

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

549

Brauer Groups

Proceedings of the Conference Held at
Evanston, October 11–15, 1975

416 109 546 200 16



Edited by D. Zelinsky



Springer-Verlag
Berlin · Heidelberg · New York 1976

Editor

Daniel Zelinsky
Northwestern University
Department of Mathematics
Evanston, Il. 60091/USA



Library of Congress Cataloging in Publication Data

Main entry under title:

Brauer groups.

(Lecture notes in mathematics ; 549)
"Sponsored by Northwestern University."
Bibliography: p.
Includes index.

1. Brauer group--Congresses. 2. Separable algebras--Congresses. I. Zelinsky, Daniel. II. Northwestern University, Evanston, Ill. III. Series: Lecture notes in mathematics (Berlin) ; 549.

QA3.L28 no. 549 [QA251.3] 510'.8s [512'.24]
76-46416

AMS Subject Classifications (1970): 13A20, 16A16, 18H20, 14C20,
14H99, 14L15, 18D10

ISBN 3-540-07989-0 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-07989-0 Springer-Verlag New York · Heidelberg · Berlin

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically those of translation, re-printing, re-use of illustrations, broadcasting, reproduction by photocopying machine or similar means, and storage in data banks.

Under § 54 of the German Copyright Law where copies are made for other than private use, a fee is payable to the publisher, the amount of the fee to be determined by agreement with the publisher.

© by Springer-Verlag Berlin · Heidelberg 1976

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.

CONTENTS

Lindsay N. Childs	
On Brauer groups of some normal local rings.....	1
F. R. DeMeyer	
The Brauer group of affine curves.....	16
M.-A. Knus, M. Ojanguren, D. J. Saltman	
On Brauer groups in characteristic p	25
Gerald S. Garfinkel	
A module approach to the Chase-Rosenberg-Zelinsky sequences	50
Daniel Zelinsky	
Long exact sequences and the Brauer group.....	63
Andy R. Magid	
The Picard sequence of a fibration.....	71
George Szeto	
The Pierce representation of Azumaya algebras.....	86
William Gustafson	
A remark on class groups.....	92
Philip LaFollette	
Splitting of Azumaya algebras over number rings.....	100
H. F. Kreimer, Jr.	
Abelian p -extensions and cohomology.....	104
Bodo Pareigis	
Non-additive ring and module theory IV. The Brauer group of a symmetric monoidal category.....	112
Morris Orzech	
Brauer groups of graded algebras.....	134
Stephen U. Chase	
On a variant of the Witt and Brauer group.....	148

The Conference on Brauer Groups was originally titled Conference on Brauer and Picard Groups. The present title is more nearly representative of the contents of the conference and these proceedings.

The conference was sponsored by Northwestern University and was held there (Evanston, Illinois) from October 11 to 15, 1975.

The list of participants which follows gives the university of each participants at the time of the conference (Department of Mathematics in each case). Professor Chase had to cancel his attendance but kindly submitted his manuscript for these Proceedings.

Besides the papers published here, the following were read:

- | | |
|---------------|---|
| R. T. Hoobler | How to construct U. F. D.'s |
| M. Ojanguren | Generic splitting rings |
| S. Rosset | Some solvable group rings are domains |
| D. Haile | Generalization of involution for central
simple algebras of order m in the Brauer
group |
| S. A. Amitsur | Cyclic splitting of generic division algebras |
| G. Szeto | Lifting modules and algebras |

CONFERENCE ON BRAUER GROUPS

List of Participants

AMITSUR, S. A.
Hebrew University
Jerusalem, Israel

AUSLANDER, Bernice
Univ. of Massachusetts
Boston, Massachusetts

CHASE, Stephen U.
Cornell University
Ithaca, New York

CHILDS, Lindsay N.
SUNY at Albany
Albany, New York

COOK, P. M.
Michigan State Univ.
East Lansing, Michigan

DEMEYER, Frank R.
Colorado State Univ.
Ft. Collins, Colorado

ELGETHUN, Edward
Univ. of N. Florida
Jacksonville, Florida

GARFINKEL, Gerald S.
New Mexico St. Univ.
Les Cruces, New Mexico

GUSTAFSON, William
Indiana University
Bloomington, Indiana

HAILE, Darrell
Northwestern Univ.
Evanston, Illinois

HOOBLER, Raymond
City College, CUNY
New York, New York

INGRAHAM, Edward
Michigan State Univ.
East Lansing, Michigan

KNUS, Max-Albert
ETH
Zurich, Switzerland

KREIMER, H. F.
Florida State Univ.
Tallahassee, Florida

LAFOLLETTE, Philip
Indiana Univ.
Bloomington, Indiana

LEE, Hei-Sook
Queens University
Kingston, Ontario

MAGID, Andy R.
Univ. of Illinois
Urbana, Illinois

OJANGUREN, Manuel
Westfälische-Wilhelms
University
Münster, Germany

ORZECH, Morris
Queens University
Kingston, Ontario

PAREIGIS, Bodo
Ludwig-Maximilians
University
Munchen, Germany

REINER, Irving
Univ. of Illinois
Urbana, Illinois

ROSSET, Shmuel
Tel Aviv University
Tel Aviv, Israel

SALTMAN, David J.
Yale University
New Haven, Conn.

SMALL, Charles
Queens University
Kingston, Ontario

SZETO, George
Bradley University
Peoria, Illinois

ZELINSKY, Daniel
Northwestern Univ.
Evanston, Illinois

NON-ADDITIVE RING AND MODULE THEORY IV

The Brauer Group of a Symmetric Monoidal Category

Bodo Pareigis

In [5], [6] and [7] we introduced general techniques in the theory of a monoidal category, i.e. of a category \mathcal{C} with a bifunctor $\boxtimes: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, an object $I \in \mathcal{C}$ and natural isomorphisms $\alpha: A \boxtimes (B \boxtimes C) \cong (A \boxtimes B) \boxtimes C$, $\lambda: I \boxtimes A \cong A$ and $\rho: A \boxtimes I \cong A$ which are coherent in the sense of [3, VII. 2]. In this paper we want to introduce the notion of a Brauer group of \mathcal{C} . For this purpose we are going to assume that \mathcal{C} is symmetric, i.e. that there is a natural isomorphism $\gamma: A \boxtimes B \cong B \boxtimes A$ which is coherent with α , λ and ρ [3]. One of the main models for such a category \mathcal{C} is, apart from the category of k -modules for a commutative ring k , the dual of the category of \mathcal{C} -comodules for a cocommutative coalgebra \mathcal{C} . This category is a symmetric monoidal category, but it is not closed.

Another type of monoidal categories, which are not symmetric but which allow the construction of Brauer groups, are for example categories of bimodules over a commutative, cocommutative Hopf algebra [2]. Their general theory will be discussed elsewhere.

In many special cases of symmetric monoidal categories the basic object I turns out to be projective, i.e. the functor $\mathcal{C}(I, -)$ preserve epimorphisms. In the general situation, however, it turns out that there may be constructed two Brauer groups $\mathcal{B}_1(\mathcal{C})$ and $\mathcal{B}_2(\mathcal{C})$ and a group-homomorphism $\mathcal{B}_2(\mathcal{C}) \longrightarrow \mathcal{B}_1(\mathcal{C})$, which is an isomorphism if I projective

We will construct these two Brauer groups and discuss under which condition for a functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ we get an induced homomorphism

$$\mathcal{B}_i(F): \mathcal{B}_i(\mathcal{C}) \longrightarrow \mathcal{B}_i(\mathcal{D}) .$$

Preliminaries

In [7] we proved analogues of the Morita Theorems which will be used in this paper. For the convenience of the reader we will collect the most important definitions and facts of [5], [6] and [7].

