Non-Archimedean Hilbert like spaces*

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Abstract

Let $\mathbb{K}$ be a non-Archimedean, complete valued field. It is known that the supremum norm $\|\cdot\|_\infty$ on c_0 is induced by an inner product if and only if the residual class field of $\mathbb{K}$ is formally real. One of the main problems of this inner product is that c_0 is not orthomodular, as is any classical Hilbert space. Our goal in this work is to identify those closed subspaces of c_0 which have a normal complement. In this study we also involve projections, adjoint and self-adjoint operators.

1 Introduction

Two of the most useful and beautiful mathematical theories in real or complex functional analysis have been Hilbert spaces and continuous linear operators. These theories have exactly matched the needs of certain branches of physics, biology, etc.

The importance of Hilbert space over real or complex fields have been an inspiration for many researchers to extend these ideas for non-Archimedean fields. Since 1945, several attempts have been made to define an appropriate non-Archimedean inner product. One of the most recent papers about non-Archimedean Hilbert spaces is that of L. Narici and E. Beckenstein [4]. They define a non-Archimedean inner product over a vector space X as a non-degenerate $\mathbb{K}$-function in $X \times X$, which is linear in the first variable and satisfies what they call the Cauchy-Schwarz type inequality. The main problem that these researchers have faced is the orthomodular property, that is, for any subspace M of X; $M = M^{\perp\perp} \Leftrightarrow X = M \oplus M^{\perp}$. It is well known that real and complex Hilbert spaces are orthomodular. The existence

*This work was partially supported by Proyecto DIUC No.2005.013.024-1.0 and Proyecto DIN 02/2006

2000 *Mathematics Subject Classification :* Primary 46C50; Secondary 46S10.

Key words and phrases : Non-archimedean fields, inner products, normal complemented subspaces, projections, adjoint and selfadjoint operators.

of infinite-dimensional non-classical orthomodular spaces was an open question until the following interesting theorem was proved by M. P. Solér [6]: "Let X be an orthomodular space and suppose it contains an orthonormal sequence $e_1, e_2, \cdots$ (in the sense of the inner product). Then the base field is $\mathbb{R}$ or $\mathbb{C}$". Based on the result of Solér, if $\mathbb{K}$ is a non-Archimedean, complete valued field, then the space $(c_0, \|\cdot\|_\infty)$ is not an orthomodular space, where c_0 is the space of all null sequences $x = (a_n)$, $a_n \in \mathbb{K}$, and $\|x\|_\infty = \sup\{|a_n| : n \in \mathbb{N}\}$.

It was proved in [4] that there is an "inner product", $\langle\cdot,\cdot\rangle$ on c_0 which induces $\|\cdot\|_\infty$ if and only if the residue class field of $\mathbb{K}$ is formally real. Unlike classical Hilbert spaces, however, as previously mentioned, c_0 is not orthomodular. We identify those closed subspaces of c_0 which have normal complement and investigate projections, adjoint and self-adjoint operators on c_0.

2 Preliminary and definitions

Throughout $\mathbb{K}$ will be a field with a non-Archimedean valuation $|\cdot|$, for which $\mathbb{K}$ is a complete.

We remind the reader that the residue class field of $\mathbb{K}$ is the field

$$\Bbbk = B(0,1)/B^-(0,1)$$

where $B(0,1) = \{\lambda \in \mathbb{K} : |\lambda| \le 1\}$ and $B^-(0,1) = \{\lambda \in \mathbb{K} : |\lambda| < 1\}$.

Let X be a vector space over $\mathbb{K}$. By a non-Archimedean inner product we mean a map $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{K}$ which satisfies for all $a, b \in \mathbb{K}$ and $x, y, z \in X$

I.1 $x \ne 0 \Rightarrow \langle x, x\rangle \ne 0$;

I.2 $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$;

I.3 $|\langle x, y\rangle|^2 \le |\langle x, x\rangle|\,|\langle y, y\rangle|$

A vector space X with $\langle\cdot,\cdot\rangle$ is called a non-Archimedean inner product space. If $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in X$, then $\langle\cdot,\cdot\rangle$ is called a symmetric inner product.

In what follows we omit "non-Archimedean" in "non-Archimedean inner product" in order to simplify the reading of this article.

