Non-binary and linear precoded faster-than-Nyquist signaling

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Non Binary and Precoded Faster Than Nyquist Signaling

Fredrik Rusek and John B. Anderson

***Abstract*—Faster than Nyquist (FTN) signaling is an important method of narrowband coding. The concept is extended here to non binary signal constellations; these are much more bandwidth efficient than binary ones. A powerful method of finding the minimum distance for binary and non binary FTN is presented. Precoding FTN transmissions with short linear filters proves to be an effective way to gain distance. A Shannon limit to bit error rate is derived that applies for FTN. Tests of an M-algorithm receiver are performed and compared to this limit.**

***Index Terms*—Coded modulation, Mazo limit, faster than Nyquist, bandwidth efficient coding.**

I. Introduction

THE concept of Faster Than Nyquist (FTN) signaling is well established. If a pulse amplitude modulation (PAM) signal $\sum a[n]v(t-nT)$ is based on an orthogonal pulse $v(t)$, the pulses can be packed closer than the Nyquist rate 1/T without suffering any distance loss. In a bandpass system two quadrature signals can be used.

The result is a much more bandwidth-efficient coding system. Mazo showed [1] that for binary sinc pulses the symbol time can be reduced to $0.802T$ without suffering any loss in minimum Euclidean distance. We refer to this value as the Mazo limit. An introduction to the philosophy of Mazo signaling has been given in [2]. At first, FTN signaling seems to contradict the Nyquist limit and so it is useful to review how it works. Nyquist pulse signal design is based on *orthogonality*. There exist about $2W\mathcal{T}$ orthogonal signals in W positive Hertz and $\mathcal{T}$ seconds. By means of filters matched to each one, data values that modulate each can be maximum likelihood detected independently, and therefore about $2W\mathcal{T}$ symbols can be transmitted. If v is $\sqrt{1/T}\mathrm{sinc}(t/T)$ and there are N pulses each in the I and Q baseband channels, the product $2W\mathcal{T}$ tends in ratio to $2(1/T)(NT)=2N$. The sinc pulses thus carry as many symbols as any orthogonal pulse train can carry.

If the aim is to achieve asymptotically the same *error rate*, without necessarily using orthogonal pulses, then the sinc pulses can arrive faster than $1/T$. A more complex maximum likelihood sequence estimation (MLSE) receiver is required because of intersymbol interference. A similar phenomenon occurs for other T-orthogonal pulses, and the limits for root raised cosine (RC) pulses with excess bandwidth were derived in [3]. Efficient receivers for FTN signaling were presented there for the first time. Methods of computing the minimum distance of FTN signaling can be found in [4] and [5]. Mazo-type limits can be derived for pulse shapes that are not orthogonal for any T [6]. Mazo limit phenomena turn up in other places as well, for example, in constant-envelope coded modulation; see [7] and references therein. Precoding strategies for FTN were studied in [8] and [9].

The non binary case has not been studied as much, and its minimum distances are still an open problem. In this paper we develop an efficient method of finding minimum distances for non binary FTN. Distances for short (4–8 tap) optimal precoding filters with quaternary as well as binary FTN are also studied.

The paper is organized as follows. In section II we give the system model and in section III we derive the algorithm used to search for the minimum Euclidean distance. In section IV a method to find optimal precoding filters is presented. Numerical results and capacity calculations are given in section V and VI. Decoding is discussed in section VII.

II. System Model

Consider a baseband PAM system based on a T-orthogonal pulse $\psi(t)$. We are mostly interested in $\psi(t)$ being a root RC pulse with excess bandwidth α. When $\alpha=0$, $\psi(t)$ is a sinc pulse. The one sided bandwidth of $\psi(t)$ is $W=(1+\alpha)/(2T)$. The transmitted signal is

$$s_{\mathbf{a}}(t)=\sum_{n=-\infty}^{\infty}a[n]\psi(t-n\tau T),\quad \tau\leq 1 \tag{1}$$

where $a[n]$ are independent identically distributed data symbols and $1/\tau T$ is the signaling rate. We assume $\psi(t)$ to be unit energy, i.e. $\int_{-\infty}^{\infty}|\psi(t)|^2dt=1$. For T-orthogonal pulses the system will not suffer from intersymbol interference (ISI) when $\tau=1$. For $\tau<1$ we say that we have FTN signaling, and ISI is unavoidable. The normalized bandwidth consumption is

$$\mathrm{nbw}=\tau T\frac{1+\alpha}{2T}\frac{1}{\log_2 M_a}\qquad \mathrm{Hz/bit/s}, \tag{2}$$

where M_a is the data alphabet size.

The optimum receiver should filter the received signal $s_{\mathbf{a}}(t)+n(t)$, where $n(t)$ is additive white Gaussian noise (AWGN) with spectral density $N_0/2$, with a filter matched to $\psi(t)$ [3]. This should be followed by sampling every τT

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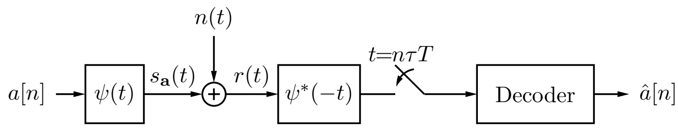

Fig. 1. Overall system of faster-than Nyquist signaling. Consecutive data symbols $a[n]$ are spaced every τT seconds.

second and a decoding algorithm to mitigate the effects of the ISI. The system model is illustrated in figure 1. For MLSE reception, it can be shown that there exist constants K_1 and K_2 such that the probability of a symbol error P_s can be bounded by [16]

$$K_1 Q\Big(\sqrt{d^2_{\min}\frac{E_b}{N_0}}\Big) \leq P_s \leq K_2 Q\Big(\sqrt{d^2_{\min}\frac{E_b}{N_0}}\Big). \qquad (3)$$

These inequalities are tight for large E_b/N_0, and $d^2_{\min}$ thus drives the asymptotic error probability and is a measure of a systems noise immunity. The square Euclidean distance, henceforth simply called "distance", between the (real) data sequences $\mathbf{a}$ and $\mathbf{a}'$ is

$$\begin{aligned} d^2(\mathbf{a},\mathbf{a}') &= \frac{1}{2E_b}\int_{-\infty}^{\infty} |s_{\mathbf{a}}(t) - s_{\mathbf{a}'}(t)|^2 dt \\ &= \frac{1}{2E_b}\int_{-\infty}^{\infty} |\sum_{n=-\infty}^{\infty}(a[n]-a'[n])\psi(t-n\tau T)|^2 dt \\ &= \int_{-\infty}^{\infty} |\sum_{n=-\infty}^{\infty} e[n]\psi(t-n\tau T)|^2 dt \\ &= \sum_{m,n=-\infty}^{\infty} e[m]\rho_\psi[n-m]e[n] = d^2(\mathbf{e}), \qquad (4) \end{aligned}$$

where

$$\rho_\psi[n] = \int_{-\infty}^{\infty} \psi(t)\psi(t+n\tau T)dt \qquad (5)$$

is the autocorrelation of the continuous pulse $\psi(t)$ at samples spaced τT seconds, and $e[n] = (a[n]-a'[n])/\sqrt{2E_b}$. An important fact is that (4) takes the linear form

$$d^2(\mathbf{e}) = \sum_{n=-\infty}^{\infty} r_{\mathbf{e}}[k]\rho_\psi[n] \qquad (6)$$

where

$$r_{\mathbf{e}}[n] = \sum_{k=-\infty}^{\infty} e[n]e[n+k]. \qquad (7)$$

By concatenating an outer code to the FTN signals the $d^2_{\min}$ here can be significantly increased, which will reduce BER. There may else be a bandwidth expansion, so it will be important to compare systems with similar bandwidth in what follows.