If P is an object of C we denote by $P(X)$ the set $C(X, P)$ for $X \in C$. Elements in $P \otimes Q(X)$ will often be denoted by $p \otimes q$. If the functor $C(P \otimes -, Q)$ is representable then the representing object is $[P, Q]$, so that $C(P \otimes X, Q) \cong C(X, [P, Q]) = [P, Q](X)$. The "evaluation" $P(X) \times [P, Q](Y) \rightarrow Q(X \otimes Y)$, induced by the composition of morphisms, is denoted by $P(X) \times [P, Q](Y) \ni (p, f) \mapsto \langle p \rangle f \in Q(X \otimes Y)$. Thus the "inner morphism sets" $[P, Q]$ operate on P from the right. In [5, Proposition 3.2] we prove that any natural transformation $P(X) \rightarrow Q(X \otimes Y)$, natural in X , is induced by a uniquely determined element of $[P, Q](Y)$, if $[P, Q]$ exists.

We call an object $P \in C$ finite or finitely generated projective if $[P, I]$ and $[P, P]$ exist and if the morphism $[P, I] \otimes P \rightarrow [P, P]$ induced by $P(X) \times [P, I](Y) \times P(Z) \ni (p, f, p') \mapsto \langle p \rangle fp' \in P(X \otimes Y \otimes Z)$ is an isomorphism. For $[P, I] \otimes P \rightarrow [P, P]$ to be an isomorphism it is necessary and sufficient that there is a "dual basis" $f_0 \otimes p_0 \in [P, I] \otimes P(I)$ such that $\langle p \rangle f_0 p_0 = p$ for all $p \in P(X)$ and all $X \in C$. The difference between finite and finitely generated projective objects, as discussed in [8], does not appear here.

A finite object P is called faithfully projective if the morphism $P \otimes_{[P, P]} [P, I] \rightarrow I$, induced by the evaluation, is an isomorphism. This is equivalent to the existence of $p_1 \otimes_{[P, P]} f_1 \in P \otimes_{[P, P]} [P, I](I)$ with $\langle p_1 \rangle f_1 = 1 \in I(I)$. If there exists an element $p_1 \otimes_{[P, P]} f_1 \in P \otimes_{[P, P]} [P, I](I)$ with $\langle p_1 \rangle f_1 = 1$, then P is called a progenerator. Now $P \otimes_{[P, P]} [P, I] \rightarrow P \otimes_{[P, P]} [P, I]$ is an epimorphism; if I is projective, then P is faithfully projective iff P is a progenerator.

Let A^C denote the category of A -objects in C with A a monoid.

Then a functor $A^C \ni X \mapsto P \boxtimes_A X \in {}_B C$ with ${}_B P_A$ a left B right A biobject is a category equivalence iff ${}_B P$ is faithfully projective and $A \cong {}_B [P, P]$ as has been proved in [7]. For this Morita equivalence all the usual conclusions hold, in particular the centers ${}_A [A, A]_A$ and ${}_B [B, B]_B$ of A resp. B are isomorphic monoids if they exist.

The Brauer group $B_1(C)$.

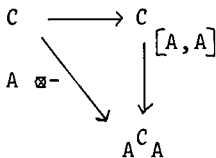
Let C be a symmetric monoidal category. A monoid A in C is called 1-Azumaya if $C \ni X \mapsto A \boxtimes X \in {}_A^C A$ is an equivalence of categories. Thus the Morita Theorems, in particular [7, Theorem 5.1] can be applied.

Proposition 1: A monoid A is 1-Azumaya iff $A \in C$ is faithfully projective and

$\psi: A \boxtimes A(X) \ni a \boxtimes b \mapsto (A(Y) \ni c \mapsto acb \in A(X \boxtimes Y)) \in [A, A](X)$
is an isomorphism.

Proof: Let A be faithfully projective and ψ be an isomorphism. Let A^{op} be the monoid on A with inverse multiplication. Then $\psi: A^{op} \boxtimes A \rightarrow [A, A]$ is an isomorphism of monoids. Thus the categories ${}_A^C A \cong C [A, A]$ are equivalent with the functor $M \mapsto M$. Furthermore $C \ni X \mapsto A \boxtimes X \in C [A, A]$ is an equivalence by [7, Theorem 5.4] hence $C \ni X \mapsto A \boxtimes X \in {}_A^C A$ is an equivalence and A is 1-Azumaya.

Conversely if A is 1-Azumaya the morphism ψ which exists for any monoid A induces a commutative diagram



Hence $A \in \mathcal{C}$ is faithfully projective in \mathcal{C} by [7, Theorem 5.1] and so ψ must induce a category isomorphism and must even be an isomorphism [7, Theorem 5.1 d)] . Q.E.D.

Recall that the center of a monoid A is the object ${}_A[A, A]_A$, if it exists. ${}_A[A, A]_A$ is the representing object of the functor $\mathcal{C} \in X \longmapsto {}_A \mathcal{C}_A(A \boxtimes X, A) \in \mathcal{S}$. Observe that for this definition we need the symmetry of the monoidal category \mathcal{C} .

Let A be a monoid in \mathcal{C} . Since we have ${}_A[A, A]_A(X) \subseteq A(X)$, the inclusion given by the isomorphism ${}_A[A, A] \cong A$ using the multiplication with A from the right, it is easy to see that $a \in {}_A[A, A]_A(X)$ iff $ab = ba$ for all $b \in A(Y)$ and all $Y \in \mathcal{C}$. Since $ab = ba$ for all $a \in \text{Im}(\varphi(X): I(X) \longrightarrow A(X))$ and all $b \in A(Y)$, all $X, Y \in \mathcal{C}$, we get that $\varphi(X)$ maps $I(X)$ into ${}_A[A, A]_A(X)$. If this morphism is an isomorphism then A is called a central monoid.

Let A be 1-Azumaya. Then $[A, A]$ is Morita equivalent to I hence the center I of I coincides with the center of $[A, A]$ via the morphism $\varphi: I \longrightarrow [A, A]$ [7, Corollary 6.3]. Thus the morphism

$$I \xrightarrow{\varphi} A \xrightarrow{\varphi} A \overset{\text{op}}{\boxtimes} A \xrightarrow{\psi} [A, A]$$

is injective. φ is defined by $\varphi(a) = 1 \boxtimes a$. Hence $I(X)$ is contained in the center ${}_A[A, A]_A(X)$. Now let $a \in A(X)$ such that $ab = ba$ for all $b \in A(Y)$, all $Y \in \mathcal{C}$. Then $(1 \boxtimes a)(b \boxtimes c) = b \boxtimes ac = b \boxtimes ca = (b \boxtimes c)(1 \boxtimes a)$ for all $b \boxtimes c \in A \overset{\text{op}}{\boxtimes} A(Y)$. Thus $\varphi(a)$ is in the center of $A \overset{\text{op}}{\boxtimes} A$ or $\psi\varphi(a)$ in the center of $[A, A]$, which was I . Furthermore φ is a monomorphism, even a section with retract $\mu: A \boxtimes A \longrightarrow A$. Thus $a \in \text{Im}(\varphi(X): I(X) \longrightarrow A(X))$, so that I is the center of A .

Corollary 2: If A is 1-Azumaya then A is central.

Proposition 3: Let A, B be 1-Azumaya then $A \boxtimes B$ is 1-Azumaya.