If $X = c_0$, then there is a natural symmetric bilinear form $\langle\cdot,\cdot\rangle : c_0 \times c_0 \to \mathbb{K}$, defined by

$$\langle x, y\rangle = \sum_{n\in\mathbb{N}} x_n y_n. \tag{2.1}$$

It is easy to see that $|\langle x, y\rangle| \le \|x\|_\infty \|y\|_\infty$; in other words, it satisfies the Cauchy-Schwarz type inequality. But, it may happen that $|\langle x, x\rangle| < \|x\|_\infty^2$ for some $x \in c_0$. In order to avoid the last strict inequality, we need an extra algebraic condition on $\mathbb{K}$.

Definition 1. *A field F is said to be formally real if for any finite subset $\{a_1, \cdots, a_n\}$ of F, $\sum_{i=1}^n a_i^2 = 0$ implies each $a_i = 0$.*

According to this definition, $\mathbb{R}$ is formally real and $\mathbb{C}$ is not. But, for example, in the non-Archimedean case $\mathbb{Q}_p$ and $\mathbb{C}_p$ are not formally real; however, the Levi Civita field is formally real. Now, in the non-Archimedean context, we have the following result: $\Bbbk$ is formally real if, and only if, for each finite subset $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of $\mathbb{K}$,

$$\left|\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2\right| = \max\left\{\left|\lambda_1^2\right|, \left|\lambda_2^2\right|, \cdots, \left|\lambda_n^2\right|\right\}.$$

The following theorem was one of the main results proved in [4], (Th. 6.1, p. 194):

Theorem 1. *The symmetric bilinear form given in* (2.1) *is an inner product on* c_0 *which induces the original norm if and only if the residue class field* $\Bbbk$ *of* $\mathbb{K}$ *is formally real.*

From this point on the Banach X will be c_0 provided with the inner product given in (2.1) and we will assume that the residue class field $\Bbbk$ of $\mathbb{K}$ is formally real.

Definition 2. *A subset* D *of* c_0 *such that for all* $x, y \in D$, $x \neq y \Rightarrow \langle x, y\rangle = 0$ *is called a normal family. A countable normal family* $\{x_n : n \in \mathbb{N}\}$ *of unit vectors is called an orthonormal sequence.*

If $A \subset c_0$, then $[A]$ and $cl\,[A]$ will denote the linear and the closed linear span of A, respectively. The Gram-Schmidt procedure is also proved in [4] and it says:

Theorem 2. *If* (x_n) *is a sequence of linearly independent vectors in* c_0, *then there exists an orthonormal sequence* (y_n) *such that* $[\{x_1, \cdots, x_n\}] = [\{y_1, \cdots, y_n\}]$.

We will now prove some statements about the existence of countable bases on closed subspaces of c_0. We remind the reader that a non-Archimedean Banach space E is said to be of countable type if it contains a countable set whose linear span is dense.

Lemma 1. *Let* E *be a normed space of countable type and let* Y *be a subset of* E *such that* $cl\,[Y] = E$. *Then, there exists a countable subset* Z *of* Y *such that* $cl\,[Z] = E$.

Proof. If we take $x \in E = cl\,[Y]$, then there exists a sequence (y_n) in $[Y]$ such that $y_n \to x$. Now, each y_n can be written as $\sum_{i \in F_n} \alpha_{i,n} x_{i,n}$, where F_n is a finite subset of $\mathbb{N}$, $\alpha_{i,n} \in \mathbb{K}$ and $x_{i,n} \in Y$. Therefore, $Z_x = \cup_{i,n=1}^{\infty} \{x_{i,n} : i \in F_n\}$ is countable and $x \in cl\,[Z_x]$. Since E is of countable type, there exists $\{e_n : n \in \mathbb{N}\} \subset E$ such that $cl\,[\{e_n : n \in \mathbb{N}\}] = E$. If $x = e_n$, then $Z_{e_n} \subset Y$, Z_{e_n} is countable and $e_n \in cl\,[\cup_{n=1}^{\infty} Z_{e_n}] = cl\,[Y] = E$. ∎