Mazo's claim that it is possible to transmit at $0.802/T$ for $\alpha = 0$ without distance loss was proven rigorously in [4]. The results for $\alpha > 0$, given in [3], were obtained by an exhaustive search out to 14 error symbols. For the nonbinary case little is known. Finding minimum distances by searching is very hard for large alphabet sizes since there are $|\mathcal{E}|^L$ error events of length L for the error symbol alphabet $\mathcal{E}$. For 8 PAM, $|\mathcal{E}| = 15$, and searching out to length 14 as in [3] gives 3×10^{16} error sequences which is beyond our computation capability. Therefore symmetry properties of the error events are important. In [8] the following hypothesis was stated: If $|\rho_\psi[1]| >> |\rho_\psi[n]|,\ n > 1$ then the error event causing the minimum distance should be one where the symbols alternate in sign. Another hypothesis is that for low enough bandwidth the worst error event is a zero sum event, i.e. $\sum_n e[n] = 0$. Low enough bandwidth means typically nbw$<$ .2 Hz/bit/s; see [7] and references therein. Both these hypotheses heavily reduce the computation. However, we can never be sure that they are valid and in our numerical results we give an example where a search based on the first hypothesis does not produce the minimum distance of the system. We will show that the second assumption gives too small reduction to be really useful.

We therefore take a new look at the problem of efficiently finding the minimum distance for non binary alphabets. The notation used is as follows:

$\mathbf{e}$	discrete vector of symbols, with nth element $e[n]$
$\rho_\psi[n]$	τT-sampled autocorrelation of a continuous pulse $\psi(t)$
$g_{\mathbf{b}}[n]$	autocorrelation of a discrete sequence $b[n]$
$\mathcal{T}_N\mathbf{e}$	Truncation to the first $N+1$ symbols of $\mathbf{e}$
$d^2_{\rho_\psi}(\mathbf{e})$	distance of $\mathbf{e}$ calculated by (4) using $\rho_\psi[n]$
$\mathbf{u}\star\mathbf{v}$	convolution of $\mathbf{u}$ and $\mathbf{v}$
$u^*[n]$	complex conjugate of $u[n]$
supp($\mathbf{e}$)	support of $\mathbf{e}$

III. Finding the Minimum Distance

We start by describing a different but closely related problem; we will then transform our original ISI problem into the new one. Consider the finite causal ISI tap sequence $b[n]$. The transmitted signal for data symbols $\mathbf{a}$ and generator sequence $\mathbf{b}$ is

$$x_{\mathbf{a}|\mathbf{b}}[k] = \sum_{n=-\infty}^{\infty} a[n]b[k-n]. \qquad (8)$$

The distance between two data signals is

$$\begin{aligned} d^2(\mathbf{a},\mathbf{a}') &\triangleq \frac{1}{2E_b}\sum_{n=-\infty}^{\infty} |x_{\mathbf{a}|\mathbf{b}}[n] - x_{\mathbf{a}'|\mathbf{b}}[n]|^2 \\ &= \sum_{m,n=-\infty}^{\infty} e[m]g_{\mathbf{b}}[n-m]e[n] = d^2(\mathbf{e}), \qquad (9) \end{aligned}$$

with

$$g_{\mathbf{b}}[k] = \sum_{n=-\infty}^{\infty} b[n]b[n+k] \qquad (10)$$

It can be shown [10] that the $\mathcal{Z}$ transform of $g_{\mathbf{b}}[k]$ can be written as

$$G_{\mathbf{b}}(z) = cc^* \prod_{i=1}^{N_z}(1-\zeta_i z^{-1})(1-\zeta_i^* z), \qquad (11)$$

where ζ_i and ζ_i^* are the zeros of $G_b(z)$ and c is a normalization constant. From (11) we see that it is always possible to construct

$$H(z) = c\prod_{i=1}^{N_z}(1-\zeta_i z^{-1}), \qquad (12)$$

such that

$$H^*(1/z^*) = c^* \prod_{i=1}^{N_z} (1 - \zeta_i^* z), \tag{13}$$

and

$$G_{\mathbf{b}}(z) = H(z)H^*(1/z^*). \tag{14}$$

Let $h[n]$ be a sequence obtained by the inverse z-tranform of $H(z)$, i.e. $h[n] = \mathcal{Z}^{-1}\{H(z)\}$; note that since there is a great degree of freedom when constructing $H(z)$ we have in general $b[n] \neq h[n]$. We say that $h[n]$ and $H(z)$ are obtained from the spectral factorization of $G_{\mathbf{b}}(z)$ [10]. Since $h^*[-n] = \mathcal{Z}^{-1}\{H^*(1/z^*)\}$ it follows that

$$g_{\mathbf{b}}[k] = \sum_{n=-\infty}^{\infty} h[n]h[n+k]. \tag{15}$$

If $H(z)$ is obtained from the factorization with $|\zeta_i| \leq 1,\ \forall i$, the sequence obtained by the inverse z-transform is minimum phase and is denoted $h_{\text{mp}}[n]$. The minimum distance of all $h[n]$ including $b[n]$ and $h_{\text{mp}}[n]$ are equal since they have the same autocorrelation sequence. But more effective bounds will stem from $h_{\text{mp}}[n]$ since it is minimum phase. Henceforth we construct all tap sequences such that they are minimum phase, and $b[n]$ will be taken as the factorization $h_{\text{mp}}[n]$.

An efficient branch and bound algorithm to find the minimum distance of an ISI sequence is given in [14]. The algorithm works as follows. For a given error event $\mathbf{e}$, a lower bound on distance for all error events starting with the same symbols as $\mathbf{e}$ is found. This lower bound is then compared to an upper bound for $d^2_{\min}$; when the lower bound is larger than the upper, the whole tree emanating from $\mathbf{e}$ is removed. The lower bound is lemma 1 below. This bound can be further sharpened but this is omitted here since we will eventually replace the lemma with another.

Lemma 1: Given a generator sequence $h[n]$ and a particular error sequence $e_s[n]$, if $\mathcal{A}_N(\mathbf{e_s})$ is the set of error sequences

$$\mathcal{A}_N(\mathbf{e_s}) = \{\mathbf{e} : \mathcal{T}_N \mathbf{e} = \mathcal{T}_N \mathbf{e_s}\}, \tag{16}$$

then a lower bound for these is

$$l_N^2(\mathbf{e_s}) \triangleq \sum_{n=0}^{N} |x_{\mathbf{e_s}|\mathbf{h}}[n]|^2 \leq \min_{\mathbf{e} \in \mathcal{A}_N(\mathbf{e_s})} \{d^2(\mathbf{e})\}. \tag{17}$$

The lemma implies that if $l_N^2(\mathbf{e_s})$ is larger than some known upper bound d^2_{ub} to $d^2_{\min}$ then all events in the set $\mathcal{A}_N(\mathbf{e_s})$ can be eliminated from the search for $d^2_{\min}$, as previously mentioned. Note that the distance of any error event gives an upper bound to $d^2_{\min}$. Based on the sequence $h[n]$ and lemma 1 it is straightforward to set up a branch and bound algorithm to solve for $d^2_{\min}$. Any $h[n]$ giving the same autocorrelation g may be used, but the minimum phase one will be most effective in curtailing the search.

