 2$, let $\overline{G} = G/H \cap G'$. Since $|\overline{G}| = 2^5$, $\overline{G} \cong$ the group of type (3). Since $|\overline{H}| = 2^2$ and $\overline{H} \ntrianglelefteq \overline{G}$, $|N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$ by the same argument as that of Case 3. Thus $|N_G(H) : H| = |N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$. If $|H \cap G'| = 2^2$, then there exists $N \leq H \cap G'$ and $|N| = 2$, such that $G/H \cap G' \cong G/N/H \cap G'/N$. Since $G/N \cong$ the group of type (3), $G/H \cap G' \cong$ the group of type (2) by the same argument as that of Case 3. Let $\overline{G} = G/H \cap G'$ and $|\overline{H}| = 2$. Since $\overline{H} \ntrianglelefteq \overline{G}$, $|N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$ by the result of Case 2. Thus $|N_G(H) : H| = |N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$.

If $|H| = 2^2$, then $|H \cap G'| = 2$. Let $\overline{G} = G/H \cap G'$. Since $|\overline{G}| = 2^5$, $\overline{G} \cong$ the group of type (3). Since $|\overline{H}| = 2$ and $\overline{H} \ntrianglelefteq \overline{G}$, $|N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$ by the result of Case 3. Thus $|N_G(H) : H| = |N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^2$.

If $|H| = 2$, since $\Omega_1(G) = \langle a^2\rangle \times \langle b^2\rangle \times \langle c^2\rangle \leq Z(G)$, $H \trianglelefteq G$, a contradiction.

The groups listed in theorem are mutually non-isomorphic, the details are omitted. □

5. Classifying $\mathcal{S}_3$

Theorem 5.1. *If G is a non-Dedekind p-group, then $G \in \mathcal{S}_3$ if and only if G is one of the following mutually non-isomorphic groups*

(1) $M_p(i+1, m)$, *where* $m \leq i+1$;
(2) $M_p(1, 1, 1) * C_{p^i}$;
(3) $D_8 * Q_8, (i = 3)$;
(4) $\langle a, b, c, d \mid a^4 = b^4 = c^4 = d^4 = 1, a^2 = d^2, b^2 = c^2, [d, b] = a^2, [b, a] = a^2, [c, a] = b^2, [d, a] = [c, b] = a^2b^2, [c, d] = 1\rangle, (i = 3)$.

Proof. $\Longrightarrow$: Case 1. $|G'| = p$.

Let N_1 be a non-normal subgroup of G with minimal order, then all of maximal subgroups of N_1 are normal in G, and N_1 is nonnormal in G. Then N_1 cannot be generated by its maximal subgroups, the maximal subgroup of N_1 is unique, thus N_1 is cyclic. Let $N_1 = \langle b\rangle$. Since N_1 is non-normal in G, there exists $a \in G$ such that $[a, b] \neq 1$. Because $|G'| = p$, we have $\langle a, b\rangle$ is a minimal nonabelian subgroup of G. Let $H = \langle a, b\rangle$. By Lemma 2.10, we obtain $G = H * C_G(H)$. Since H is minimal nonabelian, we have $C_H(N_1) \lessdot H$. And $C_G(N_1) \geq C_G(H)$, $C_G(N_1) \geq C_H(N_1)$, thus $C_G(N_1) = N_G(N_1) \lessdot G$. By $G \in \mathcal{S}_3$, we can get N_1 is a nonnormal subgroup of G with maximal order, it follows that all nonnormal subgroups of G are of same order.

If all nonnormal subgroups of G are of order p, by Lemma 2.9, G is one of following groups:
(1) $M_p(i+1, 1)$;
(2) $M_p(1, 1, 1) * C_{p^i}$;
(3) $D_8 * Q_8$(i=3).

If all of nonnormal subgroups of G are of order p^m, where $m \geq 2$, then $\Omega_1(G) \leq Z(G)$. When $p > 2$, by Lemma 2.8, we get $G \cong M_p(i+1, m)$, where $m \leq i+1$. When $p = 2$, since $G = H * C_G(H)$, $G \in \mathcal{S}_3$, so is H, we can get $H \cong M_2(i+1, m)$, where $m \leq i+1$. Assume that $C_G(H) \nleq H$, then there exists $c \in C_G(H) \backslash H$. Let $H = \langle a\rangle \rtimes \langle b\rangle$ and $c^{2^n} = a^{2^s} b^{2^t}$, $n, s, t \geq 1$. We get a contradiction, thus $C_G(H) \leq H$, $G \cong M_2(i+1, m)$, where $m \leq i+1$.

