http://www.ictp.trieste.it/~pub_off

IC/97/195

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

# NON-ABELIAN ANALOGUES OF ABELS' THEOREM

Andrei Tyurin
Algebra Section, Steklov Math Institute,
Ul. Gubkina 8, Moscow, GSP-1, 117966, Russian Federation
and
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We compute some complex Casson invariants of Calabi-Yau threefolds and prove some analogues of the Abel's theorem for Calabi-Yau threefolds. We discuss related geometrical constructions, arising from the Mirror symmetry and the symplectic geometry.

MIRAMARE – TRIESTE

November 1997

E-mail addresses: Tyurin@tyurin.mian.su; Tyurin@Maths.Warwick.Ac.UK; Tyurin@mpim-bonn.mpg.de

The main aim of this paper is the slight developing of the Donaldson-Thomas program (see [D-T]) for odd dimensional case. In this case it is possible to reduce the main constructions of this program to the "conventional" finite dimensional simplectic or algebraic geometry. The original Abelian integral has the Jacobian of a curve as its target space. Similarly the analogue of Abelian integral for complex threefolds has an intermediate Jacobian (see [G]) as its target space.

## Ch.1. The classical real case.

### §1 The integral of the Chern-Simons form.

Let $Y$ be a compact, smooth, 3-dimensional manifold and $E_h$ be a Hermitian vector bundle of fixed topology type $(r, c_1)$, where $r$ is the rank of the bundle and $c_1 \in H^2(Y, \mathbb{Z})$ is the first Chern class of it. As usual let $\mathcal{A}(E_h)$ be the affine space (over the vector space $\Omega^1(EndE_h)$) of Hermitian connections and let $\mathcal{G}_h$ be the Hermitian gauge group. Then the space of orbits

$$\mathcal{B}_Y(E_h) = \mathcal{A}(E_h)/\mathcal{G}_h \tag{1.1.1}$$

is a Banach orbifold of infinite dimension.

Sending a connection to its curvature tensor defines the $\mathcal{G}_h$- equivariant map

$$F : \mathcal{A}(E_h) \to \Omega^2(EndE_h) \tag{1.1.2}$$

Thus the zero set of this map is defined in the orbit space $\mathcal{B}_Y(E_h)$ correctly and

$$(F)_0 = Rep(\pi_1(Y)) \in \mathcal{B}_Y(E_h) \tag{1.1.3}$$

is the space of classes of $U(r)$- representations of the fundamental group of $Y$. It is the finite dimensional orbifold which can be fibered

$$det : Rep(\pi_1(Y)) \to H^1(Y, \mathbb{R})/H^1(Y, \mathbb{Z}) = Alb(Y). \tag{1.1.4}$$

sending a representation to the determinant of it and using the standard exact sequence for characters of the fundamental group.

On the orbit space $\mathcal{B}_Y(E_h)$ one has 1-form of Chern-Simons defined by the $\mathcal{G}$ - invariant 1-form on $\mathcal{A}(E_h)$ a value of which on a tangent vector $\omega \in \Omega^1(EndE_h)$ at a point $a$ is given by the formula

$$\delta CS_Y(\omega) = \int_Y tr(\omega \wedge F_a), \tag{1.1.5}$$

where the 2-form $F_a$ is the curvature form of $a$.

This form can be integrated along any path in the orbit space (1.1.1) and, in particular, along paths coming from the affine space of connections $\mathcal{A}(E_h)$. Here any pair of connections $(a_0, a_0 + \omega)$ can be joined by the interval $[0, 1]$ of the line $a_0 + t \cdot \omega$. So integrating the form (1.5) along this interval one gets the function on the space $\Omega^1(EndE_h)$ (identified with $\mathcal{A}(E_h)$ by the choice of $a_0$) :

$$CS_Y(a_0 + \omega) = \int_Y tr(\omega \wedge F_{a_0} - 2/3 \cdot \omega \wedge \omega \wedge \omega) \tag{1.1.6}$$

If our vector bundle is trivial then

$$\mathcal{G}_h = Map(Y, U(r)) \tag{1.1.7}$$

and the Chern-Simons function (1.8) subjects to relation

$$CS_Y(g(a)) = CS_Y(a) + \alpha_Y \cdot \deg g \tag{1.1.8}$$

where $g$ is any element of the gauge group which one considers as a map. It is easy to see that one has the formula of the same type for any vector bundle.

So this integral gives the continuous map

$$CS_Y : \mathcal{B}_Y(E_h) \to \mathbb{R}/\alpha_Y \cdot \mathbb{Z} \tag{1.1.9}$$

defined up to rotation of the circle $\mathbb{R}/\alpha_Y \cdot \mathbb{R}$ by another choice of the beginning point $a_0$.

It is natural to call this constant the *period* of a 3-fold $Y$ and the circle

$$J^3(Y) = \mathbb{R}/\alpha_Y \cdot \mathbb{Z} \tag{1.1.10}$$

as the *intermediate Jacobian* of $Y$.

The geometrical meaning of periods is the following: let us consider the universal connection $A$ on the direct product $Y \times \mathcal{B}_Y(E_h)$ and the cohomology class

$$ch_2(A) = [trF_A \wedge F_A] \in H^4(Y \times \mathcal{B}_Y(E_h)) \tag{1.1.11}$$

and the Kunnet component of it

$$ch_2(A)^{3,1} : H_3(Y) \to H^1(\mathcal{B}_Y(E_h)) \tag{1.1.12}$$

Then the value of the image of the fundamental form of $Y$ on the natural generator of $H^1(\mathcal{B}_Y(E_h))$ is the period of $Y$.

Thus the maps (1.9) and (1.4) define (component-wise) the map:

$$CS_Y \cdot det : Rep(\pi_1(Y)) \to J^3(Y) \times Alb(Y). \tag{1.1.13}$$

We will call this map the *Chern-Simons-Jacobi map.*

If the moduli space $Rep(\pi_1(Y))$ is smooth as an orbifold (in particular, it has "expected dimension") and $Rep(\pi_1(Y))^{irr}$ is the subspace of irreducible representations, then the projection $det$ restricted to $Rep(\pi_1(Y))^{irr}$ is proper (see [T]) and the degree of projection $det$ is called the *Casson invariant* of $Y$.

**Main question.** *Is* $(CS \cdot det)^{irr}$ *embedding?*

We will call the positive answer the *analogue of the Abel theorem.*

For simplicity consider now the very known case when $Y$ is a homological sphere and the gauge group is $SU(2)$. In this case the expected behavior of the zero set of the Chern-Simons differential in $Rep(\pi_1(Y))^{irr}$ is the finite set of points:

$$(\delta CS_Y)_0^{irr} = \{a_1, ...., a_{n_+}\} \cup \{b_1, ...., b_{n_-}\} \tag{1.1.14}$$

besides the trivial representation, where $\{a_i\}$ and $\{b_j\}$ are sets of irreducible representations with the same orientations. Then the Casson invariant of $Y$ is

$$Cas(Y) = \tfrac{1}{2}(n_+ - n_-) \tag{1.1.15}$$

(see [T] for explanations of the coefficient $\frac{1}{2}$). On the other hand, $Alb(Y) = 0$ and $CS_Y \cdot det = CS_Y$ maps this finite set of points to the circle $J^3(Y)$. If $CS_Y(a_i) = CS_Y(b_j)$, one has to cancel both of them from our sets. So the analogue of the Abel theorem for this case says

$$a_i \neq a_j \implies CS_Y(a_i) \neq CS_Y(a_j) \tag{1.1.16}$$

and the statement is the same for $\{b_i\}$.

**Remark.** The Casson invariant of threefold is a correlation function of the Topological Quantum Field Theory. Thus this "proof" of the Analogue of Abel theorem has to be pure topological.

At any case the topology of the fibration

$$CS_Y : \mathcal{B}_Y(E_h) \to J^3(Y). \tag{1.1.17}$$

and the topology of smooth fibers outside of critical points $\{CS(a_i)\} \cup \{CS(b_j)\}$ is very important to prove that every singular fiber can contain only one simplest singular point. Suppose we can prove that the topology structure of non-singular fiber is very simple : namely

$$H^*(CS_Y^{-1}(a), \mathbb{Q}) = \mathbb{Q}[\nu] \tag{1.1.18}$$

where $\nu$ is some special cohomology class (like in the case when a smooth fibre has a homotopy type of a cylinder). Then the monodromy around a singular point changes a sign of $\nu$. So every singular fibre can care only one simplest singular point.

Another way to investigate this problem is reducing it to the ordinary finite dimensional geometry of Lagrangian submanifolds of the standard orbifolds such as the spaces of classes of unitary representations of fundamental groups of Riemann surfaces.

## §2 The geometry of Heegard diagram.

The classical way to define the Casson invariant is to use the cutting-pasting method (see [A-M]). Consider a Riemann surface $\Sigma$ in $Y$ and cut $Y$ along this smooth surface. Then we get two 3-folds $Y_\pm$ with boundaries

$$\partial Y_\pm = \pm\Sigma \tag{1.2.1}$$

and $Y$ is a gluing of $Y_\pm$ along $\Sigma$.

Following Taubes let us describe the same construction of CS-map for 3-folds, say $M$, with a boundary $\partial M = \Sigma$ - a smooth oriented Riemann surface of genus $g > 1$. Here we have to restrict ourselves to the case when a vector bundle $E$ is trivial and we note it by $E_0$ with the Hermitian structure of the direct product. Consider again the affine space of smooth $\mathcal{A}(E_0)$ (over the vector space $\Omega^1(EndE_0)$) of Hermitian connections and the Hermitian gauge group $\mathcal{G}_0$. These spaces will be considered as Frechet spaces in $C^\infty$-topology or as pre-Hilbert manifolds with $L^2_1$-Sobolev structure or ($L^p_k$ with $p \geq 2, k \geq 1$). Then the space of orbits

$$\mathcal{B}_M(E_0) = \mathcal{A}(E_0)/\mathcal{G}_0 \tag{1.2.2}$$

has the structure of the same type (with others $p$ and $q$).

Sending a connection to its curvature tensor defines the $\mathcal{G}_0$- equivariant map

$$F : \mathcal{A}(E_0) \to \Omega^2(EndE_0) \tag{1.2.3}$$

Thus the zero set of this map is defined in the orbit space $\mathcal{B}_M(E_0)$ correctly and

$$(F)_0 = Rep(\pi_1(M)) \in \mathcal{B}_M(E_0) \tag{1.2.4}$$

is the space of classes of representations of the fundamental group of $M$. It is the finite dimensional orbifold which can be fibred

$$det : Rep(\pi_1(Y_\pm)) \to H^1(Y_\pm, \mathbb{R})/H^1(Y_\pm, \mathbb{Z}) = Alb(Y_\pm) \tag{1.2.5}$$

Let

$$\mathcal{A}(E_0)^{irr}, \mathcal{B}_M(E_0)^{irr}... \tag{1.2.6}$$

be subsets of connections with irreducible restrictions to the boundary $\Sigma$.

On the orbit spaces $\mathcal{B}_M(E_0)$ one has the Chern-Simons form $\delta CS_M$ defined by the $\mathcal{G}_0$ - invariant 1-forms on $\mathcal{A}(E_0)$ a value of which on a tangent vector $\omega \in \Omega^1(EndE_0)$ at points $a$ is given by just the same formula as (1.1.5).

Again these forms can be integrated along any path in the orbit space (1.2.2) and, in particular, along paths coming from the affine space of connections $\mathcal{A}(E_0)$. Now one can fix a connection $a_0$ on $M$ in such a way that any pair of connections $(a_0, a_0 + \omega)$ can be connected by the interval $[0, 1]$ of the line $a_0 + t \cdot \omega$ as before. So integrating the form $\delta CS_M$ along such interval one gets the CS-function on the space $\mathcal{A}(E_0)$ that is, the continious map to the circle

$$CS_M : \mathcal{B}_M(E_0) \to \mathbb{R}/\alpha_M \cdot \mathbb{Z}. \tag{1.2.7}$$

Again let us call the constant $\alpha_M$ as the *period* of 3-folds $M$ and the circle

$$J^3(M) = \mathbb{R}/\alpha_M \cdot \mathbb{Z} \tag{1.2.8}$$

as the intermediate Jacobian. But in this case the period $\alpha_M$ has a slightly different interpretation.

Now the exact excision sequence of the pair $(M, \Sigma)$ gives the equality

$$H_3(M, \Sigma) = H_2(\Sigma) \tag{1.2.9}$$

Considering as before the universal connection $A$ on the direct product $M \times \mathcal{B}_M(E_0)$ and the cohomology class

$$ch_2(A) = [trF_A \wedge F_A] \in H^4(M \times \mathcal{B}_M(E_0), \partial M \times \mathcal{B}_M(E_0)) \tag{1.2.10}$$

and the Kunnet component of it we get the homomorphism

$$ch_2(A)^{3,1} : H_3(M,\Sigma) \to H^1(\mathcal{B}_M(E_0)) \tag{1.2.11}$$

Then, using the identification (1.2.9), the value of the image of the fundamental form $[\Sigma] \in H_2(\Sigma)$ of $\Sigma$ on the natural generator of $H^1(\mathcal{B}_M(E_0))$ given by the degree of $g \to SU(2)$ is the period of $M$.

So the intermediate Jacobian of $M$ in this case admits the new interpretation

$$J^3(M) = H^2(\Sigma,\mathbb{R})/H^2(\Sigma,\mathbb{Z}) = J^2(\Sigma) \tag{1.2.12}$$

and we can call it as the *second Jacobian of a Riemann surface.* Now we would like to apply these constructions to the parts of the decomposition (1.2.1) that is, let $M = Y_\pm$ with $\partial Y_\pm = \pm\Sigma$, that is the Riemann surface with opposite orientations,

$$E_0 = E_\pm, \quad \mathcal{B}_M(E_0)) = \mathcal{B}_{Y_\pm}(E_\pm) \quad \text{and so on...} \tag{1.2.13}$$

Then (1.2.9) gives

$$H_3(Y_\pm,\Sigma) = \pm H_2(\Sigma). \tag{1.2.14}$$

Now if we note $\mathcal{B}_Y(E_h)^0 \subset \mathcal{B}_Y(E_h)$ the subspace of connections with irreducible restriction to $\Sigma$ (like in (1.2.6)) we get the "exact sequence"

$$\mathcal{B}_Y(E_h)^{irr} \underset{I}{\to} \mathcal{B}_{Y_+}(E_+)^{irr} \times \mathcal{B}_{Y_-}(E_-)^{irr} \underset{J}{\to} \mathcal{B}_\Sigma(E|_\Sigma)^{irr} \times B_\Sigma(E|_\Sigma)^{irr} \tag{1.2.15}$$

where $I$ is induced by the natural embeddings and $J$ is induced by restrictions to the boundary. The "exactness" means that $I$ is an embedding, $J$ is a submersion and the image

$$Im(I) = J^{-1}(\Delta) \tag{1.2.16}$$

where $\Delta$ is the diagonal. (For the proof and details see §4 of [T].)

Now if our fix connection $a_0$ is in $\mathcal{B}_Y(E_h)$ then it is easy to see that the periods are related by the equality

$$\alpha_{Y_+} - \alpha_{Y_-} = \alpha_Y. \tag{1.2.17}$$

Moreover, from the excision sequence of the pair $(Y,\Sigma)$ one can see that the relative cohomology group

$$H_3(Y,\Sigma) = H_3(Y) \oplus H_2(\Sigma) \tag{1.2.18}$$

contains the couple of special elements $([Y], \pm[\Sigma])$. Thus in parallel to (1.2.15) for intermediate Jacobians we have the exact sequence

$$J^3(Y) \to J^3(Y_+) \times J^3(Y_-) \to J^2(\Sigma) \times J^2(\Sigma) \tag{1.2.19}$$

(see (1.2.12)).

Thus one can identify the intermediate Jacobians naturally:

$$J^3(Y_+) = J^3(Y_-) = J^2(\Sigma) = J^3(Y) \tag{1.2.20}$$

Thus our partial CS- maps define (componentwise) Chern-Simons-Jacobi maps:

$$CS_{Y_\pm} \cdot det_\pm : Rep(\pi_1(Y_\pm)) \to J^3(Y) \times Alb(Y_\pm). \tag{1.2.21}$$

Now, the fundamental group $\pi_1(\Sigma)$ admits epimorphisms

$$\phi_\pm : \pi_1(\Sigma) \to \pi_1(Y_\pm) \tag{1.2.22}$$

Thus spaces of classes of representations

$$(F_\pm)_0 = Rep(\pi_1(Y_\pm)) \in \mathcal{B}_{Y_\pm}(E_\pm) \tag{1.2.23}$$

are embedded to the space of representations of the fundamental group of $\Sigma$:

$$\phi^*_\pm : Rep(\pi_1(Y_\pm)) \to Rep(\pi_1(\Sigma)). \tag{1.2.24}$$

Let images be

$$\phi^*(Rep(\pi_1(Y_\pm))) = \mathcal{L}_\pm. \tag{1.2.25}$$

Recall that on the orbit space $\mathcal{B}_\Sigma(E_h)$ for any Hermitian vector bundle $E_h$ on a surface $\Sigma$ there exists the canonical symplectic structure. This structure is induced by the canonical 2-form given on the tangent space $\Omega^1(EndE_h)$ of a connection $a$ by the formula

$$\Omega_0(\omega_1, \omega_2) = \int_\Sigma tr(\omega_1 \wedge \omega_2). \tag{1.2.26}$$

It's well known that this form
1) is closed (because of constant coefficients),
2) $\mathcal{G}_h$ - invariant (because of "tr"),
3) degenerated along $\mathcal{G}_h$-orbits only.

Moreover, the curvature map $F : \mathcal{A}(E_h) \to \Omega^2(EndE_h)$ (1.2) is the moment map for the gauge group action. Thus we get the canonical symplectic structure $\omega_\Sigma$ on $\mathcal{B}_\Sigma(E_h)$ by the *symplectic reduction procedure.*

On the other hand, repeating the construction (1.1.11-12) of "$\mu$-classes", one considers the universal connection $A$ on the direct product $\Sigma \times \mathcal{B}_\Sigma(E_h)$ and the cohomology class

$$ch_2(A) = [trF_A \wedge F_A] \in H^4(\Sigma \times \mathcal{B}_\Sigma(E_h)) \tag{1.2.27}$$

and the Kunnet component of it

$$ch_2(A)^{2,2} : H_2(\Sigma) \to H^2(\mathcal{B}_\Sigma(E_h)) \tag{1.2.28}$$

Then it's easy to see that the restriction of the image of the fundamental class of $\Sigma$

$$ch_2(A)^{2,2}([\Sigma])|_{Rep(\pi_1(\Sigma))} = [\omega_\Sigma]. \tag{1.2.29}$$

Now, it is easy to see that the submanifolds

$$\mathcal{L}_\pm \subset Rep(\pi_1(\Sigma)) \tag{1.2.30}$$

are Lagrangian with respect to the canonical symplectic structure $\omega_\Sigma$.