We now return to our original problem: given an arbitrary time continuous pulse $\psi(t)$, find the minimum distance $d^2_{\min}$. From (6) we see that if the error event support is limited to $L+1$ error symbols then the distance of an event only depends on $\{\rho_\psi[-L], \rho_\psi[-L+1], \ldots, \rho_\psi[L]\}$. If we only consider events of length $L+1$ we actually only find upper bounds to $d^2_{\min}$. But if L is large, say 20 or so, we are confident that the result is valid. This is motivated by the fact that the $d^2_{\min}$ achieving error events turned out to be much shorter than the search length $(L+1)$ used in forthcoming sections. In the sequel we write $d^2_{\min}$ when we mean upper bound to $d^2_{\min}$.

Since $d^2_{\min}$ for $\psi(t)$ only depends on a finite sequence of autocorrelation values there is in principle nothing that differs this problem from finding $d^2_{\min}$ for a discrete tap sequence. We could try to find a sequence $b[n]$ having an autocorrelation sequence equal to

$$g_{\mathbf{b}}[n] = \begin{cases} \rho_\psi[n], & |n| \leq L \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

But this truncated $g_{\mathbf{b}}[n]$ is in general not a valid autocorrelation sequence and consequently no sequence $b[n]$ need exist such that $g_{\mathbf{b}}[n] = b[n] \star b^*[-n]$. Thus lemma 1 cannot be used.

Note that if we truncate the pulse $\psi(t)$ to a certain length, the (finite) autocorrelation stemming from the truncation is valid; then the method to come is unnecessary. However, the obtained result is then only a (good) approximation. If we want to avoid truncations and seek distances for infinite pulse shapes we need the method below. Furthermore, using our approach the problem of escaped distance (see [7], chapter 6) is completely avoided.

The following lemma gives a sufficient and necessary condition for a sequence to be a valid autocorrelation sequence. A formal proof is found in [15], although the lemma has appeared much earlier.

Lemma 2: A sequence $g[n]$ with Hermitian symmetry is a valid autocorrelation sequence if and only if

$$G(e^{i\omega}) = \sum_{k=-\infty}^{\infty} g[k] e^{-i\omega k} \geq 0, \quad \text{for all } \omega \in (-\pi, \pi]. \tag{19}$$

Now let $g_{\mathbf{b}}$ be as in (18) and take

$$\theta = \min_{\omega \in (-\pi, \pi]} G_{\mathbf{b}}(e^{i\omega}). \tag{20}$$

The case $\theta \geq 0$ implies that a sequence $b[n]$ can be found from $g_{\mathbf{b}}$ and consequently the algorithm in [14] can be applied. Take $\theta < 0$ and define a new autocorrelation sequence $g_{\mathbf{b}'}$ such that

$$g_{\mathbf{b}'}[n] = \begin{cases} g_{\mathbf{b}}[n] - \theta, & n = 0 \\ g_{\mathbf{b}}[n], & n \neq 0. \end{cases} \tag{21}$$

From $g_{\mathbf{b}'}[n]$ it is now possible to obtain a sequence $b'[n]$ through spectral factorization since $G_{\mathbf{b}'}(\omega) \geq 0,\ \omega \in (-\pi, \pi]$. However, the distance of an error event $e[n]$ calculated using $g_{\mathbf{b}'}$ is not equal to the distance calculated using $g_{\mathbf{b}}$ (or ρ_ψ). In fact, from (6),

$$d^2_{g_{\mathbf{b}}}(\mathbf{e}) = d^2_{g_{\mathbf{b}'}}(\mathbf{e}) + \theta r_{\mathbf{e}}[0]. \tag{22}$$

Due to (22) lemma 1 does not hold and consequently the algorithm to find $d^2_{\min}$ must be modified. First let ϵ denote the largest energy among the error symbols in $\mathcal{E}$, i.e.

$$\epsilon = \max |e[n]|^2, \quad e[n] \in \mathcal{E}. \tag{23}$$

We now initialize every error event with the "distance" $\epsilon(L+1)\theta$. This corresponds to the worst case of $\theta r_e[0]$ in (22) for a given search length L. We can modify lemma 1 into

Lemma 3: Let the set $\mathcal{A}_N(\mathbf{e_s}) = \{\mathbf{e} : \mathcal{T}_N \mathbf{e} = \mathcal{T}_N \mathbf{e_s},\ \text{supp}(\mathbf{e}) \leq L+1\}$, with $N \leq L$. For $g_{\mathbf{b}}$ as in (18) and a particular error sequence $e_s[n]$, a lower bound for set $\mathcal{A}_N(\mathbf{e_s})$ is

$$\lambda_N^2(\mathbf{e_s}) \triangleq \sum_{n=0}^{N} |x_{\mathbf{e_s}|\mathbf{b}'}[n]|^2 + \epsilon(L+1)\theta + \sum_{n=0}^{N}(e_s^2[n]-\epsilon)\theta \;\le\; \min_{\mathbf{e}\in\mathcal{A}_N(\mathbf{e_s})}\{d_{g_\mathbf{b}}^2(\mathbf{e})\}. \quad (24)$$

Proof: According to (22) we can write $d_{g_\mathbf{b}}^2(\mathbf{e})$ as

$$\begin{aligned} d_{g_\mathbf{b}}^2(\mathbf{e}) &= \sum_{n=0}^{N} |x_{\mathbf{e}|\mathbf{b}'}[n]|^2 + \sum_{n=N+1}^{\infty} |x_{\mathbf{e}|\mathbf{b}'}[n]|^2 + \theta\sum_{n=0}^{L} e^2[n] \\ &= \sum_{n=0}^{N} |x_{\mathbf{e}|\mathbf{b}'}[n]|^2 + \sum_{n=N+1}^{\infty} |x_{\mathbf{e}|\mathbf{b}'}[n]|^2 + \theta(L+1)\epsilon \\ &\quad + \sum_{n=0}^{L}(e^2[n]-\epsilon)\theta \\ &= \lambda_N^2(\mathbf{e}) + \sum_{n=N+1}^{\infty} |x_{\mathbf{e}|\mathbf{b}'}[n]|^2 + \sum_{n=N+1}^{L}(e^2[n]-\epsilon)\theta. \end{aligned} \quad (25)$$

Since $\epsilon \ge e^2[n]$, $\theta \le 0$ implies that the last term of (25) is always nonnegative. $|x_{\mathbf{e}|\mathbf{b}'}[n]|^2$ being nonnegative, we have $\lambda_N^2(\mathbf{e}) \le d_{g_\mathbf{b}}^2(\mathbf{e})$. All sequences in $\mathcal{A}_N(\mathbf{e_s})$ have the same first $N+1$ components as $\mathbf{e_s}$ and we have for all $\mathbf{e}\in\mathcal{A}_N(\mathbf{e_s})$

$$\lambda_N^2(\mathbf{e_s}) = \lambda_N^2(\mathbf{e}) \le d_{g_\mathbf{b}}^2(\mathbf{e}), \quad (26)$$

which especially implies that $\lambda_N^2(\mathbf{e_s}) \le \min_{\mathbf{e}\in\mathcal{A}_N(\mathbf{e_s})}\{d_{g_\mathbf{b}}^2(\mathbf{e})\}$ and the proof is complete. ∎

We now further sharpen lemma 3. Let

$$d_{\min,g_{\mathbf{b}'}}^2[n] = \min_{\mathbf{e}\,:\,\mathrm{supp}(\mathbf{e})\le n}\{d_{g_{\mathbf{b}'}}^2(\mathbf{e})\}, \quad n \text{ integer}. \quad (27)$$

We can then prove the following lemma.