Definition 3. *Let* E *be a normed space and let* X *be a subset of* E. *We say that* X *is NA-orthogonal if for any finite subset* $\{x_1, x_2, \cdots, x_m\}$ *of* X, *the following is satisfied:*

$$\left\|\sum_{i=1}^{m} \alpha_i x_i\right\| = \max\{\|\alpha_i x_i\| : i = 1, \cdots, m\}, \quad \text{for all } \alpha_1, \alpha_2, \cdots, \alpha_m \in \mathbb{K}.$$

Proposition 1. *Let E be a normed space of countable type. Then, every NA-orthogonal system is countable.*

Proof. Let $\{e_i : i \in I\}$ be an NA-orthogonal system. By [7], Th. 3.16.v, $D = cl\left[\{e_i : i \in I\}\right]$ is also of countable type . By Lemma 1, there exists a countable subset J of I such that $D = cl\left[\{e_j : j \in J\}\right]$. We claim that I is countable. If I were uncountable, then there would be an $i \in I \setminus J$; since $e_i \in D$, we have that $e_i = \sum_{j\in J} \alpha_j e_j$ or, equivalently, $\sum_{j\in J} \alpha_j e_j + \alpha_i e_i = 0$, where $\alpha_i = -1$. By the NA-orthogonality of $\{e_i : i \in I\}$,

$$0 = \left\| \sum_{j\in J} \alpha_j e_j + \alpha_i e_i \right\| = \max\left\{\|\alpha_j e_j\| : j \in J\right\}$$

which contradicts the NA-orthogonality of the family $\{e_i : i \in I\}$. ■

According to Th. 5.9, p. 174 in [7], we conclude that every closed subspace of c_0 has an orthogonal base. Moreover, since c_0 has a countable orthogonal base, every closed subspace of c_0 has the same property. Now, using the Gram-Schmidt process, we have the following theorem:

Theorem 3. *Every closed subspace D of c_0 admits a countable orthonormal base, that is, a sequence (y_n) such that $D = cl\left[\{y_n : n \in \mathbb{N}\}\right]$, $\langle y_n, y_m\rangle = 0,\ n \neq m$, and $\|y_n\| = 1$.*

3 Normal Complemented and Riesz functional

The behavior of the symmetric inner product given in Section 2 has some differences with respect to the real or complex case, for example:

1. If the field is $\mathbb{R}$ or $\mathbb{C}$, then

$$|\langle x, y\rangle| = \|x\|\,\|y\| \Rightarrow y = ax.$$

   But this property is lost if the field is a non-Archimedean valued field. In fact, if we take

$$\underbrace{(0,1,1,0,\cdots)}_{x}, \underbrace{(1,0,1,0,\cdots)}_{y} \in c_0$$

   then

$$|\langle x, y\rangle| = \|x\|\,\|y\| = 1$$

   and clearly $y \neq ax$ for any $a \in \mathbb{K}$.

2. Given a subspace M of c_0, we will denote by M^p the subspace of all elements x of c_0 such that $\langle x, y\rangle = 0$, for all $y \in M$, that is,

$$M^p = \{x \in c_0 : \langle x, y\rangle = 0 \text{ for all } y \in M\}.$$

   Suppose that the valuation of $\mathbb{K}$ is dense and that (λ_i) is a sequence such that $1 < |\lambda_1| < |\lambda_2| < \cdots < 2$. If we consider the closed subspace

$$M = \left\{x = (x_i) \in c_0 : \sum_{i\geq 1} x_i \lambda_i = 0\right\},$$

then $M^p = \{\theta\}$. In fact, suppose that there exists a nonzero element $\varpi = (\omega_i)$ in M^p; we define $\widetilde{\varpi} = \alpha^{-1}\varpi$, where $\alpha = \sum \omega_i \lambda_i$. Now, if $x = (x_i) \in c_0$, then $x = (x - \beta\widetilde{\varpi}) + \beta\widetilde{\varpi}$, where $\beta = \sum x_i \lambda_i$. It is easy to see that $x - \beta\widetilde{\varpi} \in M$. If $b = \langle \widetilde{\varpi}, \widetilde{\varpi} \rangle^{-1}$, then