Now let

$$Rep^{irr}(\pi_1(\Sigma)) \subset Rep(\pi_1(\Sigma)) \tag{1.2.31}$$

be the subset of irreducible representations and

$$\mathcal{L}_\pm^{irr} = \mathcal{L}_\pm \cap Rep^{irr}(\pi_1(\Sigma)) \tag{1.2.32}$$

Then it can be shown that the intersection

$$\mathcal{L}_+^{irr} \cap \mathcal{L}_-^{irr} \tag{1.2.33}$$

is compact (see [A-M]) and the Casson invariant (1.1.15) is the intersection number

$$Cas(Y) = alg\,\#(\mathcal{L}_+^{irr} \cap \mathcal{L}_-^{irr}) \tag{1.2.34}$$

where the intersection points are considered with orientation (and multiplies).

This number doesn't depend on this cutting-pasting procedure (see [A-M]).

We get the classical definition of the Casson invariant via ordinary finite dimensional geometry. In parallel to this description we would like to reduce our Main question from §1 to the problem in the finite dimensional symplectic geometry.

## §3. Lagrangian geometry.

Now one has to recall the main geometrical construction for a pair ($\mathcal{L} \subset S$) where $S$ is a smooth symplectic manifold and $\mathcal{L}$ is a smooth, oriented, maximal ($= 2dim\mathcal{L} = dimS = 2 \cdot l$) Lagrangian submanifold in $S$. Let the symplectic structure on $S$ be given by the symplectic form $\omega$. Then any tangent space $TS_p$ at a point $p$ admits the symplectic form $<,>= \omega_p$ and one can consider the Lagrangian Grassmanian $\wedge_p$ of maximal Lagrangian subspaces in $TS_p$ and the double cover of it which is

$$\overline{\wedge}_p \subset Gr_\uparrow(dim\mathcal{L}, TS_p) \tag{1.3.1}$$

the Grassmanian of oriented Lagrangian subspaces.

Taking this space over every point of $S$ we get the Oriented Lagrangian Grassmanisation

$$\pi : \overline{\wedge}(S) \to S \quad ; \quad \pi^{-1}(p) = \overline{\wedge}_p \tag{1.3.2}$$

of the tangent bundle $TS$.

Let us equip $S$ with any compatible almost complex structure. Then the tangent bundle $TS$ becomes a $U(l)$-bundle and locally over every point

$$\overline{\wedge}_p = U(l)/SO(l) \tag{1.3.3}$$

This space admits the canonical map

$$det : \overline{\wedge} \to U(1) = S^1 \tag{1.3.4}$$

sending every matrix $u \in U(l)$ to $det \quad u \in U(1) = S^1$. Recall that the preimage of the fundamental class of $S^1$ on $\overline{\wedge}$ is the *Universal Maslov class.* Taking this map over every point of $S$ we get the map

$$det : \overline{\wedge}(S) \to S^1(L_{-K}) \tag{1.3.5}$$

where $S^1(L_{-K})$ is the unit circle bundle of the line bundle $\wedge^l TS = detTS$ with the first Chern class

$$c_1(detTS) = -K_S \tag{1.3.6}$$

where $K_S$ is the canonical class of $S$. Recall that as a cohomology class $K_S$ does not depend on a compatible almost complex structure.

Let us emphasize again that a choice of a compatible almost complex structure equips $TS$ and $detTS = L_{-K}$ with the Hermitian structure and unit circle bundle (1.3.5) is given in this metric. The precise choice of such metric will be described in the next §.

Now for every pair $(\mathcal{L} \subset S)$ we have the Gauss lifting of the embedding $i : \mathcal{L} \to S$ to the section

$$G(i) : \mathcal{L} \to \overline{\wedge}(S)|_{\mathcal{L}} \tag{1.3.7}$$

sending a point $p \in \mathcal{L}$ to the oriented subspace $T\mathcal{L}_p \subset TS_p$. The composition of this Gauss map with the projection (1.3.5) gives the map

$$det \cdot G(i) : \mathcal{L} \to S^1(L_{-K})|_{\mathcal{L}} \tag{1.3.8}$$

Now suppose that the cohomology class of the symplectic form is proportional to the canonical class of $S$ :

$$\kappa \cdot [\omega] = K_S; \quad \kappa \in \mathbb{Q} \tag{1.3.9}$$

then the restriction $detTS|_{\mathcal{L}}$ is trivial because the restriction of $[\omega]$ to a Lagrangian $\mathcal{L}$ is zero. Moreover there exists the canonical trivialization (given by the Levi-Civita connection of Hermitian structure induced by the choice of a compatible almost complex structure or a compatible metric on $S$ )

$$S^1(L_{-K})|_{\mathcal{L}} = \mathcal{L} \times S^1 \tag{1.3.10}$$

preserving the Hermitian form and the canonical projection

$$pr : S^1(L_{-K})|_{\mathcal{L}} \to S^1 \tag{1.3.11}$$

Now composing (1.3.8) and (1.3.11) we get the map

$$I = pr \cdot det \cdot G(i) : \mathcal{L} \to S^1 \tag{1.3.12}$$

It is easy to see that under changing of a compatible complex structure this map is changed at most by some diffeomorphism of $S^1$ and properties of this map aren't changing but in our special case $S = R_g^{irr}$ there exists the *canonical choice* of the compatible almost complex structure. We describe it in the next §.

**Mirror digression.** The property (1.3.9) satisfies automatically if $S$ is Calabi-Yau manifold ($\kappa = 0$). For example if $\mathcal{L}$ is a special Lagrangian subthreefold (SLag-cycles for short) in the sense SYZ (see [SYZ]) of a complex Calabi-Yau threefold $S$ this construction gives the map (1.3.12) $I : \mathcal{L} \to S^1$ and we have to prove that the non-singular fibre is $T^2$. In the same vein we get such map for an elliptic curve from some elliptic pencil on K3-surface with changed complex structure. In this case $S^1$ is the moduli space of SLag - 1 - cycles and the mirror symmetry sends the $S^1$-bundle over this moduli space to the dual bundle.

Now, returning to our cutting-pasting situation for the homological spheres and applying this construction to $S = Rep(\pi(\Sigma))$ containing $\mathcal{L}_\pm$ (1.2.30).

These Lagrangian subspaces can be chosen almost canonically. Recall that we restrict ourselves by the $SU(2)$ case.

Let $\Sigma$ be a compact, smooth, oriented Riemann surface of genus $g$ and let $\pi_1(g)$ be the fundamental group of it. Then $\pi_1(g)$ admits the usual presentation

$$\pi_1(g) = < a_1, ..., a_g, b_1, ..., b_g | \Pi_{i=1}^g [a_i, b_i] = id > .$$

The space $Rep(\pi_1(g))$ of classes of $SU(2)$ - representations of $\pi_1(g)$ which is smooth as an orbifold. But as a manifold

$$SingRep(\pi_1(g)) = Rep(\pi_1(g))^{red}. \tag{1.3.13}$$

The space $Rep(\pi_1(g))$ contains the special subspace of representations which are trivial on $\{b_i\}$:

$$B = \{\rho \in Rep(\pi_1(g)) | \rho(b_i) = id, i = 1, ..., g\} \tag{1.3.14}$$

As usual

$$B^{irr} \subset Rep(\pi_1(g))^{irr} \tag{1.3.15}$$

is the subset of irreducible representations.

Then $B$ is a Lagrangian suborbifold of $Rep(\pi_1(g))$ with respect to the canonical symplectic form $\omega_\Sigma$ (1.2.26), (1.2.29). One can apply the collection of maps (1.3.7), (1.3.8) and (1.3.11) to this Lagrangian submanifold $\mathcal{L} = B$ in $S = R_g$.

We can do it because of the equality

$$[\omega_\Sigma] = -4 \cdot K_{R_g} \tag{1.3.16}$$

(see [N-R], [R]).

Using the full collection of constructions (1.3.1 - 12) of Lagrangian geometry we get the map

$$I_g : B^{irr} \to S^1 = U(1) \tag{1.3.17}$$

which can be investigated in the usual way (like in [A-M]).

The subspace $B$ will be our Lagrangian space $\mathcal{L}_+$ from (1.2.30 - 34). The second Lagrangian subspace $\mathcal{L}_-$ can be constructed in the following way:

Let

$$Mod_g = Aut\pi_1(g)/Aut_{in} \tag{1.3.18}$$

be the quotient group by the group of inner automorphisms. This group can be realised as the quotient group of the diffeomorphisms group of a Riemann surface $\Sigma$ of genus $g$:

$$Mod_g = Aut\pi_1(g)/Aut_{in} = Diff^+(\Sigma)/Diff_0(\Sigma) \tag{1.3.19}$$

where $Diff_0(\Sigma)$ is the connected component of $id$ in $Diff^+(\Sigma)$.

This group acts on all our general objects (see (1.3.1 - 5))

$$\pi_1(g), R_g, \overline{\wedge}(R_g), S^1(L_{-K})...$$

in such a way that, for example, the map (1.3.5)

$$det : \overline{\wedge}(R_g) \to S^1(L_{-K}) \tag{1.3.20}$$

is $Mod_g$-equivariant.

The important point is

**Proposition 1.3.1.** *The action of the group $Mod_g$ on $S^1(L_{-K})$ is trivial.*

To show this it is enough to remark that the group $\mathcal{G}$ of automorphisms of the line bundle $L_{-K}$ is abelian. So one has

$$Mod_g/[Mod_g, Mod_g] \to \mathcal{G} \tag{1.3.21}$$

But by the Mumford theorem $Mod_g/[Mod_g, Mod_g]$ is the finite group the order of which divides 12. Using the precise description of this group we can check the triviality of action.

Now we can describe the second Lagrangian subspace $\mathcal{L}_-$ from (1.2.30 - 34) : there exists some element $T_Y \in Mod_g$ such that the second Lagrangian subspace

$$\mathcal{L}_- = T_Y(B) \in R_g \tag{1.3.22}$$

that is

$$\mathcal{L}_- = \{r \in Rep(\pi_1(g)) | r(T_Y(b_i)) = id, i = 1, ..., g\} \tag{1.3.23}$$

Obviously

$$Mod_g(R_g^{irr}) = R_g^{irr} \tag{1.3.24}$$

Moreover, spaces $B^{irr}, R_g^{irr}, ...$ are non-singular and both of our Lagrangian subspaces are oriented.

An element $T \in Mod_g$ is called *B-regular* if the intersection

$$B^{irr} \cap T(B^{irr}) \tag{1.3.25}$$

is compact and transversal. For such element we can define the Casson invariant

$$2Cas(T) = alg\#(B^{irr} \cap T(B^{irr})). \tag{1.3.26}$$

In particular, if a threefold $Y$ is given by the Heegard diagram with the transformation $T_Y$ then

$$Cas(T_Y) = Cas(Y). \tag{1.3.27}$$

Now we can use the Gauss lifting (1.3.7)

$$G(i) : B^{irr} \to \overline{\wedge}(R_g)|_{B^{irr}} \tag{1.3.28}$$

and

$$G(i) : T(B^{irr}) \to \overline{\wedge}(R_g)|_{T(B^{irr})}$$

and the image of this map

$$G(i)(T(B^{irr})) = T(G(i)(B^{irr})) \tag{1.3.29}$$

where in RHS of this equality $T$ acts on $\overline{\wedge}(R_g)$ and by the transversality

$$G(i)(B^{irr}) \cap T(G(i)(B^{irr})) = \emptyset \tag{1.3.30}$$

Now we can use the map (1.3.8)

$$det \cdot G(i) : B \to S^1(L_{-K}) \tag{1.3.31}$$

and

$$det \cdot G(i) : T(B) \to S^1(L_{-K}) \tag{1.3.32}$$

Now, by Proposition 1.3.1, every point of intersection (1.3.25)

$$r \in B^{irr} \cap T(B)^{irr} \subset R_g^{irr}$$

can be lifted uniquely to the intersection point

$$\tilde{r} \in det \cdot G(i)(B^{irr}) \cap T(det \cdot G(i)(B^{irr})) \in S^1(L_{-K}) \tag{1.3.33}$$

So we have the canonical lifting of the intersection $B^{irr} \cap T(B^{irr})$ (see (1.3.25))

$$det \cdot G(i) : B^{irr} \cap T(B^{irr}) \to S^1(L_{-K}). \tag{1.3.34}$$

Now, the trivialization (1.3.10)

$$S^1(L_{-K})|_{B^{irr}} = B^{irr} \times S^1 \tag{1.3.35}$$

is sent by $T$ to the trivialization

$$S^1(L_{-K})|_{T(B^{irr})} = T(B^{irr}) \times S^1 \tag{1.3.36}$$

Thus the projections (1.3.17) or the projection

$$T(I_g) : T(B^{irr}) \to S^1 = U(1) \tag{1.3.37}$$

defines the map

$$I_g = T(I_g) : B^{irr} \cap T(B^{irr}) \to S^1. \tag{1.3.38}$$

Now returning to the set-up of Heegard diagrams from §2 we get

**Proposition 1.3.2.** *For the transformation $T_Y \in Mod_g$ which gives the Heegard diagram (1.2.1) there are the identifications of the circle $S^1$ from (1.3.37) with circles from (1.2.20)*

$$S^1 = J^3(Y_+) = J^3(Y_-) = J^2(\Sigma) = J^3(Y) \tag{1.3.39}$$

*in such a way that the Chern-Simons map for $Y$ (1.1.15 - 9)*

$$CS_Y = I_g = T(I_g) \tag{1.3.40}$$

*on the set of irreducible flat connections on $Y$.*

For the proof one has to accurately check the full chain of identifications. Here the very important point is the description of the canonical Hermitian connection on $S^1(L_{-K})$ which is invariant with respect to the $Mod_g$ - action and to get the canonical trivialization of the canonical unit circle bundle restricted to $B$. We will do it in the next section.

So the main problem of the classical set-up is coming out from the Gauge Theory to very concrete Lagrangian finite dimensional geometry of classes of representations spaces, moreover, without any relation to the geometry of real threefolds just like in the classical theory of Casson invariant in [A-M]. Like in this theory it is very useful to investigate cases of minimal genus 2 and 3 as the minimal even and odd genus cases.

## §4. The "topological" metrics on the representations space.

The following constructions were proposed by Guruprasad and Nilakantan in much more complicated case of parabolic representations (see [G-N]).

Let $\Sigma$ be a compact, smooth, oriented Riemann surface of genus $g$ and let $\pi_1(g)$ be the fundamental group of it. This group admits the standard presentation

$$\pi_1(g) =< a_1, ..., a_g, b_1, ..., b_g | \Pi_{i=1}^g [a_i, b_i] = id > . \tag{1.4.1}$$

We will use this standard presentation of this group and the other "dual" presentation given by the following construction (see [G-N]): let

$$\begin{aligned} r_i &= \Pi_{j=1}^i [a_j, b_j]; \\ \alpha_i &= r_{i-1} b_i^{-1} r_i^{-1}; \\ \beta_i &= r_i a_i^{-1} r_{i-1}^{-1}; \end{aligned} \tag{1.4.2}$$

Then

$$\pi_1(g) =< \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g | \Pi_{j=1}^g [a_j, b_j] = 1 > \tag{1.4.3}$$

is another presentation of $\pi_1(g)$. Sending the generators $a_i, b_j$ to $\alpha_i, \beta_j$ one gets the automorphism $W$ of $\pi_1(g)$ that is $W \in Mod_g$, and it is an involution: $W^2 = id$. On

$$H^1(\Sigma, \mathbb{Z}) = \pi_1(g)/[\pi_1(g), \pi_1(g)] \tag{1.4.4}$$

this involution acts as a "complex structure":

$$W(a_i) = b_i; \quad W(b_j) = a_j \tag{1.4.5}$$

in such a way that the skew symmetrical intersection form $< \gamma_1, \gamma_2 >$ defines the symmetrical form

$$(\gamma_1, \gamma_2) =< \gamma_1, W(\gamma_2) > . \tag{1.4.6}$$

Now for the space $R_g$ one has the stratification

$$SingR_g^{red} \subset R_g red = SingR_g \subset R_g \tag{1.4.7}$$

where

$$R_g - R_g red = R_g irr$$

is the smooth part of $R_g$,

$$R_g red = T^{2g}/\{\pm id\} = K_g \tag{1.4.8}$$

is the "Kummer variety of genus $g$" of the $2g$-dimensional torus

$$T^{2g} = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}) \tag{1.4.9}$$

and $SingK_g$ is the set of $2^{2g}$ points of second order of the torus $T^{2g}$.

To describe the tangent space $(TR_g)_\rho$ recall that a $SU(2)$-representation $\rho$ makes the Lie algebra $su(2)$ into $\pi_1(g)$-modul $su(2)_\rho$ by the adjoint action and

$$(TR_g)_{[\rho]} = H^1(\pi_1(g), su(2)_\rho) \tag{1.4.10}$$

as a tangent space to a point of orbifold. Recall also, that the group of cycles of the module $su(2)_\rho$ is the group of skew homomorphisms from $\pi_1(g)$ to this module, that is

$$Z^1(su(2)_\rho) = \{u : \pi_1(g) \to su(2)_\rho | u(g_1 \cdot g_2) = u(g_1) + g_1(u(g_2))\}. \tag{1.4.11}$$

Of course, any such function $u$ can be extended to the $\mathbb{Z}$-linear map of the integral group algebra

$$u : \mathbb{Z}\pi_1(g) \to su(2) \tag{1.4.12}$$

and the boundary subspace is

$$B^1(su(2)_\rho) = \{u : \pi_1(g) \to su(2)_\rho | u(g) = g(v) - v, \quad \text{for some} \quad v \in su(2)\}. \tag{1.4.13}$$

Thus

$$(TR_g)_{[\rho]} = Z^1(su(2)_\rho)/B^1(su(2)_\rho). \tag{1.4.14}$$

Now the canonical symplectic structure on $R_g$

$$\omega_g = \omega_\Sigma \tag{1.4.15}$$

given by (1.2.26) can be defined as a skew symmetrical bilinear form on the tangent space of every class of representations $[\rho]$ by the bilinear and $PU(2)$-invariant form on $Z^1(su(2)_\rho)$ given by the formula

$$< u, v >= \sum_{i=1}^{g} [(u(r_{i-1}^{-1} - b_i^{-1} \cdot r_i^{-1}, v(a_i))+$$

$$+(u(a_i^{-1}r_{i-1}^{-1} - r_i^{-1}, v(b_i)) \tag{1.4.16}$$

where (, ) is the standard inner product on $su(2)$:

$$(m,m) = -trm^2 \tag{1.4.17}$$

The dual presentation (1.4.3) gives the inner product on the space of cycles $Z^1(su(2)_\rho)$ (1.4.10) by the formula

$$G(u,v) = \sum_{i=1}^{g}[(u(\alpha_i), v(\alpha_i)) + (u(\beta_i), v(\beta_i))] \tag{1.4.18}$$

(see [G-N]) where the more complicated "parabolic" case is considered).

**Proposition 1.4.1 [G-N].** *This inner product is non degenerate and positive.*

Now we can consider the orthogonal

$$B^1(su(2)_\rho)^\perp = (TR_g)_{[\rho]} \tag{1.4.19}$$

and the restriction of the inner product (1.4.18) to this orthogonal defines the special Riemannian metric on $R_g^{irr}$ and on $R_g$ as an orbifold. Let us call it GN-metric.

Obviously this metric is invariant with respect to $Mod_g$ action and it can be checked (see Proposition 3.1 of [G-N]) that this metric is compatible with the symplectic form (1.4.15). So we get the canonical almost complex structure on $R_g$ as on an orbifold. Now we can use it to get the canonical Hermitian structure and the canonical connection on the tangent bundle (the Levi-Civita connection) and the induced Hermitian structure and connection on the determinant line bundle $L_{-K}$. Now our map (1.3.12) is absolutely canonical.