Lemma 4: Define the set $\mathcal{B}_N(\mathbf{e_s}) = \{\mathbf{e} : \mathcal{T}_N(\mathbf{e}) = \mathcal{T}_N\mathbf{e_s},\ \mathcal{T}_N\mathbf{e} \ne \mathbf{e},\ \mathrm{supp}(\mathbf{e}) \le L+1\}$, with $N \le L$.

Let $\Delta_N(\mathbf{e}) = d_{g_{\mathbf{b}'}}^2(\mathcal{T}_N\mathbf{e}) - \lambda_N^2(\mathbf{e}) - (L-N)\epsilon\theta$. Then for $\mathbf{e}\in\mathcal{B}_N(\mathbf{e_s})$ and $\Delta_N(\mathbf{e}) < d_{\min,g_{\mathbf{b}'}}[L+1-N]$ we have

$$d_{g_\mathbf{b}}^2(\mathbf{e}) \ge d_{\min,g_{\mathbf{b}'}}^2[L+1-N] + d_{\min,g_{\mathbf{b}'}}[L+1-N]\Delta_N(\mathbf{e}) + d_{g_\mathbf{b}}^2(\mathcal{T}_N\mathbf{e}) - (L-N)\theta\epsilon \quad (28)$$

And for the case $\Delta_N(\mathbf{e}) \ge d_{\min,g_{\mathbf{b}'}}[L+1-N]$ we have

$$d_{g_\mathbf{b}}^2(\mathbf{e}) \ge \lambda_N^2(\mathbf{e_s}). \quad (29)$$

This lemma is proved in appendix B. The lemma is a modification of a lemma in [15].

It is now straightforward to set up a branch and bound algorithm to find the minimum distance of the system. For lemma 4 to be useful, it should be easy to find $d_{\min,g_{\mathbf{b}'}}^2[n]$ compared to $d_{\min,g_\mathbf{b}}^2[n]$. From our experience this is always the case; since $g_{\mathbf{b}'}$ is a valid autocorrelation sequence the search method in [14] can be applied. The root node (depth 0), should be initialized by $\epsilon(L+1)\theta$ and the branch metric at depth k is

$$|x_{\mathbf{e}|\mathbf{b}'}[k]|^2 + (e^2[k]-\epsilon)\theta. \quad (30)$$

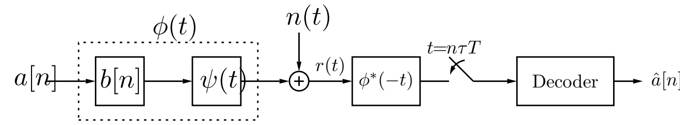

Fig. 2. System model for precoded FTN signaling. The input data symbols are spaced τT seconds apart.

If the expressions for $d_{g_\mathbf{b}}^2(\mathbf{e_s})$ in lemma 4 are larger at some node than an upper bound to $d_{\min}^2$, the entire tree beyond that node can be removed from the search.

IV. Precoded Faster Than Nyquist Signaling

In this section we improve $d_{\min}^2$ by precoding the input data sequence. In the literature there is a rich variety of coding/precoding methods for ISI and partial response signaling (PRS) channels; see [11]– [13] and references therein. Most of these methods work by applying some sort of rate decreasing outer code and possibly a precoder; sometimes the precoder is a linear filter. In [3] constraint coding was used to increase the distance, but the systems there could never achieve the full antipodal distance of an uncoded system.

The task of the precoder is usually to ease the decoding burden at the expense of the bit error rate (BER); examples are the famous Tomlinson-Harashima precoder and the Laroia-Tretter-Farvardin precoder from [12]. We will in a sense do the opposite, use a rate 1 PRS precoder in order to improve the BER, at the cost of decoding complexity. This strategy was used in [8], with an argument tracing back to Forney [16], but the results are apparently incorrect.[1] A method of designing optimal linear filters with respect to $d_{\min}^2$ is given in [14]. The scope of that paper was to design optimal PRS codes based on orthogonal interpolation pulses $\psi(t)$. A technique called partial spectrum mapping was used in [15] for designing bandlimited filters; this technique is essentially what we make use of next. But since the algorithm in section 3 was unknown it was not possible to find $d_{\min}^2$ and it could only be estimated by means of a heavy search. Furthermore, how to implement the pulses obtained via partial spectral mapping was not described; there was no concept of an underlying pulse form.

Assume that the input data sequence is convolved with a finite tap sequence $b[n]$. Alternatively this can be seen as a linear modulation of the data sequence with a different pulse $\phi(t)$; this is illustrated in figure 2. The transmitted signal is

[1]Our outcomes differ from [8] in a number of ways in the sequel, most often because the error events explored in [8] were too short. Some brief examples are as follows. (i) The distance 0.9023 marked by an asterix in table III stems from the error sequence $2,-2,0$ repeated 15 times, an event of length 44; [8] suggests 0.9778, which stems from a shorter event. (ii) Combining the duobinary pulse (37) having $\rho = 0.65$ with the 2-tap prefilter $1.4302, -1.4302$ leads not to signals with $d_{\min}^2 = 2$, but to $d_{\min}^2 = 0.3665$; it is achieved by the error event $2,-2,2,0,0,0,2,-2,2,0,0,0,2,-2,2$. Furthermore, for optimum 2-tap prefilters, $d_{\min}^2$ cannot equal 2 for any $\rho \le 0.867$. (iii) The paper proposes another system in which the transfer function is $|H(f)| = \sqrt{(\pi T/2\rho)\cos(\pi T f/\rho)}$, $|f| < (\rho/2T)$, $\rho \le 1$. It is claimed that $\rho = 0.60$ and the 3-tap prefilter $1.3727, -1.3727, 1.3727$ lead to $d_{\min}^2 = 2$, but we find that $d_{\min}^2$ is 0.7027, achieved by the event $2,-2,2,-2,0,0,2,-2,2,-2,0,0,2,-2,2,-2$. Furthermore, for the best 3-tap prefilters, $d_{\min}^2$ cannot equal 2 for $\rho \le 0.775$.

TABLE I
THE OBTAINED MINIMUM DISTANCES FOR 4 AND 8 PAM. FOR 4 PAM THE SEARCH LENGTH L WAS 25 FOR ALL α. FOR 8 PAM L WAS 20 FOR ALL α.