$$\langle x, b\widetilde{\varpi} \rangle = \langle x - \beta\widetilde{\varpi}, b\widetilde{\varpi} \rangle + \langle \beta\widetilde{\varpi}, b\widetilde{\varpi} \rangle = 0 + \beta b \langle \widetilde{\varpi}, \widetilde{\varpi} \rangle = \beta;$$

in particular, if e_i is an element of the canonical base, then $\lambda_i = \langle e_i, b\widetilde{\varpi} \rangle$ which is a contradiction, since the right hand side must converge to 0. In other words, $M \neq M^{pp}$. This proves that c_0 is not orthomodular when the valuation of the field $\mathbb{K}$ is dense.
The orthomodularity of c_0 is also lost if the valuation of the field $\mathbb{K}$ is discrete; the closed subspace

$$M = \left\{ x = (x_i) \in c_0 : \sum_{i \geq 1} x_i = 0 \right\}$$

shows that $M \neq M^{pp}$.

Definition 4. *Let M and N be two closed subspaces of c_0. We say that N is a normal complement of M if*

$$(x \in M, y \in N \Rightarrow \langle x, y \rangle = 0) \quad \textit{and} \quad c_0 = M \oplus N.$$

If such a normal complement exists for M, then we say that M is normally complemented.

Definition 5. *If a sequence $(x_i)_{i \in \mathbb{N}}$ of c_0 satisfies $\langle x, x_i \rangle \to 0$, for any $x \in c_0$, then we say that $(x_i)_{i \in \mathbb{N}}$ has the Riemann-Lebesgue Property.*

Example 1. *The most obvious example of a sequence with the Riemann-Lebesgue Property is any Schauder base (or simply base) of c_0. In particular, the canonical base $(e_i)_{i \in \mathbb{N}}$ has this property.*

The following theorem was proved in [4] :

Theorem 4. *If $S \subset c_0$ is a finite set or an orthonormal sequence with the Riemann-Lebesgue Property, then S can be extended to a base for c_0. Under this condition, the closure M of the subspace generated by S has a normal complement. This complement is M^p and $c_0 = M \oplus M^p$.*

Remark 1. *Another important space is the dual of c_0. It is well-known that $(c_0)' \cong l^\infty$. There exists many continuous linear functionals on c_0 which are not Riesz functionals, that is, functionals of the form $x \to \langle x, y \rangle$, for some $y \in c_0$. As an example of a non-Riesz functional we have any functional defined by $f(x) = \sum_{n \in \mathbb{N}} x_n a_n$, where $(a_n) \in l^\infty \setminus c_0$.*

For $f \in c_0$, we will denote by $N(f)$ the kernel or null space of f, that is, $N(f) = \{x \in c_0 : f(x) = 0\}$.

Theorem 5. *If $f \in c_0'$ is a Riesz functional, then any orthonormal base of $N(f)$ has the Riemann-Lebesgue Property.*

Proof. If $f \equiv 0$, there is nothing to prove. Suppose that f is not null and let $(x_n)_{n\in\mathbb{N}}$ be any orthonormal base of $N(f)$. It is enough to show that for $e_j \notin N(f)$, $\lim_{n\to\infty} \langle e_j, x_n\rangle = 0$. In fact, let $(f(e_n))_{n\in\mathbb{N}}=(a_n)_{n\in\mathbb{N}}$ be the sequence generated by the canonical base $(e_n)_{n\in\mathbb{N}}$ of c_0; it is clear that $(a_n)_{n\in\mathbb{N}} \in l^\infty$. Since f is a Riesz functional, there exists y in c_0 such that $f = \langle y, \cdot\rangle$. If we consider $y - \langle y,y\rangle a_j^{-1} e_j$, then

$$\left\langle y, y - \langle y,y\rangle a_j^{-1} e_j \right\rangle = \langle y,y\rangle - \langle y,y\rangle a_j^{-1} \langle y, e_j\rangle = \langle y,y\rangle - \langle y,y\rangle a_j^{-1} a_j = 0,$$

that is, $y - \langle y,y\rangle a_j^{-1} e_j \in N(f)$ .Thus,

$$0 = \langle y, x_n\rangle = \left\langle y - \langle y,y\rangle a_j^{-1} e_j, x_n \right\rangle + \langle y,y\rangle a_j^{-1} \langle e_j, x_n\rangle$$

implies that

$$\lim_{n\to\infty} \langle e_j, x_n\rangle = -\lim_{n\to\infty} \frac{a_j}{\langle y,y\rangle} \left\langle y - \langle y,y\rangle a_j^{-1} e_j, x_n \right\rangle = 0,$$

since $(x_n)_{n\in\mathbb{N}}$ is an orthonormal base of $N(f)$ and $y - \langle y,y\rangle a_j^{-1} e_j \in N(f)$. ■