**Remark.** We saw that the canonical symplectic form (1.4.16) on the representations space $R_g$ is the restriction of the canonical symplectic structure $\Omega_0$ (1.2.26) on the orbit space $\mathcal{B}_\Sigma(E_h)$. We can ask the question of the same type for the GN-metric.

**Question 1.4.1.** *Can we get the GN-metric as the restriction of the canonical metric on the orbit space $\mathcal{B}_\Sigma(E_h)$ ?*

Of course, we can construct a metric on $\mathcal{B}_\Sigma(E_h)$ the restriction of which to $R_g$ is the GN-metric. For this it is enough to lift the involution $W$ (see (1.4.6)) to a diffeomorphism

$$\widetilde{W} \in Diff^+(\Sigma_g) \tag{1.4.20}$$

and to define the metric by the inner product

$$G(\omega_1, \omega_2) = \int_\Sigma tr(\omega_1 \wedge \widetilde{W}^*(\omega_2)). \tag{1.4.21}$$

It is easy to see that the GN-metric is the restriction of this metric to $R_g$. But a priori this metric depends on the lifting $\widetilde{W}$ (1.4.20).

# Ch II. The complex case.

## §1. Spaces of orbits.

We have to add a new notion to the standard collection of algebro-geometrical - the notion of *connection* in the case of complex varieties. There is a beautiful text about this subject - the book [D-K]. We have to extract from it some small parts.

Let $X$ be a complex variety (smooth or with an orbifold structure) and let $E$ be a complex vector bundle on it. Let $\mathcal{A}$ be the space of connections on this bundle. Every connection $a \in \mathcal{A}$ is given by the *covariant derivative*

$$\nabla_a : \Gamma(E) \to \Gamma(E \otimes T^*X) \tag{2.1.1}$$

as a differential operator of degree 1 with the ordinary derivative $d$ as the principal symbol. The complex structure gives the decomposition $d = \partial + \overline{\partial}$, so any covariant derivative can be decomposed as

$$\nabla_a = \partial_a + \overline{\partial}_a \tag{2.1.2}$$

where

$$\partial_a : \Gamma(E) \to \Gamma(E \otimes \Omega^{1,0})$$

and

$$\overline{\partial}_a : \Gamma(E) \to \Gamma(E \otimes \Omega^{0,1}) \tag{2.1.3}$$

Thus in the complex case the space of connections admits the decomposition

$$\mathcal{A} = \mathcal{A}' \times \mathcal{A}'' \tag{2.1.4}$$

and $\mathcal{A}'$ is the affine space over $\Omega^{1,0}(EndE)$ and $\mathcal{A}''$ is the one over $\Omega^{0,1}(EndE)$.

Now the group of all automorphisms $\mathcal{G}$ of $E$ acts as the group of gauge transformations and the projection

$$pr : \mathcal{A} \to \mathcal{A}'' \tag{2.1.5}$$

is equivariant with respect to $\mathcal{G}$-action. We will call the space $\mathcal{A}''$ as the *space of $\overline{\partial}$-operators* on $E$ (and the space $\mathcal{A}'$ as the space of $\partial$ - operators).

Equipping $E$ with a Hermitian structure $h$ one gets the subspace $\mathcal{A}_h \subset \mathcal{A}$ of Hermitian connections. It is well known that the restriction of the projection (2.1.5) to $\mathcal{A}_h$ is one-to-one (see, for example, [D-K]) and this identification

$$\mathcal{A}_h = \mathcal{A}'' \tag{2.1.6}$$

defines the section of the fibration (2.1.5). The subgroup $\mathcal{G}_h \subset \mathcal{G}$ preserves this section. To consider the full group $\mathcal{G}$ as the complexification of $\mathcal{G}_h$

$$\mathcal{G}_h^{\mathbb{C}} = \mathcal{G} \tag{2.1.7}$$

we have to recall precisely how the complex group $\mathcal{G}$ acts on the direct product (2.1.4). Of course we have to use the Hermitian structure $h$ on $E$. For any element $g \in \mathcal{G}$ this Hermitian metric gives the element

$$\tilde{g} = (g^*)^{-1} \tag{2.1.8}$$

(remark, that $*$ is defined by $h$) and

$$\tilde{g} = g \equiv g \in \mathcal{G}_h. \tag{2.1.9}$$

Now, the action $\mathcal{G}$ on the component $\mathcal{A}''$ is the standard : for $g \in \mathcal{G}$

$$\overline{\partial}_{g(a)} = g \cdot \overline{\partial}_a \cdot g^{-1} = \overline{\partial}_a - (\overline{\partial}_a g) \cdot g^{-1}; \tag{2.1.10}$$

But on the first component $\mathcal{A}'$ of $\partial$-operators the action is

$$\partial_{g(a)} = \tilde{g} \cdot \partial_a \cdot \tilde{g}^{-1} = \partial_a - ((\overline{\partial}_a g) \cdot g^{-1})^*. \tag{2.1.11}$$

It is easy to see that if $g = \tilde{g}$ then this element acts as an element of $\mathcal{G}_h$ on the full covariant derivative $\nabla_a$ (2.1.2).

On the other hand, it is easy to see directly that described in (2.1.10 - 11) action preserves unitary connections:

$$\mathcal{G}(\mathcal{A}_h) = \mathcal{A}_h \tag{2.1.12}$$

and the identification (2.1.6) is equivariant with respect to described action.

It is easy to see (because $(d')^2 = 0$) that

$$\overline{\partial}_a^2 \in \Omega^{0,2}(EndE) \tag{2.1.13}$$

is a tensor and under the identification (2.1.6)

$$\overline{\partial}_a^2 = F_a^{0,2} \tag{2.1.14}$$

is the $(0,2)$ -Hodge component of the curvature tensor.

The analogue of the orbit space (1.1.1) is the orbit space

$$\mathcal{D}''_X(E) = \mathcal{A}''/\mathcal{G} = \mathcal{A}_h \ \mathcal{G} \tag{2.1.15}$$

which does not have such as $\mathcal{B}_X(E_h)$. We will look out for the "good" subloci of the space (2.1.15) imitating the zero level loci of moment maps.

At any case one has the map

$$p : \mathcal{B}_X(E_h) \to \mathcal{D}''_X(E) \tag{2.1.16}$$

and in a number of constructions on $\mathcal{D}''_X(E)$ we will construct something on the good space $\mathcal{B}_X(E_h)$ and check that it is $\mathcal{G}$-invariant.

The diagonal action of $\mathcal{G}$ on the direct product $\mathcal{A}'' \times \Omega^{0,2}(EndE)$ defines the vector bundle

$$\overline{\Omega^{0,2}(EndE)} = (\mathcal{A}'' \times \Omega^{0,2}(EndE))/\mathcal{G} \to \mathcal{D}''_X(E) \tag{2.1.17}$$

with the infinite-dimensional space $\Omega^{0,2}(EndE))$ as a fibre.

Sending an $\overline{\partial}$-operator $\overline{\partial}_a$ to its square $(\overline{\partial}_a)^2$defines the $\mathcal{G}$ - equivariant map

$$F : \mathcal{A}'' \to \Omega^{0,2}(EndE))$$

that is the section

$$F = (\overline{\partial})^2 : \mathcal{D}''_X(E) \to \overline{\Omega^{0,2}(EndE)} \tag{2.1.18}$$

of this bundle. The zero set of this section

$$(F)_0 = ((\overline{\partial})^2)_0 = \cup \mathcal{M}_i \tag{2.1.19}$$

is the union of all components of moduli spaces of holomorphic bundles on $X$ of the topological type $E$. This union doesn't admit any good structure because it contains all non stable vector bundles. It shows that this section isn't transversal in any sense.

In the same vein we can use the standard orbit space $\mathcal{B}_X(E_h)$ of Hermitian connections and consider the diagonal action of $\mathcal{G}_h$ on the direct product $\mathcal{A}_h \times \Omega^2(EndE_h)$ which defines the vector bundle

$$\overline{\Omega^2(EndE_h)} = (\mathcal{A}_h \times \Omega^2(EndE_h))/\mathcal{G}_h \to \mathcal{B}_X(E_h) \tag{2.1.20}$$

with the infinite-dimensional space $\Omega^2(EndE_h))$ as a fibre.

Sending a Hermitian connection $a$ to $(0,2)$-Hodge component of its curvature defines the section

$$F^{0,2} : \mathcal{B}_X(E_h) \to \overline{\Omega^2(EndE_h)} \tag{2.1.21}$$

of this bundle. The zero set of this section is the union of infinite-dimensional components and it is easy to see that the vector bundle contains preimage with respect to the map $p$ (2.1.16) of the bundle (2.1.17) and

$$(F^{0,2})_0 = p^{-1}(((\overline{\partial})^2)_0). \tag{2.1.22}$$

To make this subspace finite-dimensional one has to consider the *Hitchin - Kobayashi conditions.* One has to fix a polarisation $H$ of $X$ with the Kahler form $\omega_H$ and hence, some Kahler metric and consider the subspace

$$\mathcal{A}_{asd} = \{a \in \mathcal{A}_h | F_a \cdot \omega_H = 0\} \tag{2.1.23}$$

It is easy to see that $\mathcal{A}_{asd}$ is $\mathcal{G}_h$ - invariant (but non $\mathcal{G}$-equivariant) and the intersection

$$(F^{0,2})_0 \cup S_h/\mathcal{G}_h = \cup \mathcal{M}^s_i \tag{2.1.24}$$

is the union of all components of moduli spaces of $H$-stable holomorphic bundles on $X$ of the topological type $E$. This union admits quite a good structure and this section is transversal in any sense. It was just a brief reminding of results and constructions of Ch.6 of [D-K].

## §2. Holomorphic differentials on orbit spaces.

Consider a smooth, compact, complex-analytical threefold $X$ and let us return to the complex orbit space $\mathcal{D}''_X(E)$. Here every holomorphic (3,0)-form $\theta$ on $X$ defines the complex analogue of the Chern-Simons 1-form on the orbit space $\mathcal{D}''_X(E)$ by almost the same formula as (1.1.5). Namely, one defines $\mathcal{G}$-invariant 1-form on $\mathcal{A}''$ a value of which on a tangent vector $\omega \in \Omega^{0,1}(EndE)$ at a $\overline{\partial}$-operator $\overline{\partial}_a$ is given by the formula

$$D_\theta(\omega) = \int_X tr(\omega \wedge (\overline{\partial}_a)^2) \wedge \theta \tag{2.2.1}$$

We will note by the same symbol the form on the orbit space $\mathcal{D}''_X(E)$. Thus one gets the linear isomorphism of the complex space $H^{3,0}(X)$ to the complex vector space $H^{1,0}(\mathcal{D}''_X(E))$ (we don't want to discuss any Hodge meaning of this notation but of course it's easy to see that in some imprecise sense these differential forms are holomorphic). Now, suppose that the intersection of zero sets of these forms

$$\cap_{\theta \in H^{3,0}(X)} (\theta)_0 = \emptyset \tag{2.2.2}$$

is empty. This means that the canonical complete linear system is base points free.

**Proposition 2.2.1.** *Under condition (2.2.2) the intersection*

$$\cap_{\theta \in H^{3,0}(X)} (D_\theta)_0 = \cup \mathcal{M}_i \tag{2.2.3}$$

*is the union of all components of moduli spaces of holomorphic bundles on $X$ of the topological type $E$ as in (2.1.11).*

For the proof it is enough to remark that conditions (2.2.2) and LHS of (2.2.3) for $\overline{\partial}$-operator $\overline{\partial}_a$ implies the equality $(\overline{\partial}_a)^2 = 0$ which gives the integrability condition for holomorphic structure.

Now, one can integrate these forms along any path in the orbit space (2.1.8) and in particular along paths coming from the affine space of connections $\mathcal{A}''$. Here any pair of $\overline{\partial}$-operators $(\overline{\partial}_{a_0}, \overline{\partial}_{a_0} + \omega)$ can be connected by the interval $[0,1]$ of the line $\overline{\partial}_{a_0} + t \cdot \omega$.

Consider some basis $\theta_1, ..., \theta_{h^{3,0}}$ of the space $H^{3,0}$ of holomorphic differential forms and the vector

$$\begin{pmatrix} \theta_1 \\ \dots \\ \theta_{h^{3,0}} \end{pmatrix}$$

This vector defines the vector 1-forms on $\mathcal{D}''_X(E)$

$$\begin{pmatrix} D_{\theta_1} \\ \dots \\ D_{\theta_{h^{3,0}}} \end{pmatrix} \tag{2.2.4}$$

Then integrating this vector of forms along such interval one gets the map of the space $\Omega^{0,1}(EndE)$ (identified with $\mathcal{A}''$ by the choice of $a_0$) to the vector space $\mathbb{C}^{h^{3,0}}$. This map is locally $\mathcal{G}$-equivariant and defines the map of the universal cover of $\mathcal{D}''_X(E)$

$$I_X : \widetilde{\mathcal{D}''_X(E)} \to \mathbb{C}^{h^{3,0}}. \tag{2.2.5}$$

with the covering group at most $H^3(X, \mathbb{Z})$.

**Definition 2.2.1.** *The restriction of this map to any component $\mathcal{M}_i$ (2.1.11) of moduli space of holomorphic bundles on $X$ is called the analogue of the Abel integral (AAI for short).*

This situation suggests that as soon as the right interpretation of the Abel integral is the map to the Jacobian of curve in the same vein AAI can be extended to the map to the intermediate Jacobian of $X$. To realise this program let us return to the "Hermitian" situation.

For our 3-fold $X$ consider the vector space $H^{3,0} \oplus H^{2,1}$ as the space of differential 3-forms on $X$ and let $\theta_1, ..., \theta_{h^{3,0}}, \Omega_1, ..., \Omega_{h^{2,1}}$ be some basis of this space which we will consider as the vector

$$\begin{pmatrix} \theta_1 \\ \dots \\ \theta_{h^{3,0}} \\ \Omega_1 \\ \dots \\ \Omega_{h^{2,1}} \end{pmatrix} \tag{2.2.6}$$

This vector defines the vector of forms on $\mathcal{A}''$ given on a tangent vector $\omega \in \Omega^1(EndE_h)$ by the formula

$$\begin{pmatrix} D_{\theta_1}(\omega) = \int_X tr(\omega \wedge F_a) \wedge \theta_1 \\ \dots \\ D_{\theta_{h^{3,0}}}(\omega) = \int_X tr(\omega \wedge F_a) \wedge \theta_{h^{3,0}} \\ D_{\Omega_1}(\omega) = \int_X tr(\omega \wedge F_a) \wedge \Omega_1 \\ \dots \\ D_{\Omega_{h^{2,1}}}(\omega) = \int_X tr(\omega \wedge F_a) \wedge \Omega_{h^{2,1}} \end{pmatrix} \tag{2.2.7}$$

components of which are $\mathcal{G}_h$ - invariant and define the vector of 1-form ( with the same symbol ) on the orbit space $\mathcal{B}_X(E_h)$.

**Remark.** We have to repeat the definition for holomorphic components because we are sitting on the other space of connections. We don't distinguish notations of these components of integrals.

Now it is easy to check that the collection of path integrals of these forms defines the map

$$J_E : \mathcal{B}_X(E_h) \to J^3(X) = H^3(X, \mathbb{R})/H^3(X, \mathbb{Z}). \tag{2.2.8}$$

(we use here the Poincare duality in middle cohomology of $X$).

**Definition 2.2.2.** *The restriction of this map to any component $\mathcal{M}_i^s$ (2.1.15) of the moduli space of stable holomorphic bundles on $X$ is called the Abel - Jacobi map.*

Again in parallel to the abelian case we would like to ask

**Question 2.2.1.** *Are $J_E$- images of disjoint components of moduli spaces of holomorphic bundles disjoint in $J^3(X)$?*

Moreover if the restriction of $J_E$ to one of the components is embedding we can call this result the *Non-Abelian analogue of the Abel theorem* .

In the next section we will explain the cohomological correspondence which gives this construction.

## §3. Cohomological correspondence.

The construction of the cohomological correspondence which sends 3 - classes of $X$ to 1 - classes of $\mathcal{B}_X(E_h)$ can be done precisely like in (1.1.11 - 13) of §1 of Ch.I.: let us consider the universal connection $A$ on the direct product $X \times \mathcal{B}_X(E_h)$ and the cohomology class

$$ch_2(A) = [trF_A \wedge F_A] \in H^4(X \times \mathcal{B}_X(E_h)) \tag{2.3.1}$$

and the (3,1) - Kunnet component of it

$$ch_2(A)^{3,1} : H_3(Y), \mathbb{Z}) \to H^1(\mathcal{B}_X(E_h), \mathbb{Z}). \tag{2.3.2}$$

This cohomological correspondence induces the homomorphism of tori

$$(ch_2(A)^{3,1})^* : Alb(\mathcal{B}_X(E_h)) = H^1(\mathcal{B}_X(E_h), \mathbb{R})/H^1(\mathcal{B}_X(E_h), \mathbb{Z}) \to$$

$$\to J^3(X) = H_3(Y), \mathbb{R})/H_3(Y), \mathbb{Z}) \tag{2.3.3}$$

**Remark.** The formula of the Chern character contains some divisions by 2. It is easy to see the way to lift this construction to integers.

Now, as usual, the integration of the collection of 1-forms gives the Jacobi map

$$J : \mathcal{B}_X(E_h) \to Alb(\mathcal{B}_X(E_h)) \tag{2.3.4}$$

and the composition of (2.3.3) and (2.3.4) gives the map

$$\mathcal{B}_X(E_h) \to J^3(X) \tag{2.3.5}$$

restrictions of which to components of moduli spaces of holomorphic vector bundles give $J_E$ (2.2.8).

Now, the intermediate Jacobian $J(X)$ admits the complex structure, defined by Griffiths (see, for example [G]). On the other hand, any component of moduli space $\mathcal{M}_i$ (2.2.3) admits the complex structure too.

**Proposition 2.3.1.** *The map $J_E$ is holomorphic with respect to these complex structures.*

To prove this statement one has to recall that the complex structure on $J(X)$ is defined by the identification

$$H^3(X, \mathbb{R}) = H^{3,0} \oplus H^{2,1} = (TJ^3(X))_0 \tag{2.3.6}$$

which is given by the projection of the complexification $H^3(X, \mathbb{C})$ to the sum of Hodge components. Now, on the Hodge component $H^{3,0}(X)$ the map $J_E$ goes through the space $\mathcal{D}''_X(E)$ and is defined by AAM (2.2.4). It is holomorphic because the complex structure on $\mathcal{M}_i$ is induced by the natural holomorphic structure on $\mathcal{D}''_X(E)$. To check holomorphity on the $H^{2,1}$-component one has to give another interpretation of this map.