$d^2_{\min}$ for M_a PAM

α	10%		20%		30%	
$\tau \diagdown M_a$	4	8	4	8	4	8
.80	4/5	2/7	4/5	2/7	4/5	2/7
.75	.708	.204	4/5	2/7	4/5	2/7
.70	.593	.131	.677	.203	.791	.282
.65	.358	.0885	.547	.127	.642	.184
60	.151	.0479	.254	.091	.437	.114
.55	.129		.151	.0393	.198	.0708
.50			.131	.0212	.147	.0331
.45			.102		.128	.0160

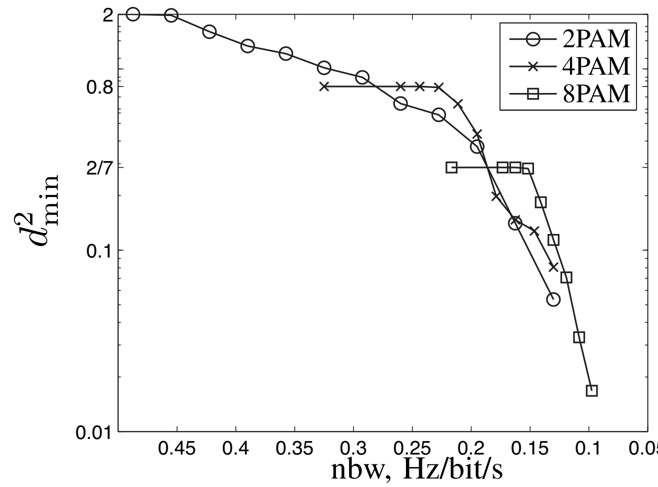

Fig. 3. Comparison of $d^2_{\min}$ for 2-, 4- and 8PAM.

given by

$$s_{\mathbf{a}}(t) = \sum_{n=-\infty}^{\infty} a[n]\phi(t - n\tau T), \tag{31}$$

where $\phi(t)$ is

$$\phi(t) = \sum_{n=0}^{L_b-1} b[n]\psi(t - n\tau T), \tag{32}$$

and L_b is the support of $b[n]$. It can be shown that the autocorrelation $\rho_\phi[n]$ is

$$\rho_\phi[n] = g_{\mathbf{b}}[n] \star \rho_\psi[n]. \tag{33}$$

It can be shown that the distance of an error event is

$$d^2(\mathbf{e}) = \sum_{n=1-L_b}^{L_b-1} g_b[n]\mu_{\mathbf{e}}[n], \tag{34}$$

where

$$\mu_{\mathbf{e}}[n] = \sum_{k=-\infty}^{\infty} r_{\mathbf{e}}[k]\rho_\psi[k-n], \tag{35}$$

with $r_{\mathbf{e}}[k]$ defined in (7). The energy of $\phi(t)$ should equal 1; this is equivalent to $\rho_\phi[0] = 1$, or from (33)

$$\sum_{k=1-L_b}^{L_b-1} g_{\mathbf{b}}[k]\rho_\psi[k] = 1. \tag{36}$$

We now have linear equations both for distance (34) and energy normalization (36). Together with the linear condition given in lemma 2 we can solve for the optimal sequence $b[n]$ for a given pulse $\psi(t)$. For the procedure used we refer to [14]. To find $d^2_{\min}$ we use the branch and bound algorithm in section 3.

V. Numerical Results

We start with $d^2_{\min}$ for the uncoded root RC FTN case in table I. We present results for 4 and 8 PAM and $\alpha = 10, 20$ and 30%. Recall that the matched filter bounds are .8 and $2/7$ for 4 and 8 PAM. The Mazo limits, i.e. the τ where $d^2_{\min}$ falls below the matched filter bound for the first time, are the same for 2,4 and 8 PAM and are $\tau = .7032$ for $\alpha = 30\%$.

A comparison between 2,4 and 8 PAM is shown in figure 3 for $\alpha = 30\%$. It is seen that there is an optimal alphabet size for each bandwidth. For example, at nbw 0.15 Hz/bit/s there is roughly 3 dB gain by using 8 PAM instead of 4 PAM. A similar result for Butterworth pulses is reported in [6].

TABLE II
MINIMUM DISTANCES FOR PRECODED 2 AND 4 PAM WITH DIFFERENT LENGTHS L_b OF THE PRECODING FILTER. SEARCH LENGTH L WAS $L = 25$ FOR 2 PAM, $L = 20$ FOR 4 PAM WITH $L_b = 4$ AND $L = 18$ FOR 4 PAM WITH $L_b = 6$. THE NO-PRECODING CASE IS INCLUDED FOR COMPARISON.

$d^2_{\min}$ for precoded M_a PAM

	uncoded		$L_b = 4$		$L_b = 6$		$L_b = 8$	
$\tau \diagdown M_a$	2	4	2	4	2	4	2	4
.65	1.60	.64	2	4/5	2	4/5	2	
.625	1.47	.59	2	.759	2	4/5	2	
.58	1.27	.35	1.59	.578	1.86	.67	2	
.55	1.21	.20	1.46	.445	1.66	.61		
.50	1.01	.15	1.31	.193				
.46	.93		1.14					

We give some results for the precoded FTN signaling next. We only consider only $\alpha = 30\%$. Results for 2 PAM with $L_b = 4, 6, 8$ and 4 PAM with $L_b = 4, 6$ are given in table II. Especially note the 4.8 dB gain by using a precoding filter of support 6 for 4 PAM and $\tau = .55$.

We also found $d^2_{\min}$ for frequency scaled versions of the duobinary pulse, as proposed for binary transmission in [8]. The transfer function of the duobinary pulse is

$$|\Psi(f)| = \sqrt{2T/\rho}\cos(\pi T f/\rho), \quad |f| < (\rho/2T), \quad \rho \le 1. \tag{37}$$

The normalized bandwidth is $\mathrm{nbw} = \rho/2\log_2 M_a$ Hz/bit/s. Results are given in table III. Some of these differ from [8].

To see the strength of the $d^2_{\min}$ algorithm we compare the effort of finding $d^2_{\min}$ for 8 PAM, $\alpha = 10\%$, $\tau = .60$ with an exhaustive search. We searched over all events with support ≤ 20. For an exhaustive search this implies testing $15^{20} \approx 3.33 \times 10^{23}$ events but our algorithm only considered $\approx 6 \times 10^7$. Using generating functions one can show that the number of error events fullfilling the zero sum assumption for M_a

TABLE III
MINIMUM DISTANCES FOR THE DUOBINARY PULSE, 2,4 AND 8 PAM. SEARCH LENGTH L WAS 20 FOR ALL THREE ALPHABETS. THE VALUE MARKED BY AN ASTERIX IS DIFFERENT FROM THAT IN [8].

$d^2_{\min}$ for the duobinary pulse

ρ	2 PAM	4 PAM	8 PAM
1	2	4/5	2/7
.95	1.648	.659	.236
.90	1.49	.595	.207
.85	1.37	.531	.139
.80	1.281	.361	.109
.75	1.19	.233	.078
.70	1.11	.172	.031
.65	.9023*	.143	.021
.60	.723	.094	
.55	.628	.076	
.50	.568	.055	
.45	.420		
.40	.336		

PAM and support L is

$$\sum_{0\le i\le \frac{(M_a-1)L}{(2M_a-1)}} \binom{L}{i}(-1)^i \binom{M_aL+i(1-2M_a)-1}{(M_a-1)L+i(1-2M_a)}.$$

Evaluating this for $M_a = 8$ and $L = 20$ gives $\approx 6.81 \times 10^{21}$ error events; thus the zero sum reduction is less than 100-fold and is not very helpful.

In general the worst case error events are rather short, 4–12 error symbols, but we found events up to 18 error symbols long. In the lower region ($\tau < .6$) of table I, all events for 8 PAM have support > 10 symbols, and the largest support is 14. We found many examples of worst case error events that contradict the hypothesis in [8].[2] The zero sum assumption was never violated in any of the tables.