Corollary 1. *If $f \in c_0'$ is a Riesz functional, then $N(f)$ has a normal complement and it is $N(f)^p = \{y \in c_0 : \langle x, y\rangle = 0; x \in N(f)\}$.*

Remark 2. *The converse of Theorem 5 is also true and it was proved in* [4].

Now we define the linear functional $e_i' : c_0 \to \mathbb{K}$; $x \to e_i'(x) = x_i$, where $x = \sum_{j\geq 1} x_j e_j$. It is clear that $e_i' \in c_0' = l^\infty$ and that

$$\|e_i'\| = \sup_{j\geq 1} \frac{|e_i'(e_j)|}{\|e_j\|} = \sup_{j\geq 1} |\delta_{ij}| = 1.$$

Moreover, e_i' is a Riesz functional, since $\langle e_i, x\rangle = x_i = e_i'(x)$.

Proposition 2. $cl\,[\{e_i' : i \geq 1\}] \cong c_0$.

Proof. We define the linear mapping $\Phi : [\{e_i' : i \geq 1\}] \to c_0$ by $\Phi(e_i') = e_i$. It is easy to see that Φ is an isometry and its extension $\widehat{\Phi}$ to the closure maintains the same property. Finally, we need to prove that $\widehat{\Phi}$ is onto; in fact, if (a_n) is an element of c_0, then

$$x' = \sum_{n\geq 1} a_n e_n' = \lim_{n\to\infty} \sum_{i=1}^{n} a_i e_i' \in cl\,[\{e_i' : i \geq 1\}]$$

since $\lim_{n\to\infty} |a_n| \, \|e_n'\| = \lim_{n\to\infty} |a_n| = 0$. Now, by the continuity of $\widehat{\Phi}$, we conclude that $\widehat{\Phi}(x') = (a_n)$. ■

Proposition 3. $f \in cl\,[\{e_i' : i \geq 1\}]$ *if and only if f is a Riesz functional.*

Proof. ($\Rightarrow$)For $f = \sum_{k=1}^{n} \alpha_k e_i' \in [\{e_i' : i \geq 1\}]$, its associated element y of c_0 is $\sum_{k=1}^{n} \alpha_k e_k$, that is, $f = \langle \sum_{k=1}^{n} \alpha_k e_k, \cdot \rangle$. Suppose that $f \in cl\,[\{e_i' : i \geq 0\}]$; hence

$$f = \sum_{k=1}^{\infty} \alpha_k e_k' = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k e_k' = \lim_{n\to\infty} \left\langle \sum_{k=1}^{n} \alpha_k e_k, \cdot \right\rangle$$

with $\lim_{k\to\infty} |\alpha_k| \, \|e_k'\| = 0$. Now, since

$$\lim_{k\to\infty} |\alpha_k| = \lim_{k\to\infty} |\alpha_k| \, \|e_k'\| = 0,$$

we have $y = \sum_{k=1}^{\infty} \alpha_k e_k \in c_0$, and $f = \langle y, \cdot \rangle$.
($\Leftarrow$) If $f = \langle y, \cdot \rangle$, for a fixed $y = \sum_{k=1}^{\infty} \alpha_k e_k \in c_0$, then

$$\begin{aligned} f(x) = \langle y, x \rangle &= \left\langle \sum_{k=1}^{\infty} \alpha_k e_k, x \right\rangle = \sum_{k=1}^{\infty} \alpha_k \langle e_k, x \rangle \\ &= \sum_{k=1}^{\infty} \alpha_k x_k = \sum_{k=1}^{\infty} \alpha_k e_k'(x) \end{aligned}$$

Therefore, $f \in cl\,[\{e_i' : i \geq 1\}]$. ∎

4 Normal Projections

In Section 3, we identified the null spaces $N(f)$, $f \in c_0'$, which admit complements. In Section 4 we extend this identification for arbitrary closed subspaces of c_0.