If $s \geq 2$, we have $c_1 = c^{-2^{n-1}} a^{2^{s-1}} b^{2^{t-1}} \notin H$ with order p, then $\langle b, c_1\rangle \ntrianglelefteq G$ and $|\langle b, c_1\rangle| \neq |\langle b\rangle|$, which is contrary to that all of nonnormal subgroups of G are of same order.

If $s = 1$ and $t \geq 2$, we have $c_1 = c^{-2^{n-1}} ab^{2^{t-1}} \notin H$ with order p, then $\langle c_1\rangle \ntrianglelefteq G$ and $|\langle c_1\rangle| \neq |\langle b\rangle|$, which is contrary to that all nonnormal subgroups of G are of same order.

If $s = 1$ and $t = 1$, let $K = \langle H, c\rangle = \langle a^{2^{i+1}} = b^{2^m} = 1, [a, b] = a^{2^i}, c^{2^n} = a^2 b^2, [c, a] = [c, b] = 1\rangle$, where $m \leq i+1$. If $i+1 \geq 3$, we have $c_1 = c^{-2^{n-1}} aba^{2^i} \notin H$ with $c_1 = c^{-2^{n-1}} aba^{2^i} \notin H$ with order p, then $\langle c_1\rangle \ntrianglelefteq G$ and $|\langle c_1\rangle| \neq |\langle b\rangle|$, which is contrary to that all nonnormal subgroups of G are of order $|N_1|$. If $i+1 = m = 2$, since $G \in \mathcal{S}_3$ and $\langle b\rangle \ntrianglelefteq G$, we have $n \geq 2$ thus $\langle ca\rangle \ntrianglelefteq G$, and $|\langle ca\rangle| \neq |\langle b\rangle|$, which is contrary to that all nonnormal subgroups of G are of same order.

Case 2. $|G'| \geq p^2$.

We use induction on $|G|$. If $|G| = p^5$ and $G \in \mathcal{S}_3$, then all nonnormal subgroups of G are of order p, by Lemma 2.9, we can get $G \cong M_p(4, 1)$, $M_p(1, 1, 1) * C_{p^3}$ or $D_8 * Q_8$. The conclusion is true. Assume the conclusion is true for groups of order $< |G|$. Since G is a p-group, there exists $N \leq G' \cap Z(G)$ and $|N| = p$. By Lemma 2.13 and $|G/N| < |G|$, G/N is the group of listed in Theorem by induction hypothesis.

If $G/N \cong M_p(i+1, m) = \langle \overline{a}, \overline{b} \mid \overline{a}^{p^{i+1}} = 1, \overline{b}^{p^m} = 1, [\overline{a}, \overline{b}] = \overline{a}^{p^i}\rangle$, then it follows by $|(G/N)'| = |G'N/N| = |G'/G' \cap N| = |G'/N| = p$ that $|G'| = p^2$. By Lemma 2.3, $G \cong \langle a, b | a^{p^{i+2}} = 1, b^{p^m} = 1, [a, b] = a^{p^i}\rangle$, where $m \geq 2$. Let $H = \langle b^p\rangle$. Obviously, $H \ntrianglelefteq G$. But $N_G(H) \geq \langle a^p, b\rangle$. Thus $|N_G(H) : H| \geq p^{i+2}$, a contradiction.

If $G/N \cong M_p(1,1,1) * C_{p^i} = \langle \overline{a}, \overline{b}, \overline{c} \rangle = \langle \overline{a}^{p^i} = 1, \overline{b}^p = 1, \overline{c}^p = 1, [\overline{b}, \overline{c}] = \overline{a}^{p^{i-1}}, [\overline{a}, \overline{b}] = 1, [\overline{a}, \overline{c}] = 1 \rangle$, then, letting $N = \langle x \rangle$, $G = \langle a, b, c \mid a^{p^i} = x^s, b^p = x^j, c^p = x^k, [a,b] = x^l, [a,c] = x^m, [b,c] = a^{p^{i-1}} x^n, x^p = 1, [x,a] = [x,b] = [x,c] = 1 \rangle$. Since $|(G/N)'| = |G'N/N| = |G'/G' \cap N| = |G'/N| = p$, $|G'| = p^2$. Thus G is metaabelian. Since $b^p \in Z(G)$, $1 = [b^p, c] = [b,c]^p [b,c,b]^{\binom{p}{2}} = (a^{p^{i-1}} x^n)^p [a^{p^{i-1}} x^n, b]^{\binom{p}{2}} = a^{p^i} [a^{p^{i-1}}, b]^{\binom{p}{2}} = a^{p^i}$. Thus $o(a) = p^i$. Since $\overline{G}' = \overline{G'} = \langle \overline{a^{p^{i-1}}} \rangle$, $G' = \langle a^{p^{i-1}}, x \rangle \cong C_p \times C_p$. Moreover, l, m are not zero in the same time. Assume $G = \langle a, b, c \mid a^{p^i} = 1, b^p = x^j, c^p = x^k, [a,b] = x^l, [a,c] = x^m, [b,c] = a^{p^{i-1}} x^n, x^p = 1, [x,a] = [x,b] = [x,c] = 1 \rangle$. Let $H = \langle a \rangle$. Since l, m are not zero in the same time, $H \ntrianglelefteq G$. But $|N_G(H)| \le p^{i+2}$. Thus $|N_G(H) : H| \le p^2 \ne p^i$, a contradiction.