VI. CAPACITY

In this section we derive the capacity of schemes like FTN and the Shannon bound to bit error rate for them. In section VII the BER bound for FTN will be compared to both actual decoding performance and to BER bounds for trellis-coded modulation (TCM) schemes based on RC pulses. The capacity calculation has appeared in the literature (see e.g. [21]) but the BER bound has apparently not. It is an important tool in the evaluation of FTN-like coding schemes, since it includes the effect of both energy and spectral density and it directly relates to an easily measured quantity.

According to classical Shannon theory, signals with W positive hertz, uniform power spectral density (PSD) and P Watts have capacity $\mathcal{C}_W = W\log_2[1+P/N_0W]$, where $N_0/2$ is the noise density. Elementary calculus extends this brickwall result to signals with an arbitrary PSD $|H(f)|^2$; the outcome is

$$\mathcal{C}_H = \int_0^\infty \log_2[1+\frac{2|H(f)|^2}{N_o}]\,df \qquad \text{(bit/s)} \tag{38}$$

[2] An example is uncoded binary transmission with $\alpha = 30\%$, $\tau = .45$. Then $\rho_\psi[1] = .6868$ and $\max_{n>1}|\rho_\psi[n]| = .1796$ so the conditions in the hypotheses are fullfilled, but the worst event is $2, -2, 0, 2, -2, 0, 2, -2, 0$.

in which P is now $\int |H(f)|^2 df$. Some calculations with (38) show that the stopband of $H(f)$ has a major effect on the capacity of narrowband signaling, even though its power is small.

A Shannon limit to BER gives the lowest BER of coding schemes with a given PSD, as a function of E_b/N_0. It is derived as follows. Consider a coded modulation with PSD $|H(f)|^2$ that carries $R_{\rm ber}$ binary data bits/s. If $R_{\rm ber} \ge \mathcal{C}_H$, standard rate-distortion theory tells us that $R_{\rm ber}$ can be compressed to $\mathcal{C}_H$ in the ratio

$$\frac{R_{\rm ber}}{\mathcal{C}_H} = 1 - h_B(\beta), \tag{39}$$

where β is the resulting error rate of the compression and $h_B(\,)$ is the binary entropy function. The channel carries the compressed data nearly perfectly at rate $\mathcal{C}_H$. We now fix the system rate $R_{\rm ber}$ and scale $|H(f)|^2$ by a parameter $\gamma > 0$. This scales $E_b/N_0 = P/N_0R_{\rm ber}$ to $\gamma E_b/N_0$, and eqs. (38)–(39) yield a BER β for the new $\gamma E_b/N_0$. We thus obtain a relationship between β and E_b/N_0 that is parameterized in γ. The highest allowed E_b/N_0 is the one for which $\mathcal{C}_H$ is the given $R_{\rm ber}$.

The Shannon BER limit is the ultimate limit for any coding scheme having PSD $|H(f)|^2$, but some coded modulations may be bounded away from this limit. Consider TCM, multilevel coding and many other types of coded modulation, in which the signals have the form $s(t) = \sum a_n v(t-nT)$ with T-orthogonal $v(t)$. The $\{a_n\}$ can be thought of as real valued code letters. If the $\{a_n\}$ are uncorrelated and zero mean, the PSD of $s(t)$ has the same shape as $|V(f)|^2$. As shown by Nyquist, useful orthogonal pulse PSDs obey a symmetry condition about the frequency $W = 1/2T$. The most common pulse is the root RC, with nominal passband $[0, W]$ Hz and stopband $[W, (1+\alpha)/2T]$. The most narrowband orthogonal pulse is $\mathrm{sinc}(t/T)$, with flat PSD and $\alpha = 0$.

It can be shown that $\mathcal{C}_V$ in (38) always *increases* compared to the brickwall $\mathcal{C}_W$ when a pulse with Nyquist's symmetry and $\alpha > 0$ is substituted for $\mathrm{sinc}(t/T)$. This is the fundamental reason why orthogonal-pulse schemes can be bounded away from their limit: The BER of these does not depend on α but only on the fact that $v(t)$ is orthogonal; in particular, α can be zero, so these schemes must achieve a Shannon BER limit derived from the brickwall capacity $\mathcal{C}_W$. The general BER limit derived from a non-sinc $|V(f)|^2$ via eqs. (38)–(39) must lie strictly to the left in a plot of BER vs. E_b/N_0 like those in the next section. This opens an avenue for FTN schemes to perform better than orthogonal ones.

VII. DECODING

Now we discuss decoding of the coded modulation schemes and give some receiver tests to verify the obtained minimum distances. The optimal strategy is a full MLSE, but this is out of the question, since the complexity of MLSE is M_a^L where L, the length of the ISI, is in theory infinite and in practice very long. Thus a reduced sequence estimation (RSE) is necessary. However, a detector that gives essentially the MLSE performance is needed, otherwise the bandwidth gains are not exploited; this implies that all forms of linear

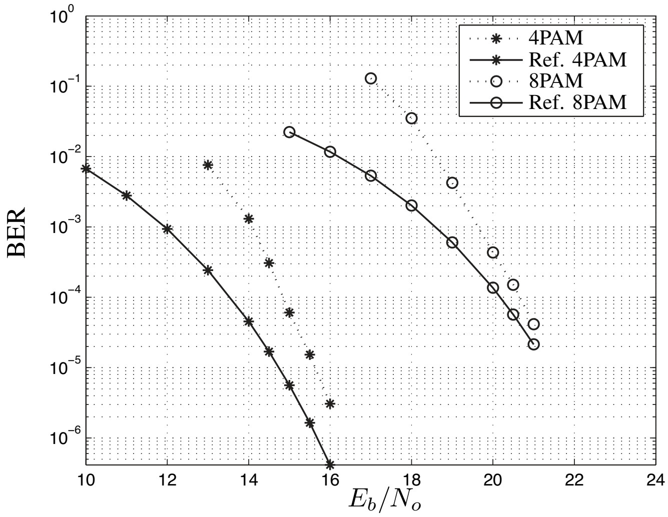

Fig. 4. M-algorithm receiver tests of 4- and 8PAM systems with parameters $\tau = 0.55$, $L_b = 6$, $\alpha = 0.30$. Bandwidths are 0.1788 and 0.1192 Hz/bit/s for 4- and 8PAM. Curves marked by Ref. are the references $Q(\sqrt{d^2_{\min}E_b/N_0})$ for 4PAM and $K_1Q(\sqrt{d^2_{\min}E_b/N_0})$ for 8PAM, K_1 is the multiplicity of the $d^2_{\min}$ achieving error event. M is 32 and 100 for 4- and 8PAM. The actual precoder used is $\{2.2418, -3.6121, 4.2673, -3.5962, 2.3436, -.9159\}$.

