University of Nebraska at Omaha
DigitalCommons@UNO

Student Work

8-1-1996

Chaotic Behaviour in Three Dimensional Quadratic Systems.

Fu Zhang

Follow this and additional works at: https://digitalcommons.unomaha.edu/studentwork

Recommended Citation

Zhang, Fu, "Chaotic Behaviour in Three Dimensional Quadratic Systems." (1996). *Student Work*. 3521.
https://digitalcommons.unomaha.edu/studentwork/3521

This Thesis is brought to you for free and open access by DigitalCommons@UNO. It has been accepted for inclusion in Student Work by an authorized administrator of DigitalCommons@UNO. For more information, please contact unodigitalcommons@unomaha.edu.

Chaotic Behaviour in Three Dimensional Quadratic Systems

A Thesis

Presented to the

Department of Mathematics and the

Faculty of the Graduate College

. University of Nebraska

in Partial Fulfillment of the Requirements for

the Degree of

Master of Arts

University of Nebraska at Omaha

by

Fu Zhang

August, 1996

UMI Number: EP74719

All rights reserved

INFORMATION TO ALL USERS
The quality of this reproduction is dependent upon the quality of the copy submitted.

In the unlikely event that the author did not send a complete manuscript
and there are missing pages, these will be noted. Also, if material had to be removed,
a note will indicate the deletion.

UMI
Dissertation Publishing

UMI EP74719

Published by ProQuest LLC (2015). Copyright in the Dissertation held by the Author.

Microform Edition © ProQuest LLC.
All rights reserved. This work is protected against
unauthorized copying under Title 17, United States Code

ProQuest

ProQuest LLC.
789 East Eisenhower Parkway
P.O. Box 1346
Ann Arbor, MI 48106 - 1346

THESIS ACCEPTANCE

Acceptance for the faculty of the Graduate College,
University of Nebraska, in partial fulfillment of
the requirements for the degree of Master of Arts,
University of Nebraska of Omaha.

Committee

Name	Department/School
Glenn P. Sowell	Physics / A&S
John P. Maloney	Math / A&S
Jack Heidel	Math / A&S

Chairperson of Supervisory Committee

Chairperson ____________________

Date 8-7-96

Abstract: This thesis presents a part of the proof of that there is no chaos in three dimensional autonomous quadratic systems of four terms with one nonlinear term and without constant terms. The first step of the proof is identifying equivalent systems from all possible systems by the 3rd order permutation group and as a result 138 unequivalent patterns are found(refer to appendix A). And then for the solvable systems we show how to solve them. Some of the nonsolvable systems turn out to be 2nd order autonomous systems and they are resolved by analyzing the monotonicity of the solutions and/or using the Poincaré-Bendixon theorem. The main and difficult part of the proof is to prove that the nonsolvable 3rd order systems have no chaos. This thesis introduces a general theory of analyzing the behavior of the solutions of higher dimensional autonomous systems($n \geq 3$) qualitatively in the phase space. Some sufficient conditions for systems to have no chaos are concluded by this theory. We also found out that there is a very close relation between coupled loops and the properties of the positive bounded limit sets and chaotic behaviour of a system. Because of limited time, only the seven dissipative nonsolvable 3rd order patterns are studied in this thesis. As an application of the theory, we proved in this thesis that three of them have no chaos. Two of the remaining four have coupled loops and thus the behavior of these two patterns is determined. The last two can be resolved by analyzing the monotonicity of the solutions. A system known to have chaos is also analyzed by this theory and this is a typical example of a chaotic system with coupled loops and with more than one positive bounded limit set.

Acknowledgement

I must express my gratefulness to Professor Jack Heidel who spent countless hours discussing with me many details about this thesis. He made many enlightning suggestions which helped this thesis get completed. I am also very grateful to the support from Professor John Maloney and Professor Glenn Sowell. And I would like to thank Professor Yi-Hsin Liu who's been encouraging me to do research in Mathematics.

Contents

Chapter 1. Introduction

Up to now, chaotic behaviour has been found in many simple three dimensional autonomous dynamical systems. The well known examples are the Lorenz attractor[1] and the Rössler attractor[2]. J. C. Sprott showed in his paper[3] that chaotic behaviour can occur in even simpler three dimensional autonomous systems. Here the simplicity refers to the algebraic representation rather than to the physical process described by the equations. All of the systems discussed in Sprott's paper are quadratic systems and characterized by five terms with two quadratic nonlinear terms or six terms with one quadratic nonlinear term. In 1995, J. Heidel conjectured [4] that there is no chaos in three dimensional autonomous quadratic systems of five terms with one nonlinear term or of less than five terms with any number of nonlinear terms.

The difficulty of solving a differential equation analytically is that there is no straight forward way to find the flows for the given vector fields. In other words, there is no general way to determine whether any two points in the phase space are on the same solution or not. Numerical methods are a powerful tool in solving differential equations. However, it is usually hard to determine if a numerical scheme works for a particular problem. Actually, there is no general definition for chaos mathematically. Chaotic behaviour is found in practice either by finding a solution numerically, or by calculating the Lyapunov exponents or calculating the dimension of the strange attractor of a system. It is still hard to say exactly when a particular system is chaotic, but in general systems with any one of the following properties are not chaotic: (i) there are sufficient analytic solutions for the system (or the system is integrable); (ii) each solution of a system is either a limit cycle or a critical point or it asymptotically goes to a limit cycle or a fixed point or it is eventually monotone; (iii) the system is a linear system or can be linearized.

We are interested in what the simplest chaotic system is and what is the simplest nonlinearity which can cause chaos. The systems in the conjecture are simple nonlinear systems which consist of systems of five terms with one quadratic nonlinear term and systems of four terms with one, two, three or four quadratic nonlinear terms and systems of three terms with one, two or three quadratic nonlinear terms. In this thesis, only systems of four terms with one quadratic nonlinear term and without constant terms are studied. The reason we start from this point is that almost all the three-term systems and four-term systems with constant terms are easy to resolve.

The thesis consists of three parts. The first part is chapter 2. In this chapter , all the systems except the nonsolvable systems are resolved by giving the analytic solutions of the systems. The second part is chapter 3. It gives a proof that there is no chaos in the systems each of which turns out to be a 2nd order autonomous system by analyzing the monotonicity of the solutions and/or using the Poincaré-Bendixon theorem. The third part is chapter 4 and chapter 5. In chapter 4 a general theory of analyzing the global behavior of higher dimensional autonomous systems is introduced by the author. The benefit we get from this theory will be seen in chapters 4 and 5. In chapter 5 an example given in Sprott's paper which has chaos is analyzed by this theory.

Chapter 2. Three Dimensional Autonomous Systems in $\dot{S}[4;1;0]$

Quadratic systems are the simplest systems of all the nonlinear dynamical systems. However there are still a huge number of systems that belong to this type. Here a classification of quadratic systems will be given. The general form of an n-dimensional quadratic system is:

$$\dot{\mathbf{x}}=f(\mathbf{x})=\mathbf{a}+\sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n}\sum_{j=i}^{n} c_{i,j}x_i x_j \tag{2.0}$$

where $\mathbf{x}\in R^n$ is a real n-dimensional phase space variable, $f=(f_1,\ldots,f_n)$ is an n-dimensional second degree polynomial, and **a**, **b** and **c** are real n-dimensional coefficient vectors.

Definition2.1.1 $S[p;q;r]$ is the set of systems in the form (2.0) with p terms on the right hand side, q of them are quadratic terms and r of them are constant terms.

Definition2.1.2 Considering the distribution of p, q, r in each dimension, $S[(p_1,\ldots,p_n);(q_1,\ldots,q_n);(r_1,\ldots,r_n)]$, $S[(p_1,\ldots,p_n);(q_1,\ldots,q_n);r]$ and $S[(p_1,\ldots,p_n);q;r]$ are defined as subsets of $S[p;q;r]$ where $p=\sum_{i=1}^{n} p_i$, $q=\sum_{i=1}^{n} q_i$ and $r=\sum_{i=1}^{n} r_i$ and there are p_i terms, q_i quadratic terms and r_i constant terms on the right hand side of f_i.

Notation We denote $S[(p_1,\ldots,p_n);q;0]$ and $S[(p_1,\ldots,p_n),(q_1,\ldots,q_n);0]$ as $S[(p_1,\ldots,p_n);q]$ and $S[(p_1,\ldots,p_n);(q_1,\ldots,q_n)]$ respectively

The systems that are studied in this thesis are three-dimensional S[4;1;0] systems., and $(x_1,x_2,x_3)=(x,y,z)$, and $(f_1,f_2,f_3)=(f,g,h)$.

2.1 The Total Number of 3-D Systems in S[4;1;0]

It can be verified that

$$S[4;1;0]=S[(2,1,1);1;0]\cup S[(1,2,1);1;0]\cup S[(1,1,2);1;0]$$
$$S[(2,1,1);1;0]=S[(2,1,1);(1,0,0)]\cup S[(2,1,1);(0,1,0)]\cup S[(2,1,1);(0,0,1)]$$
$$S[(1,2,1);1;0]=S[(1,2,1);(1,0,0)]\cup S[(1,2,1);(0,1,0)]\cup S[(1,2,1);(0,0,1)]$$
$$S[(1,1,2);1;0]=S[(1,1,2);(1,0,0)]\cup S[(1,1,2);(0,1,0)]\cup S[(1,1,2);(0,0,1)]$$

All possible f(x,y,z)'s, g(x,y,z)'s and h(x,y,z)'s of the above 9 subsets of S[4;1;0] are:

1. S[(2,1,1);(1,0,0)]

$\dot{x}=$ ax^2+bx, ax^2+by, ax^2+bz, ay^2+bx, ay^2+by, ay^2+bz, az^2+bx, az^2+by, az^2+bz, $axy+bx$, $axy+by$, $axy+bz$, $axz+bx$, $axz+by$, $axz+bz$, $ayz+bx$, $ayz+by$, $ayz+bz$

$\dot{y}=$ cx, cy, cz

$\dot{z}=$ rx, ry, rz

2. S[(2,1,1);(0,1,0)]

$\dot{x}=$ ay+bz, ay+bx, az+bx

$\dot{y}=$ cx^2, cy^2, cz^2, cxy, cxz, cyz

$\dot{z}=$ rx, ry, rz

3. S[(2,1,1);(0,0,1)]

$\dot{x}=$ ay+bz, ay+bx, az+bx

$\dot{y}=$ cx, cy, cz

$\dot{z}=$ rx^2, ry^2, rz^2, rxy, rxz, ryz

4. S[(1,2,1);(1,0,0)]

$\dot{x}=$ ax^2, ay^2, az^2, axy, axz, ayz

$\dot{y}=$ bz+cy, by+cx, bz+cx

$\dot{z}=$ rx, ry, rz

5. S[(1,2,1);(0,1,0)]

$\dot{x}=$ ax, ay, az

$\dot{y}=$ bx^2+cx, bx^2+cy,bx^2+cz, by^2+cx, by^2+cy, by^2+cz, bz^2+cx, bz^2+cy, bz^2+cz, bxy+cx, bxy+cy, bxy+cz, bxz+cx,bxz+cy,bxz+cz, byz+cx, byz+cy, byz+cz

$\dot{z}=$ rx, ry, rz

6. S[(1,2,1);(0,0,1)]

$\dot{x}=$ ax, ay, az

$\dot{y}=$ bz+cy, by+cx, bz+cx

$\dot{z}=$ rx^2, ry^2, rz^2, rxy, rxz, ryz

7. S[(1,1,2);(1,0,0)]

$\dot{x}=$ ax^2, ay^2, az^2, axy, axz, ayz

$\dot{y}=$ bx, by, bz

$\dot{z}=$ cy+rz, cy+rx, cz+rx

8. S[(1,1,2);(0,1,0)]

$\dot{x}=$ ax, ay, az

$\dot{y}=$ bx^2, by^2, bz^2, bxy, bxz, byz

$\dot{z}=$ cy+rz, cy+rx, cz+rx,

9. S[(1,1,2);(0,0,1)]

$\dot{x}=$ ax, ay, az

$\dot{y}=$ bx, by, bz

$\dot{z}=$ cx^2+rx, cx^2+ry, cx^2+rz, cy^2+rx, cy^2+ry, cy^2+rz, cz^2+rx, cz^2+ry, cz^2+rz, cxy+rx, cxy+ry, cxy+rz, cxz+rx, cxz+ry, cxz+rz, cyz+rx, cyz+ry, cyz+rz

where a, b, c, r are arbitrary nonzero constants and we call each possible combination of f(x,y,z), g(x,y,,z), h(x,y,z) a pattern regardless of the constants a, b, c, r.

For example the system:

$$\begin{cases} \dot{x} = axy + bx \\ \dot{y} = cz \\ \dot{z} = ry \end{cases}$$

is a pattern in $S[(2,1,1);(1,0,0)]$

Theorem 2.1.5 The number of all the possible patterns in $S[4;\ 1;\ 0]$ is 810.

Proof: The possible patterns in $S[(2,1,1);(1,0,0)]$ is $18\times3\times3$=162;
in $S[(2,1,1);(0,1,0)]$ is $3\times6\times3$=54; in $S[(2,1,1);(0,0,1)]$ is $3\times3\times6$=54;
in $S[(1,2,1);(1,0,0)]$ is $6\times3\times3$=54; in $S[(1,2,1);(0,1,0)]$ is $3\times18\times3$=162;
in $S[(1,2,1);(0,0,1)]$ is $3\times3\times6$=54; in $S[(1,1,2);(1,0,0)]$ is $6\times3\times3$=54;
in $S[(1,1,2);(0,1,0)]$ is $3\times6\times3$=54; in $S[(1,1,2);(0,0,1)]$ is $3\times3\times18$=162.
Thus the possible patterns in $S[4;\ 1;\ 0]$ is $162\times3+54\times6=810$.

Studying 810 systems one by one is a lot of work. Fourtunately the systems can be classified by the 3rd order permutation group There will be approximately one sixth of the 810 patterns left which are not equivalent by the permutation group. The parameters a, b, c of a system can be transformed to all ones or partly ones by a scalar transformation.

2.2 Eliminating Equivalent Patterns by the 3rd Order Permutation Group

Definition2.2.1: The 3rd order permutation group G consists of six 3 by 3 matrices $P_i \in G$, i=1,...,6:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad P_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad P_6 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Theorem 2.2.2 (a) There is a one-to-one correspondence between the patterns in $S[(2,1,1);1]$ and the patterns in $S[(1,2,1);1]$ by the transformation P_2 and none of the patterns in either $S[(2,1,1);1]$ or $S[(1,2,1);1]$ can be transformed to each other by P_2 . (b) There is a one-to-one correspondence between the patterns $S[(2,1,1);1]$ and the patterns in S[(1,1,2);1] by the transformation P_3 and none of the patterns in either S[(2,1,1);1] or S[(1,1,2);1] can be transformed to each other by P_3.

Proof: (a) P_2 acting on one pattern is equivalent to switching x and y in the pattern. We can see that after switching x and y in S[(2,1,1);1], the patterns in S[(2,1,1);(1,0,0)] become exactly the patterns in S[(1,2,1);(0,1,0)], the patterns in S[(2,1,1);(0,1,0)] become exactly the patterns in S[(1,2,1);(1,0,0)] and the patterns in S[(2,1,1);(0,0,1)] become exactly the patterns in S[(1,2,1);(0,0,1)]. And this tells us that there is a one-to-one correspondence between S[(2,1,1);1] and S[(1,2,1);1]. It is clear that after switching x and y in the patterns in S[(2,1,1);1], the pattern has to become one of those in S[(1,2,1);1]. So it can't remain in S[(2,1,1);1] Similarly after switching x and y, patterns in S[(1,2,1);1] can't can't remain in S[(1,2,1);1]. The proof of (b) is similar to the proof of (a).

With the above linear equivalence we can choose any one of the three subsets S[(2,1,1);1], S[(1,1,2);1] or S[(1,2,1);1] of S[4;1;0] and ignore the other two. Here we choose S[(2,1,1);1]. In S[(2,1,1);1] we have:

Theorem2.2.3 The patterns in S[(2,1,1);(0,1,0)] and the patterns in S[(2,1,1);(0,0,1)] and in S[(2,1,1);(1.0.0)], the patterns in (2.2.1) and in (2.2.2), the patterns in (2.2.3) and in (2.2.4), and the patterns in (2.2.5) and in (2.2.6) have a one-to-one correspondence by P_4 and none of the patterns in any of (2.2.1), (2.2.2), (2.2.3), (2.2.4), (2.2.5), (2.2.6), S[(2,1,1);(0,1,0)] or S[(2,1,1);(0,0,1)] are equivalent by P_4.

$$\begin{aligned}\dot{x} &= ax^2+by,\ ay^2+bx,\ ay^2+by,\ ay^2+bz,\ axy+bx,\ axy+by,\ axy+bz,\ ayz+by\\ \dot{y} &= cx,\ cy,\ cz\\ \dot{z} &= rx,\ ry,\ rz\end{aligned} \tag{2.2.1}$$

and

$$\begin{aligned}\dot{x} &= ax^2+bz,\ az^2+bx,\ az^2+bz,\ az^2+by,\ axz+bx,\ axz+bz,\ axz+by,\ ayz+bz\\ \dot{y} &= cx,\ cy,\ cz\\ \dot{z} &= rx,\ ry,\ rz\end{aligned} \tag{2.2.2}$$

and the patterns

$$\begin{aligned}\dot{x} &= ax^2+bx,\ ayz+bx\\ \dot{y} &= cy,\ cz\\ \dot{z} &= rx\end{aligned} \tag{2.2.3}$$

and

$$\begin{aligned}\dot{x} &= ax^2+bx,\ ayz+bx\\ \dot{y} &= cx\end{aligned} \tag{2.2.4}$$

$\dot{z}=$ rz, ry

and the patterns

$$\begin{aligned}\dot{x}&= ax^2+bx,\ ayz+bx\\ \dot{y}&= cy\\ \dot{z}&= ry\end{aligned} \tag{2.2.5}$$

and

$$\begin{aligned}\dot{x}&= ax^2+bx,\ ayz+bx\\ \dot{y}&= cz\\ \dot{z}&= rz\end{aligned} \tag{2.2.6}$$

Proof: The proof is similar to the proof of Theorem 2.2.2.

Similarly we can eliminate the patterns (2.2.2), (2.2.4), (2.2.6), in S[(2,1,1);(0,0,1)].

Corollary 2.2.4 The biggest subset of S[4;1;0] in which all the patterns are not equivalent under the permutation group consists the following 138 patterns:

(a) $\dot{x}=$ ax^2+by, ay^2+bx, ay^2+by, ay^2+bz, axy+bx, axy+by, axy+bz, ayz+by

$\dot{y}=$ cx, cy, cz

$\dot{z}=$ rx, ry, rz

(b) $\dot{x}=$ ax^2+bx, ayz+bx

$\dot{y}=$ cx, cy, cz

$\dot{z}=$ rx

(c) $\dot{x}=$ ax^2+bx, ayz+bx

$\dot{y}=$ cy

$\dot{z}=$ ry, rz

(d) $\dot{x}=$ ax^2+bx, ayz+bx

$\dot{y}=$ cz

$\dot{z}=$ ry

(e) $\dot{x}=$ ay+bz, ay+bx, az+bx

$\dot{y}=$ cx^2, cy^2, cz^2, cxy, cxz, cyz

$\dot{z}=$ rx, ry, rz.

Proof: The patterns in S[(2,1,1);(1,0,0)] that are not listed in theorem 2.2.3 are:

$$\begin{aligned}\dot{x}&= ax^2+bx,\ ayz+bx\\ \dot{y}&= cx\\ \dot{z}&= rx\end{aligned}\ ,\quad \begin{aligned}\dot{x}&= ax^2+bx,\ ayz+bx\\ \dot{y}&= cz\\ \dot{z}&= ry\end{aligned}\quad \text{and}\quad \begin{aligned}\dot{x}&= ax^2+bx,\ ayz+bx\\ \dot{y}&= cy\\ \dot{z}&= rz\end{aligned}$$

We can see that each of these patterns becomes itself after exchanging y and z in it. This six patterns plus the patterns in S[(2,1,1);(0,1,0)] and the patterns in (2.2.1), (2.2.3), (2.2.5) are exactly what is listed in this theorem. The total number of these patterns is $8\times3\times3+2\times3\times1+2\times1\times2+2\times1\times1+3\times6\times3=138$.

Most of the 138 patterns are solvable. We classify the solvable patterns into 6 types according to the way that is suitable for solving the system. The patterns that can be solved directly by separation of variables belong to type I. Some patterns are either first order linear equations or second order linear equations and they are type II. Systems of type III are the 1st order Ricatti equations. The solutions of type IV systems are elliptic functions and type V are a form of Rayleigh equations that are solvable. Each of the above pattern will be either solved in the next section except for the very easy ones or the ones that have a formula solution. There is one more type of patterns whose solution is obvious and so the solution will be listed in Appendix A directly and they are type S. For the nonsolvable patterns, they are type VI if they are essentially 2nd order autonomous systems and type VII if they are either 3rd order nonlinear autonomous systems or 2nd order nonlinear and nonautonomous systems. All the 138 patterns are listed in appendix A which tells the type of the system, their solvable forms if they are solvable, their equivalent scalar form if they are not solvable or it is hard to determine their sovability.