Let $CH^2(X)$ be the space of 1-dimensional algebraic cycles codimension 2 on X of the topology type $ch_2(E)$ up to rational equivalence. For it there exists the Abel-Jacobi map

$$a : CH^2(X) \to J^3(X). \tag{2.3.7}$$

sending a class $c - c_0$ where $c_0$ is a fixed class to the vector

$$\begin{pmatrix} \int_\delta \theta_1 \\ \dots \\ \int_\delta \theta_{h^{3,0}} \\ \int_\delta \Omega_1 \\ \dots \\ \int_\delta \Omega_{h^{2,1}} \end{pmatrix} \tag{2.3.8}$$

where $\delta$ is any 3-cycle with the boundary

$$\partial \delta = c - c_0 \tag{2.3.9}$$

The following statement is well known

**Proposition 2.3.2.** *If algebraic cycles $c$ and $c_0$ can be joint by algebraic curve $C$ then locally the image $a(C)$ is contained in $H^{2,1}$-component .*

Now, every component $\mathcal{M}_i$ of moduli spaces of holomorphic bundles admits the map:

$$c : \mathcal{M}_i \to CH^2(X) \tag{2.3.10}$$

sending a vector bundle $F$ to the class of rational equivalence of the algebraic class $ch_2(F)$. Because the map $J_E$ is given by $ch_2$ of the quasi universal bundle (see (2.3.1)) it is easy to see that this map up, to shift, has the form of composition of two maps

$$c \cdot a : \mathcal{M}_i \to J^3(X) \tag{2.3.11}$$

Thus $J_E$ is holomorphic.

Moreover one can see that the map $J_E$ is defined correctly on any component of moduli space of holomorphic vector bundles (2.2.3), not on components of stable bundles (2.1.15) only.

The following problem is natural in this situation: one can see that the map $J_E$ on $H^{3,0}$ part is a restriction of AAM defined on $\mathcal{D}''_X(E)$. On the other hand, on the other Hodge component $H^{2,1}$ the map $J_E$ is defined a priori on the subsets $\mathcal{M}_i$ of $\mathcal{D}''_X(E)$ only.

**Question 2.3.1.** *Can the map $c$ (2.3.8) be extended to the map*

$$C : \mathcal{D}''_X(E) \to CH^2(X)?$$

Now consider a local lifting $\widetilde{J_E(\mathcal{M}_i)}$ of the image of the $J_E$ map of some component $\mathcal{M}_i$ to the universal cover $H^{3,0} \oplus H^{2,1}$ of the intermediate Jacobian $J(X)$.

**Proposition 2.3.3.** *The image of the projection $p_{3,0}$ of the direct product $H^{3,0} \oplus H^{2,1}$ to the first component is a point. That is locally the image of component $\widetilde{J_E(\mathcal{M}_i)}$ is contained in the affine space*

$$\widetilde{J_E(\mathcal{M}_i)} \subset < p_i, H^{2,1} > \tag{2.3.12}$$

To see this statement it is enough to return to the Proposition 2.3.1.

**Example.** Let $X$ be a smooth Calabi-Yau threefold with $h^{2,1}(X) = 0$. Then $X$ is infinitesimal rigid and the intermediate Jacobian of it

$$J(X) = \mathcal{E} \tag{2.3.13}$$

is an elliptic curve. There is a number of threefolds of such type. The best example is the Barth-Nieto threefold [DvS], which is the moduli space of abelian surfaces with polarisation of type (2,6) and fixed theta structure of level 2. For threefolds of such type the image of the Abel-Jacobi map $J_E$ is a finite set of points $\{e_1, ..., e_{Cas(E)}\}$. This set is defined up to translation and one can kill this ambiguity by the condition

$$\sum_i e_i = 0 \tag{2.3.14}$$

as points on elliptic curve. The arithmetic meaning of these points is quite interesting.

## Ch.III. Holomorphic vector bundles on threefolds.

### §1. Mukai lattices.

Let $X$ be a smooth compact threefold and

$$H^{2*}(X,\mathbb{Z}) = \oplus_{i=0}^{3} H^{2i}(X,\mathbb{Z}) \tag{3.1.1}$$

is the algebra of even dimensional cohomology groups of $X$. Actually we have other algebra,

$$A(X) = \oplus_{i=0}^{3} A^i(X) \tag{3.1.2}$$

- the algebra of algebraic cycles modulo algebraic equivalence, which is related to (3.1.1) by the natural homomorphism

$$h : A(X) \to H^{2*}(X,\mathbb{Z}) \tag{3.1.3}$$

but in the 3-dimensional case it isn't an isomorphism a priori because of the negative solution of the Lefshetz conjecture and non triviality of the Griffiths group.

The involution $*$ acts component wise:

$$*|_{H^{0,4}(X,\mathbb{Z})} = id; \quad *|_{H^{2,6}(X,\mathbb{Z})} = -id \tag{3.1.4}$$

and in the same way on $A(X)$ so that the map $h$ (3.1.3) is $*$-equivariant. Of course we have the natural identification

$$H^0(X,\mathbb{Z}) = \mathbb{Z} = H^6(X,\mathbb{Z}) \tag{3.1.5}$$

So any element $u \in H^{2*}(X,\mathbb{Z})$ is a vector $u = u_0 + u_2 + u_4 + u_6$ and we denote the $i$-component of it by $[u]_i$. Let $K_X$ be the canonical class of $X$,

$$K_X \in H^2(X,\mathbb{Z}); \quad K_X = (K_X)_2 \quad \text{and} \quad k_X = c_2(TX) = c_2(X) = (k_X)_4. \tag{3.1.6}$$

Then on $H^{2*}(X,\mathbb{Z})$ we have two bilinear forms

$$< u,v >= -1/2K_X \cdot [u^* \cdot v]_4$$

$$(u,v) = -[v^* \cdot u]_6 \tag{3.1.7}$$

It is easy to see that the first is symmetrical and the second is skew-symmetrical.

Now if we consider the algebraic K-functor $K^0_{alg}$ on $X$ then the Chern-character gives the chain of homomorphisms

$$K^0_{alg} \underset{ch}{\longrightarrow} A(X) \underset{h}{\rightarrow} H^{2*}(X,\mathbb{Q}) \tag{3.1.8}$$

which is equivariant with respect to $*$ which acts on $K^0_{alg}$ by sending a vector bundle $E$ to $E^*$.

Now on $K^0_{alg}$ the following bilinear form is defined

$$-\chi(E_1,E_2) = \sum_{i=0}^{3}(-1)^{i+1} rkH^i(X, E_1^* \otimes E_2) \tag{3.1.9}$$

where cohomology spaces are coherent cohomology.

This form is the preimage of some bilinear form on $H^{2*}(X,\mathbb{Q})$. To see this recall that by the Riemann-Roch theorem

$$\chi(E_1,E_2) = [chE_2 \cdot chE_1^* \cdot td_X]_6 \tag{3.1.10}$$

where $td_X$ is the Todd class of X.

Decompose this form to symmetric and antisymmetric components:

$$\chi(E_1,E_2) = \chi_+(E_1,E_2) + \chi_-(E_1,E_2) \tag{3.1.11}$$

It is easy to compute directly that

$$td_X^* = td_X \cdot chK_X = td_X \cdot e^{K_X} \tag{3.1.12}$$

Thus

$$2td_X^{\pm} = td_X \pm td_X^* = td_X \cdot (1 \pm e^{K_X}) \tag{3.1.13}$$

and

$$\chi(E_1,E_2)_+ = [chE_2^* \cdot chE_1 \cdot td_X^-]_6;$$

$$\chi(E_1,E_2)_- = [chE_2^* \cdot chE_1 \cdot td_X^+]_6. \tag{3.1.14}$$

Moreover we have one special class in $A(X)$ (or $H^{2*}(X,\mathbb{Q})$) :

$$\sqrt{td_X^+} = 1 + \frac{k_X}{24} = \sqrt{td_X^-/K_X} \tag{3.1.15}$$

defined uniquely by the conditions $[\sqrt{td_X^+}]_0 = 1$.

Following Mukai we can correct slightly the Chern character homomorphism (3.1.8)

$$m(E) = chE \cdot \sqrt{td_X^+} \qquad (3.1.16)$$

Let us call this vector the *Mukai vector* of $E$.

Now we can recompute the "non standard" forms (3.1.14) in terms of the standard bilinear forms (3.1.7)

$$\chi(E_1, E_2)_+ = < m(E_1), m(E_2) >$$

and

$$\chi(E_1, E_2)_- = (m(E_1), m(E_2)) \qquad (3.1.17)$$

The precise formula for a Mukai vector is

$$m(E) = ch(E) + rkE \cdot \frac{k_X}{24} + \frac{c_1(E) \cdot k_X}{24}. \qquad (3.1.18)$$

At last

$$\chi(E_1, E_2)_+ = \frac{1}{2} \cdot K_X \cdot [c_1(E_1) \cdot c_1(E_2) - \frac{1}{2} \cdot rkE_2 \cdot (c_1(E_1)^2 - c_2(E_1)) -$$
$$-\frac{1}{2} \cdot rkE_1 \cdot (c_1(E_2)^2 - c_2(E_2)) - rkE_1 \cdot rkE_2 \cdot \frac{k_X}{12}]; \qquad (3.1.19)$$

and

$$\chi(E, E)_+ = \frac{1}{2} \cdot K_X[c_1^2 - rk(c_1^2 - 2c_2) - rk^2 \cdot \frac{k_X}{12}] = rkK_X[(c_2 - \frac{(rk-1) \cdot c_1^2}{2 \cdot rk}) - rk \cdot \frac{k_X}{24}]$$

Summing these calculations we have

**Proposition 3.1.1.** *The virtual (expected) dimension of the moduli space of simple vector bundles $E$ is given by the formula*

$$v.dim\mathcal{M}_E = rk(E)((c_2(E) - \frac{(rk(E)-1) \cdot c_1(E)^2}{2 \cdot rk(E)}) \cdot (-K_X) - (rk(E)^2 - 1) \cdot \chi(\mathcal{O}_X) \qquad (3.1.20)$$

*where* $\chi(\mathcal{O}_X) = 1 - h^{1,0} + h^{2,0} - h^{3,0}$ *as usual.*

Recall again that by the Riemann-Roch theorem

$$\chi(\mathcal{O}_X) = \frac{k_X \cdot K_X}{24}. \qquad (3.1.21)$$

Using the standard notation:

$$c_2(E) - \frac{rk(E) - 1}{2 \cdot rk(E)} \cdot c_1(E)^2 = \Delta(E) \qquad (3.1.22)$$

recall that for the Bogomolov stability of $E$ we need the equality

$$\Delta(E) \cdot H > 0 \qquad (3.1.23)$$

for any polarisation $H$ of $X$.

Now suppose that the canonical class $K_X$ of $X$ is defined and we have three cases (just like for Riemann surfaces case):

$K_X < 0$, so $X$ is Fano variety;

$K_X = 0$, so $X$ is Calabi-Yau variety (CY for short);

$K_X > 0$, in this case we call $X$ a *canonical general type variety* (CG for short).

**Calabi-Yau case.**

$$td_X = td_X^+; \tag{3.1.24}$$

and the bilinear form (3.1.10)

$$\chi = \chi_-; \tag{3.1.25}$$

and we will define a topological type of $E$ by the Mukai vector $m = m(E) \in H^{2*}(X, \mathbb{Q})$.

**Corollary 3.1.0.** *In the Calabi-Yau case virtual dimensions of moduli spaces are zero. For stable bundles we can expect that for every topological type $m$ the moduli space $\mathcal{M}_X^s(m)$ is compact (more precisely admits the structure of a zero dimensional scheme). As the length of this scheme the number*

$$CD(X, m) = deg\mathcal{M}_X(m) = \#\{\mathcal{M}_X(m)\} \tag{3.1.26}$$

*is well defined.*

We will call this number the *Casson-Donaldson invariant* (CD for short) because on the one hand it is the obvious analogue of the Casson invariant from Ch. I proposed in [D-T ] and on the other hand it is the obvious analogue of the Donaldson polynomial of degree 0 for four manifolds.

**Mirror digression.** By the Striminger-Yau-Zaslow conjecture (see [SYZ]) for any CY-threefold $X$ there exists its mirror partner $X'$ which is CY-threefold again and the isomorphism

$$mir : H^{2*}(X, \mathbb{Q}) \to H^3(X', \mathbb{Q}) \tag{3.1.27}$$

such that the skew symmetrical form $\chi$ (3.1.9), (3.1.10) becomes the intersection form :

$$\chi(m_1, m_2) = mir(m_1) \cdot mir(m_2). \tag{3.1.28}$$

Remark that the LSH of (3.1.27) admits the graduation (3.1.1). Conjecturally the same graduation of the RSH of (3.1.27) is given by the monodromy transformation around the large complex structure limit point. More precisely, transformations of Mukai lattice given by twistings by divisor classes can be described in terms of monodromy transformations around boundary divisors passing through the large complex structure limit point (see [Gr]).

If the fundamental class of a 3-cohomology class $mir(m)$ can be realised as a SLag-cycle (see the Mirror digression in §3 of Ch.1 ) which is a smooth oriented Lagrangian 3-submanifold $Y$, subjecting the *calibration condition*, then the tangent space to the moduli space $\mathcal{M}_{X'}(mir(m))$ of all deformations of such realisations at point $Y$ is

$$(T\mathcal{M}_{X'}(mir(m)))_Y = H^1(Y, \mathbb{R}) \tag{3.1.29}$$

(where $H^1(Y, \mathbb{R}$ is realised as the space of harmonic forms). Thus any SLag-realisation of $mir(m)$ by a *homological sphere* $Y$ is rigid. We have the finite set of such realisations $\mathcal{M}_{X'}^0(mir(m))$.

Moreover, every such $Y$ is carrying the finite set of flat $SU(2)$-bundles and the number of *supercycles* of such type, that is, the set of pair $(Y_i, a_j)$, where $Y_i$ is a SLag-homological sphere and $a_j$ is an irreducible flat connection, is finite. So one can associate to every 3-homology class $\sigma$ of $X'$ the number

$$CD(X', \sigma) = \sum_i Cas(Y_i) \tag{3.1.30}$$

The natural quite speculative conjecture is

$$CD(X, m) = CD(X', mir(m)) \tag{3.1.31}$$

To see this we have to perform two steps. The first step is in the general set-up of the theory of calibrated cycles in any Calabi-Yau threefold: we have to consider the 3-dimensional component $\mathcal{M}^3_{X'}(\sigma)$ of the moduli space of SLag-cycles which are 3-tori. Now if there exists some good compactification of this moduli space then one can see (using the trick, which currently became usual) that

$$CD(X', \sigma) = Cas(\overline{\mathcal{M}^3_{X'}(\sigma)}).$$

The second step is to use the "twistor construction" decribed in the section 5 of Ch. 4 to link flat bundles on $\mathcal{M}^3_{X'}(\sigma)$ and holomorphic bundles on $X$ (of course if you believe the mirror conjecture).

Let us return to the complex case. Of course there exists the special case of vector bundles when this invariant is 1:

$$CD(X, m(L)) = 1$$

for any line bundle $L$ on $X$ (if $h^{1,0}(X) = 0$).

On the other hand, there exists a lot of sheaves such that the moduli space $M_X(m)$ is compact and smooth. For example $F = \mathcal{O}_p$ where $p \in X$ is a point. In this case, the deformation to the normal cone predicts that

$$CD(X, m(\mathcal{O}_p)) = \chi_{top}(X) = \sum_i (-1)^i rk H^i(X, \mathbb{Z}) \tag{3.1.32}$$

Moreover, one can see from §2 of Ch.2 that $M_X(m)$ is the zero set of the holomorphic differential (2.2.1) on the space $\mathcal{D}''_X(E)$. Recall that this space depends on the topological type of $E$ only, that is on the Chern character $ch(E)$ or on the Mukai vector $m(E)$. So in parallel to (3.1.8) one has the interpretation

$$CD(X, m(E)) = \chi_{top}(\mathcal{D}''_X(E)). \tag{3.1.33}$$

in spite of the fact that the space $\mathcal{D}''_X(E)$ is infinite dimensional.

Now suppose we have the transversal situation, that is, there are finite sets of infinitesimal rigid, simple vector bundles

$$\{E_1, ..., E_{CD(m)}\} \tag{3.1.34}$$

of the topology type $m = m(E)$. Then the Abel-Jacobi map (2.2.8) sends this collection of bundles to the intermediate Jacobian of $X$:

$$\{J_m(E_1), ..., J_m(E_{CD(m)})\} \subset J(X) \tag{3.1.35}$$

This collection is defined up to translation and we can kill this ambiguity by the condition

$$\sum_i J_m(E_i) = 0 \tag{3.1.36}$$

In some cases we can prove that

$$i \neq j \implies J_m(E_i) \neq J_m(E_j). \tag{3.1.37}$$

In parallel to the *Clemens conjecture* about rational curves on generic quintic in $\mathbb{P}^4$ we can propose the following

**Conjecture 3.1.1.** *On generic quintic in $\mathbb{P}^4$ every stable rank 2 vector bundle $E$ is infinitesimally rigid that is $H^1(adE) = 0$.*

**Remark.** It is well known that for rank 3 vector bundles this statement isn't true: the tangent bundle isn't rigid.

**Fano case.** The number

$$\Delta(E) \cdot (-K_X) = D_X(E) > 0 \tag{3.1.38}$$

has to be positive and the virtual dimension of stable vector bundles

$$v.dim\mathcal{M}_E^s = rk(E) \cdot D_X(E) - (rk(E)^2 - 1) \tag{3.1.39}$$

because

$$\frac{k_X \cdot (-K_X)}{24} = \chi(\mathcal{O}_X) = 1 \tag{3.1.40}$$

One can see that there exists the infinite collection of stable moduli spaces of increasing dimensions.

Now suppose for simplicity that

$$H^2(X, \mathbb{Z}) = \mathbb{Z} \cdot H. \tag{3.1.41}$$

Then the geometrical situation is described by one positive constant - the index of Fano variety :

$$-K_X = i_X \cdot H. \tag{3.1.42}$$

Then we can define Chern classes of $E$ as constants:

$$c_1(E) = c_1 \cdot P.D.(H); \quad c_2(E) = c_2 \cdot H \tag{3.1.43}$$

and

$$D_X(E) = (c_2 - \frac{rk-1}{2\cdot rk}c_1^2)\cdot i_X \tag{3.1.44}$$

and the virtual dimension of the moduli space

$$v.dim\mathcal{M}_E = i_X \cdot rk \cdot ((c_2 - \frac{rk-1}{2\cdot rk}\cdot c_1^2) - (rk^2-1). \tag{3.1.45}$$

We can consider the case when

$$0 \leq c_1 < rk. \tag{3.1.46}$$

Then

$$v.dim\mathcal{M}_E > i_X \cdot rk \cdot c_2 - \frac{(rk-1)(rk+3)}{2}. \tag{3.1.47}$$

One can see all possibilities for positive dimensional moduli spaces.

Now, if $S$ is an algebraic surface then there are formulas of the same type for dimension of moduli spaces of vector bundles: on the even cohomology ring $H^{2*}(S,\mathbb{Q})$ we have the standard bilinear form $(u,v) = (u^*\cdot v)_4$ and the non-standard" bilinear form

$$-\chi(E_1,E_2) = \sum_{i=0}^{2}(-1)^{i+1}rkH^i(X,E_1^*\otimes E_2) \tag{3.1.48}$$

with the symmetrical $\chi(E_1,E_2)_+$ and skew symmetrical $\chi(E_1,E_2)_-$ parts. The same correction of the Chern character :

$$m(E) = chE\cdot\sqrt{td_S^+}$$

defines a Mukai vector and reduces the non-standard" bilinear form to the standard form (for details of this machinery see [Ty]).