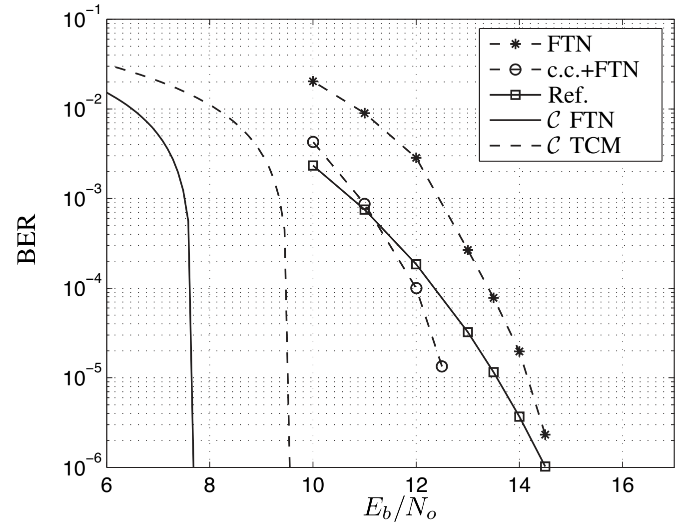

Fig. 5. Comparison of M-algorithm receiver tests for rate 2 uncoded and convolutionally encoded signals having $\tau = 0.7$ and $\alpha = 0.30$. The uncoded system, denoted FTN, is a 4PAM system with $\tau = 0.7$, $M = 8$; curve marked by Ref. is $Q(\sqrt{d^2_{\min}E_b/N_0})$ for this system. Curve marked c.c.+FTN denotes a convolutionally encoded 8PAM system with $\tau = 0.7$, $M = 80$; encoder is the (23,40) convolutional code; $\mathcal{C}$ FTN denotes Shannon BER limit (38). Curve marked $\mathcal{C}$ TCM denotes limit for TCM and related methods having the same normalized bandwidth.

equalization and nonlinear methods such as straightforward decision feedback are ordinarily ruled out.

Over the past 30 years much research has gone into finding good RSE strategies for the AWGN channel; see [17], [18] and references therein. For example a common strategy is reduced state sequence estimation (RSSE); this method works with a considerably smaller trellis than the original and obtains close to optimal performance as E_b/N_0 increases.

An efficient receiver structure well suited to binary FTN was recently proposed in [3]; the structure was based on the Ungerboeck observation model [19]. This structure could be generalized to M_a-ary signaling. But the first part of the receiver is a soft output truncated Viterbi algorithm [20], whose complexity grows as $M_a^{L_v}$ where L_v denotes the truncation length of the ISI. Since we have precoded signals and significant FTN complexity, a large L_v is probably needed in order to avoid too much residual ISI; therefore we believe that this receiver is in general too complex for quaternary and octal signaling with precoders as long as 4–6 taps.

In this paper a different strategy is tested, the simple M-algorithm. If the Ungerboeck model is used the M-algorithm is observed to work badly for large alphabets, such as 8 PAM. Therefore the whitened matched filter (WMF) model [16] is assumed. Similar to [3] we found it hard to work with the WMF model when the impulse response of the root RC pulse is long, e.g. $80T$. Therefore we have done the following: a front end whitened matched receiver filter was determined for root RC pulses of length $20T$; all receiver tests are done with pulses of length $80T$ but the receiver filter is for the $20T$ pulse. This of course makes the decoder mismatched, but the noise variance emanating from the mismatch is small compared to the AWGN variance. We now have to decode backwards, which might be a drawback, since the decoding cannot start until the whole block has arrived. If the receiver filter is set up for forward decoding, it is no longer stable; this is due to the mismatching of the filters.

This receiver is simple and shows good performance. At each level in the trellis the M-algorithm keeps only the M most promising paths. As usual the symbols are released with some delay (the decision depth); see [22] for a study of decision depths for ISI channels.

We have performed receiver tests for both 4- and 8PAM for uncoded, precoded and convolutionally encoded FTN systems; the tests are shown in figures 4–6. By convolutionally encoded FTN signaling we mean a scheme where 1 out of k input bits are first encoded by a rate $1/2$ convolutional code, and these $k+1$ bits are then mapped onto a 2^{k+1} PAM signal set. This is followed by ordinary FTN signaling. The minimum distance of these systems has not been conclusively determined.

Figure 4 compares M-algorithm error rates for 4 and 8PAM systems with optimal precoding to their $d^2_{\min}$-based Q-function estimate. The signal generation parameters are $\tau = 0.55, L_b = 6, \alpha = 0.3$. Note that precoded 8PAM FTN is not included in Table 2 because it is very hard to find $d^2_{\min}$. In the test we therefore used the precoder constructed for 4PAM also for 8PAM. The required M is approximately 32 and 100 for 4- and 8PAM, and these are used in the tests. The normalized bandwidth of e.g. 8PAM is $[(1 + 0.3)/2]0.55/3 = 0.1192$ Hz/bit/s. The Q-function estimates lie 1–2 dB to the left of the test results. This reference is based solely on $d^2_{\min}$ for 4PAM and only applies asymptotically. For 8PAM we included the multiplicity K_1, see eq. (3), of the $d^2_{\min}$ achieving error event [16] [23]. This reference becomes tight.

Figure 5 shows coding systems at rate 2 bits/channel use. An uncoded 4PAM system (decoder $M = 8$) is compared to a (23,40) convolutionally encoded 8PAM system (decoder

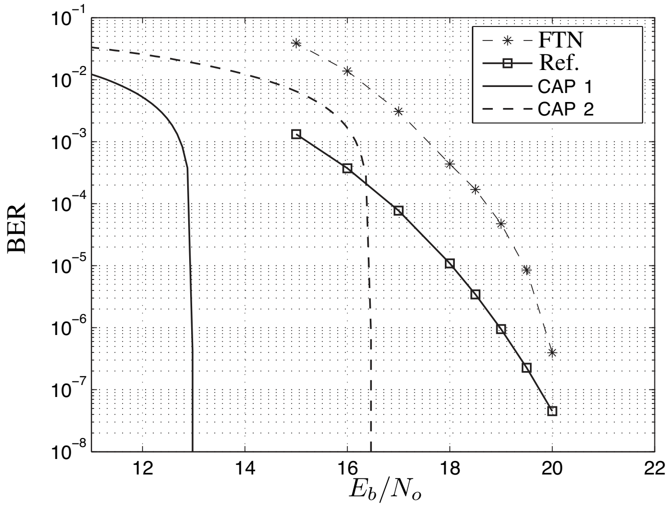

Fig. 6. M-algorithm receiver tests of a rate 3 uncoded 8PAM uncoded FTN system with $\tau = 0.7$, $\alpha = 0.30$ and $M = 16$. Curve marked Ref. is $Q(\sqrt{d^2_{\min} E_b/N_0})$. Curve marked $\mathcal{C}$ FTN denotes FTN Shannon BER bound (38); $\mathcal{C}$ TCM denotes bound for TCM and related methods having the same normalized bandwidth.

$M = 80$). Signal parameters are $\tau = 0.7$ and $\alpha = 0.3$. The convolutionally coded system is about 2 dB better at all E_b/N_0 tested, although it requires a decoder with 10 times the complexity. The figure shows the $d^2_{\min}$ reference for the uncoded 4PAM system, and it is a tight estimate at high E_b/N_0. The figure also shows Shannon BER limits both for the FTN pulse PSD and for 30% root RC-based TCM systems with the same bandwidth (dashed curve).

Figure 6 shows an uncoded rate 3 system with the same signal parameters as Figure 5. Once again, performance is compared to the $d^2_{\min}$ reference and the two Shannon BER limits. Agreement with the reference is again good. In both figures 5 and 6 we see that the uncoded FTN signaling lies roughly 6 dB from its Shannon limit at BER 10^{-5} but only 3–4 dB from the Shannon limit for competing methods such as TCM and multilevel coding. The convolutionally encoded scheme (figure 5) actually gets within 1.5 dB of the competing method Shannon limit at BER 10^{-3}.