For a linear operator $T : c_0 \to c_0$, we denote by $N(T)$ the null subspace $\{x \in c_0 : T(x) = 0\}$ and by $R(T)$ the range subspace of T

$$\{y \in c_0 : y = T(x);\ \text{for some } x \in c_0\}.$$

We also denote by $\mathcal{L}(c_0)$ the space of all continuous linear operators T defined on c_0.

Definition 6. *A linear operator P on c_0 is said to be a Normal Projection if*

1. $P^2 = P$ $(\Rightarrow c_0 = N(P) \oplus R(P))$
2. *P is continuous and*
3. $\langle x, y \rangle = 0$, *for $x \in N(P)$ and $y \in R(P)$*
   (notation: $\langle N(P), R(P) \rangle = 0$).

Theorem 6. *Let P be a normal projection. If $\{x_n : n \in \mathbb{N}\}$ is an orthonormal base of $N(P)$, then it has the Riemann-Lebesgue Property.*

Proof. Take an arbitrary $x \in c_0$. We have to prove that $\lim_{n\to\infty} \langle x, x_n \rangle = 0$. If $x \in N(P)$, then $\lim_{n\to\infty} \langle x, x_n \rangle = 0$, since $\{x_n : n \in \mathbb{N}\}$ is a base on $N(P)$.
Suppose that $x \notin N(P)$; since P is a normal projection, we have $x - Px \in N(P)$ and $\langle Px, x_n \rangle = 0$. From this fact, we have

$$\begin{aligned} \langle x, x_n \rangle &= \langle x - Px, x_n \rangle + \langle Px, x_n \rangle \\ &= \langle x - Px, x_n \rangle, \end{aligned}$$

and then

$$\lim_{n\to\infty} \langle x, x_n \rangle = \lim_{n\to\infty} \langle x - Px, x_n \rangle = 0.$$ ∎

Corollary 2. *If P is a normal projection on c_0, then $I - P$ is also a normal projection. Moreover, $N(P)^p = R(P) = N(I-P)$ and $R(P)^p = N(P) = R(I-P)$.*

Proof. Let $x \in N(P)^p$. Then there exists $(u,v) \in N(P) \times R(P)$ such that $x = u+v$. Since $\langle x, u \rangle = 0$ and $\langle v, u \rangle = 0$, we have that $\langle u, u \rangle = 0$, which implies that $u = \theta$. Thus, $x = v \in R(P)$ and $N(P)^p \subseteq R(P)$. The other inclusion follows from the fact that $\langle N(P), R(P) \rangle = 0$. The rest of the proof is routine matter and we omit it. ■

Proposition 4. *Let M be a closed subspace of c_0. If M has an orthonormal base with the Riemann-Lebesgue Property, then there exists a normal projection P such that $M = N(P)$.*

Proof. By Cor. 8.2 of [4], we have $c_0 = M \oplus M^p$. If $x \in c_0$, then there exists a unique pair $(u,v) \in M \times M^p$ such that $x = u + v$. If we define $P(x) = v$, then P is the normal projection that satisfies the statement. ■

Corollary 3. *Let M be an infinite dimensional closed subspace of c_0. Then, the following statements are equivalent:*

1. *M has a normal complement.*
2. *M has an orthonormal base with the Riemann-Lebesgue Property.*
3. *There exists a normal projection P such that $N(P) = M$.*

Corollary 4. *Let M be a closed subspace c_0. Then,*

1. *If M is of finite dimension or it has an orthonormal base with the Riemann-Lebesgue Property, then M^p has an orthonormal base with the Riemann-Lebesgue Property or is of finite dimension.*
2. *If M of c_0 has an orthonormal base with the Riemann-Lebesgue Property, then any other orthonormal base has the same property.*

Proof. These statements are direct consequence of Theorem 6, Corollaries 2 and 3 and Proposition 4. ■

Definition 7. *A closed subspace M of c_0 is said to have the Riemann-Lebesgue property if M is of finite dimension or it has an orthonormal base with the Riemann-Lebesgue Property.*