In parallel to (3.1.19)

$$\chi(E_1,E_2)_+ = c_1(E_1)\cdot c_1(E_2) - \frac{1}{2}\cdot rkE_2\cdot(c_1(E_1)^2 - c_2(E_1)) -$$

$$-\frac{1}{2}\cdot rkE_1\cdot(c_1(E_2)^2 - c_2(E_2)) - rkE_1\cdot rkE_2\cdot\chi(\mathcal{O}_S); \tag{3.1.49}$$

and

$$\chi(E,E)_+ = 2rk\cdot(\Delta(E) - rk\cdot\frac{\chi(\mathcal{O}_S)}{2})$$

**Proposition 3.1.2.** *The virtual (expected) dimension of moduli space of simple vector bundles $E$ on $S$ is given by the formula*

$$v.dim\mathcal{M}_E = 2[rk(E)\cdot\Delta(E) - (rk(E)^2-1)\cdot\frac{\chi(\mathcal{O}_S)}{2}] \tag{3.1.50}$$

*where $\chi(\mathcal{O}_S) = 1 - h^{1,0} + h^{2,0}$ as usual.*

Now if $S \in |-K_X|$ is a smooth K3-surface from the anticanonical linear system on $X$ and $\chi(\mathcal{O}_X) = 1$ (as for example for a quasi Fano variety, see below), then

$$D_X(E) = \Delta(E|_S) \quad \text{and} \quad \frac{\chi(\mathcal{O}_S)}{2} = \chi(\mathcal{O}_X) \tag{3.1.51}$$

Thus we get

**Proposition 3.1.3.** *If $S \in |-K_X|$ is a smooth K3-surface from the anticanonical linear system on $X$ , then for any pair $E_1, E_2$ of vector bundles on $X$*

$$\chi_+(E_1, E_2) = \frac{\chi(E_1|_S, E_2|_S)}{2}$$

*and for any simple vector bundle on $X$*

$$v.dim\mathcal{M}_E = \frac{v.dim\mathcal{M}_{E|_S}}{2} \tag{3.1.52}$$

One can see from the next section that it isn't just a simple numerical coincidence.

**Canonical general type case.** The Bogomolov stability condition says that

$$\Delta(E) \cdot (-K) < 0 \tag{3.1.53}$$

and the formula for the virtual dimension has the form

$$v.dim\mathcal{M}^s_E = (rk(E)^2 - 1) \cdot \chi(\mathcal{O}_X) - rk(E) \cdot D_X(E) \cdot K_X. \tag{3.1.54}$$

One gets

**Corollary 3.1.1.** *The virtual (expected) dimension of moduli space of simple vector bundles $E$ on $X$ is non negative iff*

$$(rk(E)^2 - 1) \cdot \chi(\mathcal{O}_X) - rk(E) \cdot degD_X(E) \geq 0 \tag{3.1.55}$$

*where deg is with respect to the canonical polarisation $K_X$.*

So the collection of components is a priori bounded.

As an illustration consider the case (3.1.36) with the pair of positive constants

$$K_X = d \cdot H. \tag{3.1.56}$$

Then using constants (3.1.40) we can write down

$$v.dim\mathcal{M}_E = (rk(E)^2 - 1) \cdot \chi(\mathcal{O}_X) - rk(E)((c_2 - \frac{rk - 1}{2 \cdot rk} \cdot c_1^2) \cdot d. \tag{3.1.57}$$

Again we can consider the constrain (3.1.46) and under it one has

$$v.dim\mathcal{M}_E \leq (rk(E)^2 - 1) \cdot \chi(\mathcal{O}_X) + \frac{(rk - 1)^2}{4} \cdot d - rk(E) \cdot c_2 \cdot d \tag{3.1.58}$$

and you can see that the virtual dimension of the moduli spaces is non negative iff

$$0 < c_2 \leq \frac{(rk(E)^2 - 1) \cdot \chi(\mathcal{O}_X) + \frac{(rk-1)^2}{4} \cdot d}{rk(E) \cdot d}. \tag{3.1.59}$$

Thus we have the finite set of solutions to these inequalities.

## §2. Polynomials of positive degrees.

The results of §1 show that one may expect the existence of moduli spaces of positive dimensions in Fano and CG- cases only. All our computations are preserved if we consider a little more large classes of varieties:

**Definition 3.2.1.** *1) A variety $Y$ is called a quasi Fano iff the anticanonical linear system contains a smooth K3-surface and*

$$\chi(\mathcal{O}_Y) = 1 \tag{3.2.1}$$

*2) A variety $X$ is called a variety of quasi canonical general type (quasi CG for short) if the canonical class $K_X$ is nef and*

$$h^{0,1}(X) = h^{0,2}(X) = 0 \quad \textit{and} \quad h^{0,3} > 0. \tag{3.2.2}$$

A good example of quasi Fano variety is the blow up of a classical Fano variety.

Suppose we have the transversal situation for a Mukai vector $m$ that is there is the finite set of components of correct dimension of the moduli space $\mathcal{M}^s_X(m)$

$$\{\mathcal{M}^s_X(m)_1, ..., \mathcal{M}^s_X(m)_N\} \tag{3.2.3}$$

of the topological type $m = m(E)$. Then in parallel to the classical Donaldson polynomial for the real 4-dimensional case one can define the collection of polynomials $D_X(m)_j$ of degree

$$d = m^2 + \chi(\mathcal{O}_X) \tag{3.2.4}$$

on the truncated ring (3.1.1)

$$H^{2*}(X, \mathbb{Q})' = H^0(X, \mathbb{Q}) \oplus H^2(X, \mathbb{Q}) \tag{3.2.5}$$

using intersections of $\mu$-classes on the compactification of $\mathcal{M}^s_X(m)_j$. It is quite expectable construction:

$$D^m_j(\sigma) = \mu(\sigma)^d \tag{3.2.6}$$

on $\overline{\mathcal{M}^s_X(m)_j}$ for any $\sigma \in H^2(X, \mathbb{Q})'$.

Now the Abel-Jacobi map (2.2.8) sends every component $\mathcal{M}^s_X(m)_j$ of the moduli space to the intermediate Jacobian of $X$:

$$J_m(\mathcal{M}^s_X(m)_j) \subset J^3(X). \tag{3.2.7}$$

Let the codimension of the image be

$$codim_{J^3(X)} J_m(\mathcal{M}^s_X(m)_j) = cd^m_j. \tag{3.2.8}$$

Then one can define the new symmetrical polynomial on $H^2(J^3(X), \mathbb{Z})$ of degree $cd = cd^m_j$ :

$$A^m_j(\sigma) = (\sigma)^{cd} \cdot [J_m(\mathcal{M}^s_X(m)_j)]. \tag{3.2.9}$$

in the cohomology ring of the torus $J^3(X)$.

**Remark.** The image $J_m(\mathcal{M}^s_X(m)_j)$ admits the natural compactification in the intermideal Jacobian, so the fundamental class $[J_m(\mathcal{M}^s_X(m)_j)]$ is defined correctly.

Now the restriction of this polynomial to the subspace

$$\wedge^2 H^1(J^3(X), \mathbb{Z}) \subset H^2(J^3(X), \mathbb{Z}) \tag{3.2.10}$$

defines the tensor

$$A^m_j(X) \in S^{cd}(\wedge^2 H^3(X, \mathbb{Z}). \tag{3.2.11}$$

**Remark.** The amazing fact is that the number of such polynomials for Mukai vectors of vector bundles with fixed rank is finite. It is quite easy to see for quasi CG-varieties (see (3.1.52) and (3.1.56). But for the quasi Fano case it is easy to see that if v.dim of the component of moduli space is big enough then the Abel-Jacobi map is surjective.

Now Proposition 2.3.1 shows

**Proposition 3.2.1.** *If $X$ is quasi CG-variety and $\mathcal{M}^s_X(m)_j$ is a regular component then the invariant $A^m_j(X)$ (3.2.11) is non trivial.*

Indeed in this case the Abel-Jacobi map isn't surjective.

In the quasi Fano and CG realm it is quite reasonable to define the Casson - Donaldson invariant in the following way:

**Definition 3.2.2.** *For a variety $X$ and a Mukai vector $m$ the Casson-Donaldson invariant*

$$CD(X, m) = \#\ \text{of connected components of } \mathcal{M}^s_X(m) \tag{3.2.12}$$

It is quite reasonable because formally the Casson-Donaldson invariant admits the interpretation of the same type as (3.1.29) by Proposition 2.3.1.

In the transversal situation there is the finite set of right dimensional components of $\mathcal{M}^s_X(m)$

$$\{\mathcal{M}^s_X(m)_1, ..., \mathcal{M}^s_X(m)_{CD(m)}\} \tag{3.2.13}$$

of the topological type $m = m(E)$. Then the Abel-Jacobi map (2.2.8) sends this collection of spaces to the intermediate Jacobian of $X$:

$$\{J_m(\mathcal{M}^s_X(m)_1), ..., J_m(\mathcal{M}^s_X(m)_{CD(m)})\} \subset J(X) \tag{3.2.14}$$

This collection is defined up to translation and we can kill this ambiguity by the condition

$$\sum_i pr_{H^{3,0}} \cdot J_m(\mathcal{M}^s_X(m)_i) = 0 \tag{3.2.15}$$

where $pr_{H^{3,0}}$ is the projection described in Proposition 2.3.1.

In some cases which are very close to trivial we can prove that

$$i \neq j \implies J_m(\mathcal{M}^s_X(m)_i) \cap J_m(\mathcal{M}^s_X(m)_j) = \emptyset. \tag{3.2.16}$$

In parallel to the classical Casson invariant which can be computed using the cutting-pasting procedure given by Heegard diagrams ( see Ch.I and [A-M]). Donaldson and Thomas [D-T] proposed to compute complex Casson-Donaldson invariants (3.1.6) using the analogical complex "cutting-pasting" procedure.

## §3. Geometry of vector bundles on flags.

As usual for computations it is quite productive to use the degeneration principle considering our main problems for special type of degenerations of Calabi-Yau threefolds: $X = Y_+ \cup_S Y_-$, where $Y^{\pm}$ are smooth threefolds transversally intersecting along non-singular surface $S$ of K3-type from the anticanonical systems $S \in | - K_{Y_\pm}|$.

Before going to the problem of gluing of two quasi Fano flags to the virtual Calabi-Yau threefold let us describe (in real parallel to §2 of Chapter I) the geometry of vector bundles on a pair $(S \subset Y)$ where $S \in |-K_Y|$ is a smooth K3-surface of the anticanonical linear system of variety $Y$.

First of all, returning to topological invariants of vector bundles on $Y$ it is easy to see from (3.1.17), (3.1.9) and (3.1.19) that the symmetric form $\chi_+(E_1, E_2)$ depends on three first components of the cohomology ring (3.1.1).

Now,

$$\mathcal{O}_Y(K_X) = \mathcal{O}_Y(-S). \tag{3.3.1}$$

As any invertible sheaf this line bundle defines the automorphism $T_{K_Y}$ of the lattice $K^0_{top} = H^{2*}(Y, \mathbb{Z})$ (3.1.1) which we consider as the Mukai lattice with the symmetric bilinear form (3.1.16). Then

$$T_{K_Y}(m) = m \cdot e^{K_Y} \tag{3.3.2}$$

and the restriction map defines the map of Mukai lattices

$$res : M_Y \to M_S = H^*(S, \mathbb{Z}). \tag{3.3.3}$$

So the image in the Mukai-lattice of $S$ is the image of operator

$$im(id - T_{-S}) = M_S. \tag{3.3.4}$$

More precisely,

$$(id - T_{-S})(u_0, u_1, u_2, u_3) = (0, -K_X \cdot u_0, u_1 \cdot S, u_2 \cdot S) = (u_0, u_1 \cdot S, u_2 \cdot S) \in H^*(S) \tag{3.3.5}$$

From (3.1.7) one can see that the bilinear

$$<,> = \frac{1}{2} res^*(,) \tag{3.3.6}$$

is the preimage of the standard symmetric bilinear form

$$(u, v) = -[v^* \cdot u]_4 \tag{3.3.7}$$

on $H^*(S, \mathbb{Z})$.

Now the restriction (=transformation (3.3.5)) of the root (3.1.15)

$$(id - T_{-S})(\sqrt{td^+_X}) = (1, 0, 1) = \sqrt{td_S} \tag{3.3.8}$$

is the root of the Todd class of K3-surface, because on a quasi Fano variety $\frac{k_Y \cdot K_Y}{24} = 1$. Thus for any Mukai vector (3.1.16) of a vector bundle on $Y$

$$(id - T_{-S})(m(E)) = m(E|_S) = ch(E|_S) \cdot \sqrt{td_S} \tag{3.3.9}$$

is the classical Mukai vector of vector bundle $E|_S$ on K3-surface $S$ (see [M]).

Moreover, the symmetrical bilinear form $\chi(E_1, E_2)_+$ (3.1.17) on $Y$ is the preimage of the classical symmetric form

$$\chi(E_1|_S, E_2|_S) = (m(E_1|_S), E_2(|_S)) \tag{3.3.10}$$

where $(,)$ is (3.3.7). (We saw it already in the previous section.)

Now using the Bertini theorem one has

$$H^2(Y, \mathbb{Z}) = H^{1,1}(S) \cap H^2(S, \mathbb{Z}) \tag{3.3.11}$$

and by Mukai theorem for any primitive vector $m = (u_0, u_2, u_4)\ inH^{2*}(S, \mathbb{Z})$, satisfying

$$u_0 > 0; \quad u_2 \in H^{1,1}(S); \quad m^2 \geq -2 \implies \mathcal{M}_S(m) \neq \emptyset \tag{3.2.12}$$

that is there exists a stable bundle $E_0$ such that $m = m(E)$. It is easy to see that then

$$dim \mathcal{M}_S(m) \geq m^2 + 2. \tag{3.3.13}$$

Suppose there exists a $(-K_Y)$ - stable vector bundle $E$ on $Y$ such that $E_0 = E|_S$. Then

$$\mathcal{M}_Y(m) \neq \emptyset \quad \text{and} \quad dim \mathcal{M}_Y(m) \geq \tfrac{1}{2}m^2 + 1 \tag{3.3.14}$$

(see [M]).

**Definition 3.3.1.** *A vector bundle $E$ on $Y$ is called regular if*

$$H^2(adE) = 0 \tag{3.3.15}$$

*An irreducible component $\mathcal{M}_Y(m)_0$ is called regular if a generic bundle $E$ of it is regular.*

By the deformation theory one has in this case

$$dim \mathcal{M}_Y(m)_0 = v.dim \mathcal{M}_Y(m)_0 = \tfrac{1}{2}m^2 + 1 \tag{3.3.16}$$

Now the restriction map

$$r : \mathcal{M}_Y(m)_0 \to \mathcal{M}_S(m) \tag{3.3.17}$$

is immersion on bundles $E$ with condition (3.3.15).

Both of these statements follow from the short exact sequence

$$0 \to H^1(adE) \underset{dr}{\longrightarrow} H^1(adE|_S) \to H^2(adE(K_X)) \to 0 \tag{3.3.18}$$

of the restriction sequence for $adE$:

$$0 \to adE(K_X) \to adE \to adE|_S \to 0. \tag{3.3.19}$$

Moreover, the continuation of the cohomological sequence (3.3.18)

$$0 \to H^2(adE|_S) \to H^3(adE(K_X)) \tag{3.3.20}$$

shows that $H^2(adE|_S)$ if $E$ is simple :

$$H^0(adE) = 0 \implies H^3(adE(K_X)) = H^0(adE)^* = 0 \tag{3.3.21}$$

Thus for a simple bundle on $Y$

$$E \quad \text{is regular} \implies E|_S \quad \text{is regular.} \tag{3.3.22}$$

and one has

**Proposition 3.3.1.** *The restriction map $r$ (3.3.17) is generically an immersion to the regular component $\mathcal{M}_S(m)_0$ and*

$$dim\mathcal{M}_S(m)_0 = 2dim\mathcal{M}_Y(m)_0 \tag{3.3.23}$$

Now every regular component $\mathcal{M}_S(m)_0$ of vector bundles on $S$ admits the Mukai holomorphic symplectic structure

$$\omega_S : T\mathcal{M}_S(m)_0 \to T^*\mathcal{M}_S(m)_0 \tag{3.3.24}$$

which can be defined in parallel to the canonical symplectic form on Riemann surface $\Sigma$ (1.2.26-29).

Namely for an algebraic surface $S$ consider a topological vector bundle $E$ with the Mukai vector $m$ and the space $\overline{\partial}$-operators on $E$. Consider the 2-form on this space given on the tangent space $\Omega^{0,1}(EndE)$ of a $\overline{\partial}$-operator $\overline{\partial}_a$ by the formula

$$\omega_S(\omega_1, \omega_2) = \int_\Sigma tr(\omega_1 \wedge \omega_2) \wedge \theta. \tag{3.3.25}$$

where $\theta$ is any (2,0)-form on $S$.

Again this form

1) is closed,

2) $\mathcal{G}$ - invariant (because of "tr"),

3) degenerate along $\mathcal{G}$-orbits only.

From this we get the (2,0)-form on the orbit space $\mathcal{D}''_S(E)$ and by restriction of it to

$$((\overline{\partial})^2)_0 = \cup\mathcal{M}_i \supset \mathcal{M}_S(m)_0 \tag{3.3.26}$$

we get the symplectic structure $\omega_S$ on any regular component $\mathcal{M}_S(m)_0$ on $S$.

**Proposition 3.3.2.** *The image $r(\mathcal{M}_Y(m)_0) \in \mathcal{M}_S(m)_0$ is a maximal symplectic subvariety of $\mathcal{M}_S(m)_0$ with respect to $\omega_S$.*

Indeed, the tangent space of $\mathcal{M}_S(m)_0$ at any regular bundle $r(E)$ is $H^1(adE|_S)$ and we can consider the monomorphism of (3.3.18) as the differential of the restriction map:

$$0 \to (T\mathcal{M}_Y(m)_0)_E \underset{dr}{\longrightarrow} (T\mathcal{M}_S(m)_0)_{E|_S} \tag{3.3.27}$$

On the other hand the epimorphism of this exact sequence

$$(T^*\mathcal{M}_S(m)_0)_{E|_S} \underset{dr}{\longrightarrow} (T^*\mathcal{M}_Y(m)_0)_E \to 0 \tag{3.3.28}$$

we can consider as the codifferential conjugate to the differential monomorphism using the identification

$$(T\mathcal{M}_S(m)_0)_{E|_S} = H^1(adE|_S) = H^1(adE|_S)^* = (T^*\mathcal{M}_S(m)_0)_{E|_S} \tag{3.3.29}$$

given by Serre-duality which is the restriction of the symplectic structure $\omega_S$ on the fibre of the tangent bundle (see [M]). But the sequence (3.3.18) is exact thus

$$\omega_S|_{(T\mathcal{M}_Y(m)_0)_E} = 0 \tag{3.3.30}$$

and we are done. Moreover, in the regular case the normal bundle

$$N_{r(\mathcal{M}_Y(m)_0)\in\mathcal{M}_S(m)_0} = T^*\mathcal{M}_Y(m)_0 \tag{3.3.31}$$

is the cotangent bundle of the moduli space of vector bundles on $Y$.