Proof: Let $f_0 \otimes a_0$ resp. $g_0 \otimes b_0$ be a dual basis for A resp. B . Then $f_0 \otimes g_0 \otimes a_0 \otimes b_0 \in [A \otimes B, I] \otimes A \otimes B(I)$ is a dual basis for $A \otimes B$ where we identified $[A, I] \otimes [B, I]$ with $[A \otimes B, I]$. Furthermore we have $[A, A] \otimes [B, B] \cong [A \otimes [A, I], [B, B]] \cong [A \otimes B \otimes [A, I], B] \cong [A \otimes B, A \otimes B]$ since A and B are finite [8, Theorem 1.2]. Hence $[A \otimes B, I]$ and $[A \otimes B, A \otimes B]$ exist.

Let $A^e = A \overset{\text{op}}{\otimes} A$ and $A' \in {}_{A^e} C$ be the dual $[A, I]$ of A . With the analogous notation for B we get

$$(A \otimes B) \overset{A^e}{\otimes} \overset{B^e}{\otimes} (A' \otimes B') \cong (A \overset{A^e}{\otimes} A') \otimes (B \overset{B^e}{\otimes} B') \cong I$$

since A, B are faithfully projective. Hence $A \otimes B$ is faithfully projective in C .

Finally since $[A, A] \otimes [B, B] \cong [A \otimes B, A \otimes B]$ we get that $\psi: A \otimes B \otimes A \otimes B \longrightarrow [A \otimes B, A \otimes B]$ is an isomorphism.

Proposition 4: Let P be faithfully projective then $[P, P]$ is 1-Azumaya.

Proof: We know that $[P, I] \otimes P \cong [P, P]$ as $[P, P]$ - $[P, P]$ - objects. Furthermore $C \ni X \longmapsto [P, I] \otimes X \in {}_{[P, P]} C$ and $C \ni X \longmapsto X \otimes P \in C$ are equivalences. Hence $C \ni X \longmapsto [P, P] \otimes X \cong [P, I] \otimes X \otimes P \in {}_{[P, P]} C$ is an equivalence, since $[P, P] \otimes C \ni Y \longmapsto Y \otimes P \in {}_{[P, P]} C$ is also an equivalence.

Proposition 5: Let P, Q be faithfully projective, then $P \otimes Q$ is faithfully projective and $[P, P] \otimes [Q, Q] \cong [P \otimes Q, P \otimes Q]$ as monoids.

Proof: Since P and Q are finite we get for all $X, Y \in C$ that $\varphi: [P, X] \otimes [Q, Y] \ni f \otimes g \longmapsto (p \otimes q \longmapsto \langle p \rangle f \otimes \langle q \rangle g) \in [P \otimes Q, X \otimes Y]$ is an isomorphism and the right side exists. In particular

$[P \otimes Q, I]$ and $[P \otimes Q, P \otimes Q]$ exist. Furthermore we have $\langle p \otimes q \rangle \mathcal{F}(f \otimes g) \mathcal{F}(f' \otimes g') = \langle p \rangle f f' \otimes \langle q \rangle g g' = \langle p \otimes q \rangle \mathcal{F}(f f' \otimes g g')$ and $\mathcal{F}(\text{id}_P \otimes \text{id}_Q) = \text{id}_{P \otimes Q}$, hence $\mathcal{F}: [P, P] \otimes [Q, Q] \longrightarrow [P \otimes Q, P \otimes Q]$ is an monoid isomorphism.

If $f_0 \otimes p_0$ resp. $g_0 \otimes q_0$ are dual bases of P and Q then $\mathcal{F}(f_0 \otimes g_0) \otimes (p_0 \otimes q_0)$ is a dual basis of $P \otimes Q$ for $\langle p \otimes q \rangle \mathcal{F}(f_0 \otimes g_0)(p_0 \otimes q_0) = \langle p \rangle f_0 p_0 \otimes \langle q \rangle g_0 q_0 = p \otimes q$. Hence $P \otimes Q$ is finite.

Now let $p_1 \otimes_{[P, P]} f_1$ resp. $q_1 \otimes_{[Q, Q]} g_1$ be elements such that $\langle p_1 \rangle f_1 = 1$ and $\langle q_1 \rangle g_1 = 1$. Then $(p_1 \otimes q_1) \otimes_{\mathcal{B}} \mathcal{F}(f_1 \otimes g_1)$ with $\mathcal{B} = [P \otimes Q, P \otimes Q]$ has the property $\langle p_1 \otimes q_1 \rangle \mathcal{F}(f_1 \otimes g_1) = \langle p_1 \rangle f_1 \langle q_1 \rangle g_1 = 1$. Thus $P \otimes Q$ is faithfully projective.

Now we can define the Brauer group $\mathcal{B}_1(C)$ of a symmetric monoidal category C . Let A be the (illegitimate) set of isomorphism classes of 1-Azumaya monoids A in C . Then we define an equivalence relation on A by $\bar{A} \sim \bar{B}$ iff there exist faithfully projective $P, Q \in C$ such that $A \otimes [P, P] \cong B \otimes [Q, Q]$ as monoids. Denote the set of equivalence classes by $\mathcal{B}_1(C)$. $\mathcal{B}_1(C)$ becomes a commutative group in the usual way by $[A][B] = [A \otimes B]$ with unit element $[I]$ and inverse $[A^{\text{op}}]$ for $[A]$, where A^{op} is the 1-Azumaya monoid A with inverse multiplication $A \otimes A \xrightarrow{\gamma} A \otimes A \xrightarrow{\mu} A$.

Separable monoids

Let A be a monoid in C . A is called a separable monoid if the multiplication $\mu: A \otimes A \longrightarrow A$ has a splitting $\sigma: A \longrightarrow A \otimes A$ in A^C_A such that $\mu \sigma = \text{id}_A$. Observe that $A(X) \ni a \longmapsto a \otimes 1 \in A \otimes A(X)$ is a splitting for μ in A^C but it is no A -right-morphism.

Proposition 6: Let $A \in \mathcal{C}$ be a monoid. Equivalent are

a) A is separable

b) There is an element $a \boxtimes b \in A \boxtimes A(I)$ such that

i) $\forall c \in A(X): ca \boxtimes b = a \boxtimes bc \in A \boxtimes A(X)$,

ii) $ab = 1 \in A(I)$.

Proof: a) \Rightarrow b): Define $a \boxtimes b := \sigma(1)$. Then $1 = \mu\sigma(1) = \mu(a \boxtimes b) = ab$ which is condition (ii). Since σ is an A - A -morphism we have $ca \boxtimes b = c\sigma(1) = \sigma(c) = \sigma(1)c = a \boxtimes bc$ for all $a \in A(X)$.

b) \Rightarrow a): Let $\sigma: A(X) \rightarrow A \boxtimes A(X)$ be defined by $\sigma(c) = ca \boxtimes b$.

By (i) σ is an A - A -morphism. By (ii) we get

$$\mu\sigma(c) = \mu(ca \boxtimes b) = cab = c \text{ , hence } \mu\sigma = \text{id}_A \text{ .}$$

Observe that (i) does not depend on a symmetry in \mathcal{C} , since $a \boxtimes b \in A \boxtimes A(I)$ and $I \boxtimes X \cong X \boxtimes I$ even without a symmetry. The element $a \boxtimes b$ will be called a Casimir element.