2.3. Scalar Transformations

The parameters a, b, c, r in each of the above 138 patterns can be eliminated or partly eliminated by a scalar transformation $x=\alpha X$, $y=\beta Y$, $z=\gamma Z$, $t=\delta T$, for nonsolvable cases, we require that $\delta>0$, because in some systems with chaotic behavior when $t\to\infty$, the chaotic behavior may disappear when $t\to-\infty$. Here are three exapmles of the transformations:

Example 1. System (24) can be transformed by a scalar transformation T to a system without parameters:

$$\begin{cases}\dot{x}=axy+bx\\ \dot{y}=cx\\ \dot{z}=ry\end{cases} \xrightarrow{\ \mathrm{T}\ } \begin{cases}\dot{x}=xy+x\\ \dot{y}=x\\ \dot{z}=y\end{cases}$$

Proof: Let $x=\alpha X$, $y=\beta Y$, $z=\gamma Z$, $t=\delta T$ then

$$\frac{d}{dt}=\frac{d}{dT}\frac{dT}{dt}=\frac{1}{\delta}\frac{d}{dT},\quad \frac{dx}{dt}=\alpha\frac{dX}{dt}=\frac{\alpha}{\delta}\frac{dX}{dT},\quad \frac{dy}{dt}=\frac{\beta}{\delta}\frac{dY}{dT},\quad \frac{dz}{dt}=\frac{\gamma}{\delta}\frac{dZ}{dT}$$

and thus we have:

$$\begin{cases} \dfrac{\alpha}{\delta}\dfrac{dX}{dT} = a\alpha\beta XY + b\alpha X \\ \dfrac{\beta}{\delta}\dfrac{dY}{dT} = c\alpha X \\ \dfrac{\gamma}{\delta}\dfrac{dZ}{dT} = r\beta Y \end{cases} \qquad \text{or} \qquad \begin{cases} \dfrac{dX}{dT} = \delta a\beta XY + \delta bX \\ \dfrac{dY}{dT} = \dfrac{\delta c\alpha}{\beta}X \\ \dfrac{dZ}{dT} = \dfrac{\delta r\beta}{\gamma}Y \end{cases}$$

since this is a solvable system, set $\delta a\beta=1$, $\delta b=1$ $\dfrac{\delta c\alpha}{\beta}=1$, $\dfrac{\delta r\beta}{\gamma}=1$

then $\alpha=b^2/(a\,c)$, $\beta=b/a$, $\gamma=r/a$ and $\delta=1/b$

Example 2. System (8) can be transformed by a scalar transformation T to a system with one parameter:

$$\begin{cases} x = ayz + bx \\ \dot{y} = cy \\ \dot{z} = rx \end{cases} \xrightarrow{\text{T}} \begin{cases} \dot{x} = yz + \dfrac{b}{c}x \\ \dot{y} = y \\ \dot{z} = x \end{cases}.$$

Proof: Let x= αX, y= βY, z= γZ, t= δT then

$$\frac{d}{dt} = \frac{d}{dT}\frac{dT}{dt} = \frac{1}{\delta}\frac{d}{dT}, \quad \frac{dx}{dt} = \alpha\frac{dX}{dt} = \frac{\alpha}{\delta}\frac{dX}{dT}, \quad \frac{dy}{dt} = \frac{\beta}{\delta}\frac{dY}{dT}, \quad \frac{dz}{dt} = \frac{\gamma}{\delta}\frac{dZ}{dT}$$

and thus we have

$$\begin{cases} \dfrac{\alpha}{\delta}\dfrac{dX}{dT} = a\beta\gamma YZ + b\alpha X \\ \dfrac{\beta}{\delta}\dfrac{dY}{dT} = c\beta Y \\ \dfrac{\gamma}{\delta}\dfrac{dZ}{dT} = r\alpha X \end{cases} \qquad \text{or} \qquad \begin{cases} \dfrac{dX}{dT} = \dfrac{\delta a\beta\gamma}{\alpha}YZ + \delta bX \\ \dfrac{dY}{dT} = \delta cY \\ \dfrac{dZ}{dT} = \dfrac{\delta r\alpha}{\gamma}X \end{cases}$$

since δb. and δc can not be 1 at the same time if b≠c, without loss of generality, since this is a solvable system , we make $\delta c=1$ and set $\dfrac{\delta a\beta\gamma}{\alpha}=1$, $\dfrac{\delta r\alpha}{\gamma}=1$,

then $\alpha = c\gamma / r, \beta = c^2 / ar, \gamma = \gamma, \delta = 1/c$.

Example 3. System (54) can be transformed by scalar transformations $T_{\pm}$ to systems without parameters:

$$\begin{cases} \dot{x} = ay^2 + bx \\ \dot{y} = cz \\ \dot{z} = rx \end{cases} \xrightarrow{\text{T}_{\pm}} \begin{cases} \dot{x} = y^2 \pm x \\ \dot{y} = z \\ \dot{z} = x \end{cases}$$

Let x=αX, y=βY, z=γZ, t=δT then

$$\frac{d}{dt}=\frac{d}{dT}\frac{dT}{dt}=\frac{1}{\delta}\frac{d}{dT},\quad \frac{dx}{dt}=\alpha\frac{dX}{dt}=\frac{\alpha}{\delta}\frac{dX}{dT},\quad \frac{dy}{dt}=\frac{\beta}{\delta}\frac{dY}{dT},\quad \frac{dz}{dt}=\frac{\gamma}{\delta}\frac{dZ}{dT}$$

$$\begin{cases}\frac{\alpha}{\delta}\frac{dX}{dT}=a\beta^2Y^2+b\alpha X\\ \frac{\beta}{\delta}\frac{dY}{dT}=c\gamma Z\\ \frac{\gamma}{\delta}\frac{dZ}{dT}=r\alpha X\end{cases}\quad or \quad \begin{cases}\frac{dX}{dT}=\frac{\delta a\beta^2}{\alpha}Y^2+b\delta X\\ \frac{dY}{dT}=\frac{\gamma\delta}{\beta}cZ\\ \frac{dZ}{dT}=\frac{\delta r\alpha}{\gamma}X\end{cases}$$

Since this is a nonsolvable system, to assure $\delta>0$, set $\frac{\delta a\beta^2}{\alpha}$=1, $b\delta$=1 if b>0 ($b\delta$=-1, if b<0), $\frac{\gamma\delta}{\beta}c$=1, $\frac{\delta r\alpha}{\gamma}$=1 and we get $T_{\pm}$= { $\alpha=-b^5/(ac^2r^2)$, $\beta=-b^3/(acr)$, $\gamma=-b^4/(ac^2r)$, $\delta=1/b$ if $b>0(-1/b$, if $b<0)$ }

In all that followed, we will express the scalar transformation as T={ $\alpha,\beta,\gamma,\delta$ }. And for each necessary scalar transformation we will give the transformation T={ $\alpha,\beta,\gamma,\delta$ } directly, not the procedure of finding the T.

2.4 Solvable Systems

101 patterns out of the 138 patterns have analytic solutions and thus there is no chaos in these systems. In this section we will give solutions of all of them unless there is a general formula solution for them.

2.4.1 Type I: Separation of Variables

The general form of the equation can be represented as:

$$f_1(x)g_1(y)dx+f_2(x)g_2(y)dy=0$$

28 out of the 101 patterns can be solved by separation of variables. Except the four marked by * in the table in the Appendix A, whose solutions depend on several integrals that have no compact forms, all the rest can be solved completely. Here we will give solutions of three of them which are fairly complicated and then list the integrals that the solutions of the above four patterns depend on.

1. Consider system (138):

$$\begin{cases}\dot{x}=ay+bz\\ \dot{y}=cyz\\ \dot{z}=rz\end{cases}$$

There is no chaotic solution in this system.

Proof: $\dot{z} = rz \Rightarrow z(t) = Ae^{rt}$, $\frac{\dot{y}}{y} = cAe^{rt} \Rightarrow \frac{d}{dt}(\ln y) = cAe^{rt} \Rightarrow \ln y = \frac{cA}{r}e^{rt} + \ln B$

$$\Rightarrow y = Be^{\frac{cA}{r}e^{rt}},$$

and thus $\dot{x} = aBe^{\frac{cA}{r}e^{rt}} + bAe^{rt}$

or $\quad x(t) = \int (aBe^{\frac{cA}{r}e^{rt}} + bAe^{rt})dt + C = \frac{cA}{r}e^{rt} + aB\int e^{\frac{cA}{r}e^{rt}} dt + C$

Let $\quad s = \frac{cA}{r}e^{rt}, \; ds = cAe^{rt}dt = rsdt,$

then $\quad x(t) = \frac{cA}{r}e^{rt} + \frac{aB}{r}\int \frac{e^s}{s}ds + C$

thus $\quad x(t) = \frac{cA}{r}e^{rt} + \frac{aB}{r}\left(\ln s + s + \frac{s^2}{2\times 2!} + \frac{s^3}{3\times 3!} + \dots \right) + C$

$$= \frac{cA}{r}e^{rt} + \frac{aB}{r}\left(\ln(\frac{cA}{r}) + rt + \frac{cA}{r}e^{rt} + \frac{(\frac{cA}{r}e^{rt})^2}{2\times 2!} + \frac{(\frac{cA}{r}e^{rt})^3}{3\times 3!} + \dots \right) + C$$

where A, B, C are integral constants.

2.Consider the system (24):

$$\begin{cases} \dot{x} = axy + bx \\ \dot{y} = cx \\ \dot{z} = ry \end{cases}$$

There is no chaotic solution in this system.

Proof: From example1 in section 2.3, system (24) can be transformed to: $\dot{x} = xy + x$, $\dot{y} = x$, $\dot{z} = y$.

thus $\quad \dot{x} = xy + x \;\; \dot{y} = x \Rightarrow x = \frac{1}{2}y^2 + y + A,$

$$\frac{dy}{dt} = \frac{1}{2}y^2 + y + A \;\Rightarrow\; t + B = \int \frac{2dy}{y^2 + 2y + 2A}$$

Case (a): If $A < \frac{1}{2}$, by appendix B(3), then

$$t + B = \frac{1}{\sqrt{1-2A}} \ln\left| \frac{y + 1 - \sqrt{1-2A}}{y + 1 + \sqrt{1-2A}} \right|$$

Let $\quad A' = \sqrt{1-2A}$

(a.1) If $\frac{y+1-A'}{y+1+A'} > 0$, then

$$e^{A'(t+B)} = \frac{y+1-A'}{y+1+A'} \quad \text{or} \quad y(t) = \frac{(1+A')e^{A'(t+B)}+A'-1}{1-e^{A'(t+B)}}.$$

$$x(t) = \dot{y}(t) = \frac{(1+A')A'e^{A'(t+B)}}{1-e^{A'(t+B)}} + \frac{((1+A')e^{A'(t+B)}+A'-1)A'e^{A'(t+B)}}{(1-e^{A'(t+B)})^2}$$

$$z(t) = \int \frac{(1+A')e^{A'(t+B)}+A'-1}{1-e^{A'(t+B)}}dt$$

Let $s = 1-e^{A'(B+t)}$, $ds = -A'e^{A'(B+t)}dt$ then

$$z(s) = -\int \frac{2A'-s(1+A')}{A's(1-s)}ds + C$$

$$= -\frac{2}{1+A'}\ln\frac{s}{1-s} + \frac{1}{A'\ln(1-s)+C}$$

thus $z(t) = -2\ln(1-e^{\sqrt{1-2A}(t+B)}) + (3\sqrt{1-2A}+1)(t+B)+C$

(a.2): If $\frac{y+1-A'}{y+1+A'}$<0, then

$$e^{A'(t+B)} = -\frac{y+1-A'}{y+1+A'}$$

or y= $x(t) = \dot{y}(t) = \frac{(1+A')A'e^{A'(t+B)}}{1-e^{A'(t+B)}} + \frac{((1+A')e^{A'(t+B)}+A'-1)A'e^{A'(t+B)}}{(1-e^{A'(t+B)})^2}$

$$x(t) = \dot{y}(t) = \frac{-A'(1+A')e^{A'(t+B)}}{1+e^{A'(t+B)}} - \frac{(A'-1-(1+A')e^{A'(t+B)})A'e^{A'(t+B)}}{(1+e^{A'(t+B)})^2}$$

Let $s = 1+e^{A'(B+t)}$, $ds = A'e^{A'(B+t)}dt$

$$z(s) = -\int \frac{2A'-s(1+A')}{A's(s-1)}ds + C = \frac{A'-1}{A'}\ln\frac{s}{s-1} - \frac{A'+1}{A'}\ln s + C$$

thus $z(t) = 2\ln(1+e^{\sqrt{1-2A}(t+B)}) + (1-\sqrt{1-2A})(t+B)+C$

Case (b):If $A > \frac{1}{2}$, then

$$t+B = \frac{2}{\sqrt{2A-1}}\arctan\frac{y+1}{\sqrt{2A-1}} \quad \text{or} \quad y(t) = \sqrt{2A-1}\tan(\frac{\sqrt{2A-1}}{2}(t+B))$$

thus $x(t) = \dot{y}(t) = \frac{1}{2}\sec^2(\frac{\sqrt{2A-1}}{2}(t+B))$

$$z(t) = \sqrt{2A-1}\int \tan(\frac{\sqrt{2A-1}}{2}(t+B))dt + C = 2\ln\left|\sec(\frac{\sqrt{2A-1}}{2}(t+B))\right| + C$$

Case (c) : If $A = \frac{1}{2}$, then

$$t+B = -\frac{2}{y+1} \quad \text{or} \quad y(t) = -1-\frac{2}{t+B}$$

$$x(t) = \frac{2}{(t+B)^2} \quad \text{and} \quad z(t) = -t - 2\ln|t+B| + C$$

where A, B and C are integral constants.

3. Consider system (136):

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cyz \\ \dot{z} = rz \end{cases}$$

There is no chaotic solution in this system.

Proof: $\dot{z} = rz \Rightarrow z = Ae^{rt}, \dot{y} = cyAe^{rt} \Rightarrow y = Be^{\frac{cA}{r}e^{rt}}$

and $\quad \dot{x} = ax + bBe^{\frac{cA}{r}e^{rt}}$

thus $\quad (xe^{-at})' = e^{-at}(bBe^{\frac{cA}{r}e^{rt}})$ or $x = bBe^{at}\int e^{-at}e^{\frac{Ac}{r}e^{rt}}dt + C$.

where A, B and C are integral constants.

Considering system (109), $z = (\frac{3r}{2c}y^2 + \frac{3A}{c})^{\frac{1}{3}}$, $\dot{y}(t) = c(\frac{3r}{2c}y^2 + \frac{3A}{c})^{\frac{2}{3}}$

and thus

$$\int \frac{dy}{c(\frac{3r}{2c}y^2 + \frac{3A}{c})^{\frac{2}{3}}} = t + C \ , \qquad (2.4.1.1)$$

where C is an integration constant. And $\dot{x} = ax + by$, thus $x(t)$ depends on the inverse relation of (2.4.1.1). Systems (110) and (111) are similar to system(109). The three systems are also two dimensional autonomous systems and thus the behaviour of the solutions of these systems can also be analyzed by the method in chapter 3.

2.4.2 Type II: Linear Systems

The general form of the nth-order linear ordinery differetial equations can be represented as:

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = \varphi(x)$$

where $a_i(x), i = 0, 1, 2, \dots, n-1$, is continuous functions.

43 out of the 138 pqtterns turned out to be linear ODEs. 29 of them are 1st order linear ODEs, 8 of them are 2nd order linear OEDs with constant coefficients and 6 of them are 2nd order linear ODEs with variable coefficients. There is a general method to solve the first two kinds of the equations. For the last 6 patterns, we will solve one by one in this section , since there is not a general method to solve them.

1. Consider system (8):

$$\begin{cases} \dot{x} = ayz + bx \\ \dot{y} = cy \\ \dot{z} = rz \end{cases}$$

There is no chaotic solution in this system.

Proof: $\dot{y} = cy \Rightarrow y = Ae^{ct}$, $x = \frac{1}{r}\dot{z}$, $\dot{x} = \frac{1}{r}\ddot{z} \Rightarrow$

$$\ddot{z} - b\dot{z} - arAe^{ct}z = 0 \tag{2.4.2.1}$$

From Appendix B(4) this equation belongs to the general form:

$$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + (\beta e^{\lambda x} + \delta)y = 0$$

where $\alpha = -b, \beta = arA, \lambda = c, \delta = 0$, the general solution of (2.4.2.1) is:

$$z(t) = e^{\frac{bt}{2}}\left[C_1 J_v\left(\frac{2\sqrt{arA}}{c}e^{\frac{ct}{2}}\right) + C_2 Y_v\left(\frac{2\sqrt{arA}}{c}e^{\frac{ct}{2}}\right)\right], \qquad v = \frac{\sqrt{\alpha^2 - 4\delta}}{\lambda} = \frac{|b|}{c}$$

$$x(t) = \frac{dz(t)}{dt}$$

$$= \frac{b}{2}e^{\frac{bt}{2}}\left[C_1 J_v\left(\frac{2\sqrt{arA}}{c}e^{\frac{ct}{2}}\right) + C_2 Y_v\left(\frac{2\sqrt{arA}}{c}e^{\frac{ct}{2}}\right)\right] \pm \frac{1}{t}\left[vC_1 J_v\left(\frac{2\sqrt{arA}}{c}e^{\frac{ct}{2}}\right)\right.$$

$$\left. + vC_2 Y_v\left(\frac{2\sqrt{arA}}{c}e^{\frac{ct}{2}}\right) - C_1 J_{v\pm1}\left(\frac{2\sqrt{arA}}{c}e^{\frac{ct}{2}}\right) - C_2 Y_{v\pm1}\left(\frac{2\sqrt{arA}}{c}e^{\frac{ct}{2}}\right)\right]$$

where J_v *and* Y_v are Bessel functions and A, C_1, C_2 are integral constants.

2. Consider System (32):

$$\begin{cases} \dot{x} = axz + by \\ \dot{y} = cx \\ \dot{z} = rz \end{cases}$$

There is no chaotic solution in this system.

Proof: The scalar transformation T transforms (32) to a simpler form:

$\dot{x} = xz + b'y$, $\dot{y} = x$, $\dot{z} = z$ where $b' = \frac{bc}{r^2}$

and $T = \left\{\alpha = 1, \beta = \frac{c}{r}, \gamma = \frac{r}{a}, \delta = \frac{1}{r}\right\}$, $\dot{z} = z \Rightarrow z(t) = Ae^t \Rightarrow$

$$\ddot{y} - Ae^t\dot{y} - b'y = 0 \tag{2.4.2.2}$$

equation (2.4.2.2) belongs to form (5) in Appendix B:

$$f_2(x)y''(x) + f_1(x)y'(x) + f_0(x)y = 0$$

where $f_0(x) = -b$, $f_1(x) = -Ae^t$, $f_2(x) = 1$, $f(x) = -b - \frac{A^2}{4}e^{2t} + \frac{A}{2}e^t$

and let $$y(t) = u(t)e^{\frac{A}{2}e^t}$$

$$u''(t) - \left(\frac{A^2}{4}e^{2t} - \frac{A}{2}e^t + b\right)u(t) = 0 \qquad (2.4.2.3)$$

equation (2.4.2.3) is the canonical form (6) in Appendix B:

$$y''(x) - (\alpha e^{2\lambda x} + \beta e^{\lambda x} + \gamma)y(x) = 0$$

where $\lambda = 1,\ \alpha = \frac{A^2}{4},\ \beta = -\frac{A}{2},\ \gamma = b,\ k = \sqrt{b}$,

after the transformtion $\eta(t) = e^t,\ w(\eta) = \eta^{-\sqrt{b}u(t)}$, we have

$$\eta w''(\eta) + \left(1 + 2\sqrt{b}\right)w'(\eta) + (-\frac{A^2}{4}\eta + \frac{A}{2})w(\eta) = 0 \qquad (2.4.2.4)$$

And equation (2.4.2.4) belongs to the general form (7) in Appendix B:

$$(a_2x + b_2)y''(x) + (a_1x + b_1)y'(x) + (a_0x + b_0)y(x) = 0 \qquad (2.4.2.5)$$

where $a_2 = 1,\ b_2 = 0,\ a_1 = 0,\ b_1 = 1 + 2\sqrt{b},\ a_0 = -\frac{A^2}{4},\ b_0 = \frac{A}{2}$

Let $D^2 = a_1^2 - 4a_0a_2 = A^2,\ h = \frac{D - a_1}{2a_2} = \frac{A^2}{2},\ A(h) = 2a_2h + a_1 = A^2,\ \sigma = -\frac{a_2}{A(h)} = -\frac{1}{A^2}$

$\mu = -\frac{b_2}{a_2} = 0,\ B(h) = b_2h^2 + b_1h + b_0 = \frac{A^2}{2}(1 + 2\sqrt{b}) + \frac{A}{2},\ \xi = \frac{x - \mu}{\sigma} = -A^2\eta,$

$a' = \frac{B(h)}{A(h)} = \frac{1}{2}(1 + 2\sqrt{b} + A^{-1}),\ b' = (a_2b_1 - a_1b_2)a_2^{-2} = 1 + 2\sqrt{b}$

then the general solution of the equation (2.4.2.5) can be written as:

$$w(\eta) = e^{\frac{A^2}{2}}\Gamma(\frac{1}{2}(1 + 2\sqrt{b} + A^{-1}), 1 + 2\sqrt{b}; -A^2\eta)$$

here $\Gamma(a', b'; \xi)$ is an arbitrary solution of the degenerate hypergeometric equation:

$$xy''(x) + (b - x)y' - ay = 0$$

$$\eta(t) = e^t,\ \ln(w(\eta)) = \ln(\eta^{-\sqrt{b}u(t)}) = -\sqrt{b}u(t)\ln(\eta(t)) = -\sqrt{b}tu(t)$$

$$u(t) = -\frac{1}{\sqrt{b}t}\ln(w(\eta)) = -\frac{1}{\sqrt{b}t}\ln(e^{\frac{A^2}{2}}\Gamma(\frac{1}{2}(1 + 2\sqrt{b} + A^{-1}), 1 + 2\sqrt{b}; -A^2e^t))$$

thus $$y(t) = -\frac{1}{\sqrt{b}t}\ln(e^{\frac{A^2}{2}}\Gamma(\frac{1}{2}(1 + 2\sqrt{b} + A^{-1}), 1 + 2\sqrt{b}; -A^2e^t))e^{\frac{A}{2}e^t}$$

$$x(t) = \dot{y}(t) = -\frac{1}{\sqrt{b}t}\frac{d}{dt}\left[\ln(e^{\frac{A^2}{2}}\Gamma(\frac{1}{2}(1 + 2\sqrt{b} + A^{-1}), 1 + 2\sqrt{b}; -A^2e^t))e^{\frac{A}{2}e^t}\right]$$

3. Consider the system (33):

$$\begin{cases} \dot{x} = ayz + bz \\ \dot{y} = cx \\ \dot{z} = rz \end{cases}$$

There is no chaotic solution in this system.

Proof: $\dot{z} = rz \Rightarrow z(t) = Ae^{rt}$. $x = \frac{1}{c}\dot{y} \Rightarrow \dot{x} = \frac{1}{c}\ddot{y}$ thus

$$\ddot{y} - aAe^{rt}\dot{y} - cbe^{rt} = 0 \qquad (2.4.2.6)$$

Let $w = \dot{y}$ then (2.4.2.6) becomes

$$\dot{w} - aAe^{rt}w - cbe^{rt} = 0 \qquad (2.4.2.7)$$

Then the solution of (2.4.2.7) is:

$$w(t) = e^{\frac{aA}{r}e^{rt}}\left(\int e^{-\frac{aA}{r}e^{rt}} cbe^{rt}dt + B\right),$$

thus $x = \frac{1}{c}w(t) = \frac{1}{c}e^{\frac{aA}{r}e^{rt}}\left(\int e^{-\frac{aA}{r}e^{rt}} cbe^{rt}dt + B\right)$

and thus the solution of $y(t) = y(t) = \int \left(e^{\frac{aA}{r}e^{rt}}\left(\int e^{-\frac{aA}{r}e^{rt}} cbe^{rt}dt + B\right)\right)dt + C$

4. Consider the system (44):

$$\begin{cases} \dot{x} = ayz + bz \\ \dot{y} = cy \\ \dot{z} = rx \end{cases}$$

There is no chaotic solution in this system.