Now recall that every moduli space $\mathcal{M}_S(m)$ defines the $\mu$-map

$$\mu : H^*(S,\mathbb{Z}) \to H^2(\mathcal{M}_S(m)) \oplus H^4(\mathcal{M}_S(m)). \tag{3.3.32}$$

by the slant-product on $c_2$ of the quasi universal bundle on $S\times\mathcal{M}_S(m)$. Thus one gets the polynomial on $H^2(S,\mathbb{Z})$ :

$$D_{S\subset Y}[m]^{m^2+1}(\sigma) = \mu(\sigma)^{\frac{1}{2}m^2+1}\cdot[r(\mathcal{M}_Y(m))] \tag{3.3.33}$$

just like Donaldson polynomial for 4-fold.

We don't want to discuss here technical problems like compactness, correct definitions of the fundamental cycle $[r(\mathcal{M}_S(m))]$ and so on. This can be done by the usual procedure.

But much more interesting is another integer invariant of a Mukai vector $m$ on a pair $(S\subset Y)$:

$$CD_{(S\subset Y)}(m) = [r(\mathcal{M}_Y(m))]^2 \tag{3.3.34}$$

which we call the *relative Casson-Donaldson invariant of a pair* $(S\subset Y)$.

In the compact and non-singular case this number can be computed as the top Chern class

$$CD_{(S\subset Y)}(m) = c_{top}(N_{r(\mathcal{M}_Y(m))\subset\mathcal{M}_S(m)}) \tag{3.3.35}$$

of the normal bundle.

Suppose that we are in the regular case. Then from (3.3.31)

$$N_{r(\mathcal{M}_Y(m))\subset\mathcal{M}_S(m)} = T^*\mathcal{M}_Y(m) \tag{3.3.36}$$

is the cotangent bundle of the moduli space and

$$CD_{(S\subset Y)}(m) = \chi_{top}\mathcal{M}_Y(m) \tag{3.3.37}$$

We can use this relative Casson-Donaldson invariant for computation of Casson-Donaldson invariants in Calabi-Yau case.

## §4. Deformations of flags and vector bundles.

The deformation theory of flags or pairs is quite parallel to the deformation theory for complex manifolds : one has the vector bundle (or the coherent sheaf) $T(S \subset Y)$ such that $H^1(T(S \subset Y))$ is the space of formal linear deformations, $H^2(T(S \subset Y))$ is the space of obstructions for such deformations and there exists the connecting Kuranishi map

$$\Phi : H^1(T(S \subset Y)) \to H^2(T(S \subset Y)). \tag{3.4.1}$$

Then locally $\Phi^{-1}(0)$ is modelling the moduli space of deformations.

It is easy to construct such sheaf $T(S \subset Y)$ for deformations of flags (or pairs) $(S \subset Y)$ : consider the restriction sequence for the tangent bundle

$$0 \to TY(-S) \to TY \to TY|_S \to 0 \tag{3.4.2}$$

and the standard exact sequence on $S$

$$0 \to TS \to TY|_S \to N_{(S \subset Y)} \to 0, \tag{3.4.3}$$

where the last line bundle is the normal sheaf of the surface in the threefold.

The kernel of the composition of epimorphisms of these sequences is the required sheaf and we get the sequence

$$0 \to T(S \subset Y) \to TY \to N_{(S \subset Y)} \to 0. \tag{3.4.4}$$

The kernel of this epimorphism is just our sheaf of vector fields (3.3.1) for deformations of pairs. Recall that for the classical Kuranishi model one has to use the tangent bundle $TY$ for deformations $Y$ or $TS$ - for deformations $S$. We suppose (and it is reasonable for (K3 $\subset$ quasi Fano) flags) that

$$H^1(S, N_{(S \subset Y)}) = 0. \tag{3.4.5}$$

Then parts of long exact sequence (3.3.4) can be joined by Kuranishi maps

$$\begin{array}{cccc} 0 \to & H^0(N_{(S\subset Y)}) \to & H^1(T(S \subset Y)) \to & H^1(TY) \to 0 \\ & \downarrow & \downarrow & \downarrow \\ & 0 \to & H^2(T(S \subset Y)) \to & H^2(TY) \to 0 \end{array} \tag{3.4.6}$$

From this we immediately get

**Proposition 3.4.1.** *Deformations of $Y$ are unobstructed $\implies$ deformations of the pair $(S \subset Y)$ are unobstructed.*

From the definition of $T(S \subset Y)$ one has the exact sequence

$$0 \to TY(-S) \to T(S \subset Y) \to TS \to 0 \tag{3.4.7}$$

If our pair is a (K3 $\subset$ Fano)- flag then we have as the long cohomological sequence of (3.4.7)

$$0 \to H^1(\wedge^2 \Omega Y) \to H^1(T(S \subset Y)) \to H^1(\Omega S)^* \to H^1(\Omega Y)^* \to$$

$$\to H^2(T(S \subset Y)) \to 0. \tag{3.4.8}$$

We are using here the equality $E \otimes \wedge^3 E^* = \wedge^2 E^*$ if $rkE = 3$ and the Serre-duality. Consider the restriction map. It is easy to see that the homomorphism

$$H^1(\Omega S)^* \to H^1(\Omega Y)^*$$

of the exact sequence (3.4.8) is dual to the restriction map

$$r : H^{1,1}(Y) \to H^{1,1}(S). \tag{3.4.9}$$

(under the Dolbeault isomorphism).

Thus we get

**Proposition 3.4.2.** *The obstructions space*

$$H^2(T(S \subset Y)) = (ker \quad r)^* \subset H^{2,2}(Y) \tag{3.4.10}$$

*In particular $PicY = \mathbb{Z} \implies$ deformations of pairs $(S \subset Y)$ are unobstructed and thus by (3.3.6) deformations of $Y$ are unobstructed too.*

Indeed, the homomorphism $H^1(\Omega S)^* \to H^1(\Omega Y)^*$ is non trivial so under the condition it has to be an epimorphism.

Moreover, the space

$$H^1(TY(K_Y)) = H^1(\wedge^2 \Omega Y) \tag{3.4.11}$$

is the space of infinitesimal deformations of the pair preserving complex structure on $S$. On the other hand this space is the tangent space at 0 of the intermediate Jacobian of $Y$:

$$H^1(\wedge^2 \Omega Y) = (TJ_Y)_0 \tag{3.4.12}$$

This fact is quite expected from the point of view of deformations of reducible Calabi-Yau threefolds.

Recall that there exists the collection of obstructions for the equivalence of nth-order thickening of our K3-surface $S$ in the quasi Fano $Y$ and the flat model of it. The first obstruction is the class

$$\omega_1 \in H^1(TS \otimes N^*_{(S \subset Y)}). \tag{3.4.13}$$

and from the standard exact sequence

$$H^1(TS \otimes N^*_{(S \subset Y)}) = H^1(\wedge^2 \Omega Y|_S) \tag{3.4.14}$$

On the other hand, for our special case by the Serre-duality

$$H^1(TS \otimes N^*_{(S \subset Y)}) = H^1(TS \otimes N_{(S \subset Y)})^*. \tag{3.4.15}$$

We use these indentifications for the gluing procedure.

## Ch. IV. Cutting-pasting in the complex case.

### §1. Cutting-pasting in the almost complex case.

We can apply the constructions of Ch.II to a real 6-fold with almost complex structures. But in this general case we can't control the finiteness (or compactness) of spaces of solutions. In spite of the fact that some other serious technical problems like "stability conditions" can be avoided in the general case (see [B-T]) it is productive to restrict ourselves now by the algebro-geometrical case. The interpretation of a *complex orientation* as a holomorphic trivialization of the determinant of the tangent bundles sends us to the realm of Calabi-Yau varieties (for the general picture see [D-T]). In this realm degenerations of Calabi-Yau threefolds are very important for one of the approaches to the description of "mirror symmetries" of varieties of such type. On the other hand a large complex structure limit points of the compactification family of Calabi-Yau threefolds play an important role for comparing the Yakawa coupling with the (1,1)-topological coupling of the mirror. For the investigation of vector bundles on CY-threefolds we will use (following [D-T]) the special type of degenerations of Calabi-Yau threefolds like :

$$X_0 = Y_+ \cup_S Y_- \tag{4.1.1}$$

where $(S \subset Y_\pm)$ is a considered pair of a smooth K3-surface $S$ from the anticanonical system of a quasi Fano threefold $Y$. There are three levels of the problem of deformations of such singular object to the non-singular one:

1) as an almost complex variety;
2) as a complex variety but without a polarisation;
3) as an algebraic variety with the polarisation.
4) with smooth total space of deformation.

**Remark.** The last condition is quite important for the investigation of vector bundles on fibres of deformations.

The full collection of these deformation problems was investigated in [D-F], [L]. These deformations problems are quite different but for every of them the main question is the same:

**Question 4.1.1.** *How many topological types of smooth deformations of threefolds (3.3.1) can one get by this construction?*

Remark, that the set of types of quasi Fano threefolds is bounded and finite, so we can expect that we get the finite set of topological types of 6-folds.

Starting with the configuration (4.1.1) first of all we get on the K3-surface $S$ two normal bundles $\mathcal{O}_S(-K_{Y_\pm}) = N_{S\subset Y_\pm}$ a priori of quite different genus. Now our cutting-pasting procedure is quite parallel to the real case : first of all cut a small neighbourhood of the singular locus $S$ in $X_0$ considering small tubes

$$S^1(N_{S\subset Y_\pm}) \subset N_{S\subset Y_\pm} = L_{-K_{Y_\pm}} \tag{4.1.2}$$

in normal bundles

$$N_{S\subset Y_\pm} = L_{-K_{Y_\pm}}. \tag{4.1.3}$$

So we are removing a small disc-bundles

$$D^2(N_{S\subset Y_\pm}) \quad \text{with the boundary} \quad \partial D^2(N_{S\subset Y_\pm}) = -S^1(N_{S\subset Y_\pm}) \tag{4.1.4}$$

from quasi Fano variety $Y_\pm$ and get the open singular threefold

$$V_0 = D^2(N_{S\subset Y_+}) \cup_S D^2(N_{S\subset Y_-}) \tag{4.1.5}$$

the union of a couple of disc-bundles gluing along zero-sections of normal disc-bundles.

Thus one gets three 6-manifolds with boundaries:

$$V_0; \quad \text{with the boundary} \quad \partial V_0 = S^1(N_{S\subset Y_+}) \cup S^1(N_{S\subset Y_-});$$

$$Y_\pm^0 = Y_\pm - D^2(N_{S\subset Y_\pm}) \quad \text{with boundaries} \quad S^1(N_{S\subset Y_\pm}) \tag{4.1.6}$$

and $X_0$ is the gluing of these three pieces along corresponding boundaries.

Now we can deform slightly the singular 6-fold $V_0$ with the preserving of the boundary $\partial V_0$ using the following construction (see [D-F]): consider the quadratic map of bundles on $S$

$$q : N_{S\subset Y_+} \oplus N_{S\subset Y_-} \to N_{S\subset Y_+} \otimes N_{S\subset Y_-} = L_{-K_{Y_+}-K_{Y_-}}. \tag{4.1.7}$$

Considering any section $s \in H^0(L_{-K_{Y_+}-K_{Y_-}})$ as the embedding

$$i_s : S \to L_{-K_{Y_+}-K_{Y_-}}. \tag{4.1.8}$$

our base $S$ to the body of the line bundle. Then $s$ defines the threefold

$$V_s = q^{-1}(i_s(S)) \cap D^2(N_{S\subset Y_+}) \times_S D^2(N_{S\subset Y_-}) \tag{4.1.9}$$

It can be shown that if the neighbourhoods are small enough the boundary is diffeomorphic to

$$\partial V_s = S^1(N_{S\subset Y_+}) \cup S^1(N_{S\subset Y_-}) \tag{4.1.10}$$

Now one can glue $V_s$ with $Y_\pm^0$ along components of boundary and get new compact 6-manifold $X_s$.

Now if the zero set

$$(s)_0 = C \in |-K_{Y_+} - K_{Y_-}| \tag{4.1.11}$$

of the section $s$ is a smooth curve on $S$ then $V_s$ (4.1.9) is non-singular and we have some topomodel of Calabi-Yau threefold. If $C$ admits some simple singularities then $V_s$ is singular in these points but the small resolution of these singular points gives the topomodel of Calabi-Yau threefold of another topological type. It will be very useful to get the full list of topomodels of CY-threefolds which can be done by this procedure. In particular, we can consider topomodels which are coming from deformations when $(s)_0 = C$ is a rational curve with double points only.

**Remark.** We will get, by this construction, the topomodel of the Barth-Nieto-van Straten rigid CY-threefold as a deformation of the reducible threefold of type (4.4.1).

It is easy to see that we can do this smoothing surgery procedure with almost complex structure.

Now we have to describe deformations of complex structure along described deformations of topological types for a singular variety of type (4.4.1). We will use the canonical procedure of deformation theory.

Recall from [D-F], [L], ... that in our case there exists the sheaf $T(Y_+ \supset S \subset Y_-)$ which can be constructed from $T(S \subset Y_\pm)$ (4.4.1) such that the space $H^1(T(Y_+ \supset S \subset Y_-))$ is the space of infinitesimal deformations of reducible threefolds of the same type and the space $\mathcal{H}^1$ of infinitesimal deformations is more complicated:

$$0 \to H^1(T(Y_+ \supset S \subset Y_-)) \to \mathcal{H}^1 \to H^0(S, N_{(S\subset Y_+)} \otimes N_{(S\subset Y_-)}) \to 0 \qquad (4.1.12)$$

but the space of obstructions has precisely the same type

$$\mathcal{H}^2 = H^2(T(Y_+ \supset S \subset Y_-)) \qquad (4.1.13)$$

(see 5.1 of [D-F]).

To describe $T(Y_+ \supset S \subset Y_-)$ consider the disjoint union of two flags $(S_\pm \subset Y_\pm)$ and the identification maps

$$n_\pm : S_\pm \to S = Sing X_0 \qquad (4.1.14)$$

Using the exact sequence (3.4.7) for every flag component $(S_\pm \subset Y_\pm)$ one gets the required sheaf from the exact sequence

$$0 \to T(Y_+ \supset S \subset Y_-) \to T(S_+ \subset Y_+) \oplus T(S_- \subset Y_-) \xrightarrow{\frac{1}{2}(n_+ + n_-)_*} TS \to 0. \qquad (4.1.15)$$

Thus in parallel to (3.3.7) for $T(Y_+ \supset S \subset Y_-)$ we have the exact sequence

$$0 \to \oplus_\pm TY_\pm(-S - \pm) \to T(Y_+ \supset S \subset Y_-) \to TS = ker \tfrac{1}{2}(n_+ + n_-)_* \to 0 \qquad (4.1.16)$$

and the long cohomology sequence

$$0 \to \oplus_\pm H^1(\wedge^2 \Omega Y_\pm) \to H^1(T(Y_+ \supset S \subset Y_-)) \to H^1(\Omega S)^* \to \oplus_\pm H^1(\Omega Y)^* \to$$
$$\to H^2(T(Y_+ \supset S \subset Y_-)) \to 0. \qquad (4.1.17)$$

The sum of compositions $(r_\pm \cdot (n_\pm)_*)$ gives the map

$$(r_+ \cdot (n_+)_*) + (r_- \cdot (n_-)_*) : H^{1,1}(Y_+) \oplus H^{1,1}(Y_-) \to H^{1,1}(S) \qquad (4.1.18)$$

**Proposition 4.1.1.** *Then the space of obstructions*

$$H^2(T(Y_+ \supset S \subset Y_-)) = (ker(r_+ \cdot (n_+)_*) + (r_- \cdot (n_-)_*))^* \subset H^{1,1}(Y_+)^* \oplus H^{1,1}(Y_-)^*. \qquad (4.1.19)$$

Returning to (4.1.12) one can see that the deformations complex with the Kuranishi map is

$$0 \to H^1(T(Y_+ \supset S \subset Y_-)) \to \mathcal{H}^1 \to$$
$$\to H^0(S, N_{(S\subset Y_+)} \otimes N_{(S\subset Y_-)}) \xrightarrow{\Psi} H^2(T(Y_+ \supset S \subset Y_-)) = \qquad (4.1.20)$$
$$= (ker(r_+ \cdot (n_+)_*) + (r_- \cdot (n_-)_*))^* \subset H^{1,1}(Y_+)^* \oplus H^{1,1}(Y_-)^* =$$
$$= H^{2,2}(Y_+) \oplus H^{2,2}(Y_-).$$

For the precise description of $\Psi$ see [D-F].

**Corollary 4.1.1.** *1) If $PicY_{\pm} = \mathbb{Z}$ and $N_{(S\subset Y_{+})} \otimes N_{(S\subset Y_{-})}$ generated by sections and non trivial then the dimension of space $M_{X_0}$ of nonsingular deformations is*

$$v.dimM_{X_0} = h^{1,2}(Y_{+}) + h^{1,2}(Y_{-}) + h^{0}(N_{(S\subset Y_{+})} \otimes N_{(S\subset Y_{-})}) - 1; \tag{4.1.21}$$

*2) if*

$$N_{(S\subset Y_{+})} \otimes N_{(S\subset Y_{-})} = \mathcal{O}_S \tag{4.1.22}$$

*then the deformations are unobstructed and*

$$v.dimM_{X_0} = h^{1,2}(Y_{+}) + h^{1,2}(Y_{-}) + 1 \tag{4.1.23}$$

*and the body of the deformation family is smooth.*

The second statement is quite known (see [D-F], and [L]). To show the first statement we consider the 1st order jet that is consider the first obstruction classes of pairs $(S_{\pm} \subset Y_{\pm})$ (3.4.13)

$$\omega_1^{\pm} \in H^1(TS \otimes N^*_{(S_{\pm}\subset Y_{\pm})}). \tag{4.1.24}$$

and the natural homomorphism

$$H^0(N_{(S_{+}\subset Y_{-})} \otimes N_{(S_{-}\subset Y_{-})}) \otimes H^1(TS \otimes N^*_{(S_{\pm}\subset Y_{\pm})}) \xrightarrow{c}$$

$$\xrightarrow{c} H^1(TS \otimes N_{(S_{\pm}\subset Y_{\pm})}) = H^1(TS \otimes N^*_{(S_{\pm}\subset Y_{\pm})})^*. \tag{4.1.25}$$

(see (3.4.15) ). Now it is easy to see that 1-extension of the deformation given by a section $s \in H^0(N_{(S_{+}\subset Y_{-})} \otimes N_{(S_{-}\subset Y_{-})})$ is constrained by only one condition:

$$\omega_1^{-}(c(s \otimes \omega_1^{+})) = 0. \tag{4.1.26}$$

From this it is easy to prove statement 1).

Suppose the K3-surface $S$ of a quasi Fano flag $(S \subset Y)$ admits an involution

$$i : S \to S \tag{4.1.27}$$

such that

$$i^*(K_Y|_S) \neq K_Y|_S. \tag{4.1.28}$$

Consider the reducible CY-threefold $X_0$ given by gluing maps (4.4.14) with

$$n_{+} = id \quad \text{and} \quad n_{-} = i. \tag{4.1.29}$$

**Corollary 4.1.2.** *Then this double $X_0$ can be deformed to a smooth CY-threefold.*

For the proof remark that this situation is precisely the same as the gluing the twistor space along the non-singular quadric in [D-F]. All arguments in [D-F] for proving the existence of a smooth deformation work in our case too.