Let $\{e_i : i \in \mathbb{N}\}$ be the canonical base of c_0. From [2] we remind the reader the following facts about linear operators: $e'_j \otimes e_i : c_0 \to c_0$ is defined by $e'_j \otimes e_i(x) = x_j e_i$, for $x = \sum_{i \in \mathbb{N}} x_i e_i$. This operator is continuous, furthermore, $\left\| e'_j \otimes e_i \right\| = 1$ and if $u \in \mathcal{L}(c_0)$, then u can be written in the unique pointwise convergent sum

$$u = \sum_{i,j \in \mathbb{N}} \alpha_{ij} e'_j \otimes e_i = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots \\ \vdots & & & \ddots & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & \cdots & \alpha_{ij} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \ddots \\ 0 & 0 & 0 & & 0 & \end{pmatrix}, \tag{4.1}$$

where $\alpha_{ij} \in \mathbb{K}$, and $\lim_{i\to\infty} |\alpha_{ij}| = 0$ for any fixed $j \in \mathbb{N}$, that is, each column converges to 0 (see 4.1).

Proposition 5. *Let $u \in \mathcal{L}(c_0)$, with $u = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e'_j \otimes e_i$. Then the following statements are equivalent:*

1. $u^2 = u$
2. $\sum_{k\in\mathbb{N}} \alpha_{ik}\alpha_{kj} = \alpha_{ij}$, *for any* $i, j \in \mathbb{N}$.

Proof. It is routine matter and we omit it. ■

Remark 3. *If u is a normal projection, then $\langle x - u(x), u(x)\rangle = 0$, for any $x \in c_0$; furthermore, for any canonical element e_k,*

$$\begin{aligned} \langle e_k - u(e_k), u(e_k)\rangle &= 0 \\ \left\langle \left(0, \cdots, \underset{kth}{1}, \cdots\right) - (\alpha_{1k}, \cdots), (\alpha_{1k}, \cdots)\right\rangle &= 0 \\ -\alpha_{1k}^2 - \cdots \alpha_{(k-1)k}^2 + \alpha_{kk}(1-\alpha_{kk}) - \alpha_{(k+1)k}^2 - \cdots &= 0 \end{aligned}$$

which implies $\sum_{i\in\mathbb{N}} \alpha_{ik}^2 = \alpha_{kk}$.

Proposition 6. *Let $u = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e'_j \otimes e_i \in \mathcal{L}(c_0)$. Then the following statements are equivalent*

1. *u is a normal projection*
2. $\sum_{i\in\mathbb{N}} \alpha_{ik}^2 = \alpha_{kk}$ *for any* $k \in \mathbb{N}$, *and* $\sum_{j\in\mathbb{N}} \alpha_{ij}\alpha_{jk} = \alpha_{ik}$ *for any* $i, k \in \mathbb{N}$.
3. $\sum_{i\in\mathbb{N}} \alpha_{ik}^2 = \alpha_{kk}$ *for any* $k \in \mathbb{N}$, *and if* $x = (x_i) \in N(u)$, *then* $\sum_{i\in\mathbb{N}} \alpha_{ij} x_i = 0$ *for any* $j \in \mathbb{N}$.

5 Adjoint and self-adjoint operator

We adopt the following definition of adjoint operator given by B. Diarra [2] : Let $u, v \in \mathcal{L}(c_0)$. We say that v is an adjoint of u (with respect to the inner product $\langle\cdot,\cdot\rangle$) if, for any pair $x, y \in c_0$, $\langle u(x), y\rangle = \langle x, v(y)\rangle$. Furthermore, this definition is equivalent to say that u admits an adjoint operator if and only if for any $i \in \mathbb{N}$,

$$\lim_{j\to\infty} |\alpha_{ij}| = 0.$$

In terms of matrixes, this means

$$u = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1j} & \cdots & \to 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2j} & \cdots & \to 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3j} & \cdots & \to 0 \\ \vdots & & & \ddots & & & \\ \alpha_{i1} & \alpha_{i2} & \alpha_{i3} & \cdots & \alpha_{ij} & \cdots & \to 0 \\ \vdots & & & & & \ddots & \end{pmatrix}. \tag{5.1}$$

We also say that a continuous linear operator u is self-adjoint if $\alpha_{ji} = \alpha_{ij}$ for every $i, j \in \mathbb{N}$; in other words, the matrix is symmetric.