Proof: $\dot{y} = cy \Rightarrow y(t) = Ae^{ct}$, $x = \frac{1}{r}\dot{z} \Rightarrow$

$$\ddot{z} - r(aAe^{ct} + b)z = 0 \qquad (2.4.2.8)$$

since the equation (2.4.2.8) belongs to canonical form:

$$y''(x) + (ae^{x} - b)y(x) = 0 \qquad (2.4.2.9)$$

the general solution of equation (2.4.2.9) is:

$$y(x) = C_1 J_{2\sqrt{b}}(2\sqrt{a}e^{\frac{x}{2}}) + C_2 Y_{2\sqrt{b}}(2\sqrt{a}e^{\frac{x}{2}})$$

where J_v and Y_v are Bessel function. let $c = 1$, $\alpha = -raA$, $\beta = rb$. Then the solution of equation (2.4.2.8) can be written as

$$z(t) = C_1 J_{2\sqrt{rb}}(2\sqrt{-raA}e^{\frac{t}{2}}) + C_2 Y_{2\sqrt{rb}}(2\sqrt{-raA}e^{\frac{t}{2}})$$

$$x(t) = \frac{1}{r}\dot{z}(t) = \frac{1}{r}\frac{d}{dt}\{C_1 J_{2\sqrt{rb}}(2\sqrt{-raA}e^{\frac{t}{2}}) + C_2 Y_{2\sqrt{rb}}(2\sqrt{-raA}e^{\frac{t}{2}})\}$$

5. Consider the system (135):

$$\begin{cases}\dot{x} = ay + bz \\ \dot{y} = cxz \\ \dot{z} = rz\end{cases}$$

There is no chaotic solution in this system.

Proof: $\dot{z} = rz \Rightarrow z(t) = Ae^{rt}$, $\ddot{x} = a\dot{y} + rbAe^{rt} \Rightarrow$

$$\ddot{x} - acAe^{rt}x - rbAe^{rt} = 0 \tag{2.4.2.10}$$

(2.4.2.10) can be solved by the variation of parameters. First we find the solution $x_h(t)$ of the homogeneous equation:

$$\ddot{x} - acAe^{rt}x = 0 \tag{2.4.2.11}$$

(2.4.2.11) belongs to a cononical form $y''(x) + \alpha e^{\lambda x}y(x) = 0$, its general solution can be written as:

$$y(x) = C_1J_0(\frac{2\sqrt{\alpha}}{\lambda}e^{\frac{\lambda x}{2}}) + C_2Y_0(\frac{2\sqrt{\alpha}}{\lambda}e^{\frac{\lambda x}{2}})$$

where J_0 and Y_0 are Bessel function. Then the solution $x_h(t)$ of (2.4.2.11) is:

$$x_h(t) = C_1x_1(t) + C_2\,x_2(t)$$

where $\quad x_1(t) = J_0(2\frac{\sqrt{-acA}}{r}e^{\frac{rt}{2}}),\ x_2(t) = Y_0(2\frac{\sqrt{-acA}}{r}e^{\frac{rt}{2}})$

thus a particular solution of (2.4.2.10) is

$$x_p(t) = x_1(t)\int\frac{W_1(t)}{W(t)}rbAe^{rt}dt + x_2(t)\int\frac{W_2(t)}{W(t)}rbAe^{rt}dt$$

where $\quad W(t) = \frac{2}{\pi t},\ W_1(t) = -Y_0,\ W_2(t) = J_0$

and thus the general solution of (2.4.2.10) is $x(t) = x_h(t) + x_p(t)$

2.4.3 Type III. First-order Riccati equations

The general first-order Riccati equation can be written as:

$$\dot{x}(t) + p(t)x + q(t)x^2(t) = \varphi(t), \qquad '\cdot' = \frac{d}{dt} \tag{2.4.3.1}$$

where $p(t)$ and $q(t)$ are arbitrary functions. Now let $x(t) = \frac{\dot{\psi}(t)}{q(t)\psi(t)}$, and we get the second order linear o.d.e :

$$\ddot{\psi}(t) + (p(t) - \frac{\dot{q}(t)}{q(t)})\dot{\psi}(t) - \varphi(t)q(t)\psi(t) = 0 \tag{2.4.3.2}$$

Six patterns out of the138 patterns turn out to be first order Riccati equations. Here we are gonig to solve patterns (37) and (61), patterns (43), (45), (69), and (77) are similar to pattern (37).

1. Consider system (37):

$$\begin{cases} \dot{x} = ax^2 + by \\ \dot{y} = cy \\ \dot{z} = rx \end{cases}$$

There is no chaotic solution in this system.

Proof: $\dot{y} = cy \Rightarrow y = Ae^{ct}$, then $\dot{x} - ax^2 = bAe^{ct}$

This is a first order Riccati equation (2.4.3.1) with $p(t) = 0$, $q(t) = -a$, and $\varphi(t) = bAe^{ct}$

let $x(t) = -\frac{\dot{\psi}(t)}{a\psi(t)}$ thus we have

$$\ddot{\psi}(t) - abAe^{ct}\psi(t) = 0 \tag{2.4.3.3}$$

This is a second order linear equation with variable coefficients. From [10], the solution of (2.4.3.3) is:

$$\psi(t) = C_1 J_0(\frac{2\sqrt{-abA}}{c}e^{ct/2}) + C_2 Y_0(\frac{2\sqrt{-abA}}{c}e^{ct/2})$$

and thus

$$x(t) = -\frac{1}{a}\frac{[C_1 J_0(\frac{2\sqrt{-abA}}{c}e^{ct/2}) + C_2 Y_0(\frac{2\sqrt{-abA}}{c}e^{ct/2})]'}{C_1 J_0(\frac{2\sqrt{-abA}}{c}e^{ct/2}) + C_2 Y_0(\frac{2\sqrt{-abA}}{c}e^{ct/2})}$$

$$z(t) = \int x(t)dt + B = \int -\frac{1}{a}\frac{\dot{\psi}}{\psi}dt + B = -\frac{1}{a}\ln\psi + B$$

where C_1, C_2 and B are integral constants. .

2. Consider system (61):

$$\begin{cases} \dot{x} = ax^2 + by \\ \dot{y} = cz \\ \dot{z} = ry \end{cases}$$

There is no chaotic solution in this system.

Proof: $\dot{y} = cz, \dot{z} = ry \Rightarrow y = C_1 e^{\sqrt{rc}t} + C_2 e^{-\sqrt{rc}t}, z = C_1\sqrt{\frac{r}{c}}e^{\sqrt{rc}t} - C_2\sqrt{\frac{r}{c}}e^{-\sqrt{rc}t}$

and thus $\dot{x} - ax^2 = b(C_1 e^{\sqrt{rc}t} + C_2 e^{-\sqrt{rc}t})$

this is a 1st order Riccati equation with

$$p(t) = 0,\ q(t) = -a,\ \text{and}\ \varphi(t) = b(C_1 e^{\sqrt{rc}t} + C_2 e^{-\sqrt{rc}t})$$

after the transformation

$$x(t) = -\frac{\dot{\psi}(t)}{a\psi(t)}$$

we have

$$\ddot{\psi}(t) - ab(C_1 e^{\sqrt{rct}} + C_2 e^{-\sqrt{rct}})\psi(t) = 0 \qquad (2.4.3.4)$$

where C_1, C_2 and B are integral constants. This is a linear system with variable coefficients. and thus there is no chaos in this system. But we couldn't find the analytic solution to (2.4.3.4).

2.4.4 Type IV: Elliptic Equations

Lemma2.4.4.1 The following identity is true

$$\frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}} = -\frac{1}{2}\sqrt{(\alpha-\gamma)}\left[\frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)}}\right] \qquad (2.4.4.1)$$

Proof: The transformation: $w^2 = \dfrac{\alpha-\gamma}{y-\gamma}$, $k^2 = \dfrac{\beta-\gamma}{\alpha-\gamma}$ transforme the left hand side of (2.4.4.1) to the right hand side.

1. Consider systems (23)

$$\begin{cases} \dot{x} = ay^2 + by \\ \dot{y} = cx \\ \dot{z} = ry \end{cases}$$

There is no chaotic solution in this system.

Proof:
$$\frac{1}{c}\ddot{y} = ay^2 + by \Rightarrow \dot{y}\ddot{y} = acy^2\dot{y} + bcy\dot{y}$$

$$\Rightarrow \frac{1}{2}\dot{y}^2 = \frac{ac}{3}y^3 + \frac{bc}{2}y^2 + A \Rightarrow \dot{y} = \pm\sqrt{\frac{2ac}{3}y^3 + bcy^2 + 2A}$$

or
$$t+B = \int \frac{dy}{\pm\sqrt{\frac{2ac}{3}y^3 + bcy^2 + 2A}} \qquad (2.4.4.2)$$

where $\frac{2ac}{3}y^3 + bcy^2 + 2A$ can always be factored as $(y-\alpha)(y-\beta)(y-\gamma)$, from Lemma 2.4.4.1, (2.4.4.2) can be expressed by the Elliptic function of the first kind.

$$\frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}$$

where $w^2 = \dfrac{\alpha-\gamma}{y-\gamma}$, $k^2 = \dfrac{\beta-\gamma}{\alpha-\gamma}$

thus
$$x = \pm\frac{1}{c}\sqrt{\frac{2ac}{3}y^3 + bcy^2 + 2A}$$

and z can be expressed by the inverse relation of (2.4.4.2) as:

$$z = \int ry(t)dt + C$$

where A and C are integral constants.

2. Consider the system (60)

$$\begin{cases} \dot{x} = ayz + bz \\ \dot{y} = cz \\ \dot{z} = rx \end{cases}$$

There is no chaotic solutions in this system.

Proof: Likewise, we have $\ddot{y} = \frac{ra}{2}y^2 + rby + A$ and t+C= $\int \frac{dy}{\pm\sqrt{\frac{ar}{3}y^3 + rby^2 + 2Ay + B}}$

$x = \frac{1}{c}(\frac{a}{2}y^2 + by + \frac{A}{r})$ and $z = \int rxdt + C = \int \frac{1}{c}(\frac{a}{2}y^2 + by + \frac{A}{r})dt$+C.

2.4.5 Type V: The Rayleigh Equations

Some systems come up with a form of the Rayleigh equation:

$$y''(x) + \alpha(y')^2 + \beta y = 0$$

which has the general solution:

$$x = C_2 \pm \alpha \int [C_1 \alpha^2 e^{-2\alpha y} + \beta(\frac{1}{2} - \alpha y)]^{-\frac{1}{2}} dy.$$

1. Consider the systems (13)

$$\begin{cases} \dot{x} = ax^2 + by \\ \dot{y} = cx \\ \dot{z} = rx \end{cases}$$

There is no chaotic solution in this system.

Proof: $\dot{x} = ax^2 + by, \dot{y} = cx \Rightarrow$

$$\ddot{y} - \frac{a}{c}y^2 - bcy = 0 \qquad (2.4.5.1)$$

which is the Rayletgh equation of the above form with

$$\alpha = -\frac{a}{c}, \beta = -bc$$

and thus the solution to (2.4.5.1) is

$t = C_2 \,\mathrm{m}\, \frac{a}{c}\int [C_1(-\frac{a}{c})^2 e^{-2-\frac{a}{c}y} - bc(\frac{1}{2} + \frac{a}{c}y)]^{-\frac{1}{2}} dy$, $z = \frac{r}{c}y + A$ and $x = \frac{1}{c}\dot{y}$.

The procedure of solving systems (21) and (26) are similar to this.

Chapter3 Proof of the Nonsovable 2nd Order Autonomous Systems

13 out of the 138 patterns turn out to be 2nd order autonomous systems. We will prove in this chapter that there is no chaos in these systems by analyzing the monotonicity of all solutions in each system and/or by using the Poincaré-Bendixon theorem.

3.1 Preliminaries

Consider a general autonomous vector field

$$\dot{x} = f(x) \qquad x \in R^n \qquad (3.1.1)$$

where $f(x)$ is C^r, $r \geq 1$. The solution of (3.1.1) that passes through the point x_0 at t=t_0 is denoted by x(t, t_0, x_0). The solutions of (3.1.1) form a one parameter family of C^r, $r \geq 1$, diffeomorphisms of the phase space. This is refer to as *a phase flow* or *a flow* denoted as $\phi(t,x)$ or $\phi_t(x)$.

Definition3.1.1 A point x_0 is a *fixed point* or a *critical point* of (3.1.1) if $f(x_0) = 0$.

Definition3.1.2 A set $S \subset R^n$ is said to be *invariant* under the vector field (3.1.1) if for any $x_0 \in S$ at t=t_0 we have x(t, t_0, x_0)$\in S$ for all $t \in R$. If $t \geq 0$ then S is refer to as *positively invariant set*, for negative time, as a *negatively invariant set*. The symbol .M is understood to be a positively invariant compact set in the phase space.[6]

Definition3.1.3 A point $x_0 \in R^n$ is called an ω *limit point* of $x \in R^n$, denoted as ω(x), if there exists a sequence $\{ t_i \}$, $t_i \to \infty$, such that

$$\phi(t_i, x) \to x_0$$

an α *limit point* is defined similarly by taking a sequence $\{ t_i \}$, $t_i \to -\infty$. The set of all ω limit points of a flow is called the ω limit set. The α limit set is similarly defined[6].

Theorem3.1.4(Poincaré-Bendixon) Let M be a positively invariant region for the vector field containing a finite number of fixed points. Let $p \in$ M, and consider $\omega(p)$. Then one of the following possibilities holds.

i) $\omega(p)$ is a fixed point;
ii) $\omega(p)$ is a closed orbit;
iii) $\omega(p)$ consists of a finite number of fixed points $p_1, \ldots, p_n$ and orbits γ with $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$.

Proof: Refer to [6].

Theorem3.1.5 If an autonomous differential equation $\dot{x} = f(x)$, $x \in R^n$ has a saddle at x_0, then there are precisely two trajectories that tend to x_0 as $t \to \infty$, that together with x_0 form a smooth curve tangent at x_0 to the line of eigenvectors with negative eigenvalues(the stable direction) and precisely two trajectories that tend to x_0 as $t \to \infty$, that together with

x_0 form a smooth curve at x_0 to the line of eigenvectors with positive eigenvalue(the unstable direction)

Proof: Refer to [6].

Thoerem3.1.6 Suppose the solutions of a system (3.1.1) which are not fixed points are monotone in a connected region. Then for a solution, one of the following possibilities holds:

i) the solution approaches to one of the fixed points or approaches asymptotically to an orbit on the boundary of the region as $t \rightarrow +\infty$.

ii) the solution goes to infinity as $t \rightarrow +\infty$, if there is no boundary in the direction that the solution develops

iii) the solution reaches the boundary of the region at some later time.

Proof: Obvious.

Corollary3.1.7 Suppose the solutions of a system (3.1.1) which are not fixed points are monotone in a connected region. Then for a solution, one of the following possibilities holds:

i) the solution approaches to one of the fixed points or approaches asymptotically to an orbit on the boundary of the region as $t \rightarrow -\infty$.

ii) the solution goes to infinity as $t \rightarrow -\infty$, if there is no boundary in the direction that the solution develops from.

iii) the solution comes from the boundary of the region at some earlier time.

Proof: Obvious.

3.2 Proof by the Poincaré-Bendixon Theorem

1. Consider system(7)

$$\begin{cases} \dot{x} = ayz + bx \\ \dot{y} = cx \\ \dot{z} = rx \end{cases}$$

There is no chaotic solution in this system.

Proof: The system can be transformed to $\dot{x} = yz + x,\ \dot{y} = x,\ \dot{z} = x$ by the scalar transformation T={ $\alpha = \frac{b^3}{acr},\ \beta = \frac{b^2}{ar},\ \gamma = \frac{b^2}{ac},\ \delta = \frac{1}{b}$ } if b>0.

Thus $\dot{y} = x,\ \dot{z} = x \Rightarrow$

$$y = z + A \tag{3.2.1}$$

where A is a constant. Substitute (3.2.1) into the first equation of the system leads to a 2nd order system about x and z with an arbitrary constant A:

$$\begin{cases} \dot{x} = x + Az + z^2 = (z + \frac{A}{2})^2 + x - \frac{A^2}{4} \\ \dot{z} = x \end{cases} \tag{3.2.2}$$

(3.2.2) will be resolved in 3 cases here:

(a) If A=0, then system (3.2.2) becomes:

$$\begin{cases} \dot{x} = x + z^2 \\ \dot{z} = x \end{cases} \tag{3.2.3}$$

As shown in Fig3.2.1(a): The horizontal isocline $L1 = \{(x,\ z):\ x + z^2 = 0\}$ and the vertical isocline $L2 = \{(x,\ z):\ x = 0\}$ intersect at P1=$\{(x,z): x = 0, z = 0\}$ which is the fixed point of this system. The fixed point cuts L1 into two parts L1(1)= $\{(x,z): x + z^2 = 0, z > 0\}$ on which $\dot{x} = 0$ and $\dot{z} < 0$ and L1(2)= $\{(x,z): x + z^2 = 0, z < 0\}$ on which $\dot{x} = 0$ and $\dot{z} < 0$ and cuts L2 into two parts $L2(1) = \{(x,z): x = 0, z > 0\}$ on which $\dot{x} > 0$ and $\dot{z} = 0$ and $L2(2) = \{(x,z): x = 0, z < 0\}$ on which $\dot{x} > 0$ and $\dot{z} = 0$. The complement set of the isoclines in the phase plane are four disjiont regions each of which are connected and satisfy Q1=$\{ (x,z): x < 0, z > 0, x + z^2 > 0 \}$ on which $\dot{x} > 0,\ \dot{z} < 0$, Q2=$\{ (x,z): x > 0 \}$ on which $\dot{x} > 0,\ \dot{z} > 0$, Q3=$\{ (x,z): x < 0, x + z^2 < 0 \}$ on which $\dot{x} < 0,\ \dot{z} < 0$ and Q4= $\{ (x,z): x < 0, z < 0, x + z^2 > 0 \}$ on which $\dot{x} > 0,\ \dot{z} < 0$. Thus in each of the connected sets P1, L1(1), L1(2), L2(1), L2(2), Q1, Q2, Q3, Q4, shown in Fig. 3.2.1(a), the solutions are monotone.

From the monotonicity of the solutions in each of the above connected monotone regions, the possible paths of the solutions starting from each of the sets are: ***1***. The solutions start from any point on L1(1) will get to Q3. ***2***. The solutions start from any point on L1(2) will get to Q4. ***3.*** The solutions start from any point on L2(1) will get to Q2. ***4.*** The solutions start on L2(2) will get to Q2. ***5.*** The solutions start from any point in Q1 will either reach L1(1) or L2(1) or will approach P1.***6*** . The solutions start from any point in Q2 will approach positive infinity in both directions as $t \to \infty$. ***7.*** The solutions start from any point in Q3 will remain in Q3 as $t \to \infty$ or reach L1(2) at some later time. ***8.*** The solutions start from any point in Q4 will reach L2(2) at some later time. The diagram of the above possible paths is shown in Fig3.2.1(b). From the diagram we can see that there is no loop in the diagram which means that all solutions are eventually monotone.

(b) If A>0, then system (3.2.2) becomes:

$$\begin{cases} \dot{x} = (z + \frac{|A|}{2})^2 + x - \frac{A^2}{4} \\ \dot{z} = x \end{cases} \qquad (3.2.4)$$

Similarly, as shown in Fig.3.2.1(c) the horizontal isocline L1={(x, z): $(z+\frac{|A|}{2})^2 + x - \frac{A^2}{4} = 0$} and the vertical isocline L2 = {(x, z): $x = 0$} intersect at P1= $\{(x,z): x = 0,$ $\{(x,z): x = 0, z = -|A|\}$ and P2=$\{(x,z)$: $x = 0, z = 0\}$ which are the fixed points of system (3.2.4). The two fixed points cut L1 into three parts L1(1)= $\{(x,z)$: $(z+\frac{|A|}{2})^2 + x - \frac{A^2}{4} = 0, x < 0, z > 0\}$ on which $\dot{x} = 0$ and $\dot{z} < 0$ and L1(2)=$\{(x,z)$: $(z+\frac{|A|}{2})^2 + x - \frac{A^2}{4} = 0,$ $x > 0\}$ on which $\dot{x} = 0$ and $\dot{z} > 0$ and L1(3)=$\{(x,z)$: $(z+\frac{|A|}{2})^2 + x - \frac{A^2}{4} = 0, x < 0, z < 0\}$ on which $\dot{x} = 0$ and $\dot{z} < 0$ and cut L2 into three parts L2(1)= $\{(x,z): x = 0, z > 0\}$ on which $\dot{x} > 0$ and $\dot{z} = 0$ and L2(2)=$\{(x,z)$: $x = 0, -A < z < 0\}$ on which $\dot{x} > 0$ and $\dot{z} = 0$. and $L2(3) = \{(x,z): x = 0, z < -A\}$ on which $\dot{x} > 0$ and $\dot{z} = 0$. The complement set of the isoclines in the phase plane are five disjiont regions each of which are connected and satisfy Q1={ $(x,z): x < 0, z > 0, (z+\frac{|A|}{2})^2 + x - \frac{A^2}{4} > 0$ } on which $\dot{x} > 0, \dot{z} < 0$, Q2={ $(x,z): x > 0,$ $(z+\frac{|A|}{2})^2 + x - \frac{A^2}{4} > 0$ } on which $\dot{x} > 0, \dot{z} > 0$, Q3={ $(x,z): x > 0, (z+\frac{|A|}{2})^2 + x - \frac{A^2}{4} < 0$ } on which $\dot{x} < 0, \dot{z} > 0$, Q4={ $(x,z): x < 0, z < 0, (z+\frac{|A|}{2})^2 + x - \frac{A^2}{4} > 0$ } on which $\dot{x} > 0, \dot{z} < 0$, and Q5={ $(x,z): x < 0,$ $(z+\frac{|A|}{2})^2 + x - \frac{A^2}{4} < 0$ } on which $\dot{x} < 0, \dot{z} < 0$ Thus in each of the connected sets P1, P2, L1(1), L1(2), L1(3), L2(1), L2(2), L2(3),Q1, Q2, Q3, Q4 and Q5, as shown in Fig.3.2.1(c), the solutions are monotone .

From the monotonity of the solutions in each of the above connected sets, the possible paths of the solutions starting from each of the sets are: ***1***. The solutions start from any point on L1(1) will get to Q5. ***2***. The solutions start from any point on L1(2) will get to Q2 when z>-A/2 or Q3 when z<-A/2. ***3***. . The solutions start from any point on L1(3) will get to Q4. ***4***. . The solutions start from any point on L2(1) will get to Q2. ***5***.. The solutions start from any point on L2(2) will get to Q5. ***6***.. The solutions start from any point on L2(3) will get to Q2. ***7***. The solutions start from any point in Q1 will reach L1(1) or L2(1) or approach to P2. ***8***. The solutions start from any point in Q2 will reach

L1(2) at some later time or approach positive infinity in both x and y directions as $t \to \infty$. ***9***. The solutions start from any point in Q3 will reach L1(2) or L2(2) at some later time or approach to P2. ***10***.. The solutions start from any point in Q4 will reach L2(3) at some later time. ***11***. The solutions start from any point in Q5 will remain in Q5 as $t \to \infty$ or reach L1(3) at some later time. The diagram of the above possible paths is shown in Fig3.2.1(d). Except the loops all the solutions are eventuallly monotone. From Fig.3.2.1 (d) we can see that there are two possible loops in the diagram. The loop (Q3, L1(2), Q2) is not possible because all the solutions from Q3 through L1(2) to Q2 will go to infinity.