## §2. Cutting-pasting collection of CY-threefolds.

Consider all possible quasi Fano flags

$$\{(S \subset Y)\}, \tag{4.2.1}$$

all possible gluings

$$\{(n_{\pm} : S_{\pm} \to S)\}, \tag{4.2.2}$$

all possible (and non possible) deformations of reducible CY-threefolds and all small desingularisations of deformations.

**Definition 4.2.1.** *This class of CY-threefolds is called CP-class.*

Consider some reducible CY-threefold (which is of CP-type by definition)

$$X = (Y_+ \supset S), n_+ \cdot (n_-)^{-1} = g, (S \subset Y_-). \tag{4.2.3}$$

Then we have the pair of restriction maps of Mukai lattices (see (3.3.1 - 9)

$$res_+ : M_{Y_+} \to M_S \leftarrow M_{Y_-} : g^* \cdot res_- \tag{4.2.4}$$

and the intersection

$$res_+(M_{Y_+}) \cap g^* \cdot res_-(M_{Y_-}) = M_0. \tag{4.2.5}$$

From (3.3.8) one has

$$(id - T_{-S})(\sqrt{td^+_{Y_{\pm}}}) = (1, 0, 1) = \sqrt{td_S} \in M_0. \tag{4.2.6}$$

The restriction maps (4.2.4) define the pair of epimorphisms

$$p_{\pm} : M_{Y_{\pm}} \to M_0 \tag{4.2.7}$$

and the Mukai lattice of $X$ is

$$M_X = p_+^{-1}(M_0) \oplus_{M_0} \oplus p_-^{-1}(M_0) \tag{4.2.8}$$

In particular every Mukai vector is of the form

$$m = (m_+, m_-) \tag{4.2.9}$$

and using the Poincare dualities on $Y_{\pm}$ and $S$ it's easy to define the form (3.1.9) in in such a way that it has to be skew-symmetric:

$$\chi((m^1_+, m^1_-), (m^2_+, m^2_-) = < m^1_+, m^2_+ > - < m^2_-, m^1_- > . \tag{4.2.10}$$

Thus for every CY-threefold of CP-class we described the Mukai lattice that is the ring $H^{2*}(X)$.

Now if we would like to define vector bundles on $X$ the procedure is almost obvious:

**Proposition 4.2.1.** *A couple of stable vector bundles $E_\pm$ on the pair of quasi Fano threefolds $Y_\pm$ can be glued to a vector bundle on $X$ iff*

$$E_+|_S = g^*(E_-|_S) \tag{4.2.11}$$

In the same vein we can define coherent cohology spaces of a vector bundle on $X$.

Now, because the expected behaviour of a vector bundle on $X$ is to be infinitesimal rigid the problem is to deform a holomorphic bundle along some smooth deformation of $X$. From a technical point of view to reduce this problem to the problem of deformation of projectivizations of the components of vector bundles (see [D-F]) it is much more easy to consider deformations with smooth total spaces of deformations. It's easy to prove the following

**Proposition 4.2.2.** *The total space of described deformation of a reducible CY-threefold is smooth only in the situation 2) of Corollary 4.1.2 that is iff the tensor product of normal bundles is trivial.*

Returning to the description of deformations (see the previous section recall that a tangent vector $s$ to the deformation of $X$ (4.2.3) is given by a section

$$s \in H^0(S, N_{(S\subset Y_+)} \otimes N_{(S\subset Y_-)}) \tag{4.2.12}$$

and our threefold becomes smooth iff the zero set of this section

$$(s)_0 = C \subset S \tag{4.2.13}$$

is a smooth curve on our K3-surface $S$.

**Remark.** It is easy to see that if $C$ admits simplest singular points only then a general fibre of this one-dimensional deformation admits simplest singular points in these points and small resolution defines a new smooth CY-threefold.

To reduce this situation to the situation with trivial tensor product (4.2.13) one has to blow up the Fano variety $Y_+$ along this curve $C$ :

$$\sigma : \tilde{Y_+} \to Y_+ \tag{4.2.14}$$

and get the quasi Fano variety $\tilde{Y_+}$ with the anticanonical class $-K_{\tilde{Y_+}}$ presented by our surface $S \subset \tilde{Y_+}$. Now the normal bundle

$$N_{(S\subset \tilde{Y_+})} = N^*_{(S\subset Y_-)}, \tag{4.2.15}$$

the tensor product (4.2.13) is trivial and one comes to the situation 2) of Corollary 4.1.2.

**Mirror digression.** In the set up of our Mirror digression §3 of Ch. III the SYZ-conjecture suggests that on the mirror partner $X'$ of CY-threefold $X$ there exists a special Lagrangian 3-torus fibration

$$\pi' : X' \to \overline{\mathcal{M}^3_{X'}(mir(m))}, \tag{4.2.16}$$

where the base is the special compactification of the moduli space of SLag torus (3.1.29) of the 3-cohomology class $mir(m)$, and $X$ is the suitable compactification of the dual fibration:

$$\pi : X \to \overline{\mathcal{M}^3_{X'}(mir(m))}, \qquad (4.2.17)$$

A degeneration of CY-threefold to a reducible CY-threefold of described type admits the limit fibrations on both of Fano components $Y_\pm$ which can be compactified to fibrations:

$$\pi_\pm : (Y_\pm - S) \to \overline{\mathcal{M}^3_{Y_\pm}(mir(m))} \qquad (4.2.18)$$

where every base is the 3-fold with the boundary which conjecturally is the curve $C$ (4.2.13).

Thus the degeneration procedure of CY-threefold $X'$ to the reducible threefold $X'_0$ provides the representation of the base $\overline{\mathcal{M}^3_{X'}(mir(m))}$ of the SLag-torus filtration by the Heegard diagram that is as a gluing of $\overline{\mathcal{M}^3_{Y_\pm}(mir(m))}$ along $C$.

Let us return to reducible CY-threefold. The last problem is to describe odd-dimensional cycles on CP-threefold $X$ (4.2.3).

**Proposition 4.2.3.** *Let $X_{\epsilon \cdot s}$ be a smooth generic deformation of CP-threefold $X$ (4.2.3) given by a section $s$ (4.2.13) with the smooth curve $C$ (4.2.14). Then there exists (non canonical) isomorphism*

$$H^3(X_{\epsilon \cdot s}, \mathbb{Q}) = H^3(Y_+, \mathbb{Q}) \oplus H^3(Y_-, \mathbb{Q}) \oplus H^1(C, \mathbb{Q}) \oplus H^2(S, \mathbb{Q})_0 \qquad (4.2.19)$$

*where $H^2(S, \mathbb{Q})_0$ is the primitive part of $H^2(S, \mathbb{Q})$ that is*

$$H^2(S, \mathbb{Q})_0 = (H^2(S, \mathbb{Q})_0 \cap M_0)^\perp. \qquad (4.2.20)$$

These 3-cycles coming from $H^2(S, \mathbb{Q})_0$ can be observed in the part $V_s$ (4.1.9)

It is easy to see that in the situation 2) of Corollary 4.1.2 the answer is precisely the same because of the equality

$$H^3(\tilde{Y_+}, \mathbb{Z}) = H^3(Y_+, \mathbb{Z}) \oplus H^1(C, \mathbb{Z}). \qquad (4.2.21)$$

The last point is the description of the intermediate Jacobian of a CP-threefold $X$ (4.2.3). The deformation of Hodge structures along the deformation $X_{\epsilon \cdot s}$ shows that as holomorphic group

$$J^3(X) = lim_{\varepsilon \to 0} J^3(X_{\epsilon \cdot s}) =$$

$$= J^3(Y_+) \times J^3(Y_-) \times J(C) \times (\mathbb{C})^{h^2(S)_0 + 1} \qquad (4.2.22)$$

where $h^2(S)_0$ is the rank of the primitive part of $H^2(S, \mathbb{Q})$ (4.2.20).

**Remark.** There exists a much more precise description of the degeneration of the family $J^3(X_{\epsilon \cdot s})$ in terms of the Neron model of this family described in the unpublished paper of Donagi and Griffiths. Thus the coefficients $\mathbb{Q}$ can be justified.

**Example.** Consider the projective space $\mathbb{P}^4$ and the intersection of generic quadric and cubic:

$$S = Q \cap F_3 \subset \mathbb{P}^4. \tag{4.2.23}$$

Then the union

$$X = Q \cup_S F_3 \tag{4.2.24}$$

is a CY-threefold of CP-type.

Let $F_5$ be any smooth quintic such that the intersection

$$C = S \cap F_5 \tag{4.2.25}$$

is a smooth curve. Thus there exists the section

$$s \in H^0(N_{(S \subset Q)} \otimes N_{(S \subset F_3)}) = H^0(\mathcal{O}_S(5)) \tag{4.2.26}$$

such that the zero set

$$(s)_0 = C \subset S. \tag{4.2.27}$$

Then the described deformation $X_{\epsilon \cdot s}$ is following the deformation of quintics in the pencil

$$< Q \cup F_3, F_5 > \tag{4.2.28}$$

and by (4.2.20)

$$102 = dim J^3(F_5) = dim J^3(Q) + dim J^3(F_3) + g(C) + h^2(S)_0 =$$

$$= 0 + 5 + 76 + 21 = 102. \tag{4.2.29}$$

**Remark.** The rigid Barth-Nieto-van Straten quintic [DvS] is the deformation of the CY-threefold of CP-type : consider $\mathbb{P}^5$ with homogeneous coordinates $(z_0, ..., z_6)$ and the system of Newton's hypersurfaces

$$S_k = \sum_{i-0}^{5} z_i^k. \tag{4.2.30}$$

Then the pencil of quintics

$$< S_5, S_2 \cdot S_3 > \tag{4.2.31}$$

in $\mathbb{P}^4$, given by the linear equation $S_1 = 0$, contains the unique quintic with 130 nodes. It is the Barth-Nieto-van Straten quintic. For generic deformation of form (4.2.28) we can kill $J(C)$ considering a rational curve with 76 double points and to kill $J^3(F_3)$ considering the Segre cubic with 10 nodes. After that we have to kill 16 algebraic 2-cycles on K3 and we can get a new rigid smooth CY with some elliptic curve as the intermediate Jacobian.

In the next section we consider other examples of CY of CP-type.

## §3. Vector bundles on CY-threefolds of CP-type.

First of all we have a quite non transversal way to construct CY-threefolds: if we have a quasi Fano flag $S \subset Y$ like in §2 of the previous Chapter, then we can consider the *double* of $(S \subset Y)$

$$2_S Y = (Y \supset S, id, S \subset Y) \tag{4.3.1}$$

as in (4.2.3) of the previous section. In some cases this double can be deformed to a smooth CY-threefold $X$. Indeed, suppose that the complete linear system $|-2K_Y|$ contains a smooth surface $R$. Then the double cover $X$ of $Y$ with the ramification diviser $R$ is a smooth CY-threefold. The deformation the smooth surface $R$ to $2S$ defines the deformation of double covers to the double $2_S Y$.

Now any Mukai vector with $m^2 \geq 0$ from the Mukai lattice $M_Y$ defines the Mukai vector

$$m \in M_{2_S} Y \quad \text{with} \quad m^2 = 0 \tag{4.3.2}$$

of the Mukai lattice of the double (with the same symbol) (4.2.8).

Suppose there is a regular component $\mathcal{M}_Y(m)_0$ of the stable moduli space on $Y$. Then for every vector bundle $E \in \mathcal{M}_Y(m)_0$ we can construct the *double* $2_S E$ of it gluing two copies of $E$ along the restriction $E|_S$. Thus instead of the expected finite set of vector bundles we get non transversal component of the moduli space

$$\mathcal{M}_{2_S Y}(m) = \{2_S E\} \tag{4.3.3}$$

of positive dimension parametrised by the same variety $\mathcal{M}_Y(m)_0$.

This large component has to decay to a finite set of bundles after described deformation of $2_S Y$ to a smooth CY-threefold $X$. The expected cardinality of this finite set is $CD_{(S\subset Y)}(m)$ (3.2.34-37). The amazing fact is the coincidence of two topological Euler characteristics

$$\chi_{top}(\mathcal{D}''_X(2_S E)) = \chi_{top}(\mathcal{M}_Y(m)_0) \tag{4.3.4}$$

(see (3.1.33) and (3.3.37)).

**Example 1.** Consider a smooth quartic surface $S$ in $\mathbb{P}^3$ and the double $2_S\mathbb{P}^3$ of this Fano flag. Of course, it is the deformation of a smooth double cover of $\mathbb{P}^3$ with the ramification along a smooth surface of degree 8. Consider the vector bundle $E$ on $\mathbb{P}^3$ of rank 2, $c_1 = 0, c_2 = 1$ as in (3.1.43). So the Mukai vector of this bundle on the projective space is

$$m(E) = 2 - \tfrac{1}{2}P.D.(H) \tag{4.3.5}$$

and the virtual dimension of the moduli space

$$\mathcal{M}_{\mathbb{P}^3}(2, 0, -\tfrac{1}{2}P.D.(H), 0) = m^2 + 1 = 5. \tag{4.3.6}$$

It is well known that every stable vector bundle $E$ of such type is given by a section of $\Omega\mathbb{P}^3(2)$ and by the *monad*:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s(-1)} \Omega\mathbb{P}^3(-1) \to E \to 0. \tag{4.3.7}$$

Thus the compactification of the moduli space

$$\overline{\mathcal{M}_{\mathbb{P}^3}(2,0,-\tfrac{1}{2}P.D.(H),0)} = \mathbb{P}^5 = \wedge^2 \mathbb{P}^3. \tag{4.3.8}$$

The restriction of every vector bundle on our quartic $S$ has the monad with the display

$$0 \to \mathcal{O}_S(-1) \xrightarrow{s(-1)} \Omega\mathbb{P}^3(-1)|_S \to E|_S \to 0. \tag{4.3.9}$$

From this it is easy to see that the restriction map

$$res : \overline{\mathcal{M}_{\mathbb{P}^3}(2,0,-\tfrac{1}{2}P.D.(H),0)} \to \overline{\mathcal{M}_S(2,0,-2)} \tag{4.3.10}$$

is an embedding. Thus by (3.3.7) one gets

$$CD_{(S\supset\mathbb{P}^3)}(2,0,-\tfrac{1}{2}P.D.(H),0) = 6 \tag{4.3.11}$$

More geometrically, consider some general linear transformation

$$g : \mathbb{P}^3 \to \mathbb{P}^3 \tag{4.3.12}$$

and the *skew double* :

$$\mathbb{P}^3 \cup_S g(\mathbb{P}^3) = (\mathbb{P} \supset S, id, S \subset g(\mathbb{P}^3)). \tag{4.3.13}$$

Then a vector bundle $E \in \overline{\mathcal{M}_{\mathbb{P}^3}(2,0,-\frac{1}{2}P.D.(H),0)}$ defines some vector bundle on CY-threefold (4.3.13) iff

$$g^*(E) = E. \tag{4.3.14}$$

Now, the compactification $\overline{\mathcal{M}_{\mathbb{P}^3}(2,0,-\frac{1}{2}P.D.(H),0)}$ can be identified as $\wedge^2\mathbb{P}^3$ (see (4.3.8)). Let $\wedge^2 g$ be the acting of $g$ on $\wedge^2\mathbb{P}^3$. Then

$$g^*(E) = E \implies E \quad \text{is a fixed point of} \quad \wedge^2 g. \tag{4.3.15}$$

Thus the set $E_1, ..., E_6$ on $\mathbb{P}^3$

$$\{g^*(E_i) = E_i\} = \{\wedge^2 g(p) = p\} \tag{4.3.16}$$

is six edges of the simplex with vertex in fixed points of $g$ in $\mathbb{P}^3$.

**Warning.** It seems that images of these 6 vector bundles in the intermediate Jacobian are coincident.

Now consider some smooth curve

$$C = (s)_0, \quad s \in H^0(S, \mathcal{O}_S(8)) \tag{4.3.17}$$

and blow up it in $\mathbb{P}^3$:

$$\sigma : \tilde{\mathbb{P}^3} \to \mathbb{P}^3. \tag{4.3.18}$$

Then the gluing procedure gives us the reducible CY

$$\tilde{\mathbb{P}^3} \cup_S g(\mathbb{P}^3) \tag{4.3.19}$$

which can be deformed to a smooth CY-threefold $X$ with a smooth total deformation space (see (4.2.15-16)).

Then 6 vector bundles (4.3.16) give 6 doubles

$$\sigma^*(E_i) \cup_{E_i|_S} E_i. \tag{4.3.20}$$

**Proposition 4.3.1.** *These vector bundles are infinitesimally rigid.*

It can be checked directly by the gluing description of coherent cohomology spaces of

$$End(\sigma^*(E_i) \cup_{E_i|_S} E_i) = (\sigma^*(E_i) \cup_{E_i|_S} E_i)^* \otimes (\sigma^*(E_i) \cup_{E_i|_S} E_i). \tag{4.3.21}$$

**Corollary 4.3.1.** *For a smooth deformation $X$ of the reducible threefold (4.3.19)*

$$CD(X, ((2, 0, -\sigma^*(\tfrac{1}{2}P.D.(H)), 0), (2, 0, -\tfrac{1}{2}P.D.(H), 0)) = 6. \tag{4.3.22}$$

*(see (3.1.26) and (4.2.10)).*

It follows from the deformation theory of pairs $(X, E)$ considered as the deformation theory of algebraic varieries $\mathbb{P}(E)$ (see [D-F]).

Again it is easy to see that under the projection of the intermediate Jacobian of $X$ to the Jacobian of $C$ images of these six vector bundles are coincidence.

**Example 2.** Now consider the moduli space $MI_k$ of *mathematical instantons* that are stable vector bundles $E$ on $\mathbb{P}^3$ of rank 2, $c_1 = 0, c_2 = k$ under the *instanton condition*

$$h^1(E(-2)) = 0. \tag{4.3.23}$$

Such vector bundles admit the monad description from which we can see that the restriction to $S$ is embedding (see (4.3.10)). This monad description shows that for general linear transformation (4.3.12) the action

$$g^* : MI_k \to MI_k \tag{4.3.24}$$

admits the finite set

$$E_1, ...., E_N \tag{4.3.25}$$

of fixed points which give the finite set of vector bundles (4.3.20) :

$$\{\sigma^*(E_i) \cup_{E_i|_S} E_i\} \tag{4.3.26}$$

on CY-threefold (4.3.19).

**Remark.** Any general linear transformation defines a $\mathbb{C}^*$ action on $\mathbb{P}^3$ and on the instanton moduli space $MI_k$. The computation of the number $N$ of fixed points on $MI_k$ and "instanton" vector bundles on CY-threefold (4.3.19) is quite parallel to the computation of the Euler characteristic of $MI_k$ by the Bott formula.