Every Casimir element $a \boxtimes b \in A \boxtimes A(I)$ induces a map

$\text{Tr}: \mathcal{C}(M, N) \ni f \mapsto (M(X) \ni m \mapsto af(bm) \in N(X)) \in {}_A\mathcal{C}(M, N)$ for any two objects $M, N \in {}_A\mathcal{C}$. In fact for any $c \in A(Y)$ we have $ca \boxtimes b = a \boxtimes bc$ hence $c(af(bm)) = (ca)f(bm) = af(bcm) = af(b(cm))$.

This map is called the trace map.

Since the trace map is a natural transformation, natural in X ,

$$\text{Tr}: \mathcal{C}(M \boxtimes X, N) \longrightarrow {}_A\mathcal{C}(M \boxtimes X, N) \text{ ,}$$

we get $\text{Tr}: [M, N] \longrightarrow {}_A[M, N]$, if both objects exist.

Since $ab = 1$ we even get that

$${}_A\mathcal{C}(M, N) \longrightarrow \mathcal{C}(M, N) \xrightarrow{\text{Tr}} {}_A\mathcal{C}(M, N)$$

is the identity on ${}_A\mathcal{C}(M, N)$ since $\text{Tr}(f)(m) = af(bm) = abf(m) = f(m)$ and hence $\text{Tr}(f) = f$, if $f \in {}_A\mathcal{C}(M, N)$. Similarly

${}_A[M, N] \longrightarrow [M, N] \longrightarrow {}_A[M, N]$ is the identity on ${}_A[M, N]$.

If $M, N \in {}_A\mathcal{C}_A$ then we clearly get $\text{Tr}: \mathcal{C}_A(M, N) \longrightarrow {}_A\mathcal{C}_A(M, N)$ and ${}_A\mathcal{C}_A(M, N) \longrightarrow \mathcal{C}_A(M, N) \xrightarrow{\text{Tr}} {}_A\mathcal{C}_A(M, N)$ is the identity on ${}_A\mathcal{C}_A(M, N)$. The

same holds for $[M,N]_A$ and ${}_A[M,N]_A$.

If ${}_A[A,A]_A$ exists then

$${}_A[A,A]_A \longrightarrow [A,A]_A \xrightarrow{\text{Tr}} {}_A[A,A]_A$$

is the identity on ${}_A[A,A]_A$. Observe that $[A,A]_A$ exists, since $[A,A]_A \cong A$. Since the last isomorphism is an antiisomorphism of monoids, ${}_A[A,A]_A$ is the center of A and ${}_A[A,A]_A \longrightarrow A$ is a monoid homomorphism, we get

Proposition 7 [1, Prop. 1.2]: If A is a separable monoid, then the center ${}_A[A,A]_A$ (if it exists) is a "direct summand" of A .

Let $f: A \longrightarrow B$ be monoid homomorphism. $P \in {}_B\mathcal{C}$ is called (B,A) -projective if for each commutative diagram

$$\begin{array}{ccc} & P & \\ k \swarrow & \downarrow g & \\ M & \xrightarrow{h} & N \end{array}$$

with g, h in ${}_B\mathcal{C}$ and k in ${}_A\mathcal{C}$ there is a $g' \in {}_B\mathcal{C}(P, M)$ with $hg' = g$. The dual notion is that of a (B,A) -injective object [8].

Proposition 8: Let A be a separable monoid. Then every A -object is (A,I) -projective and (A,I) -injective.

Proof: Let $g \in {}_A\mathcal{C}(P, N)$, $h \in {}_A\mathcal{C}(M, N)$ and $k \in \mathcal{C}(P, M)$ be given such that $hk = g$. Then $g = \text{Tr}(g) = \text{Tr}(hk) = h \text{Tr}(k)$ and $\text{Tr}(k) \in {}_A\mathcal{C}(P, M)$, so that P is (A,I) -projective. Just by reversing the arrows one can prove that each object in ${}_A\mathcal{C}$ is (A,I) -injective.

In [8] we prove that in (\mathcal{C}, \times, E) , a monoidal category with the product as tensor-product and E a final object, there are no non-trivial finite objects. In Theorem 14 we shall show that $[P, P]$ is a

separable monoid for certain finite objects $P \in C$. So this construction will not produce examples of separable monoids in (C, x, E) . In fact, there are no non-trivial separable monoids in C at all.

Proposition 9: Let A be a separable monoid in the monoidal category (C, x, E) . Then $A \cong E$ as monoids.

Proof: Let (a, b) be the Casimir element for A . Then $(ca, b) = (a, bc)$ for all $c \in A(X)$, hence $ca = a$ and $b = bc$. Here we use $A \times A(X) = A(X) \times A(X)$ and $A(E) \rightarrow A(X)$ by the unique morphism $X \rightarrow E$. We also have $1c = c$ and $ab = 1$, hence $c = 1c = abc = ab = 1$ for all $c \in A(X)$, so that $A(X) = \{1\}$ which proves $A \cong E$. Finally observe that E has a unique monoid structure.

If A and B are monoids in C , then $A \boxtimes B$ is a monoid by $(a_1 \boxtimes b_1) \cdot (a_2 \boxtimes b_2) = a_1 a_2 \boxtimes b_1 b_2$.

Proposition 10: Let A and B be separable monoids. Then $A \boxtimes B$ is separable.

Proof: Let $a_1 \boxtimes a_2$ and $b_1 \boxtimes b_2$ be Casimir elements of A resp. B . Then $(a_1 \boxtimes b_1) \boxtimes (a_2 \boxtimes b_2)$ is a Casimir element for $A \boxtimes B$. In fact let $x \boxtimes y \in A \boxtimes B(X)$ then

$$(x \boxtimes y)(a_1 \boxtimes b_1) \boxtimes (a_2 \boxtimes b_2) = (xa_1 \boxtimes yb_1) \boxtimes (a_2 \boxtimes b_2) = (a_1 \boxtimes b_1) \boxtimes (a_2 x \boxtimes b_2 y) = (a_1 \boxtimes b_1) \boxtimes (a_2 \boxtimes b_2)(x \boxtimes y).$$

Furthermore $(a_1 \boxtimes b_1)(a_2 \boxtimes b_2) = a_1 a_2 \boxtimes b_1 b_2 = 1 \boxtimes 1$.

Proposition 11: Let A be a separable monoid with Casimir element $a \boxtimes b$. Assume that ${}_A[A, A]_A$ exists and that $I \rightarrow A$ is a monomorphism. Then A is central if and only if $axb \in I(X)$ for all $x \in A(X)$.

Proof: Since $ca \otimes b = a \otimes bc$ for all $c \in A(Y)$ we get

$c(axb) = (axb)c$ hence $axb \in {}_A[A, A]_A(X)$ for all $x \in A(X)$.

If A is central then $axb \in {}_A[A, A]_A(X) = I(X)$ for all $x \in A(X)$.

Conversely let $axb \in I(X)$ for all $x \in A(X)$. Let $x \in {}_A[A, A]_A(X)$

then $x = xab = axb \in I(X)$ hence ${}_A[A, A]_A(X) = I(X)$.

The Brauer group $B_2(\mathcal{C})$

A monoid A is called 2-Azumaya if $[A, I]$ and $[A, A]$ exist and

$\varphi: I \rightarrow A$ is a monomorphism and if there are elements

$a \otimes b \in A \otimes A(I)$ and $c \otimes d \otimes e \in A \otimes A \otimes A(I)$ such that

$$i) \forall X \in \mathcal{C} \quad \forall x \in A(X): xa \otimes b = a \otimes bx,$$

$$ii) ab = 1 \in A(I),$$

$$iii) ac \otimes dbe = 1 \otimes 1 \in A \otimes A(I),$$

$$iv) \forall X \in \mathcal{C} \quad \forall x \in A(X): axb \in I(X).$$

Clearly a 2-Azumaya monoid is a central, separable monoid which follows from i), ii), and iv) by Proposition 6 and 11. We do not know if the existence of $c \otimes d \otimes e$ with iii) follows from the other conditions.