If $G/N \cong D_8 * Q_8 = \langle a, b, c, d \mid \overline{a}^4 = 1, \overline{b}^2 = 1, \overline{c}^4 = 1, \overline{c}^2 = \overline{d}^2, \overline{a}^2 = \overline{c}^2, [\overline{a}, \overline{b}] = \overline{a}^2, [\overline{c}, \overline{d}] = \overline{c}^2, [\overline{a}, \overline{c}] = [\overline{a}, \overline{d}] = [\overline{b}, \overline{c}] = [\overline{b}, \overline{d}] = 1 \rangle$, then, letting $N = \langle x \rangle$, $G = \langle a, b, c, d \mid a^4 = x^i, b^2 = x^j, c^4 = x^k, c^2 = d^2 x^l, a^2 = c^2 x^m, [a,b] = a^2 x^n, [c,d] = c^2 x^s, [a,c] = x^t, [a,d] = x^u, [b,c] = x^v, [b,d] = x^w, x^2 = 1, [x,a] = [x,b] = [x,c] = [x,d] = 1 \rangle$. Since $d^{-2} c^2 \in Z(G)$, $[d^{-2} c^2, d] = 1$. On the other hand, $[d^{-2} c^2, d] = [c^2, d] = [c,d]^2 [c,d,c] = c^4$. So $c^4 = 1$. Since $a^2 = c^2 x^m$, $a^4 = c^4$. Assume $G = \langle a, b, c, d \mid a^4 = 1, b^2 = x^j, c^4 = 1, c^2 = d^2 x^l, a^2 = c^2 x^m, [a,b] = a^2 x^n, [c,d] = c^2 x^s, [a,c] = x^t, [a,d] = x^u, [b,c] = x^v, [b,d] = x^w, x^2 = 1, [x,a] = [x,b] = [x,c] = [x,d] = 1 \rangle$. By $\overline{G}' = \langle \overline{a^2} \rangle$, $G' = \langle a^2, x \rangle \cong C_2 \times C_2$. Moreover, $G' \le Z(G)$ and $\exp(G) = 4$. By the argument of [15, Lemma 4.5], $G = \langle a, b, c, d \mid a^4 = b^4 = 1, c^2 = a^2 b^2, d^2 = a^2, [a,b] = a^2, [c,d] = a^2 b^2, [a,c] = [b,d] = 1, [b,c] = [a,d] = b^2 \rangle$. It is easy to prove that $G \cong$ the group of type (4).

If $G/N \cong$ the group of type (4), then, letting $N = \langle x \rangle$, $G = \langle a, b, c, d \mid a^4 = x^i, b^4 = x^j, c^4 = x^k, d^4 = x^l, a^2 = d^2 x^m, b^2 = c^2 x^n, [d,b] = a^2 x^s, [b,a] = a^2 x^t, [c,a] = b^2 x^r, [d,a] = a^2 b^2 x^u, [c,b] = a^2 b^2 x^v, [c,d] = x^w, x^2 = 1, [x,a] = [x,b] = [x,c] = [x,d] = 1 \rangle$. Since $d^{-2} a^2 \in Z(G)$, $[d, d^{-2} a^2] = 1$. On the other hand, $[d, d^{-2} a^2] = [d, a^2] = [d,a]^2 [d,a,a] = (a^2 b^2 x^u)^2 [a^2 b^2 x^u, a] = a^4 [a^2, b^{-2}] b^4 [b^2, a] = [b^2, a] a^4 b^4 = a^4 b^4 [b,a]^2 [b,a,b] = b^4 [a^2, b] = b^4 [a,b]^2 [a,b,a] = b^4 a^{-4}$. So $a^4 = b^4$. It follows by $a^2 = d^2 x^m$ that $a^4 = d^4$. By $b^2 = c^2 x^n$ we have $b^4 = c^4$. Thus $a^4 = b^4 = c^4 = d^4$.