For the loop (Q5, L1(3), Q4, L2(3), Q2, L1(2), Q3, L2(2)), it is helpful to calculate the eigenvalues of the linearized system at the fixed point P2:

The linearization of (3.2.4) at P2 is $\dot{x} = x + |A|z,\ \dot{z} = x$. The eigenvalues for this linear system are $\lambda_1 = \dfrac{1+\sqrt{1+4|A|}}{2} > 0,\ \lambda_2 = \dfrac{1-\sqrt{1+4|A|}}{2} < 0$, thus P2 is a saddle of the linearized system.

From theorem3.1.5 and the monotonicity of the solutions in Q3, there exists a solution in Q3 that tend to P2 as $t \to \infty$ and from corollary3.1.7 this solution come from a point P3 on the boundary L1(2) of Q3 where -A/2<z<0. From P3 five cases could happen for this solution as t goes backward.

i) the solution either tend to P1 as $t \to -\infty$ from Q2 , as shown in Fig. 3.2.2(a) or

ii) goes back to a point P4 on L1(3) at some earlier time and then it could:

(a) come from a point P5 on L1(1) shown in Fig. 3.2.2(b) or

(b) tend to P2 as $t \to -\infty$ shown in Fig. 3.2.2(c) or

(c) come from a point P5 on L2(2) shown in Fig. 3.2.2(d) or

(d) or tend to P1 as $t \to -\infty$ in Q5 shown in Fig. 3.2.2(e).

In case (i) all solutions either tend to P1 or P2, or goes to infinity monotonically in regions Q3 or Q2 . In case(ii-a) this solution and the curve segment P5-P2 form a bounded region, all the solutions in this region remain in this region as $t \to \infty$. From poincaré-Bendixon theorem, there is no chaotic solutions in this region. And other solutions go to infinity monotonically in regions Q3 or Q2. In case (ii-b) the solution is a homoclinic orbit. This orbit plus P2 is a closed curve. Inside the curve is a bounded region. The solutions in this region remain in this region as $t \to \infty$. From the Poincaré-Bendixon theorem there is no chaotic solutions in this region. Similarly other solutions go to infinity monotonically in Q2 or Q3. In case (ii-c) except the solutions that tend to the fixed points or are fixed points themselves, all other solutions go to infinity. Similar situations happen to csae (ii-d).

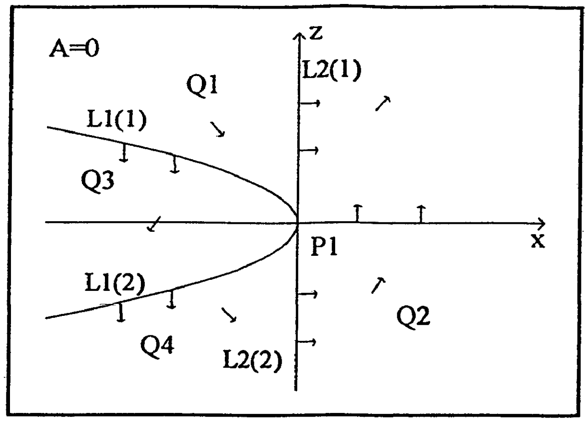


Fig.3.2.1(a) Directions of the vector field in different regions in the phase plane.

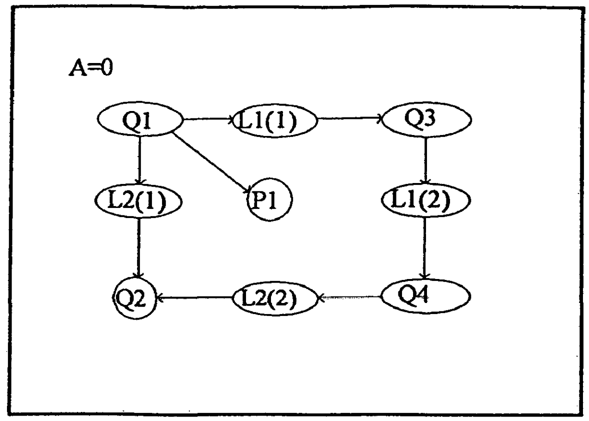


Fig.3.2.1(b) Possible paths of all the solutions of the system.

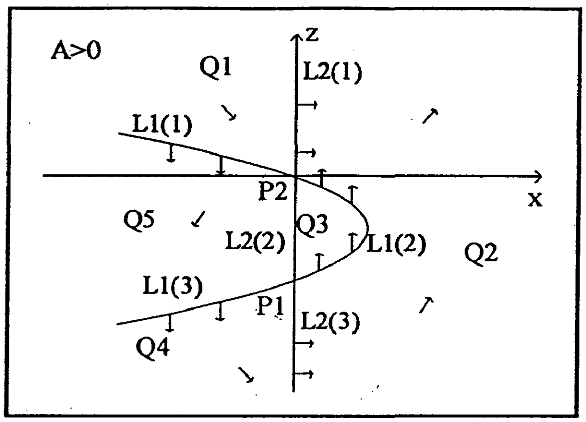


Fig.3.2.1(c) Directions of the vector field in different regions in the phase plane.

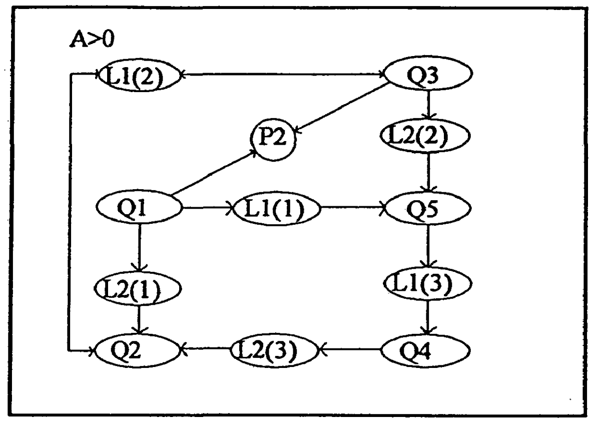


Fig.3.2.1(d) Possible paths of all the solutions of the system.

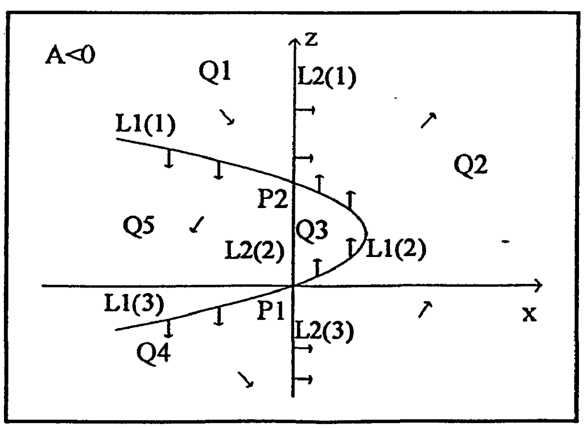


Fig.3.2.1(e) Directions of the vector field in different regions in the phase plane.

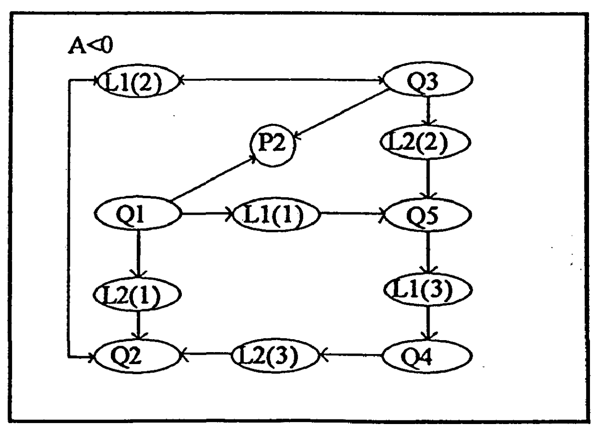


Fig.3.2.1(f) Possible paths of all the solutions of the system.

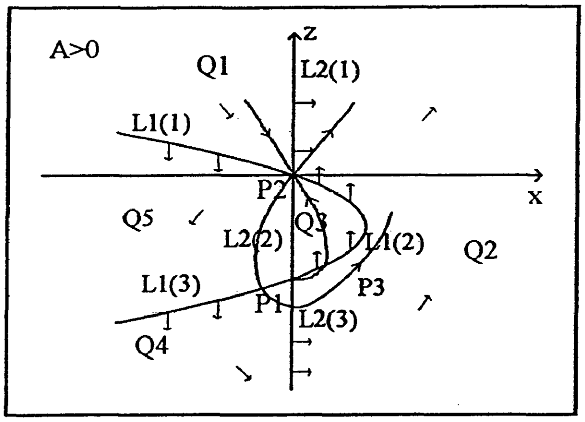


Fig.3.2.2(a)

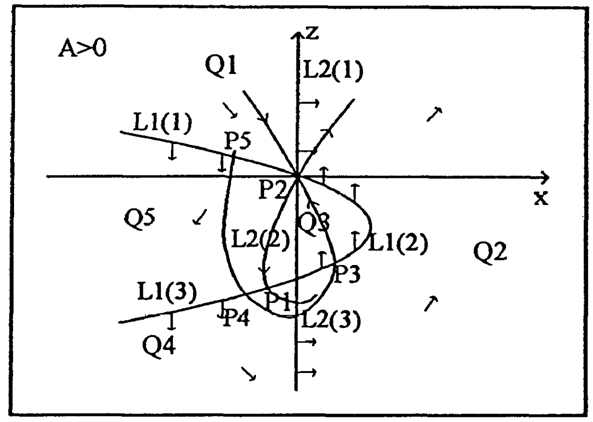


Fig.3.2.2(b)

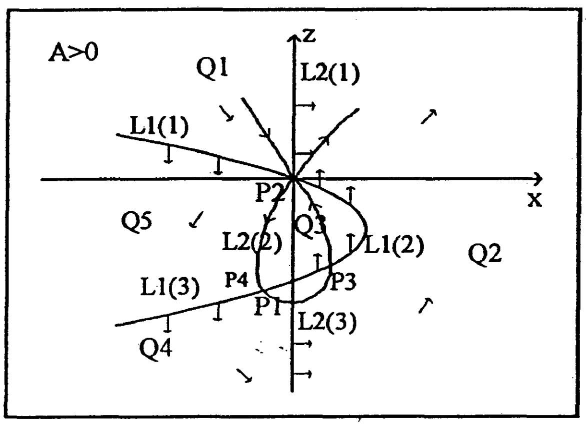


Fig.3.2.2(c) Homoclinic orbit.

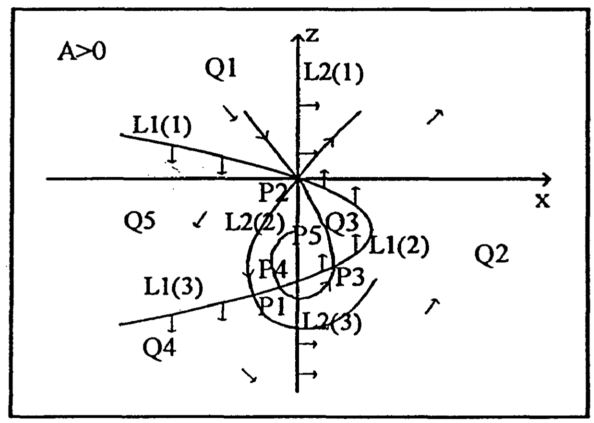


Fig.3.2.2(d)

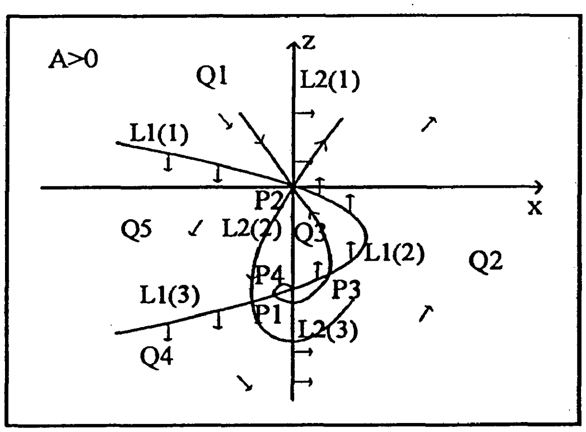


Fig.3.2.2(e)

Fig. 3.2.2 (a)--(e) Possible orbits in the eigenvector directions of saddle P2.

(c) If A<0, then the system is

$$\begin{cases} \dot{x} = (z - \frac{|A|}{2})^2 + x - \frac{A^2}{4} \\ \dot{z} = x \end{cases} \tag{3.2.5}$$

The situation is similar to that of (b). The monotone regions and the monotone diagram are shown in Fig. 3.2.1(e) and Fig.3.2.1(f).

When b<0, it can be resolved similarly. Thus there is no chaos in this system. The situations in systems(9), (19), and (97) are similar to that of this system.

2. Consider system(94):

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cxy \\ \dot{z} = rx \end{cases}$$

There is no chaotic solution in this equation

Proof: The system can be transformed to $\dot{x} = y + x$, $\dot{y} = xy$, $\dot{z} = x$ by the scalar transformation T={ $\alpha = \frac{a}{c}$, $\beta = \frac{a^2}{bc}$, $\gamma = \frac{r}{c}$, $\delta = \frac{1}{a}$} if a>0.

$\dot{y} = xy, \dot{z} = x \Rightarrow z = \ln y + C$, Then we consider the second order system $\dot{x} = x + y$, $\dot{y} = xy$.

The horizontal isocline $L1 = \{(x,y): x + y = 0\}$ and the vertical isoclines L2={(x, y): $x = 0\}$ and $L3 = \{(x, y): y = 0\}$ intersect at $P1 = L1 \cap (L2 \cup L3) = \{(x,y): x = 0, y = 0\}$ which is the only fixed point of this system. Simillar to the procecure of system (7). On $L1(1) = \{(x,y): x + y = 0, y > 0\}$ the monotonicity is $\dot{x}$=0, $\dot{y}$<0; On L1(2)={(x,y): $x + y = 0, y < 0\}$ the monotonicity is $\dot{x}$=0, $\dot{y}$<0; $L2(1) = \{(x,y): x = 0, y<0\}: \dot{x} < 0$, $\dot{y} = 0$; $L2(2) = \{(x,y): x = 0, y > 0\}: \dot{x}>0$, $\dot{y}$=0; $L3(1) = \{(x,y): y = 0, x < 0\}$: $\dot{x}$<0, $\dot{y}$=0; L3(2)= $\{(x,y): y = 0, x > 0\}$: $\dot{x}$>0, $\dot{y}$=0;. The complement set of the isoclines in the phase plane are six disjoint regions each of which are connected and satisfy Q1={ (x,y): $x < 0, y > 0, x + y < 0$ } on which $\dot{x} < 0$, $\dot{y} < 0$, Q2={ $(x,y): x < 0, y > 0, x + y > 0$ }on which $\dot{x} > 0$, $\dot{y} < 0$, Q3={ $(x,y): x > 0, y > 0$ } on which $\dot{x} > 0$, $\dot{y} > 0$, Q4={ $(x,y): x < 0, y < 0$ } on which $\dot{x} < 0$, $\dot{y} > 0$, Q5={(x,y): $x > 0, y < 0, x + y > 0$} on which $\dot{x} < 0, \dot{y} < 0$ and Q6={ $(x,y): x > 0, y > 0. x + y > 0$ } on which $\dot{x} > 0$, $\dot{y} < 0$ Thus in each of the connected sets P1, L1(1), L1(2), L2(1), L2(2), L3(1), L3(2), Q1, Q2, Q3, Q4, Q5, Q6 the solutions are monotone, as shown in Fig.3.2.3(a).

From the monotonity of the solutions in each of the above connected sets, the possible paths of the solutions starting from each of the sets are: ***1***. The solutions start on L1(1) will get to Q1. *2*. The solutions start on L1(2) will get to Q5. *3*. The solutions start on L2(1) will get to Q4. *4*. The solutions start on L2(2) will get to Q3. *5*. The solutions start on L3(1) will remain in L3(1), *6*. The solutions start on L3(2) will remain in L3(2). *7*. The solutions start in Q1 remain in Q1 as $t \to \infty$. *8*. The solutions start in Q2 will approach P1 as $t \to \infty$ or reach L1(1) or L2(2) at some later time. *9* .The solutions start in Q3 will remain in Q3 as $t \to \infty$ *.10.* The solutions start in Q4 will remain in Q4 as $t \to \infty$.*11* . The solution start from Q5 will reach L2(1) at some later time. *12* . The solutions start from Q6 will remain in Q6 as $t \to \infty$ or reach L1(2) at some later time. The diagram of the above possible paths is shown in Fig3.2.3(b). From the diagram we can see that there is no loop in the diagram which means that all solutions are eventually monotone. The situation when a>0 can analyzed similarly. Thus there is no chaos in this system.

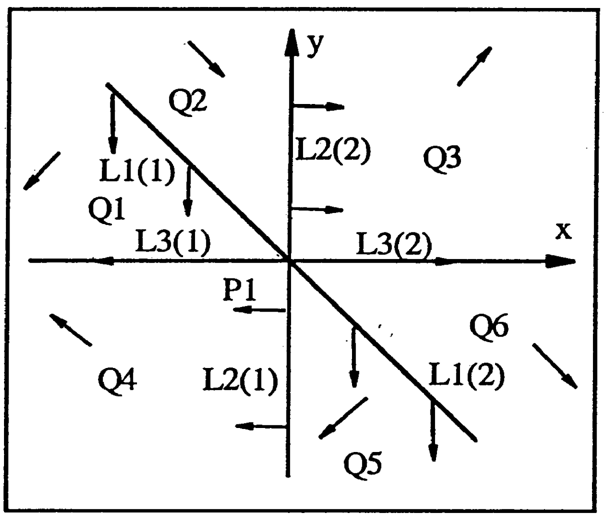


Fig.3.2.3(a) Directions of the vector field in different regions in the phase plane

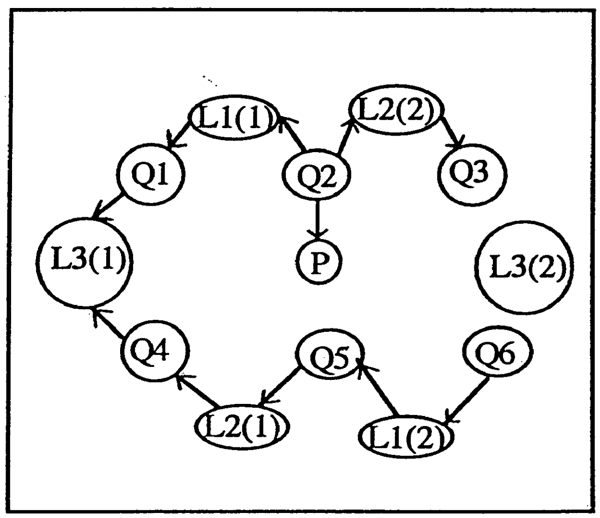


Fig.3.2.3(b) Possible paths of all the solutions of the system

3.3 Proof of Some 2nd Order Monotonic Systems

There are eight systems which are eventually monotone in the phase plane among the 138 patterns. The analysis is based on the fact that for any differentiable function $f(x)$, $x \in [a,b] \subset R$, if $f'(x) > 0$, then $f(x)$ increases; if $f'(x) = 0$, then $f(x)$=constant; if $f'(x) < 0$, then $f(x)$ decreases.

Lemme3.3.1: For the 2nd order system: $\dot{x} - x = y^2$, $\dot{y} = x$, $\exists t_e$ such that , $x(t), y(t)$ incerase as $t > t_e$and tend to infinite or a constant C as $t \to \infty$

Proof: $\dot{x} - x = y^2 \Rightarrow \dfrac{d}{dt}(xe^{-t}) = y^2$,

case1. suppose $\exists t_1,\ y(t)\equiv 0$, if $t\in[t_1,+\infty)$ then
$\frac{d}{dt}(xe^{-t})=0,\ x(t)e^{-t}=C,\Rightarrow x(t)=Ce^{t}$, C is a constant, thus $x(t)\to\infty$, as $t\to\infty$.

case2. suppose $\neg(y(t)\equiv 0)$ i.e. $\forall\tau>0,\exists t>\tau,\ y(t)\neq 0,\ as\ t\in(\tau_a,\tau_b),\ \frac{d}{dt}(xe^{-t})=y^2\geq 0$, thus $x(t)e^{-t}$ increases, and $x(t)\to\infty$, as $t\to\infty$. thus $\exists t_2$ such that $\dot{y}(t)=x(t)>0$, then $y(t)$ increases as $t>t_2$ thus $y(t)\to\infty$ or a constant C, as $t\to\infty$. Take $t_e=\max\{t_1,t_2\}$.

1. Consider system (22):

$$\begin{cases}\dot{x}=ay^2+bx\\ \dot{y}=cx\\ \dot{z}=ry\end{cases}$$

show that x(t), y(t) and z(t) go to infinity or some constants respectively as $t\to\infty$.

Proof. the scalar transformation: $T=\{\alpha=\frac{b^3}{ac^2},\ \beta=\frac{b^2}{ac},\ \gamma=\frac{br}{ac},\ \delta=\frac{1}{b}\}$ b>0 transforms (22) to $\dot{x}=y^2+x,\dot{y}=x,\dot{z}=y$. Its equivalent scalar equation is: $\ddot{y}-\dot{y}-y^2=0$.
By Lemma3.3.1. x(t) and y(t) go to infinity or constants as $t\to\infty$

case1. $y(t)\to\infty$, as $t\to\infty$. then $\exists t_1$ such that $\dot{z}(t)=y(t)>0$ as $t\in[t_1,\infty)$ z(t) increases and $z(t)\to\infty$ or C, as $t\to\infty$.

case 2. $y(t)\to C$, as $t\to\infty$. if C>0 then $\exists t_2$ such that 0<y(t)<C as $t>t_2$, and thus $z(t)$ tend to positive infinity or constant C_1, as $t\to\infty$. if C<0 then $\exists t_3$ such that y(t)<0 as If $C\leq 0,\ \exists t_3>0,\ y(t)<0$, and increase, as $t>t_3$, $\dot{z}(t)=y(t)<0$.thus $z(t)$ tend to negative infinity or constant C_2.

System (14) and (27) are completely the same as system (22), refer to the Table of appendix A. For system (14), scalar transforation: $T=\{\frac{b^3}{ac^2},\ \frac{b^2}{ac},\ \frac{b^2r}{ac^2},\ \frac{1}{b}\}$ when b>0, transforms (14) to $\dot{x}=y^2+x,\ \dot{y}=x,\ \dot{z}=x$, thus $z(t)=y(t)+A$,. For system (27), the scalar transforation: T=$\{\alpha=\frac{b^3}{ac^2},\ \beta=\frac{b^2}{ac},\ \gamma=\text{arbitrary},\ \delta=\frac{1}{b}\}$ when b>0 transforms (27) to $\dot{x}=y^2+x,\ \dot{y}=x,\ \dot{z}=\sigma z$, thus $z(t)=Ae^{\sigma t}$. The other case b<0 for the systems above can be resolved by the method introduced in chapter 4.