Theorem 12: Let A be a monoid in \mathcal{C} . Equivalent are

a) A is 2-Azumaya.

b) $A \in \mathcal{C}$ is a progenerator and the morphism

$$\psi: A \otimes A(X) \ni x \otimes y \mapsto (A(Y)) \ni z \mapsto xzy \in A(X \otimes Y) \in [A, A](X)$$

is an isomorphism.

c) A is separable and 1-Azumaya.

Proof: Let A be 2-Azumaya. Define $\varphi: A \otimes A \rightarrow [A, I]$ by

$\langle z \rangle(\varphi(x \otimes y)) := axzy$ where $a \otimes b$ is a Casimir element for A .

Then $axzy \in I(X)$ by iv) hence φ is well-defined. Now

$\varphi(c \otimes d) \otimes e$ is a dual basis for A since $\langle x \rangle (\varphi(c \otimes d)e) = acxdb = x$ for all $x \in A(X)$. So A is finite.

Now we show that $A \otimes [A, I] \longrightarrow I$, the morphism induced by the evaluation, is rationally surjective, i.e. that

$A \otimes [A, I](I) \longrightarrow I(I)$ is surjective. We have to find

$a_1 \otimes f_1 \in A \otimes [A, I](I)$ with $\langle a_1 \rangle f_1 = 1 \in I(I)$. Take $a_1 = 1$ and $f_1 = (\text{Tr}: A \longrightarrow {}_A[A, A]_A \cong I)$, the last isomorphism exists in view of Propositions 6 and 11 by the properties i), ii), and iv) of A . Then $\langle 1 \rangle f_1 = 1$, hence A is a progenerator.

To show that ψ is an isomorphism we construct the inverse morphism

$$[A, A](X) \ni \sigma \longmapsto \langle ac \rangle \sigma db \otimes e \in A \otimes A(X).$$

This morphism is in fact an inverse of ψ since

$$\begin{aligned} \langle x \rangle \psi(\langle ac \rangle \sigma db \otimes e) &= \langle ac \rangle \sigma db x e = \langle x a c \rangle \sigma db e \\ &= \langle x 1 \rangle \sigma 1 = \langle x \rangle \sigma, \end{aligned}$$

hence $\psi(\langle ac \rangle \sigma db \otimes e) = \sigma$, and

$$x a c y d b \otimes e = x \otimes a c y d b e = x \otimes 1 y 1 = x \otimes y.$$

Now assume that b) holds. By Proposition 1 the monoid A is

1-Azumaya. Let $f_0 \otimes a_0$ be a dual basis for A and

$a_1 \otimes f_1 \in A \otimes [A, I](I)$ such that $\langle a_1 \rangle f_1 = 1 \in I(I)$. Let

$g_0 \in [A, I](I)$ be defined by $\langle x \rangle g_0 = \langle x a_1 \rangle f_1$. Let $a \otimes b \in A \otimes A(I)$

be the element which corresponds to $g_0 \otimes 1 \in [A, I] \otimes A(I)$ under the

isomorphism $A \otimes A \cong [A, A] \cong [A, I] \otimes A$. Then we have

$ab = a 1 b = \langle 1 \rangle g_0 1 = \langle a_1 \rangle f_1 1 = 1 \in A(I)$. Furthermore we have

$x a y b = x \langle y \rangle g_0 = \langle y \rangle g_0 x = a y b x$ for all $y \in A(Y)$, hence

$x a \otimes b = a \otimes b x$ for all $x \in A(X)$. So $a \otimes b$ is a Casimir element

for A . Thus c) holds.

Assume that c) holds. By Proposition 1 A is faithfully projective

and ψ is an isomorphism. Construct $a_1 \otimes f_1 \in A \otimes [A, I](I)$ with

$\langle a_1 \rangle f_1 = 1 \in I(I)$ as in part one of the proof. Then b) holds.

We still have to show that b) and c) imply a). Let $f_0^1 \otimes a_0^1$ and

$f_0^2 \otimes a_0^2$ be two copies of the dual basis of A . Let $a_1 \otimes f_1 \in A \otimes [A, I](I)$ with $\langle a_1 \rangle f_1 = 1$ be given. Then define $a \otimes b$ as above corresponding to g_0 and $c \otimes d \otimes e := u \otimes v \otimes xy \in A \otimes A \otimes A(I)$, where $u \otimes v \otimes x \otimes y \in A \otimes A \otimes A \otimes A(I)$ corresponds to $f_0^1 \otimes a_0^2 \otimes f_0^2 \otimes a_0^1 \in [A, I] \otimes A \otimes [A, I] \otimes A(I)$ under the isomorphisms

$[A, I] \otimes A \otimes [A, I] \otimes A \cong [A, A] \otimes [A, A] \cong A \otimes A \otimes A \otimes A$. Then $aczdbe = auzvbxxy = \langle uzv \rangle g_0 xy = \langle \langle \langle z \rangle f_0^1 a_0^2 a_1 \rangle f_1 \rangle f_0^2 a_0^1 = \langle z \rangle f_0^1 \langle a_0^2 a_1 \rangle f_1 \langle 1 \rangle f_0^2 a_0^1 = \langle \langle 1 \rangle f_0^2 a_0^2 a_1 \rangle f_1 \langle z \rangle f_0^1 a_0^1 = z = 1z1$ for all $z \in A(Z)$, hence $ac \otimes dbe = 1 \otimes 1$. Thus iii) for a monoid to be 2-Azumaya holds. i) and ii) hold by Proposition 1, iv) by Proposition 11.

Corollary 13: Let A and B be 2-Azumaya, then $A \otimes B$ is 2-Azumaya.

Proof: In view of the equivalence of a) and c) in Theorem 12 this follows from Proposition 3 and Proposition 10.

Theorem 14: Let $P \in \mathcal{C}$ be a progenerator. Then $[P, P]$ is 2-Azumaya.

Proof: By Proposition 4 we get that $[P, P]$ is 1-Azumaya so that we only have to show that $[P, P]$ is separable. Let $f_0 \otimes p_0$ be a dual basis for P and $p_1 \otimes f_1 \in P \otimes [P, I](I)$ with $\langle p_1 \rangle f_1 = 1$. Identify $[P, P]$ with $[P, I] \otimes P$ with the multiplication $(f \otimes p)(f' \otimes p') = f \otimes \langle p \rangle f' p'$. Then define $a \otimes b := (f_0 \otimes p_1) \otimes (f_1 \otimes p_0)$. For every $g \otimes q \in [P, I] \otimes P(X)$ we have

$$(g \otimes q)(f_0 \otimes p_1) \otimes (f_1 \otimes p_0) = (g \otimes \langle q \rangle f_0 p_1) \otimes (f_1 \otimes p_0) =$$

$$(g \otimes p_1) \otimes (f_1 \otimes \langle q \rangle f_0 p_0) = (g \otimes p_1) \otimes (f_1 \otimes q) =$$

$$(f_0 \langle p_0 \rangle g \otimes p_1) \otimes (f_1 \otimes q) = (f_0 \otimes p_1) \otimes (f_1 \otimes \langle p_0 \rangle gq) = \\ (f_0 \otimes p_1) \otimes (f_1 \otimes p_0)(g \otimes q)$$

so that b) i) of Proposition 6 holds. Furthermore

$$(f_0 \otimes p_1)(f_1 \otimes p_0) = f_0 \otimes \langle p_1 \rangle f_1 p_0 = f_0 \otimes p_0 ,$$

which corresponds to $1 \in [P, P](I)$, shows b) ii) .