Assume $a^4 = b^4 = c^4 = d^4 = x$. Then it follows by $[d,b] = a^2 x^s$ that $[d,b]^a = (a^2 x^s)^a$. On the other hand,

$$\begin{aligned} [d,b]^a &= [d^a, b^a] = [da^2 b^2, ba^2] = [d, ba^2]^{a^2 b^2} [a^2 b^2, ba^2] \\ &= [d, a^2]^{a^2 b^2} [d,b]^{a^2 \cdot a^2 b^2} [a^2 b^2, a^2][a^2 b^2, b] = [d, a^2][d,b][a^2, b] \\ &= [d,a]^2 [d,a,a][d,b][a,b]^2 [a,b,a] = (a^2 b^2 x^u)^2 [a^2 b^2 x^u, a] a^2 x^s a^{-4} \\ &= a^4 [a^2, b^{-2}] b^4 [b^2, a] a^2 x^s a^{-4} = a^4 b^4 [b,a]^2 [b,a,b] a^2 x^s a^{-4} \\ &= a^4 b^4 a^4 [a^2, b] a^2 x^s a^{-4} = [a,b]^2 [a,b,a] a^2 x^s = a^2 x^s x. \end{aligned}$$

But $(a^2x^s)^a = a^2x^s$, a contradiction. So $a^4 = b^4 = c^4 = d^4 = 1$. Assume $G = \langle a, b, c, d | a^4 = b^4 = c^4 = d^4 = 1, a^2 = d^2x^m, b^2 = c^2x^n, [d, b] = a^2x^s, [b, a] = a^2x^t, [c, a] = b^2x^r, [d, a] = a^2b^2x^u, [c, b] = a^2b^2x^v, [c, d] = x^w, x^2 = 1, [x, a] = [x, b] = [x, c] = [x, d] = 1\rangle$. By calculation we have $\Omega_1(G) = \langle a^2, b^2, x\rangle \leq Z(G)$. If $G \in \mathcal{S}_3$, then by $|G| = 2^7$ we have $H \trianglelefteq G$ for any $H \leq G$ and $|H| \geq 2^4$. If $|H| = 2^3$, then $H = \Omega_1(G)$ if $\exp(H) = 2$, and H is abelian if $\exp(H) = 2^2$ (If not, $H \cong M_2(2, 1)$, this contradicts $\Omega_1(G) \leq Z(G)$). It follows that G is a meta-Hamilton p-group. But by checking the classification of meta-Hamilton p-groups [1] we know there does not exists such a group, a contradiction.

$\Longleftarrow$: By Lemmas 2.8, 2.9 we have $G \in \mathcal{S}_3$ for $G \cong$ one of the groups of type (1), (2), and (3). If $G \cong$ the group of type (4), then $G' = \langle a^2\rangle \times \langle b^2\rangle = Z(G) = \Omega_1(G)$.

It is easy to prove that all quotient groups of order 2^5 of G are isomorphic to $Q_8 * D_8$.

For any $H \ntrianglelefteq G$, we prove $|N_G(H) : H| = 2^3$ as follows. Thus $G \in \mathcal{S}_3$.

If $|H| = 2^4$, then $|H \cap \Omega_1(G)| = 2$. Let $\overline{G} = G/H \cap \Omega_1(G)$. Since $|\overline{G}| = 2^5$, $\overline{G} \cong$ the group of type (3). But $|\overline{H}| = 2^3$. It follows by Lemma 2.9 that $\overline{H} \trianglelefteq \overline{G}$. So $H \trianglelefteq G$, a contradiction.

If $|H| = 2^3$, then $|H \cap \Omega_1(G)| = 2$. Let $\overline{G} = G/H \cap \Omega_1(G)$. Since $|\overline{G}| = 2^5$, $\overline{G} \cong$ the group of type (3). But $|\overline{H}| = 2^2$. It follows by Lemma 2.9 that $\overline{H} \trianglelefteq \overline{G}$. So $H \trianglelefteq G$, a contradiction.

If $|H| = 2^2$, then $|H \cap \Omega_1(G)| = 2$. Let $\overline{G} = G/H \cap \Omega_1(G)$. Since $|\overline{G}| = 2^5$, $\overline{G} \cong$ the group of type (3). But $|\overline{H}| = 2$. It follows by $H \ntrianglelefteq G$ that $\overline{H} \ntrianglelefteq \overline{G}$. By Lemma 2.9 we have $|N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^3$. Thus $|N_G(H) : H| = |N_{\overline{G}}(\overline{H}) : \overline{H}| = 2^3$.

If $|H| = 2$, since $\Omega_1(G) = \langle a^2\rangle \times \langle b^2\rangle = Z(G)$, $H \trianglelefteq G$, a contradiction. □

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