2. Consider system (85):

$$\begin{cases}\dot{x}=ax+by\\ \dot{y}=cx^2\\ \dot{z}=rx\end{cases}$$

show that x(t), y(t) and z(t) go to infinity or certain constants respectively as $t \to \infty$.

Proof: The scalar transformation: $T = \{\alpha = \frac{a^2}{bc}, \beta = \frac{a^3}{bc^2}, \gamma = \frac{a^2 r}{b^2 c}, \delta = \frac{1}{a}\}$ a>0 transforms (85) to $\dot{x} = x + y,\ \dot{y} = x^2,\ \dot{z} = x$. Its equivalent scalar equation is: $\ddot{x} - \dot{x} - x^2 = 0$, which is same as that in Lemma 3.3.1, thus $x(t) \to \infty$, or a constant as $t \to \infty$.

For $\dot{y} = x^2$:

case1(a). $x(t) \to \infty$, as $t \to \infty$. $\exists t_1$ such that $x(t) > 0$ as $t > t_1$, thus $\dot{y}(t) = x^2(t) > 0$, and thus $y(t) \to \infty$ or constant C_1, as $t \to \infty$.

case1(b). $x(t) \to C$, as $t \to \infty$. If C>0, $\exists t_2$ such that 0<x(t)<C as $t > t_2$ thus $\dot{y}(t) = x^2(t) > 0$, $y(t) \to \infty$ or a constant C_2 as $t \to \infty$ If C≤0, then $\exists t_3$ such that $x(t) < C$ as $t > t_3$, $\dot{y}(t) = x^2(t) > 0$, thus $y(t) \to \infty$ or constant C_3, as $t \to \infty$.

For $\dot{z}(t) = x(t)$:

case2(a). $x(t) \to \infty$, as $t \to \infty$. Similar to case1a, $z(t) \to \infty$ or constant C_1, as $t \to \infty$.

case2(b). $x(t) \to C$, as $t \to \infty$.If C>0, Similar to case1b for C>0: $z(t) \to \infty$ or constant C_2, as $t \to \infty$.If C≤0, then $\exists t_3$ such that x(t)<C as $t > t_3$, $\dot{z}(t) = x(t)<0$, thus $z(t) \to -\infty$or constant C_3, as $t \to \infty$.

Systems(103) and (121) are completely the same as system (85), refer to the Table of appendix A. For system (103), the scalar transforation: $T = \{\frac{a^2}{bc}, \frac{a^3}{bc^2}, \frac{a^2 r}{b^2 c}, \frac{1}{a}\}$, transforms (103) to $\dot{x} = x + y,\ \dot{y} = x^2,\ \dot{z} = y$, Like the proof of system (85), $x(t)$, $y(t)$ and $z(t)$ go to infinity or certain constant C, as $t \to \infty$.

For system (121), the scalar transformation:T=$\{\alpha = \frac{a^2}{bc},\ \beta = \frac{a^3}{cb^2},\ \gamma,\ \delta = \frac{1}{a}\},\ \sigma = \frac{r}{a}$, transforms (121) to $\dot{x} = x + y,\ \dot{y} = x^2,\ \dot{z} = \sigma z$, thus $z(t) = Ae^{\sigma t}$.Refer to system (22).

Chapter 4. Global Analysis in Higher Dimensional Phase Space

The idea of qualitatively describing the solutions of differential equations that can not be solved analytically first came from Henri Poincaré. Poincaré and Bendixon studied two dimensional autonomous systems deeply around 1900 and concluded the Poincaré-Bendixon theorem for the qualitative behaviour of solutions in a bounded region in which there are a finite number of fixed points. Thus the labyrinth (complicated) cases can occur only when there are an infinite number of disjoint fixed point sets in a bounded region or there exist unbounded solutions. The two dimensional autonomous cases are much simpler than higher dimentional systems because each solution is a boundary of other solutions and the solutions can never reach this boundary since solutions never intersect in an autonomous system. However for three or higher dimensional systems the projection of the solutions onto a plane can intersect in any way. In this chapter a theory of higher dimensional phase space is developed and some examples that applying this theory are given.

4.1 The Main Results of the Theory

Lemma4.1.1 Given a system(3.1.1), the monotonicity of a solution at **x** must be in one of $N = 3^n = \sum_{k=0}^{n} C_n^k 2^{n-k}$ ways, where $N_k = C_n^k 2^{n-k}$, k=1,...,n is the number of monotonicities where there are exactly k dimensions that satisfy $\dot{x}_{i_1} = ... = \dot{x}_{i_k} = 0$, $1 \le i_1 \le ... \le i_k \le$ n.

Proof: The monotonicity of a solution at point **x** is determined by the signs of $\dot{x}_1, ..., \dot{x}_n$. For each dimension x_i, the sign of $\dot{x}_i$ could be positive, zero or negative. Suppose there are k zeros among $\dot{x}_i$, i=0, ..., n. Then there are C_n^k ways to choose the k $\dot{x}_{i_p} = 0, p = 1,...,k$ from $\dot{x}_i$, $i = 1,...,n$. For each of the C_n^k ways, The possible signs for the rest n-k dimensions is 2^{n-k}. Thus all the possible monotonicities when there are exactly k $\dot{x}_{i_p} = 0, p = 1,...,k$ is $N_k = C_n^k 2^{n-k}$, k=1,...,n. When k goes from 0 to n, all the possible monotonicities are covered By Newton's binomial formula $N = 3^n = \sum_{k=0}^{n} C_n^k 2^{n-k}$.

Definition 4.1.2 Given a system (3.1.1), if the phase space can be cut in to a finite number of point sets $\Omega_i, i = 1,...,m$ that satisfy the following conditions:

(i) In each of the point sets, the solutions are monotone in one of the 3^n ways.

(ii) $\Omega_i \cap \Omega_j = \varnothing$, $i \ne j$

(iii) $\Omega_1 \cup ... \cup \Omega_m = R^n$ (4.1.1)

(iv) $\Omega_i, i = 0 \ldots m$ are connected sets

$\Omega_i, i = 0 \ldots m$, is called *a monotone region* of system (3.1.1), the set W={$\Omega_1, \ldots, \Omega_m$} is called *the monotone region set* of system (3.1.1), the conditions (4.1.1) are called *monotone conditions* for system (3.1.1).

Definition4.1.3 The monotonicity of $\Omega_i, i = 0 \ldots m$ is denoted by $\Omega_i \langle \sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{i,n} \rangle$ $i = 1, \ldots, m$ where

$$\sigma_{i,k} = \begin{cases} +1 \ \text{ or } \ + & \text{if } \dot{x}_k > 0 \\ -1 \ \text{ or } \ - & \text{if } \dot{x}_k < 0 \\ 0 & \text{if } \dot{x}_k = 0 \end{cases} \tag{4.1.2}$$

If there is no subscript for Ω or there are more than one Ω, then the monotonicity of the Ω(s) at dimension x_k is denoted by $\sigma_{,k}$.

Definition4.1.4 A diagram with $\Omega_1, \ldots, \Omega_m$ of system(3.1.1) as nodes and an arrow from Ω_i to Ω_j if it is possible for a solution starting from Ω_i to approach or reach Ω_j by the monotonicity of the solution in Ω_i can be obtained. This diagram is called *the monotone diagram* of system (3.1.1).

Definition4.1.5 A loop $\Gamma = \langle \Omega_{i_1} > \Omega_{i_2} > \ldots > \Omega_{i_p} > \Omega_{i_1} \rangle$ is a closed path in the monotone diagram, where $\Omega_{i_q} > \Omega_{i_{q+1}}$ means that there is an arrow from Ω_{i_q} to $\Omega_{i_{q+1}}$. And we denote the point set of the union of $\Omega_{i_1}, \ldots, \Omega_{i_p}$ as $\Gamma^s = \Omega_{i_1} \cup \ldots \cup \Omega_{i_p}$.

Definition4.1.6 A loop Γ_i of system(3.1.1) is called an isolated loop if for any loop Γ_j, $i \neq j$ of system (3.1.1) $\Gamma_i^s \cap \Gamma_j^s = \varnothing$; Two loops Γ_i, Γ_j where $i \neq j$ are coupled if $\Gamma_i^s \cap \Gamma_j^s \neq \varnothing$; $\Gamma_i$and Γ_j, $i \neq j$ are truely coupled if there exist a solution **x**=**x**(t, 0, x_0) $x_0 \in R^n$ of system (3.1.1) for any T>0, there exist t_1, t_2>T such that **x**(t_1,0,x_0)$\in \Gamma_i^s$ **x**(t_2,0,x_0)$\notin \Gamma_i^s$ and **x**(t_2,0,x_0)$\in \Gamma_j^s$; otherwise they are not truely coupled.

Theorem4.1.7 For the system (3.1.1) and its monotone region set W, if there is no loop in the system, then there is no chaos in the system.

Proof: **f(x)** is continuous and the number of monotone sets in W is finite. Consider a solution starting from Ω_i, it either leaves this region at some later time or remain in this region as $t \to +\infty$. This solution is monotone in the latter case. If the solution leaves this region and passed through l regions $\Omega_{j_1}, \ldots, \Omega_{j_l}$, since there is no loop in the system thus thus $\Omega_{j_1}, \ldots, \Omega_{j_l}$ are different. l has to be less than the total number of monotone regions os system (3.1.1) and it has to remain in Ω_{j_l}, and thus this solution is eventually monotone. Because the solution is any solution and the monotone region is any monotone region. thus any solution in this system is eventually monotone.

Lemma 4.1.8 For a loop in system(3.1.1), if $\sigma_{j_1,k} = +1(\text{or} -1)$ in the monotone region Ω_{j_1} and $\sigma_{j_2,k} = -1(\text{or} +1)$ in the nomotone region Ω_{j_2}, Then there must be a third monotone region Ω_{j_3} in which $\sigma_{j_3,k} = 0$ in the loop.

Proof: Since the solutions in the loop that go from Ω_{j_1} to Ω_{j_2} $\dot{x}_k$ has to change from positive (negative) to negative(positive) and f(x) is a continuous function, the solution has to pass to another monotone region Ω_{j_3} in the loop where $\dot{x}_k$=0 or $\sigma_{j_3,k} = 0$.

Theorem4.1.9 For the system (3.1.1), if Ω is a monotone region which connect to every monotone region in a loop Γ, then the relations of the monotonicities of x_k (k=1,...,n) in Γ^s and the monotonicity of x_k in Ω are:

(i) if $\sigma_{,k} = +1$ & -1 & 0 in Γ^s ,then $\sigma_{,k}$=0 in Ω

(ii) if $\sigma_{,k} = +1$ & 0 in Γ^s ,then $\sigma_{,k}$=+1 or 0 in Ω

(iii) if $\sigma_{,k} = -1$ & 0 in Γ^s ,then $\sigma_{,k}=-1$ or 0 in Ω

(iv) if $\sigma_{,k} = +1$ in Γ^s ,then $\sigma_{,k}$=+1 or 0 in Ω

(v) if $\sigma_{,k} = -1$ in Γ^s ,then $\sigma_{,k}=-1$ or 0 in Ω

(vi) if $\sigma_{,k} = 0$ in Γ^s ,then $\sigma_{,k}$=+1 or -1 or 0 in Ω

proof: (i) Because **f(x)** is continuous and Ω is connected with any monotone regions in the loop Γ, $\sigma_{,k} \neq -1\,(\text{or} + 1)$ in Ω. Thus the only possibilyty for $\sigma_{,k}$=0.

The proof of (ii)-(vi) is similar to this proof.

Corollary 4.1.10 For a system (3.1.1), if Ω is a monotone region which connect to every monotone region in the loop $\Gamma = \langle \Omega_{i_1} > \Omega_{i_2} > \ldots > \Omega_{i_p} > \Omega_{i_1} \rangle$ if $\sigma_{,k}$=+1 & -1, k=1,..., n, in the loop, then Ω is a fixed point set.

4.2 General Steps of Analyzing Higher Dimensional Autonomous Systems

For a given system (3.1.1) there is a general procedure to find out all of the monotone regions that satisfy the monotone conditions (4.1.1). Let $G_1=\{(\mathbf{x}): f_1(\mathbf{x}) = 0\}$,$G_2=\{(\mathbf{x}): f_2(\mathbf{x}) = 0\}$, ... , $G_n=\{(\mathbf{x}): f_n(\mathbf{x}) = 0\}$.

Step1 Find the complement set of $G_1 \cup G_2 \cup \ldots \cup G_n$: $\neg(G_1 \cup G_2 \cup \ldots \cup G_n)$ and name each of the connected sets as : $\Omega_1,\ldots,\Omega_{m_1}$ .The solutions in each of the regions $\Omega_1,\ldots,\Omega_{m_1}$ are monotone in one of the 2^n ways.

Step2 For k=1..n$-$1, for each way of choosing k from n find the point sets $(G_{i_1} \cap \ldots \cap G_{i_k}) - (\cup G_l), (l \neq i_j, l = 1,\ldots,n; j = 1..k)$,for each k the solutions have $N_k = C_n^k 2^{n-k}$ ways of monotonicity, and name each connected subset obtained in which the solutions are monotone as $\Omega_{m_1+1,\ldots,}\Omega_{m_2}$.

Step3 Find the fixed point set $G_1 \cap G_2 \cap ... \cap G_n$ and name each of the connected subset as $\Omega_{m_2+1,...,}\Omega_m$.

For a system (3.1.1) where n=3, for convenience we will name the three dimensional connected monotone regions as Vi ; name the two dimensional connected monotone regions as Si(j), where Si is a connected surface that satisfies an algebraic equation, Si(j) is a connected monotone regions on Si.; name one dimensional regions as Li(j), where Li is a connected region that satisfies a certain algebraic equation, Li(j) are connected monotone regions on Li where i,j (<m) are integers.

4.3 Analysis of 3 systems of type VII by Global Analyzing Method in Three Dimensional Phase Space.

1. Consider system(54)

$$\begin{cases} \dot{x} = ay^2 + bx \\ \dot{y} = cz \\ \dot{z} = rx \end{cases}$$

There is no chaos in this system.

proof: System(54) can be reduced to

$$\dot{x} = y^2 - x, \dot{y} = z, \dot{z} = x \tag{4.3.1}$$

by a scalar transformation:

$$T = \{\alpha = -b^5 / (ac^2r^2), \beta = -b^3 / (acr), \gamma = b^4 / (ac^2r), \delta = -1 / b\}. \text{ b<0}$$

So all we need to do is to analyze (4.3.1).

The monotone regions can be obtained as follows:

S1={(x,y,z) | y^2-x=0}, S2={(x,y,z) | z=0}, S3={(x,y,z) | x=0}

L1=S1$\cap$S2={(x,y,z) | z=0,x=y^2}, L2=S1$\cap$S3={(x,y,z) | x=0,y=0} and L3=S2$\cap$S3 ={(x,y,z) | x=0,z=0}, P=S1$\cap$S2$\cap$S3={(x,y,z) | x=0,y=0,z=0} and L1$\cap$L2 =L1$\cap$L3 =L2$\cap$L3=P.

All the monotone regions of the system are listed here:

1. P={(x,y,z) | x=0,y=0,z=0}, $\dot{x}$=0, $\dot{y}$=0 , $\dot{z}$=0 ;
2. L1(1)={(x,y,z) | z=0, y^2-x=0, y<0}, $\dot{x}$=0, $\dot{y}$=0 , $\dot{z}$>0
3. L1(2)={(x,y,z) | z=0, y^2-x=0, y>0}, $\dot{x}$=0, $\dot{y}$=0 , $\dot{z}$>0
4. L2(1)={(x,y,z) | x=0,y=0,z<0}, $\dot{x}$=0, $\dot{y}$<0 , $\dot{z}$=0
5. L2(2)={(x,y,z) | x=0,y=0,z>0}, $\dot{x}$=0, $\dot{y}$>0 , $\dot{z}$=0
6. L3(1)={(x,y,z) | x=0,z=0,y<0}, $\dot{x}$>0, $\dot{y}$=0 , $\dot{z}$=0
7. L3(2)={(x,y,z) | x=0,z=0,y>0}, $\dot{x}$>0, $\dot{y}$=0 , $\dot{z}$=0
8. S1(1)={(x,y,z) | y^2-x=0, y<0,z<0}, $\dot{x}$=0, $\dot{y}$<0 , $\dot{z}$>0

9. $S1(2)=\{(x,y,z) \mid y^2-x=0, y>0, z<0\}, \dot{x}=0, \dot{y}<0, \dot{z}>0$
10. $S1(3)=\{(x,y,z) \mid y^2-x=0, y<0, z>0\}, \dot{x}=0, \dot{y}>0, \dot{z}>0$
11. $S1(4)=\{(x,y,z) \mid y^2-x=0, y>0, z>0\}, \dot{x}=0, \dot{y}>0, \dot{z}>0$
12. $S2(1)=\{(x,y,z) \mid z=0, y^2-x>0, x>0, y<0\}, \dot{x}>0, \dot{y}=0, \dot{z}>0$
13. $S2(2)=\{(x,y,z) \mid z=0, y^2-x<0\}, \dot{x}<0, \dot{y}=0, \dot{z}>0$
14. $S2(3)=\{(x,y,z) \mid z=0, y^2-x>0, x>0, y>0\}, \dot{x}>0, \dot{y}=0, \dot{z}>0$
15. $S2(4)=\{(x,y,z) \mid z=0, x<0\}, \dot{x}>0, \dot{y}=0, \dot{z}<0$
16 $S3(1)=\{(x,y,z) \mid x=0, y<0, z<0\}, \dot{x}>0, \dot{y}<0, \dot{z}=0$
17. $S3(2)=\{(x,y,z) \mid x=0, y>0, z<0\}, \dot{x}>0, \dot{y}<0, \dot{z}=0$
18. $S3(3)=\{(x,y,z) \mid x=0, y<0, z>0\}, \dot{x}>0, \dot{y}>0, \dot{z}=0$
19. $S3(4)=\{(x,y,z) \mid x=0, y>0, z>0\}, \dot{x}>0, \dot{y}>0, \dot{z}=0$
20. $V1=\{(x,y,z) \mid y^2-x>0, z<0, x>0, y<0\}, \dot{x}>0, \dot{y}<0, \dot{z}>0$
21. $V2=\{(x,y,z) \mid y^2-x<0, z<0, x>0\}, \dot{x}<0, \dot{y}<0, \dot{z}>0$
22. $V3=\{(x,y,z) \mid y^2-x>0, z<0, x>0, y>0\}, \dot{x}>0, \dot{y}<0, \dot{z}>0$
23. $V4=\{(x,y,z) \mid y^2-x>0, z>0, x>0, y<0\}, \dot{x}>0, \dot{y}>0, \dot{z}>0$
24. $V5=\{(x,y,z) \mid y^2-x>0, z<0, x>0, y>0\}, \dot{x}<0, \dot{y}>0, \dot{z}>0$
25. $V6=\{(x,y,z) \mid y^2-x>0, z>0, x>0, y>0\}, \dot{x}>0, \dot{y}>0, \dot{z}>0$
26. $V7=\{(x,y,z) \mid z<0, x<0\}, \dot{x}>0, \dot{y}<0, \dot{z}<0$
27. $V8=\{(x,y,z) \mid z>0, x<0,\}, \dot{x}>0, \dot{y}>0, \dot{z}$

The monotone regions of this system is shown in Fig4.3.1(a). To obtain the monotone diagram, for each of the monotone regions, we'll find all the monotone regions that connect to it and then analyze the possibility for the solutions start from the monotone region to reach its nearby monotone regions. First we denote $N[\Omega_i]$ as the set of all the monotone regions that connect with Ω_i where Ω_i, $i=1\ldots, m$ is any monotone region of the system:

1. N[P]={L1(1),...,V8}, all of the rest monotone regions in this system are connected to P, the solution start from P will stay in P, because it is a fixed point;

2. N[L1(1)]={P, S1(1), S1(3), S2(1), S2(2), V1, V2, V4, V5} , the solutions start from L1(1) can only develop along the straight line $\{(x,y,z) \mid x=x_0, y=y_0\}$ which is on S1(1) and S1(3) and because z is strictly incresing on L1(1), thus the solutions start from this region reach S1(3).

3. N[L1(2)]={P, S1(2), S1(4), S2(2), S2(3), V2, V3, V5, V6}, similarly solutions start from this region reach S1(2).

4. N[L2(1)]={P, S1(1), S1(2), S3(1), S3(2), V1, V2, V3, V7},the solutions start from L2(1) can only develop along the straight line {(x,y,z)| x=0, z=z_0} which is on S3(1) and S3(2) and because y is strictly decresing on L2(1), thus the solutions start from this region reach S3(1).

5. N[L2(2)]={P, S1(3), S1(4), S3(3), S3(4), V4, V5, V6, V8}, similarly solutions start from this region reach S3(4).

6. N[L3(1)]={P, S2(1), S2(4), S3(1), S3(3), V1, V4, V7, V8}, the solutions start from L3(1) can only develop along the straight line {y= y_0, z=0} which is on S2(1) and S2(4) and because x is strictly incresing on L3(1), thus the the solutions start from this region reach S2(1).

7. N[L3(2)]={P, S2(3), S2(4), S3(2), S3(4),V3, V6, V7, V8}, the solutions start from L3(2) reach S2(3).

8. N[S1(1))]={P, L1(1), L2(1), V1, V2}, the solutions start from this region develop only on the surface x=x_0, because y is decreasing, thus the solutions start from this region reach V1.

9. N[S1(2)]={P, L1(2), L2(1), V2, V3}, similarly the solution start from this region reach V2.

10. N[S1(3)]={P, L1(1), L2(2), V4, V5}, similarly the solutions start from this region reach V5.

11. N[S1(4)]={P, L1(2), L2(2), V5, V6}, similarly the solutions start from this region reach V6.

12. N[S2(1)]={P, L1(1), L3(1), V1,V4}, the solutions start from this region develop only on the surface y=y_0, because z is increasing the solutions reach V4.

13. N[S2(2)]={P, L1(1), L1(2), V2, V5}, similarly the solution start from this region reach V5.

14. N[S2(3)]={P, L1(2), L3(2), V3, V6}, similarly the solutions start from this region reach V6.

15. N[S2(4)]={P, L3(1), L3(2), V7, V8}, similarly the solutions start from this region reach V7.

16. N[S3(1)]={P, L3(1), L2(1), V1, V7}, the solutions start from this region can only develop on the surface z=z_0 and because $\dot{x}_0>0$ the solutions reach V1.

17. N[S3(2)]={P, L2(1), L3(2), V3, V7}, similarly thc solutions start from this region reach V3.

18. N[S3(3)]={P, L2(2), L3(1),, V4, V8} similarly the solutions start from this region reach V4.

19. N[S3(4)]={P, L2(2), L3(2), V6, V8}, similarly the solutions start from this region reach V6.

20. N[V1]={P, L1(1), L2(1), L3(1), S1(1), S2(1), S3(1)}, the solution start from this region can not reach L2(1) , L3(1),S3(1) or P because x is increasing in this region, they can not reach S1(1) from **8**, the solutions can reach S2(1) because z is increasing in this region, and it is also possible for the solutions start from this region to reach L1(1).