It may be interesting to have an explicit description of the element $c \otimes d \otimes e$ in the definition of 2-Azumaya for this case $[P, P]$. Let $f_0^i \otimes p_0^i$, $i = 1, 2, 3$ be copies of the dual basis of P . Then $c \otimes d \otimes e := (f_0^1 \otimes p_0^2) \otimes (f_0^3 \otimes p_0^1) \otimes (f_0^2 \otimes p_0^3)$ satisfies condition iii) for 2-Azumaya as is easily checked.

To define a Brauer group of 2-Azumaya monoids we need one more lemma.

Lemma 15: Let P and Q be progenerators. Then $P \otimes Q$ is a progenerator and $[P, P] \otimes [Q, Q] \cong [P \otimes Q, P \otimes Q]$ as monoids.

Proof: Let $p_1 \otimes f_1$ resp. $q_1 \otimes g_1$ with $\langle p_1 \rangle f_1 = 1$ resp. $\langle q_1 \rangle g_1 = 1$ be given. Then form the element $(p_1 \otimes q_1) \otimes \mathcal{J}(f_1 \otimes g_1) \in (P \otimes Q) \otimes [P \otimes Q, I](I)$, where $\mathcal{J}: [P, I] \otimes [Q, I] \cong [P \otimes Q, I]$ is the isomorphism used in the proof of Proposition 5 . We get $(p_1 \otimes q_1) \mathcal{J}(f_1 \otimes g_1) = \langle p_1 \rangle f_1 \langle q_1 \rangle g_1 = 1$, hence $P \otimes Q$ is a progenerator in view of Proposition 5 .

Now we can define the Brauer group $B_2(C)$, using 2-Azumaya monoids, in the same way as $B_1(C)$. Since each 2-Azumaya monoid is 1-Azumaya and since each progenerator is faithfully projective we get a group homomorphism $\xi: B_2(C) \longrightarrow B_1(C)$. Since the notions of progenerator and faithfully projective coincide, if $I \in C$ is projective, the notions of 1-Azumaya and 2-Azumaya coincide by Theorem 12, b) and Pro-

position 1 . So does the equivalence relation used in the construction of the two Brauer groups and we get

Theorem 16: The group homomorphism $\xi: \mathcal{B}_2(C) \longrightarrow \mathcal{B}_1(C)$ is the identity in case $I \in C$ is projective.

Splitting Azumaya monoids by monoidal functors.

Now we want to discuss the behaviour of the Brauer groups under a monoidal functor. Let C and \mathcal{D} be symmetric monoidal categories and $F: C \longrightarrow \mathcal{D}$ be a covariant functor. Denote the tensor products and the associativity, the symmetry and unity isomorphisms in C and \mathcal{D} by the same signs $\boxtimes, \alpha, \gamma, \lambda,$ and γ . Assume that there are natural transformations

$$\delta: FX \boxtimes FY \longrightarrow F(X \boxtimes Y)$$

$$\zeta: J \longrightarrow FI$$

such that the following diagrams commute

$$\begin{array}{ccc} FX \boxtimes FI & \xleftarrow{1 \boxtimes \zeta} & FX \boxtimes J \\ \downarrow \delta & & \downarrow \gamma \\ F(X \boxtimes I) & \xrightarrow{F(\gamma)} & FX \end{array}$$

$$\begin{array}{ccc} FI \boxtimes FX & \xleftarrow{\zeta \boxtimes 1} & J \boxtimes FX \\ \downarrow \delta & & \downarrow \lambda \\ F(I \boxtimes X) & \xrightarrow{F(\lambda)} & FX \end{array}$$

$$\begin{array}{ccccc} FX \boxtimes (FY \boxtimes FZ) & \xrightarrow{1 \boxtimes \delta} & FX \boxtimes F(Y \boxtimes Z) & \xrightarrow{\delta} & F(X \boxtimes (Y \boxtimes Z)) \\ \downarrow \alpha & & & & \downarrow F(\alpha) \\ (FX \boxtimes FY) \boxtimes FZ & \xrightarrow{\delta \boxtimes 1} & F(X \boxtimes Y) \boxtimes FZ & \xrightarrow{\delta} & F((X \boxtimes Y) \boxtimes Z) . \end{array}$$

If C and \mathcal{D} are symmetric we require in addition the commutativity of

$$\begin{array}{ccc}
 FX \boxtimes FY & \xrightarrow{\gamma} & FY \boxtimes FX \\
 \downarrow \delta & & \downarrow \delta \\
 F(X \boxtimes Y) & \xrightarrow{F(\gamma)} & F(Y \boxtimes X) .
 \end{array}$$

Such a triple (F, δ, ζ) will be called a weakly monoidal functor.

Let $\pi: X \boxtimes [X, Y] \longrightarrow Y$ and $\tau: Y \longrightarrow [X, X \boxtimes Y]$ be front and back adjunction for the adjoint pair of functors $X \boxtimes -$ and $[X, -]$, if $[X, -]$ exists. Again we use the same notation in both categories \mathcal{C} and \mathcal{D} . Let $\chi: \mathcal{C}(X \boxtimes Y, Z) \cong \mathcal{C}(Y, [X, Z])$ and $\omega: \mathcal{C}(Y, [X, Z]) \cong \mathcal{C}(X \boxtimes Y, Z)$ be the corresponding adjointness isomorphisms in \mathcal{C} resp. also in \mathcal{D} . It is an easy exercise in diagram chasing for adjoint functors to show that there is a natural transformation $\phi: F[X, Y] \longrightarrow [FX, FY]$ whenever $[X, Y]$ and $[FX, FY]$ exist, just take $\phi = \chi(F(\pi)\delta)$. Furthermore the diagrams

$$\begin{array}{ccc}
 FX \boxtimes F[X, Y] & \xrightarrow{\delta} & F(X \boxtimes [X, Y]) \\
 \downarrow 1 \boxtimes \phi & & \downarrow F(\pi) \\
 FX \boxtimes [FX, FY] & \xrightarrow{\pi} & FY \\
 \downarrow \zeta & & \downarrow \phi \\
 FI & \xrightarrow{F(i)} & F[X, X]
 \end{array}
 \quad
 \begin{array}{ccc}
 FY & \xrightarrow{F(\tau)} & F[X, X \boxtimes Y] \\
 \downarrow \tau & & \downarrow \phi \\
 [FX, FX \boxtimes FY] & \xrightarrow{[1, \delta]} & [FX, F(X \boxtimes Y)] \text{ and}
 \end{array}$$

commute. Here $i: I \longrightarrow [P, P]$ is $\chi(\gamma)$ where $\gamma: X \boxtimes I \longrightarrow X$ and j is defined analogously in \mathcal{D} .

Omitting special arrows for the associativity α we get the commutative diagram on the next page.