VIII. Conclusions

We have proposed a powerful algorithm to search for the minimum distance of nonbinary FTN signaling. We are capable of searching out to a very large search length even for large alphabets. The algorithm is the extension to infinite time signals of an existing algorithm proposed for discrete signals in [15]. Furthermore, the method can be used for a general ISI signal, and no assumptions on the worst case error events are made, except for length. We also found the optimal precoding filter for binary and quaternary FTN transmissions; the distance algorithm was a crucial tool here. M-algorithm receiver tests give a reasonable verification of these distances. A major reason for the improved performance of FTN systems is their more favorable Shannon limit. In a future paper we will present a careful study of the fTN Shannon capacity.

Appendix: Proof of Lemma 4

The proof of lemma 4 is a modification of the proof of lemma 7.4.2 in [15]. The modification is due to the extra term in (22). The proof requires lemma 5 below; except for notation, lemma 5 is identical to lemma 7.4.1 in [15] and therefore we give it without proof. Let $x_{\mathbf{e}|\mathbf{b}'}[k]$ be an error signal generated according to (8). Decompose $x_{\mathbf{e}|\mathbf{b}'}[k]$ into two parts

$$\begin{aligned} \bar{x}_{\mathbf{e}|\mathbf{b}'}[n,N] &\triangleq \sum_{n=0}^{N} e[k]b'[n-k] \\ \dot{x}_{\mathbf{e}|\mathbf{b}'}[n,N] &\triangleq \sum_{n=N+1}^{n} e[k]b'[n-k] \end{aligned} \tag{40}$$

Then lemma 7.4.1 in [15] with our notation reads

Lemma 5: If $\mathbf{e_r}$ is an error sequence such that $\mathbf{e_r} \neq \mathcal{T}_N \mathbf{e_r}$ then

$$\sum_{N+1}^{\infty} |\dot{x}_{\mathbf{e}|\mathbf{b}'}[n,N]|^2 \geq d^2_{\min,g_{\mathbf{b}'}}[L+1-N] \tag{41}$$

We can now prove lemma 4.

Proof of lemma 4. According to (22) we can write $d^2_{g_{\mathbf{b}}}(\mathbf{e})$ as shown in (42). By using $\mathcal{T}_N\mathbf{e}$ instead of $\mathbf{e}$ in (42) we obtain

$$d^2_{g_{\mathbf{b}}}(\mathcal{T}_N\mathbf{e}) = \lambda^2_N(\mathbf{e}) + \sum_{n=N+1}^{\infty} |\bar{x}_{\mathbf{e}|\mathbf{b}'}[n,N]|^2 + (L-N)\theta\epsilon \tag{43}$$

Following [15] we conclude as shown in (44). Instead of finding the minimum in (44) we lower bound it as shown in (45). The last inequality follows from the fact that there is more freedom to choose $z[n]$ than $\dot{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N]$ in the minimization. That $\|\mathbf{z}\| \geq d_{\min,g_{\mathbf{b}'}}[L+1-N]$ is clear from lemma 5. The minimization (45) is easy to solve; there are two types of solution depending whether the difference

$$\begin{aligned} \Delta^2_N(\mathbf{e_r}) &\triangleq d^2_{g_{\mathbf{b}}}(\mathcal{T}_N\mathbf{e_r}) - \lambda^2_N(\mathbf{e_r}) - (L-N)\theta\epsilon \\ &= \sum_{n=N+1}^{\infty} |\bar{x}_{\mathbf{e}|\mathbf{b}'}[n,N]|^2 \end{aligned} \tag{46}$$

is larger or smaller than $d^2_{\min,g_{\mathbf{b}'}}[L+1-N]$. The solution is

$$z[n] = -(\max\{1, d_{\min,g_{\mathbf{b}'}}[L+1-N]/\Delta_N(\mathbf{e_r})\})\,\bar{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N], \quad n > N \tag{47}$$

The value of this solution is shown in (48). Inserting (48) into (44) gives (49), and the lemma is proved. ∎

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$$\begin{aligned} d^2_{g_{\mathbf{b}}}(\mathbf{e}) &= \sum_{n=0}^{N} |x_{\mathbf{e}|\mathbf{b}'}[n]|^2 + \sum_{n=N+1}^{\infty} |\bar{x}_{\mathbf{e}|\mathbf{b}'}[n,N] + \dot{x}_{\mathbf{e}|\mathbf{b}'}[n,N]|^2 + \theta \sum_{n=0}^{L} e^2[n] \\ &= \lambda_N^2(\mathbf{e}) + \sum_{n=N+1}^{\infty} |\bar{x}_{\mathbf{e}|\mathbf{b}'}[n,N] + \dot{x}_{\mathbf{e}|\mathbf{b}'}[n,N]|^2 + \sum_{n=N+1}^{L} (e^2[n] - \epsilon)\theta \end{aligned} \tag{42}$$

$$\min_{\mathbf{e}\in\mathcal{B}_N(\mathbf{e_r})} \{d^2_{g_{\mathbf{b}}}(\mathbf{e})\} = \lambda_N^2(\mathbf{e_r}) + \min_{\mathbf{e}\in\mathcal{B}_N(\mathbf{e_r})} \Big\{ \sum_{n=N+1}^{\infty} |\bar{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N] + \dot{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N]|^2 + \sum_{n=N+1}^{L} (e_r^2[n] - \epsilon)\theta \Big\} \tag{44}$$

$$\begin{aligned} &\min_{\mathbf{e}\in\mathcal{B}_N(\mathbf{e_r})} \Big\{ \sum_{n=N+1}^{\infty} |\bar{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N] + \dot{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N]|^2 + \sum_{n=N+1}^{L} (e_r^2[n] - \epsilon)\theta \Big\} \\ &\geq \min_{\mathbf{e}\in\mathcal{B}_N(\mathbf{e_r})} \Big\{ \sum_{n=N+1}^{\infty} |\bar{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N] + \dot{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N]|^2 \Big\} + \min_{\mathbf{e}\in\mathcal{B}_N(\mathbf{e_r})} \Big\{ \sum_{n=N+1}^{L} (e_r^2[n] - \epsilon)\theta \Big\} \\ &\geq \min_{\|\mathbf{z}\|\geq d_{\min,g_{\mathbf{b}'}}[L+1-N]} \Big\{ \sum_{n=N+1}^{\infty} |\bar{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N] + z[n]|^2 \Big\} \end{aligned} \tag{45}$$

$$\min_{\|\mathbf{z}\|\geq d_{\min,g_{\mathbf{b}'}}[L+1-N]} \Big\{ \sum_{n=N+1}^{\infty} |\bar{x}_{\mathbf{e_r}|\mathbf{b}'}[n,N] + z[n]|^2 \Big\} = (\max\{0, d_{\min,g_{\mathbf{b}'}}[L+1-N] - \Delta_N(\mathbf{e_r})\})^2 \tag{48}$$

$$d^2_{g_{\mathbf{b}}}(\mathbf{e}) \geq \begin{cases} \lambda_N^2(\mathbf{e}), & \Delta_N(\mathbf{e}) \geq d_{\min,g_{\mathbf{b}'}}[L+1-N] \\ d_{\min,g_{\mathbf{b}'}}[L+1-N] + d_{\min,g_{\mathbf{b}'}}[L+1-N]\Delta_N(\mathbf{e}) & \\ \quad + d^2_{g_{\mathbf{b}}}(\mathcal{T}_N\mathbf{e}) - (L-N)\theta\epsilon, & \Delta_N(\mathbf{e}) < d_{\min,g_{\mathbf{b}'}}[L+1-N] \end{cases} \tag{49}$$

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