But this number $N$ (4.3.25) isn't CD-invariant of CY-threefold because there exists another component of the moduli space on $\mathbb{P}^3$. Recall that any holomorphic vector bundle $E$ with $c_1 = 0$ on any Fano variety $Y$ of even index $i_Y$ (see (3.1.42)) admits the Atyiah-Rees invariant

$$AR(E) = h^1(E(\tfrac{1}{2}K_Y)) \quad mod \quad 2 \tag{4.3.27}$$

which distinguish components of moduli spaces. On the other hand, if $c_1(E) = 0$ then $E$ is skew-symmetrically self dual and the Serre duality induces the non degenerated skew-symmetrical form on $H^1(E)$. Thus

$$h^1(E|_S) = 0 \quad mod \quad 2 \tag{4.3.28}$$

always. So for every instanton vector bundle

$$AR(E) = 0 \tag{4.3.29}$$

and besides $MI_k$ there exists another component $M_k$ (see the good example $k = 3$). For a small $k$ it can be checked that the restriction map embeds this component into the moduli space of vector bundles on $S$. Thus

$$res_S(MI_k) \cap res_S(M_k) = \emptyset. \tag{4.3.30}$$

So the computation of the Euler characteristic of moduli spaces by the Bott formula gives the answer

$$CD(X, ((2, 0, -\sigma^*(\frac{5}{2}P.D.(H)), 0), (2, 0, -\frac{5}{2}P.D.(H), 0)) = \chi_{top}(MI_3) + \chi_{top}(I_3). \tag{4.3.31}$$

In the next section we consider the more homogeneous case of gluing.

## §4. The genus 2 case.

Now consider $\mathbb{P}^5$ and the smooth quadric

$$Gr \in \mathbb{P}^5 \tag{4.4.1}$$

which we consider as the Grassmanian of lines in some $\mathbb{P}^3$. Then

$$H^*(Gr, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \cdot H \oplus (\mathbb{Z} \cdot p_+ \oplus \mathbb{Z} \cdot p_-) \oplus \mathbb{Z} \cdot l \oplus \mathbb{Z} \tag{4.4.2}$$

where classes $p_\pm$ realised as projective planes of lines through fixed points in $\mathbb{P}^3$ and planes of lines in fixed planes. Now there are two spinor vector bundles $E_+$ which are the tautological bundle and $E_-$ which is the antitautological bundle. These bundles are connected by the exact sequence

$$0 \to E_-^* \to \mathbb{C}^4 \otimes \mathcal{O}_{Gr} \to E_+ \to 0. \tag{4.4.3}$$

where $\mathbb{P}\mathbb{C}^4 = \mathbb{P}^3$. The Chern classes of these bundles are

$$c_1(E_\pm) = H; \quad c_2(E_\pm) = p_\pm; \tag{4.4.4}$$

so these bundles are topologically different because of $c_2$.

For a general quartic $F_4$ in $\mathbb{P}^5$ the intersection

$$X = Gr \cap F_4 \tag{4.4.5}$$

is a smooth CY-threefold.

Now restricting vector bundles $E_\pm$ to $X$ we get two vector bundles $E_\pm|_X$ which are topological identical

$$[p_+ \cdot F_4] = [p_- \cdot F_4] \in H^4(X, \mathbb{Z}). \tag{4.4.6}$$

But in spite of this by the Claire Voisin result

**Proposition 4.4.1.** *For general $F_4$ algebraic classes $(p_\pm \cdot F_4)$ are different with respect to rational equvivalence.*

**Corollary 4.4.1.** *For general $F_4$ holomorphic vector bundles $E_\pm|_X$ are different.*

Now we would like to prove that, for general quartic, images of these bundles in the intermediate Jacobian are different too. To do this consider the degeneration of $X$ of CP-type.

Let $Q_\pm$ be a pair of quadrics of generic position in $\mathbb{P}^5$ with respect to $Gr$. Then the intersection

$$S = Gr \cap Q_+ \cap Q_- \tag{4.4.7}$$

is the smooth K3-surface and intersections

$$Y_\pm = Gr \cap Q_\pm \tag{4.4.8}$$

are rational Fano threefolds of index 2. Thus

$$X_0 = Y_+ \cup_S Y_- \tag{4.4.9}$$

is the reducible CY-threefold.

For every Fano variety $Y_\pm$ consider the pencil of quadrics in $\mathbb{P}^5$ through $Y_\pm$ :

$$|2H - Y_\pm| =< Gr, Q_\pm >= \mathbb{P}^1_\pm \tag{4.4.10}$$

and six singular quadrics in these pencils

$$\{\lambda_1^\pm, ..., \lambda_6^\pm\} \subset \mathbb{P}^1_\pm. \tag{4.4.11}$$

Then double covers of $\mathbb{P}^1_\pm$ ramified in sets (4.4.11) are curves of genus 2

$$\phi_\pm : C_\pm \to \mathbb{P}^1_\pm \tag{4.4.12}$$

parametrising pairs $(E_t, Q_t)$ where $Q_t$ is a quadric from pencils (4.4.10) and $E$ is the tautological or antitautological bundle on $Q$ (see (4.4.3)). Now by

**Proposition 4.4.2 (Narasimhan and Ramanan).** *The curve $C_\pm$ is the moduli space $\mathcal{M}_{Y_\pm}$ of vector bundles on $Y_\pm$ of topologicaly type $c_1 = H, c_2 = p_\pm \cdot Q_\pm$.*

More precisely,

1) the Fano variety $Y_\pm$ (4.4.8) is the moduli spaces of rank 2 vector bundles on the curve $C_\pm$ with fixed odd determinant;

2) there exists the universal bundle $U$ on $C_\pm \times Y_\pm$ and for any point $c \in C_\pm$ the vector bundle

$$E = U|_{c \times Y_\pm} \tag{4.4.13}$$

is one of the pair of spinor bundles (4.4.3) on the quadric

$$Q = \phi_\pm(c) \in |2H - Y_\pm| = \mathbb{P}^1_\pm \tag{4.4.14}$$

restricted to $Y_\pm$.

Now consider the K3-surface $S$ (4.4.6) as the base locus of the net of quadrics

$$|2H - S| =< Gr, Q_+, Q_- >= \mathbb{P}^2. \qquad (4.4.15)$$

The set of singular quadrics of this net is the smooth plain curve

$$D \subset |2H - S| = \mathbb{P}^2. \qquad (4.4.16)$$

Then double covers of $\mathbb{P}^2$ ramified in this curve $D$ (4.4.16) is the smooth K3-surface $S_D$ again

$$\phi : S_D \to \mathbb{P}^2 \qquad (4.4.17)$$

parametrising pairs $(E_t, Q_t)$ where $Q_t$ is a quadric from the net (4.4.15) and $E$ is the tautological or antitautological bundle on $Q$ (see (4.4.3)).

**Proposition 4.4.3(Mukai [M]).** *The K3-surface $S_D$ is the moduli space $\mathcal{M}_{S_D}$ of stable vector bundles on K3-surface $S$ of topology type $c_1 = H, c_2 = 4$.*

Now pencils of quadrics (4.4.10) are embedded into the net (4.4.15)

$$< Gr, Q_+ > \cup < Gr, Q_- >\subset |2H - S| =< Gr, Q_+, Q_- >= \mathbb{P}^2, \qquad (4.4.18)$$

curves $C_\pm$ (4.4.12) are embedded into our K3-surface $S_D$:

$$i_\pm : C_\pm \to S_D \qquad (4.4.19)$$

and restrictions of doubles covers (4.4.12) are restriction of the double cover (4.4.17) to these curves:

$$i_\pm \cdot \phi_\pm = \phi|_{C_\pm}. \qquad (4.4.20)$$

So the direct interpretation shows that embeddings (4.4.19) are restrictions maps from the moduli space of vector bundles on Fano varieties to the moduli space on K3-surface:

$$i_\pm(C_\pm) = res_S(\mathcal{M}_{Y_\pm}). \qquad (4.4.21)$$

At last remark that the Mukai vector of our vector bundles on $Y_\pm$ is

$$m(E|_{Y_\pm}) = (2, H, -l) \quad \text{and} \quad m(E|_S) = (2, H, -2) \qquad (4.4.22)$$

**Corollary 4.4.2.** *The Casson-Donaldson invariant (3.3.34)*

$$CD_{(S\subset Y_\pm)}(2, H, -P.D.(H)) = g(C_\pm) = 2 \qquad (4.4.23)$$

Indeed, the genus of the curve $C_\pm = \mathcal{M}_{Y_\pm}$ is 2.

**Corollary 4.4.3.** *The Casson-Donaldson invariant (3.1.26) of CY-threefold $X_0$ (4.4.9)*

$$CD(X_0, ((2, H, -l_+), (2, H, -l_-)) = (C_+ \cdot C_-)_S = 2. \qquad (4.4.24)$$

Now, from Newstead's result the intermediate Jacobian

$$J^3(Y_\pm) = J(C_\pm) \qquad (4.4.25)$$

**Proposition 4.4.4.** *The Abel-Jacobi map (2.2.8), (3.2.7) is coincidence to the standard Abel embedding*

$$(J_{(2,H,-l_\pm)} : \mathcal{M}_{Y_\pm}(2,H,-l_\pm) \to J^3(Y_\pm) = (a : C_\pm \to J(C_\pm)) \tag{4.4.26}$$

It can be checked using the interpretation of the map (2.2.8) in terms of the Abel-Jacobi map for 2-cycles representing classes $c_2(E)$.

Now the components of the moduli space

$$\mathcal{M}_{X_0}((2,H,-l_+),(2,H,-l_-)) = \phi^{-1}(Gr) = \{E_\pm|_{X_0}\} \tag{4.4.27}$$

From our computations in §2 the intermediate Jacobian

$$J^3(X_0) = J(C_+) \times J(C_-) \times \mathbb{C}^{21} \tag{4.4.28}$$

and images of components of moduli space (4.4.27) in this intermediate Jacobian

$$\{J_m(E_\pm|_{X_0})\} = \{(\phi_+^{-1}(Gr), \phi_-^{-1}(Gr)) \tag{4.4.29}$$

are two different points.

**Corollary 4.4.4.** *For CY-threefold $X_0$ and the Mukai vector (4.4.24) the analogue of the Abel theorem is true.*

From Proposition 4.4.1 for general $F_4$ and the Mukai vector $m$ (4.4.24) we have

$$CD(Gr \cap F_4, m) \geq 2. \tag{4.4.30}$$

Now using the standard arguments of the degeneration principle we can prove

**Theorem 4.4.1.** *For general $F_4$ and the Mukai vector $m$ (4.4.24)*

$$1) \quad CD(Gr \cap F_4, m) = 2$$

*2) analogue of the Abel theorem is true.*

To use the standard arguments of deformation theory of pairs (X, E), where $E$ is an infinitesimal rigid vector bundle on $X$, that is the deformation theory of the projectivisation of $E$ as variety we need the deformation of the reducible CY-threefold $X_0$ to a smooth one with non-singular total space. To avoid this tedious procedure we would like to propose tricky geometrical arguments.

On a smooth intersection $X$ (4.4.5) consider a vector bundle $E$ of topological type $E_\pm|_X$. The coherent Euler characteristic

$$\chi(E) = 4 \tag{4.4.31}$$

Using the standard tricks it is easy to prove that if $E$ has a section, then it has 4 sections and is induced by the map of $X$ to Grassmanian which is $Gr$ and $E$ is one of the described bundles.

If $E$ hasn't any sections then

$$h^2(E) = h^1(E^*) \leq 4. \tag{4.4.32}$$

It is enough to consider the case when $h^2(E) = 4$. Consider the general hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$ and the smooth surface

$$Z = X \cap \mathbb{P}^4 = (Gr \cap \mathbb{P}^4) \cap (F_4 \cap \mathbb{P}^4). \tag{4.4.33}$$

This is a canonical surface in $\mathbb{P}^4$ that is

$$\mathbb{P}^4 = |K_Z| \tag{4.4.34}$$

and the restriction of $E$ to $Z$ is of rank 2 vector bundle with $c_1 = K_Z$ and $c_2 = 4$. The exact sequence

$$0 \to E^* \to E \to E|_Z \to 0 \tag{4.4.35}$$

and the Serre-duality give isomorphisms

$$H^0(E|_Z) = H^1(E^*) = H^2(E)^* = H^2(E|_Z)^*. \tag{4.4.36}$$

A general section $\mathcal{O}_Z \to E|_Z$ can be extended to the exact sequence

$$0 \to \mathcal{O}_Z \to E|_Z \to J_\xi(K_Z) \to 0 \tag{4.4.37}$$

where $J_\xi$ is the ideal sheaf of the zero-dimensional subscheme $\xi$ of $Z$. Thus

$$h^0(E|_Z) = 4 \implies h^0(J_\xi(K_Z)) = 3$$

and

$$< \xi >= \mathbb{P}^1 \subset \mathbb{P}^4 = |K_Z|. \tag{4.4.38}$$

But the restriction $E_\pm|_Z$ admits the same exact sequence as (4.4.7) thus

$$E|_Z = E_\pm|_Z \tag{4.4.39}$$

for any general hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$. From this it is easy to get the equality

$$E = E_\pm|_Z \tag{4.4.40}$$

for some choice of sign. We are done.

## §5. Conclusions.

One can see that

1) the gluing surgery for CY-threefold is quite parallel to the gluing surgery of twistor spaces of real fourfolds (see [D-F]). Namely, the description of twistor spaces of a connected sum of 4-folds used gluings along a complex quadric instead of K3-surfaces as in our case. The procedure of gluing of Fano varieties of index 2 described in the previous section emphasizes this parallelism because an intersection of two quadrics in $\mathbb{P}^5$ is the small desingularisation of the twistor space of the connected sum o of two copies of $\mathbb{C}P^2 = \mathbb{P}^2$ (see [D-F]);

2) on the other hand the investigation of representations of any CY-threefold as a SLag-torus fibration (4.2.17) suggests the way to construct the analogue of the "twistor space" for any compact smooth 3-fold $Y$ equipped with a Riemannian metric $g$. We finish this paper with the coarse draft of this construction for the case when a metric $g$ is general enough.

Let $g$ be a general metric on a smooth 3-fold $Y$. Consider a point $p \in Y$ the complexification and the projectivisation of the tangent space at this point

$$\mathbb{C}^3 = TY_p^{\mathbb{C}}; \quad \mathbb{P}_p^2 = projTY_p^{\mathbb{C}}. \tag{4.5.1}$$

This plane contains two conics

$$q_p \quad \text{and } q_p^{Ric} \subset \mathbb{P}_p^2 \tag{4.5.2}$$

where the first is the complexification of the metric quadric and the second is the projectivisation of the Ricci tensor of the metric at the point $p \in Y$.

The intersection of these conics is the collection of four points on $q_p$:

$$q_p \cap q_p^{Ric} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subset q_p. \tag{4.5.3}$$

At last consider the double cover

$$\phi : C_p \in q_p \tag{4.5.4}$$

ramified in the collection of four points (4.5.3). This curve $C_p$ is an elliptic curve that is 2-torus.

This double cover is called the *local invariant* of the Riemannian threefold $Y$ at point $p \in Y$.

Now consider the canonical bundle $L_{-K}$ on $C_p$ and the unit circle bundle $S^1(L_{-K}$ of this (trivial) line bundle. So the threefold

$$S^1(L_{-K}) = T^3 \tag{4.5.5}$$

The globalisation of this construction gives us:

1) $\mathbb{P}^1$-bundle over $Y$:

$$\pi : T \to Y; \qquad \pi^{-1}(p) = q_p \qquad p \in Y \tag{4.5.6}$$

which is the classical *twistor space* of the Riemannian threefold $Y$;

2)the double cover

$$\phi : \mathcal{C} \to T, \quad \phi^{-1}(\pi^{-1}(p)) = C_p \tag{4.5.7}$$

which is the cover (4.5.4) fibrewise;

3)at last we have $S^1$ bundle on $\mathcal{C}$

$$\mathcal{T}(g) = S^1(L_{-K/\pi}) \to \mathcal{C}, \quad S^1(L_{-K/\pi})|_{C_p} = S^1(L_{-K}). \tag{4.5.8}$$

So the projection

$$\overline{\pi}; \mathcal{T}(g) = S^1(L_{-K/\pi}) \to Y \tag{4.5.9}$$

is a $T^3$-bundle over points where the intersection $q_p \cap q_p^{Ric}$ is transversal. Denote the open set of such points by $Y_0$ and the restriction of the $T^3$-bundle (4.5.9) to $Y_0$ by the symbol $\mathcal{T}(g)_0$.

Now it is easy to see that $\mathcal{L}(g)_0$ is equipped with the canonical almost complex structure

$$I : T\mathcal{T}(g)_0 \to T\mathcal{T}(g)_0 \tag{4.5.10}$$

conjugating tangent spaces to fibres to tangent spaces to the base.

The question about an integrability of this almost complex structure can be reduced to the analogous questions for local deformations of calibrated 3-subtorus. The natural question is *how many CY-threefolds can we get by this construction?*.


## Acknowlegments

This is the version of my talks on the AG-Conference in Guanojuato C.I.M.A.T. (Mexico), 10-20 August 1997 and the Ramanan Conference in Bombay, June 1997. This version was written at I.C.T.P. Trieste in October-November 1997. I would like to express my gratitude to Tata Institute, C.I.M.A.T (Mexico) and I.C.T.P. Trieste and personally to S. Ramanan, X. Gometz-Mont and M.S. Narasimhan for support and hospitality. I wish to thank H. Clemens for many helpful suggestions.


## References


[A-M] S. Akbulut, J. McCarthy, *Casson's invariant for oriented homology 3-spheres.*, Michigan State Univ. (1986).

[B-T] P. de Bartolomeis, G. Tian, *Stability of complex vector bundles.*, J. Diff. Geom. 43 (1996), 231–275.

[D-F] S. K. Donaldson, R. Friedman, *Connected sums of self-dual manifolds and deformations of singular spaces*, Nonlinearity 2 (1989), 197–239.

[D-K] S. K. Donaldson, P. Kronheimer, *The geometry of four-manifolds*, Oxford UP (1990), 197–239.

[D-T] S. K. Donaldson, R. P. Thomas, *Gauge theory in higher dimensions.*, Preprint. Oxford (1996), 1–15.

[DvS] Duco van Straten, *A quintic hypersuface in $\mathbb{P}^4$ with 130 nodes.*, Topology vol.32, n4 (1993), 857–862.

[G] P. A. Griffiths, *Periods of integrals on algebraic manifolds: summary of main results and discussion of open problems.*, Bull. Amer. Math. Soc. 75 (1970), 228–296.

[Gr] M. Gross, *Special Lagrangian Fibrations !: Topology.*, Preprint alg-geom / 9710006, 1–17.

[G-N] K. Guruprasad, N. Nilakantan, *Mutually dual presentations of the surface group and symplectic structure on the parabolic moduli space.*, Preprint. Bombay (1997), 1–9.

[L] Y. Lee, *Smoothing of simple normal crossing surfaces.*, Preprint. Solt Lake. (1997), 1–11.

[M] S. Mukai, *On the moduli space of bundles on K3 surface I.*, in Vector bundles on algebraic varieties (Bombay, 1984), Tata Inst. of Fund. Res. Studies 11, Oxford Univ. Press, 1987., 341–413.

[N-R] M. S. Narasimhan, S. Ramanan, *Moduli of vector bundles on a compact Riemann surface.*, Ann. of Math. 89 (1969), 19–51.

[R] S. Ramanan, *The moduli of vector bundles over an algebraic curve.*, Math. Ann. 200 (1973), 69–84.

[SYZ] A. Strominger, S.-T. Yau, E. Zaslow, *Mirror Symmetry is T-Duality*, Nucl. Phys. B479, (1996), 243–259.

[T] C. H. Taubes, *Casson's invariant and gauge theory*, J. Diff. Geom. 31 (1990), 547–599.

[Ty] A. N. Tyurin, *Symplectic structures on the moduli spaces of vector bundles on algebraic surfaces with* $p_g > 0$., Math. Izvestya v.33 (1987), 139–177.