If we abbreviate $[1, \pi \boxtimes 1]\tau$ by $\varphi: [X, Y] \boxtimes Z \longrightarrow [X, Y \boxtimes Z]$ then the following diagram (the outer frame of the given diagram) commutes

$$\begin{array}{ccc}
 F[X, Y] \boxtimes FZ & \xrightarrow{\delta} & F([X, Y] \boxtimes Z) \xrightarrow{F(\varphi)} F([X, Y \boxtimes Z]) \\
 \downarrow \phi \boxtimes 1 & & \downarrow \phi \\
 [FX, FY] \boxtimes FZ & \xrightarrow{\varphi} & [FX, FY \boxtimes FZ] \xrightarrow{[1, \delta]} [FX, F(Y \boxtimes Z)] .
 \end{array}$$

$$\begin{array}{ccccc}
 & & F([X, Y] \boxtimes Z) & \xrightarrow{F(\tau)} & F[X, X] \boxtimes [X, Y] \boxtimes Z & \xrightarrow{F[1, \pi \boxtimes 1]} & F[X, Y] \boxtimes Z \\
 & \nearrow \delta & \downarrow \tau & & \downarrow \phi & & \downarrow \phi \\
 & & [FX, FX] \boxtimes F([X, Y] \boxtimes Z) & \xrightarrow{[1, \delta]} & [FX, F(X] \boxtimes [X, Y] \boxtimes Z) & \xrightarrow{[1, F(\pi \boxtimes 1)]} & [FX, F(Y] \boxtimes Z) \\
 & & \uparrow [1, 1 \boxtimes \delta] & & \uparrow [1, \delta] & & \\
 F[X, Y] \boxtimes FZ & \xrightarrow{\tau} & [FX, FX] \boxtimes F[X, Y] \boxtimes FZ & \xrightarrow{[1, \delta \boxtimes 1]} & [FX, F(X] \boxtimes [X, Y] \boxtimes FZ & & \\
 \downarrow \phi \boxtimes 1 & & \downarrow [1, 1 \boxtimes \phi \boxtimes 1] & & \downarrow [1, F(\pi) \boxtimes 1] & & \nearrow [1, \delta] \\
 [FX, FY] \boxtimes FZ & \xrightarrow{\tau} & [FX, FX] \boxtimes [FX, FY] \boxtimes FZ & \xrightarrow{[1, \pi \boxtimes 1]} & [FX, FY] \boxtimes FZ & &
 \end{array}$$

Theorem 17: Let $F: C \rightarrow D$ be a weakly monoidal functor. Assume that $\zeta: J \rightarrow FI$ is an isomorphism and that $\zeta: F[P, I] \otimes FP \rightarrow F([P, I] \otimes P)$ is an isomorphism for all finite objects $P \in C$. If P is finite in C and if $[FP, -]$ exists in D then FP is finite in D .

Proof: Since finiteness is equivalent to the fact that $i: I \rightarrow [P, P]$ can be factored through $[P, I] \otimes P$ the following commutative diagram shows that $j: J \rightarrow [FP, FP]$ can be factored through $[FP, J] \otimes FP$

$$\begin{array}{ccccc}
 & & FI & \xleftarrow{\zeta} & J \\
 & & \swarrow F(f_0 \otimes p_0) & & \searrow F(i) \\
 F[P, I] \otimes FP & \xrightarrow{\delta} & F([P, I] \otimes P) & \xrightarrow{F(\varphi)} & F([P, I] \otimes P) & \xrightarrow{F[1, \lambda]} & F([P, P]) & \xrightarrow{j} & \\
 \downarrow \phi \otimes 1 & & \downarrow \phi & & \downarrow \phi & & & & \\
 [FP, FI] \otimes FP & \xrightarrow{\varphi} & [FP, FI \otimes FP] & \xrightarrow{[1, \delta]} & [FP, F(I \otimes P)] & \xrightarrow{[1, F\lambda]} & [FP, FP] & & \\
 \uparrow [1, \zeta] \otimes 1 & & \uparrow [1, \zeta \otimes 1] & & \uparrow [1, \lambda] & & & & \\
 [FP, J] \otimes FP & \xrightarrow{\varphi} & [FP, J \otimes FP] & & & & & &
 \end{array}$$

thus FP is finite.

Corollary 18: Under the assumption of Theorem 17 is the morphism $\phi: F[P, X] \rightarrow [FP, FX]$ an isomorphism for all $X \in C$ and all finite $P \in C$.

Proof: Let $f_0 \otimes p_0: I \rightarrow [P, I] \otimes P$ be the dual basis for P and $\bar{f}_0 \otimes \bar{p}_0: J \rightarrow [FP, J] \otimes FP$ be the dual basis for FP . Define $\psi: [FP, FX] \rightarrow F[P, X]$ to be

$$\begin{aligned}
 [FP, FX] &\cong FI \otimes [FP, FX] \xrightarrow{F(f_0 \otimes p_0) \otimes 1} F([P, I] \otimes P) \otimes [FP, FX] \xrightarrow{\delta^{-1} \otimes 1} \\
 &F[P, I] \otimes FP \otimes [FP, FX] \xrightarrow{1 \otimes \pi} F[P, I] \otimes FX \longrightarrow F[P, X] .
 \end{aligned}$$

Omitting some of the obvious isomorphisms we get a commutative diagram

$$\begin{array}{ccccc}
 [FP, FX] & & & & \\
 \searrow & \xrightarrow{\delta^{-1} F(f_0 \otimes p_0) \otimes 1} & & & \\
 & F[P, I] \otimes FP \otimes [FP, FX] & \xrightarrow{1 \otimes \pi} & F[P, I] \otimes FX & \longrightarrow & F[P, X] \\
 & \downarrow \phi \otimes 1 \otimes 1 & & \downarrow \phi \otimes 1 & & \downarrow \phi \\
 \bar{f}_0 \otimes \bar{p}_0 \otimes 1 & & [FP, FI] \otimes FP \otimes [FP, FX] & \xrightarrow{1 \otimes \pi} & [FP, FI] \otimes FX & \cong & [FP, FX]
 \end{array}$$

where the left triangle commutes by the construction of $\bar{f}_0 \otimes \bar{p}_0$ in Theorem 17 and right square commutes in the same way as the middle of the diagram in the proof of Theorem 17 does. If we look at the lower part of our diagram we see that the morphism $[FP, FX] \longrightarrow [FP, FX]$ is the identity since $\bar{f}_0 \langle \bar{p}_0 \rangle g = g$ for all $g \in [FP, FX](Y)$. The upper part is ψ , hence $\psi = id$. Conversely the commutative diagram

$$\begin{array}{ccccccc}
 F[P, X] & \longrightarrow & F[P, I] \otimes FP \otimes F[P, X] & \xrightarrow{1 \otimes \delta} & F[P, I] \otimes F(P \otimes [P, X]) & & \\
 \downarrow \phi & & \downarrow 1 \otimes 1 \otimes \phi & & \downarrow 1 \otimes F(\pi) & & \\
 [FP, FX] & \longrightarrow & F[P, I] \otimes FP \otimes [FP, FX] & \xrightarrow{1 \otimes \pi} & F[P, I] \otimes FX & \longrightarrow & F[P, X]
 \end{array}$$

shows $\psi \phi = id$.

Corollary 19: Under the assumptions of Theorem 17 if P is a pro-generator then FP is a progenerator. If F preserves difference cokernels and P is faithfully projective then FP is faithfully projective.