Theorem 7. *The normal projections are self-adjoint.*

Proof. Let P be a normal projection, and let $x, y \in c_0$. We claim that

$$\langle x, Py \rangle = \langle Px, y \rangle .$$

In fact, there exist $u_1, u_2 \in N(P)$ and $v_1, v_2 \in R(P)$ such that $x = u_1 + v_1$, $y = u_2 + v_2$ and $\langle u_i, v_j \rangle = 0$, $i, j = 1, 2$, since $c_0 = N(P) \oplus R(P)$ and $\langle N(P), R(P) \rangle = 0$. From this,

$$\begin{aligned}\langle x, Py \rangle &= \langle u_1 + v_1, Py \rangle = \langle v_1, v_2 \rangle \\ &= \langle v_1, v_2 \rangle + \langle v_1, u_2 \rangle = \langle Px, u_2 + v_2 \rangle = \langle Px, y \rangle .\end{aligned}$$

■

Remark 4. *Let $f = \langle \cdot, y \rangle$ be a Riesz functional, where $y = (y_n) \in c_0$, $y \neq 0$. Note that $f(e_i) = y_i$, $\langle y, y \rangle = \sum_{i \in \mathbb{N}} y_i^2 = \sum_{i \in \mathbb{N}} \langle e_i, y \rangle^2 \in \mathbb{K}$, the linear operator u given by*

$$u = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is continuous, $N(u) = N(f)$ and the adjoint of u is

$$u^* = \begin{pmatrix} y_1 & 0 & 0 & 0 & \cdots \\ y_2 & 0 & 0 & 0 & \cdots \\ y_3 & 0 & 0 & 0 & \cdots \\ y_4 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

Proposition 7. *If f and u are as in Remark 4, then $P = \frac{1}{\langle y, y \rangle} u^* \circ u$ is a normal projection and $N(P) = N(u) = N(f)$.*

Proof. It is easy to see that $u^* \circ u$ is a self-adjoint operator. Now, the associated matrix of P is given as follows:

$$P = \frac{1}{\langle y, y \rangle} \begin{pmatrix} y_1^2 & y_1 y_2 & y_1 y_3 & y_1 y_4 & \cdots \\ y_1 y_2 & y_2^2 & y_2 y_3 & y_2 y_4 & \cdots \\ y_1 y_3 & y_2 y_3 & y_3^2 & y_3 y_4 & \cdots \\ y_1 y_4 & y_2 y_4 & y_3 y_4 & y_4^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

To prove that P is a normal projection, it is enough to show that $\sum_{k \in \mathbb{N}} \alpha_{ik} \alpha_{kj} = \alpha_{ij}$, for any $i, j \in \mathbb{N}$, and $\sum_{i \in \mathbb{N}} \alpha_{ik}^2 = \alpha_{kk}$, for a fixed $k \in \mathbb{N}$, where $\alpha_{ij} = \frac{1}{\langle y, y \rangle} y_i y_j$. In fact,

$$\sum_{k \geq 1} \alpha_{ik} \alpha_{kj} = \frac{1}{\langle y, y \rangle^2} \sum_{k \geq 1} (y_i y_k)(y_k y_j) = \frac{y_i y_j}{\langle y, y \rangle^2} \sum_{k \geq 1} y_k^2 = \frac{y_i y_j}{\langle y, y \rangle} = \alpha_{ij}$$

$$\sum_{i\geq 1}\alpha_{ik}^2 = \frac{1}{\langle y,y\rangle^2}\sum_{i\geq 1}(y_i y_k)^2 = \frac{y_k^2}{\langle y,y\rangle^2}\sum_{i\geq 1}y_i^2 = \frac{y_k^2}{\langle y,y\rangle} = \alpha_{kk}.$$

The last part is routine and we omit it. ■

Our sincere thanks to our friend and colleague Lawrence Narici for the suggesting the problem and the invitation to work in this fascinating subject. We wish also thank the referee for his/her valuable comments and careful reading of the paper.

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