21. N[V2]={P, L1(1), L1(2), L2(1), S2(2), S1(1), S1(2)}, the solutions start from this region can reach any regions connected to it except S1(2).

22. N[V3]={P, L1(2), L2(1), L3(2), S1(2), S2(3), S3(2)}, the solutions start from this region can not reach P, L2(1),L3(2), S3(2) because x is increasing in this region, and they can reach any other regions connected to them.

23. N[V4]={P, L1(1), L2(2), L3(1), S1(3), S2(1), S3(3)}, the solution start from this region can not reach P, L2(2), L3(1), S3(3) because x is increasing in this region, they can not reach L1(1), S2(1) either because z is increasing. V4 can reach S1(3) because x and y are increasing.

24. N[V5]={P, L1(1), L1(2), L2(2), S1(3), S1(4), S2(2)}, the solutions start from this region can not reach P, L1(1), L1(2) S2(2) because z is increasing in this region, they can not reach S1(3) either, they can reach S1(4) and L2(2).

25. N[V6]={P, L1(2), L2(2), L3(2), S1(4), S2(3), S3(4)}, The solutions start from this regionit is not possible to reach P, L2(2),L3(2),S3(4) because x is increasing they cano't reach L1(2),S2(3) because z is increasing and from **11** they can not reach S1(4), thus the solutions in this region will remain in this region.

26. N[V7]={P, L2(1), L3(1), L3(2), S2(4), S3(1), S3(2)}, the soltions start from this region can not reach P, L3(1), L3(2), S2(4) because z is decreasing, they can reach S3(!), S3(2), L2(1).

27. N[V8]={P, L2(2), L3(1), L3(2), S2(4), S3(3), S3(4)}, The solutions start in this region can reach any of the regions that connect to it.

The monotone diagram is shown in Fig4.3.1(b). we can see that there is no loop in this system. thus from theorem 4.1.7, there is no chaos in the system.

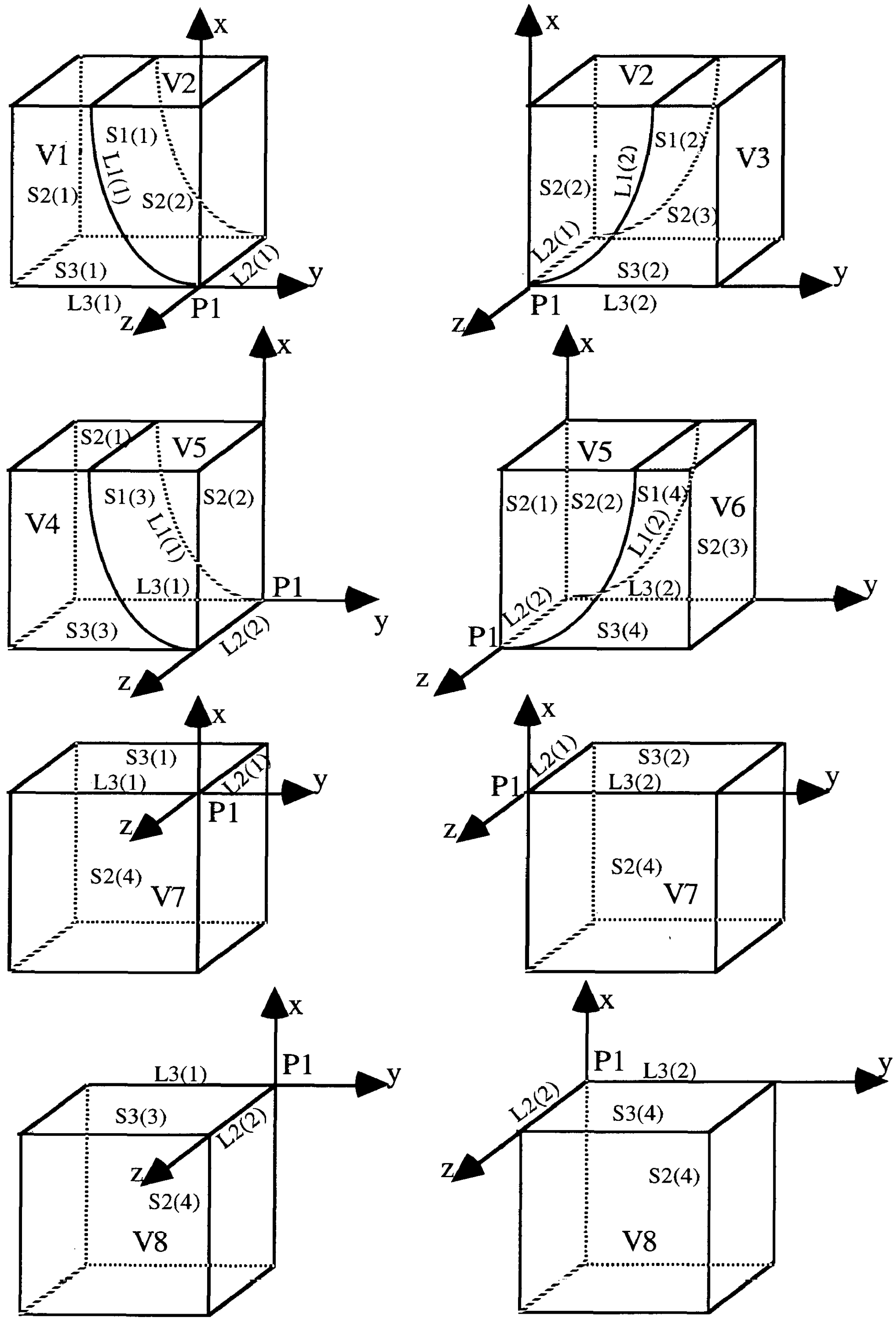


Fig. 4.3.1(a) Monotone regions of system (54)

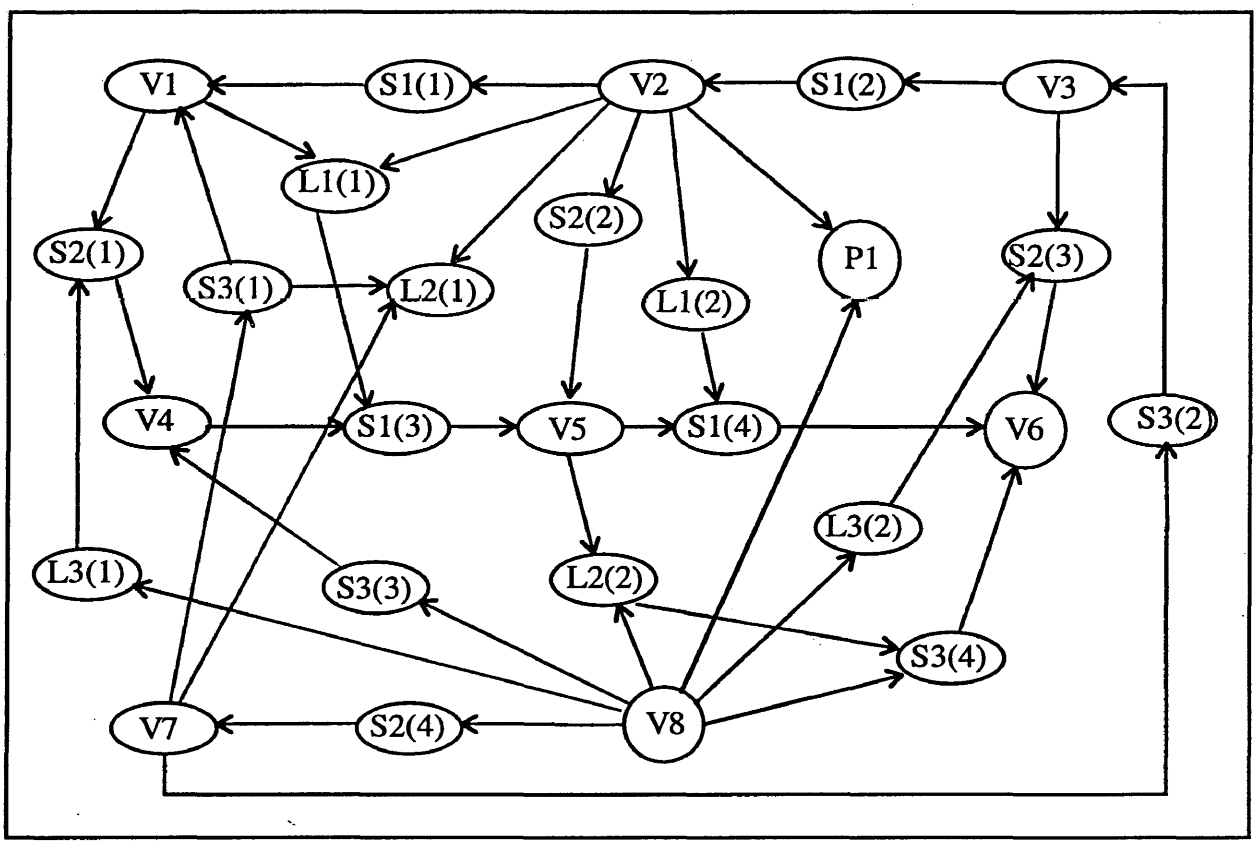


Fig. 4.3.1(b) Monotone diagram of system (54)

The case when b>0 can be proved similarly.

2. Consider system(115)

$$\begin{cases} \dot{x} = ayz \\ \dot{y} = bx + cy \\ \dot{z} = rx \end{cases}$$

There are possible coupled loops in the system.

Proof: System(115) can be reduced to

$$\dot{x} = yz, \dot{y} = x - y, \dot{z} = x \qquad (4.3.2)$$

by a scalar transformation $T = \{\alpha = -c^3/(abr), \beta = c^2/(ar), \gamma = c^2/(ab), \delta = -1/c\}$. c<0

So all we need to do is to analyze (4.3.2).

The monotone regions can be obtained as follows:

S1={(x,y,z) | y=0}, S2={(x,y,z) | z=0}, S3={(x,y,z) | x-y=0}, S4={(x,y,z) | x=0}.

L1=S1 $\cap$ S2={(x,y,z) | y=0,z=0},L2=S1 $\cap$ S3={(x,y,z) | x=0,y=0}=S1 $\cap$ S4, and

L3=S2 $\cap$ S3={(x,y,z) | z=0,x-y=0},L4=S2 $\cap$ S4={(x,y,z) | x=0,z=0}=S3 $\cap$ S4

$P=(S1 \cup S2) \cap S3 \cap S4=\{(x,y,z) \mid x=0,y=0,z=0\}$ and
$L1 \cap L2=L1 \cap L3=L1 \cap L4=L2 \cap L3=L2 \cap L4=L3 \cap L4=P$.

All the monotone regions of the system are listed here:

1. $P=\{(x,y,z) \mid x=0,y=0,z=0\}$ $\dot{x}=0,\ \dot{y}=0,\ \dot{z}=0$
2. $L1(1)=\{(x,y,z) \mid y=0,z=0,x<0\}$, $\dot{x}=0,\ \dot{y}<0,\ \dot{z}<0$
3. $L1(2)=\{(x,y,z) \mid y=0,z=0,x>0\}$, $\dot{x}=0,\ \dot{y}>0,\ \dot{z}>0$
4. $L2(1)=\{(x,y,z) \mid x=0,y=0,z<0\}$, $\dot{x}=0,\ \dot{y}=0,\ \dot{z}=0$
5. $L2(2)=\{(x,y,z) \mid x=0,y=0,z>0\}$, $\dot{x}=0,\ \dot{y}=0,\ \dot{z}=0$
6. $L3(1)=\{(x,y,z) \mid x=y,z=0,x<0\}$, $\dot{x}=0,\ \dot{y}=0,\ \dot{z}<0$
7. $L3(2)=\{(x,y,z) \mid x=y,z=0,x>0\}$, $\dot{x}=0,\ \dot{y}=0,\ \dot{z}>0$
8. $L4(1)=\{(x,y,z) \mid x=0,z=0,y<0\}$, $\dot{x}=0,\ \dot{y}>0,\ \dot{z}=0$
9. $L4(2)=\{(x,y,z) \mid x=0,z=0,y>0\}$, $\dot{x}=0,\ \dot{y}<0,\ \dot{z}=0$
10. $S1(1)=\{(x,y,z) \mid y=0,\ x<0,\ z<0\}$, $\dot{x}=0,\ \dot{y}<0,\ \dot{z}<0$
11. $S1(2)=\{(x,y,z) \mid y=0,\ x>0,\ z<0\}$, $\dot{x}=0,\ \dot{y}>0,\ \dot{z}>0$
12. $S1(3)=\{(x,y,z) \mid y=0,\ x<0,\ z>0\}$, $\dot{x}=0,\ \dot{y}<0,\ \dot{z}<0$
13. $S1(4)=\{(x,y,z) \mid y=0,\ x>0,\ z>0\}$, $\dot{x}=0,\ \dot{y}>0,\ \dot{z}>0$
14. $S2(1)=\{(x,y,z) \mid z=0,\ x<0,\ y<0,x-y>0\}$, $\dot{x}=0,\ \dot{y}>0,\ \dot{z}<0$
15. $S2(2)=\{(x,y,z) \mid z=0,\ x<0,\ y<0,x-y<0\}$, $\dot{x}=0,\ \dot{y}<0,\ \dot{z}<0$
16. $S2(3)=\{(x,y,z) \mid z=0,\ x<0,\ y<0,x-y>0\}$, $\dot{x}=0,\ \dot{y}<0,\ \dot{z}<0$
17. $S2(4)=\{(x,y,z) \mid z=0,\ x>0,\ y<0,x-y>0\}$, $\dot{x}=0,\ \dot{y}>0,\ \dot{z}>0$
18. $S2(5)=\{(x,y,z) \mid z=0,\ x>0,\ y>0,x-y>0\}$, $\dot{x}=0,\ \dot{y}>0,\ \dot{z}>0$
19. $S2(6)=\{(x,y,z) \mid z=0,\ x>0,\ y>0,x-y<0\}$, $\dot{x}=0,\ \dot{y}<0,\ \dot{z}>0$
20. $S3(1)=\{(x,y,z) \mid x-y=0,x<0,y<0,z<0\}$, $\dot{x}>0,\ \dot{y}=0,\ \dot{z}<0$
21. $S3(2)=\{(x,y,z) \mid x-y=0,x>0,y>0,z<0\}$, $\dot{x}<0,\ \dot{y}=0,\ \dot{z}>0$
22. $S3(3)=\{(x,y,z) \mid x-y=0,x<0,y<0,z>0\}$, $\dot{x}<0,\ \dot{y}=0,\ \dot{z}<0$
23. $S3(4)=\{(x,y,z) \mid x-y=0,x>0,y>0,z>0\}$, $\dot{x}>0,\ \dot{y}=0,\ \dot{z}>0$
24. $S4(1)=\{(x,y,z) \mid x=0,\ y<0,z<0\}$, $\dot{x}>0,\ \dot{y}>0,\ \dot{z}=0$
25. $S4(2)=\{(x,y,z) \mid x=0,\ y>0,z<0\}$, $\dot{x}<0,\ \dot{y}<0,\ \dot{z}=0$
26. $S4(3)=\{(x,y,z) \mid x=0,\ y<0,z>0\}$, $\dot{x}<0,\ \dot{y}>0,\ \dot{z}=0$
27. $S4(4)=\{(x,y,z) \mid x=0,\ y>0,z>0\}$, $\dot{x}>0,\ \dot{y}<0,\ \dot{z}=0$
28. $V1=\{(x,y,z) \mid x<0,\ y<0,\ z>0,\ x-y>0\}$, $\dot{x}<0,\ \dot{y}>0,\ \dot{z}<0$
29. $V2=\{(x,y,z) \mid x<0,\ y<0,\ z>0,\ x-y<0\}$, $\dot{x}<0,\ \dot{y}<0,\ \dot{z}<0$
30. $V3=\{(x,y,z) \mid x<0,\ y>0,\ z>0\}$, $\dot{x}>0,\ \dot{y}<0,\ \dot{z}<0$
31. $V4=\{(x,y,z) \mid x>0,\ y<0,\ z>0\}$, $\dot{x}<0,\ \dot{y}>0,\ \dot{z}>0$

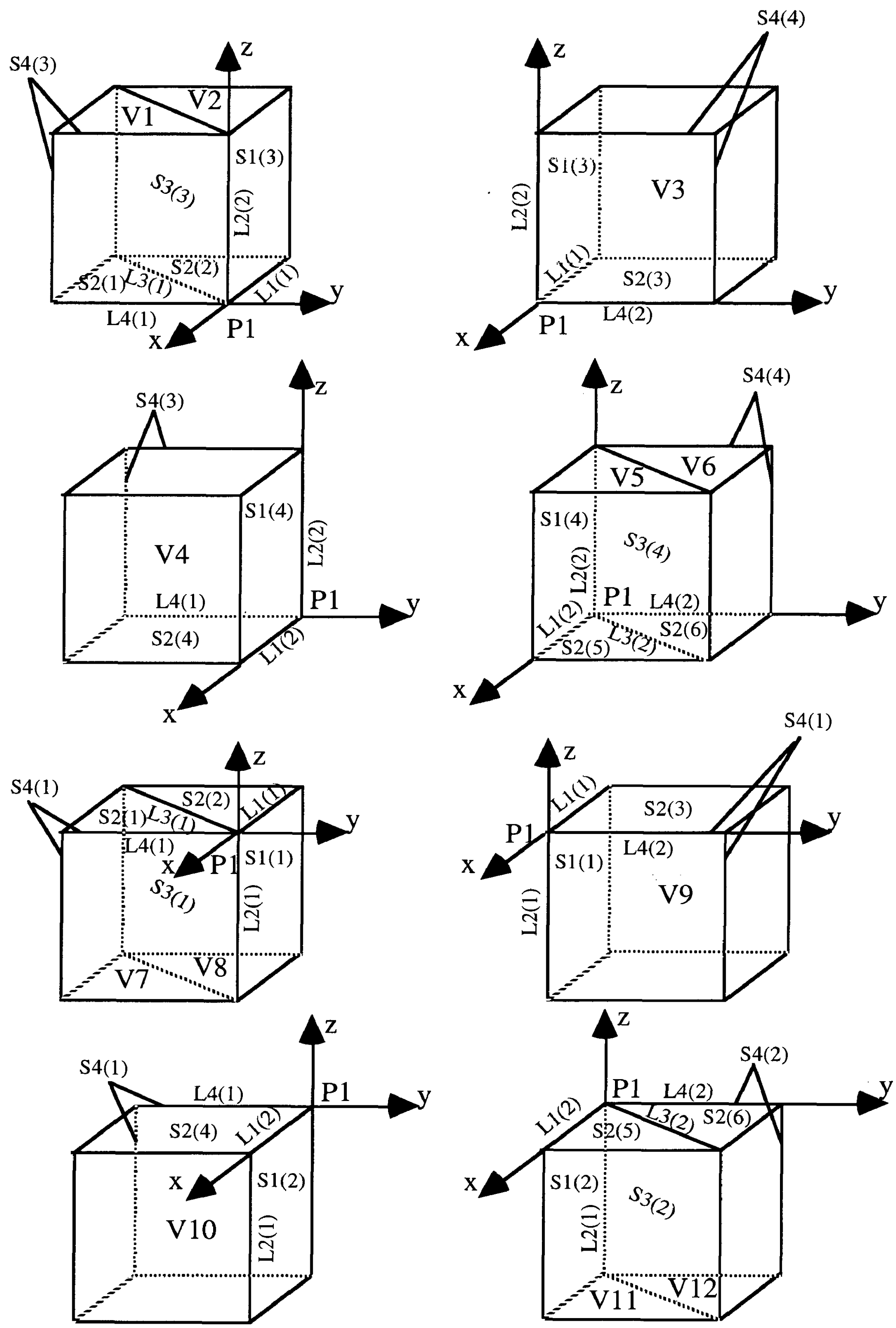


Fig. 4.3.2(a) Monotone regions of system (115)

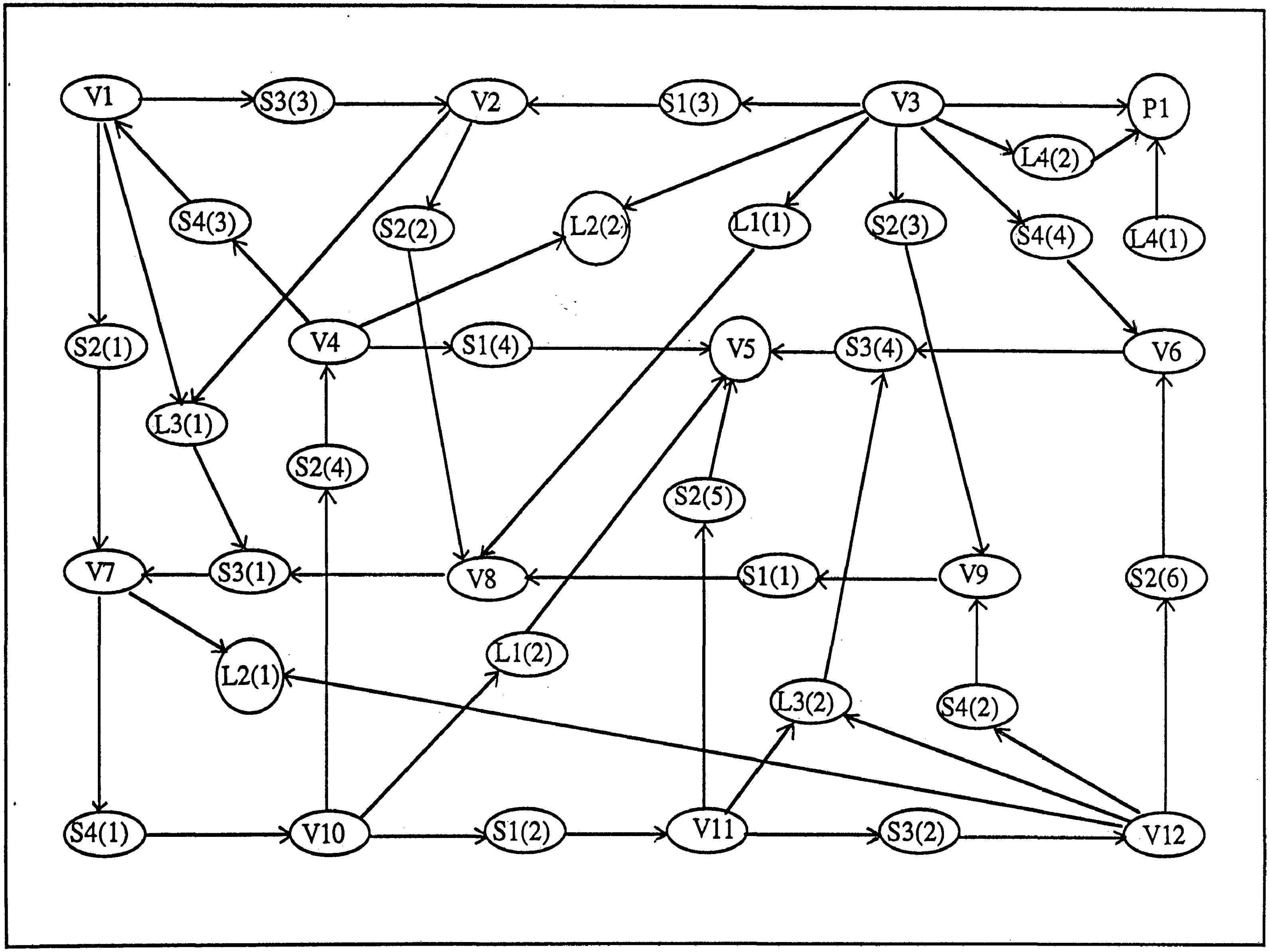


Fig. 4.3.2(b) Monotone diagram of system (115)