Proof: Let P be finite. P is a progenerator iff there is a morphism $f: I \longrightarrow P \otimes [P, I]$ such that

$$\begin{array}{ccc}
 & & I \\
 & f \swarrow & \downarrow 1 \\
 P \otimes [P, I] & \xrightarrow{\pi} & I
 \end{array}$$

commutes. Now the diagram

$$\begin{array}{ccc}
 & & J \\
 & F(f)\zeta \swarrow & \downarrow \mathbb{R}\zeta \\
 F(P \otimes [P, I]) & \xrightarrow{F(\pi)} & FI \\
 \mathbb{R}(1 \otimes [1, \zeta^{-1}])(1 \otimes \phi)\delta^{-1} & & \mathbb{R}\zeta^{-1} \\
 FP \otimes [FP, J] & \xrightarrow{\pi} & J
 \end{array}$$

commutes hence FP is a progenerator. In the case of a faithfully projective P we have to replace

$P \otimes [P, I]$ by $P \otimes [P, I]$ and $FP \otimes [FP, J]$ by $FP \otimes [FP, J] \cong F(P \otimes [P, I])$. The last isomorphism is a consequence of the fact that F preserves difference cokernels.

Theorem 20: Let $F: C \rightarrow D$ be a weakly monoidal functor such that $\zeta: J \rightarrow FI$ is an isomorphism, $\delta: FX \otimes FP \rightarrow F(X \otimes P)$ is an isomorphism for all $X \in C$ and for all finite $P \in C$, $[FP, -]$ exists for all finite $P \in C$ and F preserves difference cokernels. Then F induces homomorphisms of Brauer groups $B_i(F): B_i(C) \rightarrow B_i(D)$ for $i = 1, 2$ such that

$$\begin{array}{ccc}
 B_2(C) & \xrightarrow{B_2(F)} & B_2(D) \\
 \downarrow \xi & & \downarrow \xi \\
 B_1(C) & \xrightarrow{B_1(F)} & B_1(D)
 \end{array}$$

commutes.

Proof: Let A be a monoid in C . Then FA is a monoid in D with the multiplication $FA \otimes FA \xrightarrow{\delta} F(A \otimes A) \xrightarrow{F(\mu)} FA$ and unit $J \xrightarrow{\zeta} FI \xrightarrow{F(\eta)} FA$. If A is i -Azumaya, $i = 1, 2$, then FA is faithfully projective resp. a progenerator by Corollary 19, Proposition 1 and Theorem 12. So we only have to show that

$\psi: FA \otimes FA \rightarrow [FA, FA]$ is an isomorphism. ψ is induced by

$T: A \otimes A \otimes A(X) \ni a \otimes b \otimes c \mapsto bac \in A(X)$, so that $\psi = \chi(T)$ where $\chi: C(X \otimes Y, Z) \cong C(Y, [X, Z])$.

Now the diagram

$$\begin{array}{ccc} FA \otimes FA & \xrightarrow{\chi(T)} & [FA, FA] \\ \downarrow \delta & & \uparrow \phi \\ F(A \otimes A) & \xrightarrow{F(\chi(T))} & F[A, A] \end{array}$$

commutes, since

$$\begin{array}{ccc} FA \otimes (FA \otimes FA) & \xrightarrow{T} & FA \\ \downarrow 1 \otimes \delta & & \uparrow F(T) \\ FA \otimes F(A \otimes A) & \xrightarrow{\delta} & F(A \otimes (A \otimes A)) \end{array}$$

commutes so that by applying χ we get

$\phi \circ F(\chi(T)) \circ \delta = \chi(F(T) \circ \delta) \circ \delta = [1, F(T)] \circ \chi(\delta) \circ \delta = \chi(T)$. The first identity results from the commutativity of

$$\begin{array}{ccc} C(X \otimes Y, Z) & \xrightarrow{F} & \mathcal{D}(F(X \otimes Y), FZ) \xrightarrow{\mathcal{D}(\delta, 1)} \mathcal{D}(FX \otimes FY, FZ) \\ C(Y, [X, Z]) & \xrightarrow{F} & \mathcal{D}(FY, F[X, Z]) \xrightarrow{\mathcal{D}(1, \phi)} \mathcal{D}(FY, [FX, FZ]) \end{array}$$

Now $\phi: F[A, A] \rightarrow [FA, FA]$ is an isomorphism by Corollary 18 and $F(\chi(T)) = F(\psi)$ is an isomorphism since A is Azumaya. Since δ is an isomorphism, too, we get that $\psi = \chi(T): FA \otimes FA \rightarrow [FA, FA]$ is an isomorphism.

If $P \in C$ is faithfully projective or a progenerator in C then as above FP is faithfully projective or a progenerator in \mathcal{D} and $F[P, P] \cong [FP, FP]$ as monoids using the first commutative diagram we proved for ϕ .

Thus if A and B are i -Azumaya, then FA and FB are i -Azumaya. If A and B are equivalent w.r.t. $B_i(C)$, then so are FA and FB . Finally we have $F(A \otimes B) \cong FA \otimes FB$ and $FI = J$ so that F induces homomorphisms $B_i(F): B_i(C) \rightarrow B_i(\mathcal{D})$ such that the diagram in the theorem commutes.

If $F: C \rightarrow \mathcal{D}$ is a functor satisfying the conditions of Theorem 20 then we define the kernel of $B_i(F)$ as $B_i(C, F)$ so that we get exact sequences

$$0 \longrightarrow B_i(C, F) \longrightarrow B_i(C) \longrightarrow B_i(\mathcal{D})$$

for $i = 1, 2$. $B_i(C, F)$ contains those elements $[A]$ of $B_i(C)$ with $[FA] = [[P, P]]$ for some $P \in \mathcal{D}$ which is faithfully projective resp. a progenerator. These i -Azumaya monoids A are called F -split. From Theorem 20 follows immediately a homomorphism

$$\xi: B_2(C, F) \longrightarrow B_1(C, F) .$$

If \mathcal{C} is a symmetric monoidal closed category with difference kernels and difference cokernels and $K \in \mathcal{C}$ is commutative monoid, then ${}_K\mathcal{C}$ is again a symmetric monoidal closed category with \boxtimes_K as tensor product and K as basic object. Then the functor $\mathcal{C} \ni X \longmapsto K \boxtimes X \in {}_K\mathcal{C}$ has all properties required in Theorem 20 hence there are homomorphisms $B_i(\mathcal{C}) \longrightarrow B_i({}_K\mathcal{C})$ with kernels $B_i(K/\mathcal{C})$.

References:

- [1] Auslander, M. and Goldmann, O.: The Brauer group of a commutative ring, *Trans.Amer.Math.Soc.* 97 (1960), 367 - 409.
- [2] Long, F.W.: The Brauer group of dimodule algebras, *J. of Algebra* 30 (1974), 559 - 601.
- [3] MacLane, S.: *Categories for the working mathematician*, Graduate Texts in Mathematics. Springer New York - Heidelberg - Berlin 1971.
- [4] Orzech, M. and Small, Ch.: *The Brauer group of commutative rings*. Leisure notes in Pure and Applied Mathematics. Marcel Dekker New York 1975.
- [5] Pareigis, B.: Non-additive ring and module theory I: General theory of monoids.
To appear in: *Publicationes Mathematicae Debrecen.*
- [6] Pareigis, B.: Non-additive ring and module theory II: \mathcal{C} -categories, \mathcal{C} -functors and \mathcal{C} -morphisms.
To appear in: *Publicationes Mathematicae Debrecen.*
- [7] Pareigis, B.: Non-additive ring and module theory III: Morita theorems over monoidal categories.
To appear in: *Publicationes Mathematicae Debrecen.*
- [8] Pareigis, B.: Non-additive ring and module theory V: Projective and flat objects.
To appear in: *Algebra-Berichte.*
- [9] Fisher-Palmquist, J.: The Brauer group of a closed category, *Proc.Amer.Math.Soc.* 50 (1975), 61 - 67.