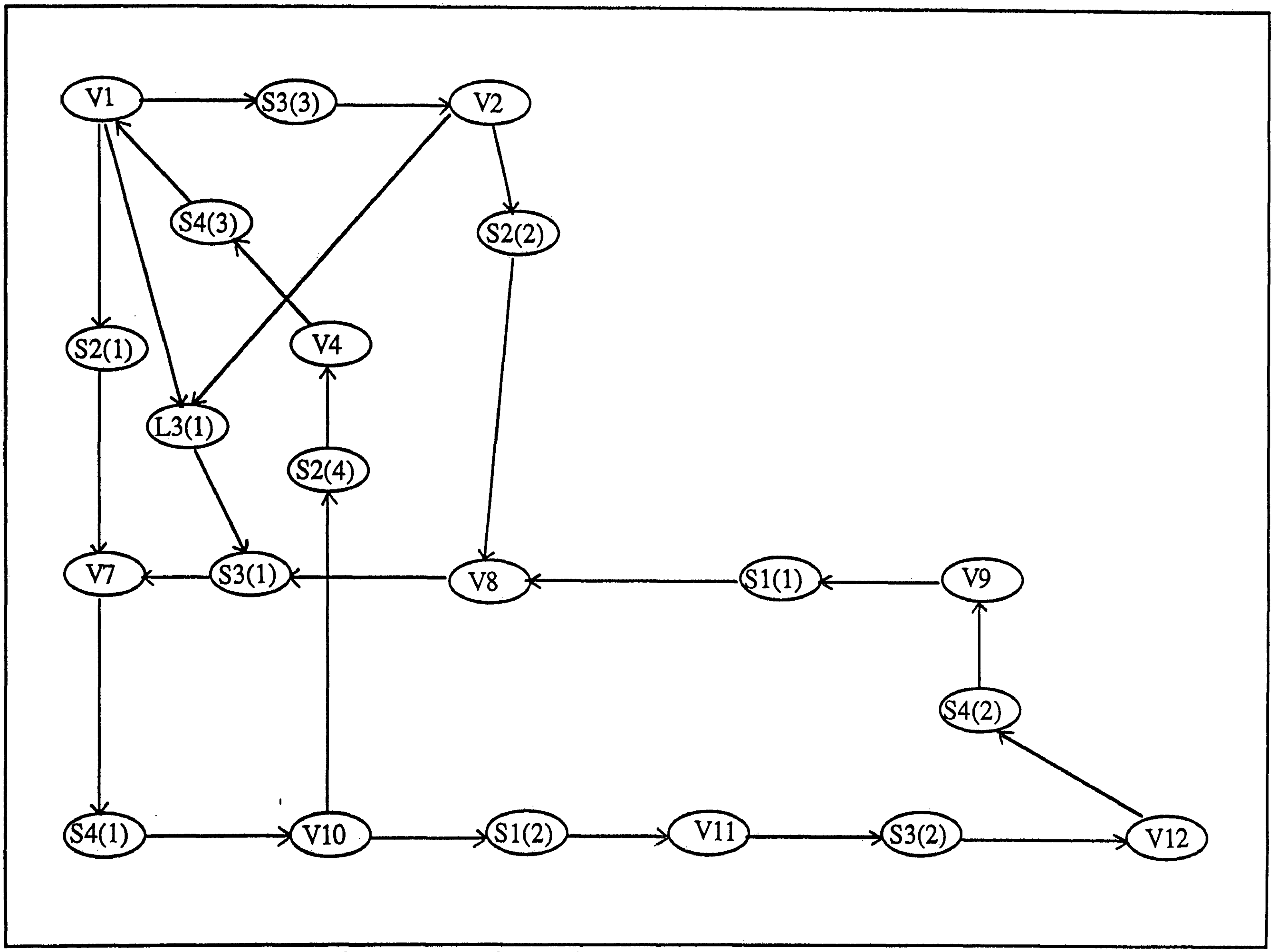


Fig. 4.3.2(c) Possble loops of all the solutions of system (115)

32. V5={(x,y,z) | x>0, y>0, z>0, x-y>0}, $\dot{x}>0$, $\dot{y}>0$, $\dot{z}>0$

33. V6={(x,y,z) | x>0, y>0, z>0, x-y<0}, $\dot{x}>0$, $\dot{y}<0$, $\dot{z}>0$

34. V7={(x,y,z) | x<0, y<0, z<0, x-y>0}, $\dot{x}>0$, $\dot{y}>0$, $\dot{z}<0$

35. V8={(x,y,z) | x<0, y<0, z<0, x-y<0}, $\dot{x}>0$, $\dot{y}<0$, $\dot{z}<0$

36. V9={(x,y,z) | x<0, y>0, z<0}, $\dot{x}<0$, $\dot{y}<0$, $\dot{z}<0$

37. V10={(x,y,z) | x>0, y<0, z<0}, $\dot{x}>0$, $\dot{y}>0$, $\dot{z}>0$

38. V11={(x,y,z) | x>0, y>0, z<0, x-y>0}, $\dot{x}<0$, $\dot{y}>0$, $\dot{z}>0$

39. V12={(x,y,z) | x>0, y>0, z<0, x-y<0}, $\dot{x}<0$, $\dot{y}<0$, $\dot{z}>0$

The monotone regions of this system is shown in Fig4.3.2(a). The monotone diagram of this system is shown as Fig4.3.2(b). From Fig.4.3.2(c) it can be recognized that there are three possible loops $\Gamma_1 = \langle V1 > S2(1) > V7 > S4(1) > V10 > S4(2) > V4 > S4(3) > V1\rangle$ $\Gamma_2 = \langle V1 > S3(3) > V2 > S2(2) > V8 > S3(1) > V7 > S4(1) > V10 > S4(2) > V4 > S4(3) > V1\rangle$ and $\Gamma_3 = \langle V7 > S4(1) > V10 > S1(2) > V11 > S3(2) > V12 > S4(2) > V9 > S1(1) > V8 > S3(1) > V7\rangle$The numerical solutions start from each three dimensional regions V1-V12 is obtained by Maple, but no coupled loops was found. which might be truely coupled. The case c>0 can also be analyzed by this method.

3. Consider system (116)

$$\begin{cases} \dot{x} = ayz \\ \dot{y} = bz + cy \\ \dot{z} = rx \end{cases}$$

There are possible coupled loops in this system

Proof: System (116) can be transformed to

$$\dot{x} = yz,\ \dot{y} = z - y, \dot{z} = x \tag{4.3.3}$$

by the scalar transformation

$$T = \{\alpha = -c^4/(abr^2), \beta = c^2/(ar), \gamma = -c^3/(abr), \delta = -1/c\} \quad c<0.$$

The monotone regions can be obtained as follows:

S1={(x,y,z) | y=0}, S2={(x,y,z) | z=0), S3={(x,y,z) | z-y=0}, S4= {(x,y,z) | x=0}

L1=S1 $\cap$ S2={(x,y,z) | y=0,z=0},S1 $\cap$ S3=L1, L2=S1 $\cap$ S4={(x,y,z) | x=0,y=0},

S2 $\cap$ S3=L1,L3=S2 $\cap$ S4={(x,y,z) | x=0,z=0},L4=S3 $\cap$ S4 ={(x,y,z) | z-y=0,x=0}

P=(S1 $\cup$ S2) $\cap$ S3 $\cap$ S4={(x,y,z) | x=0,y=0,z=0} and

L1 $\cap$ L2=L1 $\cap$ L3=L1 $\cap$ L4=L2 $\cap$ L3=L2 $\cap$ L4=L3 $\cap$ L4=P.

All the monotone regions of the system are listed here:

1. P={(x,y,z) | x=0,y=0,z=0} $\dot{x}=0$, $\dot{y}=0$, $\dot{z}=0$

2. L1(1)={(x,y,z) | y=0,z=0,x<0}, $\dot{x}=0$, $\dot{y}=0$, $\dot{z}<0$

3 $L1(2)=\{(x,y,z) \mid y=0,z=0,x>0\}$, $\dot{x}=0$, $\dot{y}=0$, $\dot{z}>0$

4. $L2(1)=\{(x,y,z) \mid x=0,y=0,z<0\}$, $\dot{x}=0$, $\dot{y}<0$, $\dot{z}=0$

5. $L2(2)=\{(x,y,z) \mid x=0,y=0,z>0\}$, $\dot{x}=0$, $\dot{y}>0$, $\dot{z}=0$

6. $L3(1)=\{(x,y,z) \mid x=0,z=0,y<0\}$, $\dot{x}=0$, $\dot{y}>0$, $\dot{z}=0$

7. $L3(2)=\{(x,y,z) \mid x=0,z=0,y>0\}$, $\dot{x}=0$, $\dot{y}<0$, $\dot{z}=0$

8. $L4(1)=\{(x,y,z) \mid x=0,z-y=0,y<0\}$, $\dot{x}>0$, $\dot{y}=0$, $\dot{z}=0$

9. $L4(2)=\{(x,y,z) \mid x=0,z-y=0,y>0\}$, $\dot{x}>0$, $\dot{y}=0$, $\dot{z}=0$

10. $S1(1)=\{(x,y,z) \mid y=0, x<0, z<0\}$, $\dot{x}=0$, $\dot{y}<0$, $\dot{z}<0$

11. $S1(2)=\{(x,y,z) \mid y=0, x<0, z>0\}$, $\dot{x}=0$, $\dot{y}>0$, $\dot{z}<0$

12. $S1(3)=\{(x,y,z) \mid y=0, x>0, z<0\}$, $\dot{x}=0$, $\dot{y}<0$, $\dot{z}>0$

13. $S1(4)=\{(x,y,z) \mid y=0, x>0, z>0\}$, $\dot{x}=0$, $\dot{y}>0$, $\dot{z}>0$

14. $S2(1)=\{(x,y,z) \mid z=0, x<0, y<0\}$, $\dot{x}=0$, $\dot{y}>0$, $\dot{z}<0$

15. $S2(2)=\{(x,y,z) \mid z=0, x<0, y>0\}$, $\dot{x}=0$, $\dot{y}<0$, $\dot{z}<0$

16. $S2(3)=\{(x,y,z) \mid z=0, x>0, y<0\}$, $\dot{x}=0$, $\dot{y}>0$, $\dot{z}>0$

17. $S2(4)=\{(x,y,z) \mid z=0, x>0, y>0\}$, $\dot{x}=0$, $\dot{y}<0$, $\dot{z}>0$

18. $S3(1)=\{(x,y,z) \mid z-y=0,x<0,y<0,z<0\}$, $\dot{x}>0$, $\dot{y}=0$, $\dot{z}<0$

19. $S3(2)=\{(x,y,z) \mid z-y=0,x<0,y>0,z>0\}$, $\dot{x}>0$, $\dot{y}=0$, $\dot{z}<0$

20. $S3(3)=\{(x,y,z) \mid z-y=0,x<0,y<0,z<0\}$, $\dot{x}>0$, $\dot{y}=0$, $\dot{z}>0$

21. $S3(4)=\{(x,y,z) \mid z-y=0,x>0,y>0,z>0\}$, $\dot{x}>0$, $\dot{y}=0$, $\dot{z}>0$

22. $S4(1)=\{(x,y,z) \mid x=0, z-y<0,y<0,z<0\}$, $\dot{x}>0$, $\dot{y}<0$, $\dot{z}=0$

23. $S4(2)=\{(x,y,z) \mid x=0, z-y>0,y<0,z>0\}$, $\dot{x}>0$, $\dot{y}>0$, $\dot{z}=0$

24. $S4(3)=\{(x,y,z) \mid x=0, z-y>0,y<0,z>0\}$, $\dot{x}<0$, $\dot{y}>0$, $\dot{z}=0$

25. $S4(4)=\{(x,y,z) \mid x=0, z-y<0,y<0,z>0\}$, $\dot{x}<0$, $\dot{y}<0$, $\dot{z}=0$

26. $S2(5)=\{(x,y,z) \mid x=0, z-y<0,y>0,z>0\}$, $\dot{x}>0$, $\dot{y}<0$, $\dot{z}=0$

27. $S2(6)=\{(x,y,z) \mid x=0, z-y<0,y>0,z>0\}$, $\dot{x}>0$, $\dot{y}>0$, $\dot{z}=0$

28. $V1=\{(x,y,z) \mid x>0, z-y<0,y<0, z<0\}$, $\dot{x}>0$, $\dot{y}<0$, $\dot{z}>0$

29. $V2=\{(x,y,z) \mid x>0, z-y>0,y<0, z<0\}$, $\dot{x}>0$, $\dot{y}>0$, $\dot{z}>0$

30. $V3=\{(x,y,z) \mid x>0, z-y>0,y<0, z>0\}$, $\dot{x}<0$, $\dot{y}>0$, $\dot{z}>0$

31. $V4=\{(x,y,z) \mid x>0, z-y>0,y<0, z<0\}$, $\dot{x}>0$, $\dot{y}<0$, $\dot{z}>0$

32. $V5=\{(x,y,z) \mid x>0, z-y<0,y>0, z>0\}$, $\dot{x}>0$, $\dot{y}<0$, $\dot{z}>0$

33. $V6=\{(x,y,z) \mid x>0, z-y>0,y>0, z>0\}$, $\dot{x}>0$, $\dot{y}>0$, $\dot{z}>0$

34. $V7=\{(x,y,z) \mid x<0, z-y<0,y<0, z<0\}$, $\dot{x}>0$, $\dot{y}<0$, $\dot{z}<0$

35. $V8=\{(x,y,z) \mid x<0, z-y>0,y<0, z<0\}$, $\dot{x}>0$, $\dot{y}>0$, $\dot{z}<0$

36. $V9=\{(x,y,z) \mid x<0, z-y>0,y<0, z>0\}$, $\dot{x}<0$, $\dot{y}>0$, $\dot{z}<0$

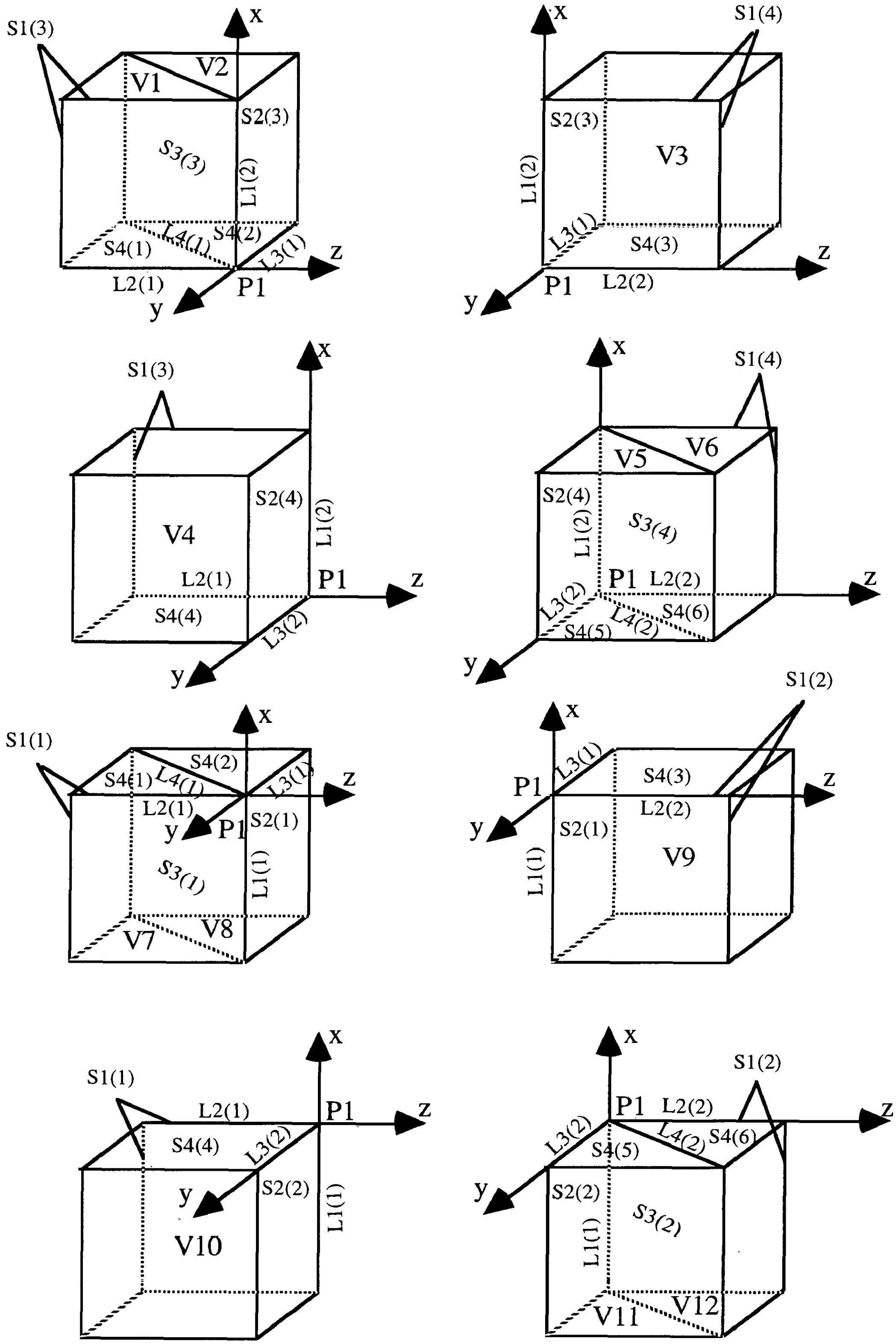


Fig. 4.3.3(a) Monotone regions of system(116)

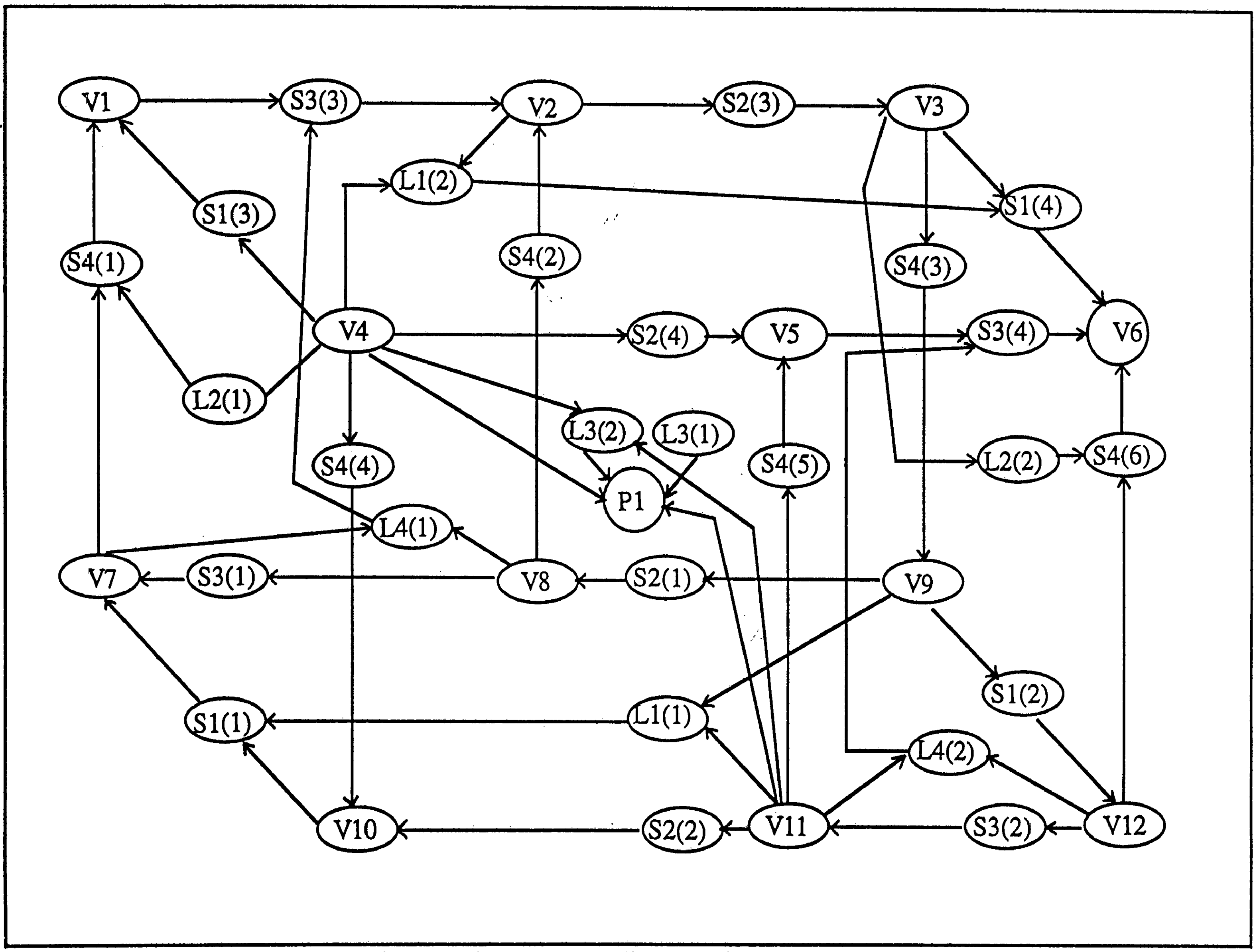


Fig. 4.3.3(b) Monotone diagram of system (116)

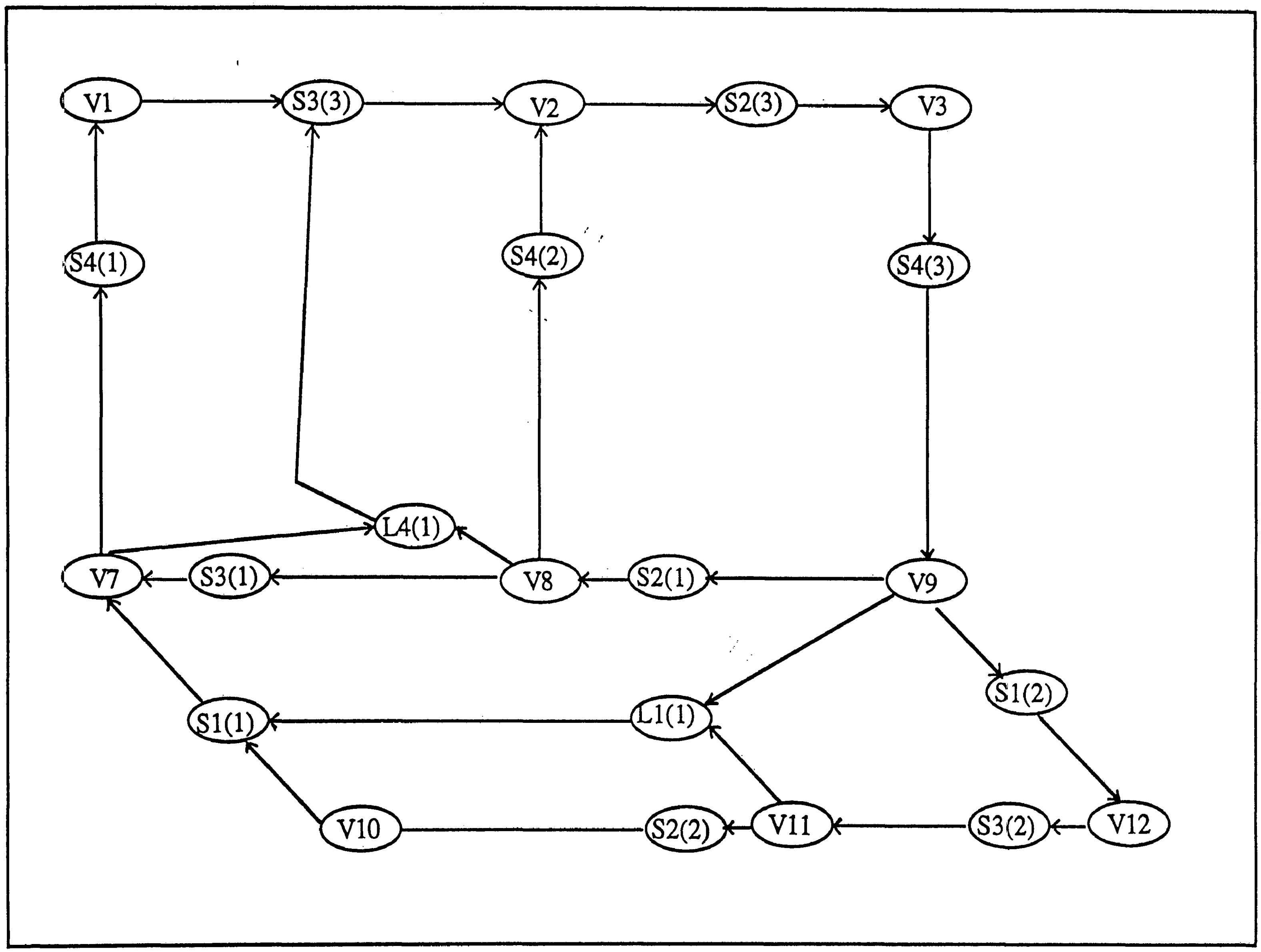


Fig. 4.3.3(c) Possible loops of all the solutions of system (116)

37. V10={(x,y,z) | x<0, z-y<0,y>0, z<0}, $\dot{x}<0$, $\dot{y}>0$, $\dot{z}<0$

38. V11={(x,y,z) | x<0, z-y<0,y>0, z>0}, $\dot{x}>0$, $\dot{y}<0$, $\dot{z}<0$

39. V12={(x,y,z) | x<0, z-y>0,y>0, z>0}, $\dot{x}>0$, $\dot{y}>0$, $\dot{z}<0$

The monotone regions of this system are shown in Fig4.3.3(a). The monotone diagram of system(116) is shown in Fig4.3.3(b). From Fig. 4.3.3(c) it can be recognized that there are three possible loops $\Gamma_1 = \langle V1 > S3(3) > V2 > S2(3) > V3 > S4(3) > V9 > S1(2) > V12 > S3(2) > V11 > S2(2) > V10 > S1(1) > V7 > S4(1) > V1 \rangle$, $\Gamma_2 = \langle V2 > S2(3) > V3 > S4(3) > V9 > S2(1) > V8 > S4(2) > V2 \rangle$ and $\Gamma_3 = \langle V1 > S3(3) > V2 > S2(3) > V3 > S4(3) > V9 > L1(1) > S1(1) > V7 > S4(1) > V1$ which might be truely coupled. The numerical solutions start from each three dimensional regions V1-V12 is obtained by Maple, but no coupled loops was found. The case c>0 can also be analyzed by this method.

Chapter 5.An Example Known to Have Chaotic Behaviour

In this chapter we are giong to study system B in sprott's paper by the method introduced in chapter 4.

System B in Sprott's paper:

$$\begin{cases} \dot{x} = yz \\ \dot{y} = x - y \\ \dot{z} = 1 - xy \end{cases}$$

The monotone regions are based on the following surfaces:

S1={(x,y,z): y=0}, S2={(x,y,z): z=0}, S3={(x,y,z):x-y=0}, S4={(x,y,z):1−xy=0, x>0, y>0}, S5={(x,y,z): 1-xy=0, x<0,y<0};

The monotone regions of this system are:

1. P1={(x,y,z): x=y=1, z=0}: $\dot{x}=\dot{y}=\dot{z}=0$
2. P2={(x,y,z): x=y=−1, z=0}: $\dot{x}=\dot{y}=\dot{z}=0$
3. P3={(x,y,z): x=0, y=0, z=0}: $\dot{x}=0$, $\dot{y}=0$, $\dot{z}>0$
4. L1(1)={(x,y,z): y=0, z=0, x<0}: $\dot{x}=0$, $\dot{y}<0$, $\dot{z}>0$
5. L1(2)={(x,y,z): y=0, z=0, x>0}: $\dot{x}=0$, $\dot{y}>0$, $\dot{z}>0$
6. L2(1)={(x,y,z): x=0, y=0, z<0}: $\dot{x}=0$, $\dot{y}=0$, $\dot{z}>0$
7. L2(2)={(x,y,z): x=0, y=0, z>0}: $\dot{x}=0$, $\dot{y}=0$, $\dot{z}>0$
8. L3(1),={(x,y,z): x=y, z=0, x>1}: $\dot{x}=0$, $\dot{y}=0$, $\dot{z}<0$
9. L3(2)={(x,y,z): x=y, z=0, 0<x<1}: $\dot{x}=0$, $\dot{y}=0$, $\dot{z}>0$

10 L3(3)={(x,y,z): x=y,z=0, −1<x<0}: $\dot{x}=0$, $\dot{y}=0$, $\dot{z}>0$

11.L3(4)={(x,y,z): x=y, z=0, x<-1}: $\dot{x}=0$, $\dot{y}=0$, $\dot{z}<0$

12. L4(1)={(x,y,z): 1−xy=0, z=0, 0<x<1, y<1}: $\dot{x}<0$, $\dot{y}=0$, $\dot{z}=0$
13. L4(2)={(x,y,z): 1−xy=0, z=0, x>1, 0<y<1}: $\dot{x}>0$, $\dot{y}=0$, $\dot{z}=0$
14. L5(1)={(x,y,z): 1−xy=0, z=0, −1<x<0, y<−1}: $\dot{x}>0$, $\dot{y}=0$, $\dot{z}=0$
15. L5(2)={(x,y,z): 1−xy=0, z=0, x<−1, −1<y<0}: $\dot{x}<0$, $\dot{y}=0$, $\dot{z}=0$
16. L6(1)={(x,y,z): x=y=1, z<0}: $\dot{x}=0$, $\dot{y}<0$, $\dot{z}=0$
17. L6(2)={(x,y,z): x=y=1, z>0}: $\dot{x}=0$, $\dot{y}>0$, $\dot{z}=0$
18. L7(1)={(x,y,z): x=y=−1, z<0}: $\dot{x}=0$, $\dot{y}>0$, $\dot{z}=0$
19. L7(2)={(x,y,z): x=y=−1, z>0}: $\dot{x}=0$, $\dot{y}<0$, $\dot{z}=0$
20. S1(1)={(x,y,z): y=0, x<0, z<0}: $\dot{x}=0$, $\dot{y}<0$, $\dot{z}>0$
21. S1(2)={(x,y,z): y=0, x>0, z<0}: $\dot{x}=0$, $\dot{y}>0$, $\dot{z}>0$
22. S1(3)={(x,y,z): y=0, x>0, z>0}: $\dot{x}=0$, $\dot{y}>0$, $\dot{z}>0$
23. S1(4)={(x,y,z): y=0, x<0, z>0}: $\dot{x}=0$, $\dot{y}<0$, $\dot{z}>0$

24. $S2(1)=\{(x,y,z): z=0,\ y>0,\ x-y<0,\ 1-xy>0\}: \dot{x}=0,\ \dot{y}<0,\ \dot{z}>0$

25. $S2(2)=\{(x,y,z): z=0,\ y>0,\ x-y<0,\ 1-xy<0\}: \dot{x}=0,\ \dot{y}<0,\ \dot{z}<0$

26. $S2(3)=\{(x,y,z): z=0,\ y>0,\ x-y>0,\ 1-xy<0\}: \dot{x}=0,\ \dot{y}>0,\ \dot{z}<0$

27. $S2(4)=\{(x,y,z): z=0,\ y>0,\ x-y>0,\ 1-xy>0\}: \dot{x}=0,\ \dot{y}>0,\ \dot{z}>0$

28. $S2(5)=\{(x,y,z): z=0,\ y<0,\ x-y>0,\ 1-xy>0\}: \dot{x}=0,\ \dot{y}>0,\ \dot{z}>0$

29. $S2(6)=\{(x,y,z): z=0,\ y<0,\ x-y>0,\ 1-xy<0\}: \dot{x}=0,\ \dot{y}>0,\ \dot{z}<0$

30. $S2(7)=\{(x,y,z): z=0,\ y<0,\ x-y<0,\ 1-xy<0\}: \dot{x}=0,\ \dot{y}<0,\ \dot{z}<0$

31. $S2(8)=\{(x,y,z): z=0,\ y<0,\ x-y<0,\ 1-xy>0\}: \dot{x}=0,\ \dot{y}<0,\ \dot{z}>0$

32. $S3(1)=\{(x,y,z): x=y,\ y>1,\ z<0\}: \dot{x}<0,\ \dot{y}=0,\ \dot{z}<0$

33. $S3(2)=\{(x,y,z): x=y,\ 0<y<1,\ z<0\}: \dot{x}<0,\ \dot{y}=0,\ \dot{z}>0$

34. $S3(3)=\{(x,y,z): x=y,\ y>1,\ z>0\}: \dot{x}>0,\ \dot{y}=0,\ \dot{z}<0$

35. $S3(4)=\{(x,y,z): x=y,\ 0<y<1,\ z>0\}: \dot{x}>0,\ \dot{y}=0,\ \dot{z}>0$

36. $S3(5)=\{(x,y,z): x=y,\ -1<y<0,\ z<0\}: \dot{x}<0,\ \dot{y}=0,\ \dot{z}>0$

37. $S3(6)=\{(x,y,z): x=y,\ y<-1,\ z<0\}: \dot{x}>0,\ \dot{y}=0,\ \dot{z}<0$

38. $S3(7)=\{(x,y,z): x=y,\ -1<y<0,\ z>0\}: \dot{x}<0,\ \dot{y}=0,\ \dot{z}>0$

39. $S3(8)=\{(x,y,z): x=y,\ y<-1,\ z>0\}: \dot{x}<0,\ \dot{y}=0,\ \dot{z}<0$

40. $S4(1)=\{(x,y,z): 1-xy=0,\ y>1,\ z<0,\ x<1\}: \dot{x}<0,\ \dot{y}<0,\ \dot{z}=0$

41. $S4(2)=\{(x,y,z): 1-xy=0,\ 0<y<1,\ z<0,\ x>1\}: \dot{x}<0,\ \dot{y}>0,\ \dot{z}=0$

42. $S4(3)=\{(x,y,z): 1-xy=0,\ y>1,\ z>0,\ x<1\}: \dot{x}>0,\ \dot{y}<0,\ \dot{z}=0$

43. $S4(4)=\{(x,y,z): 1-xy=0,\ 0<y<1,\ z>0,\ x>1\}: \dot{x}>0,\ \dot{y}>0,\ \dot{z}=0$

44. $S5(1)=\{(x,y,z): 1-xy=0,\ -1<y<0,\ z<0,\ x<-1\}: \dot{x}>0,\ \dot{y}<0,\ \dot{z}=0$

45. $S5(2)=\{(x,y,z): 1-xy=0,\ y<-1,\ z<0,\ -1<x<0\}: \dot{x}>0,\ \dot{y}>0,\ \dot{z}=0$

46. $S5(3)=\{(x,y,z): 1-xy=0,\ -1<y<0,\ z>0,\ x<-1\}: \dot{x}<0,\ \dot{y}<0,\ \dot{z}=0$

47. $S5(4)=\{(x,y,z): 1-xy=0,\ y<-1,\ z>0,\ -1<x<0\}: \dot{x}<0,\ \dot{y}>0,\ \dot{z}=0$

48. $V1=\{(x,y,z): y>0,\ z<0,\ x-y<0, 1-xy>0\}: \dot{x}<0,\ \dot{y}<0,\ \dot{z}>0$

49. $V2=\{(x,y,z): y>0,\ z<0,\ x-y<0,\ 1-xy<0\}: \dot{x}<0,\ \dot{y}<0,\ \dot{z}<0$

50. $V3=\{(x,y,z): y>0,\ z<0,\ x-y>0,\ 1-xy<0\}: \dot{x}<0,\ \dot{y}>0,\ \dot{z}<0$

51. $V4=\{(x,y,z): y>0,\ z<0,\ x-y>0,\ 1-xy>0\}: \dot{x}<0,\ \dot{y}>0,\ \dot{z}>0$

52. $V5=\{(x,y,z): y>0,\ z>0,\ x-y<0,\ 1-xy>0\}: \dot{x}>0,\ \dot{y}<0,\ \dot{z}>0$

53. $V6=\{(x,y,z): y>0,\ z>0,\ x-y<0,\ 1-xy<0\}: \dot{x}>0,\ \dot{y}<0,\ \dot{z}<0$

54. $V7=\{(x,y,z): y>0,\ z>0,\ x-y>0,\ 1-xy<0\}: \dot{x}>0,\ \dot{y}<0,\ \dot{z}<0$

55. $V8=\{(x,y,z): y>0,\ z>0,\ x-y>0,\ 1-xy>0\}: \dot{x}>0,\ \dot{y}>0,\ \dot{z}>0$

56. $V9=\{(x,y,z): y<0,\ z<0,\ x-y<0,\ 1-xy>0\}: \dot{x}>0,\ \dot{y}<0,\ \dot{z}>0$

57. $V10=\{(x,y,z): y<0,\ z<0,\ x-y<0,\ 1-xy<0\}: \dot{x}>0,\ \dot{y}<0,\ \dot{z}<0$

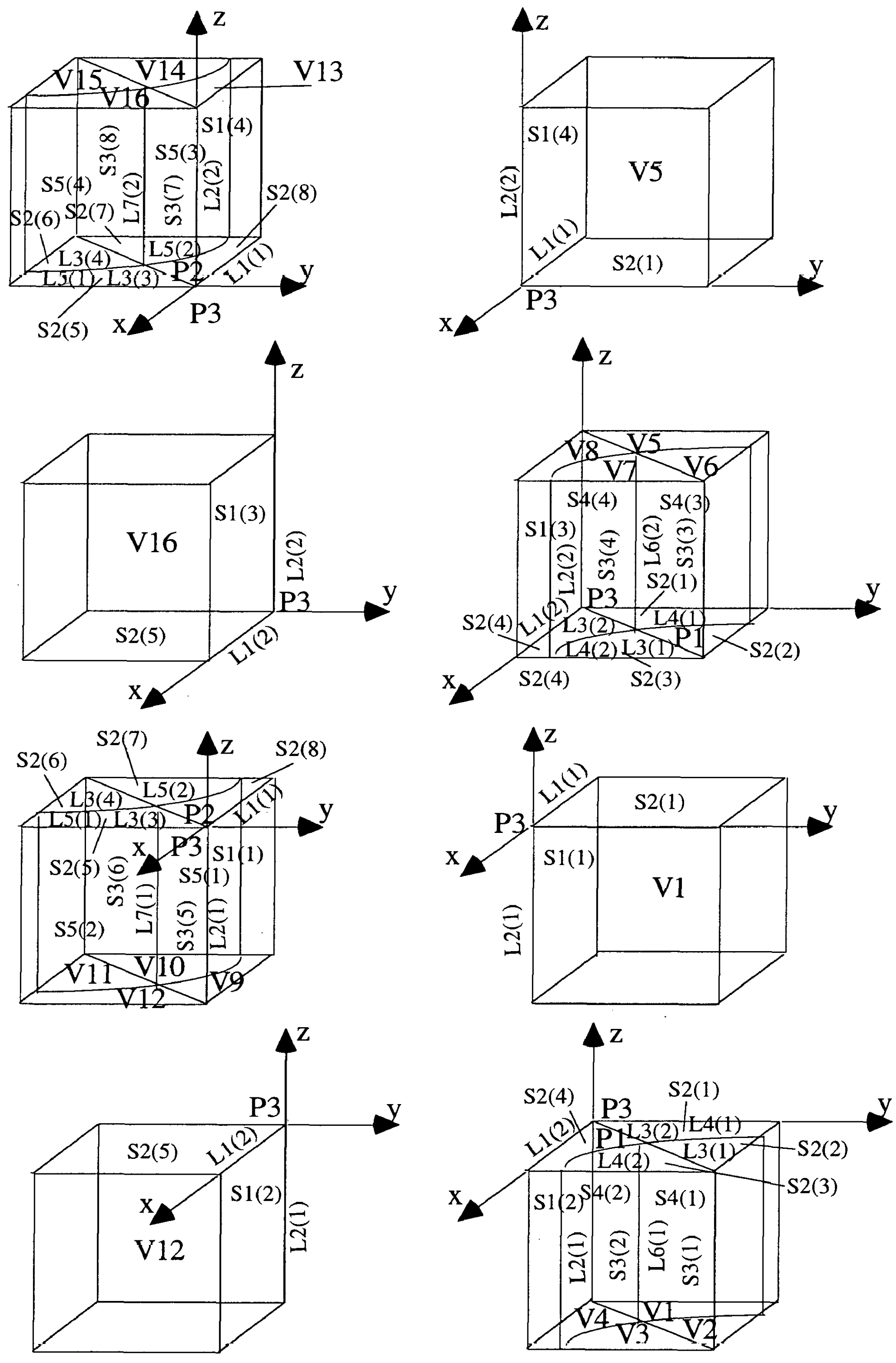


Fig.5.1.1(a) Monotone regions of sprott B

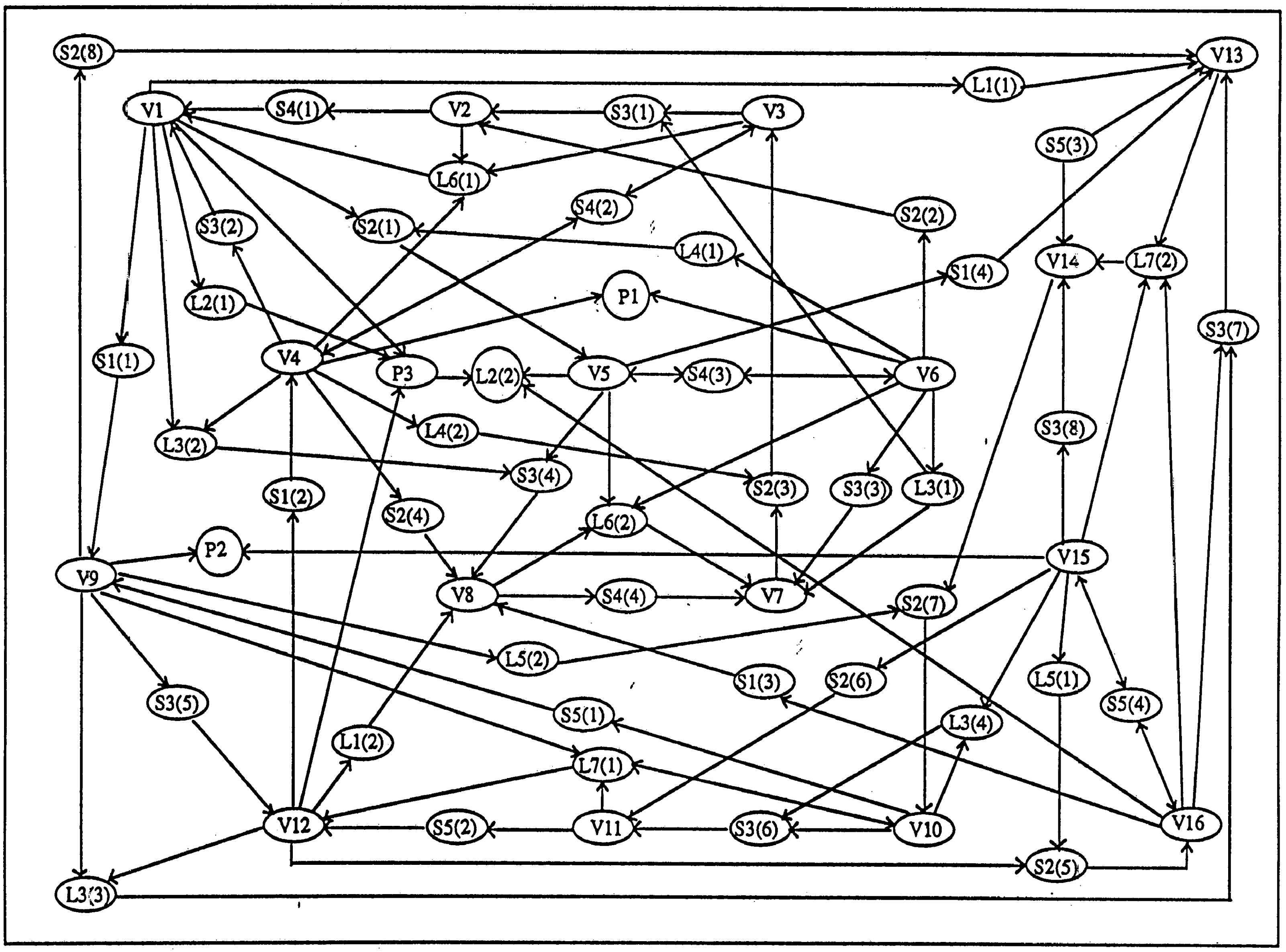


Fig. 5.1.1(b) Monotone diagram of system of Sprott B.

58. $V11=\{(x,y,z): y<0, z<0, x-y>0, 1-xy<0\}: \dot{x}>0, \dot{y}>0, \dot{z}<0$

59. $V12=\{(x,y,z): y<0, z<0, x-y>0, 1-xy>0\}: \dot{x}>0, \dot{y}>0, \dot{z}>0$

60. $V13=\{(x,y,z): y<0, z>0, x-y<0, 1-xy>0\}: \dot{x}<0, \dot{y}<0, \dot{z}>0$

61. $V14=\{(x,y,z): y<0, z>0, x-y<0, 1-xy<0\}: \dot{x}<0, \dot{y}<0, \dot{z}<0$

62. $V15=\{(x,y,z): y<0, z>0, x-y>0, 1-xy<0\}: \dot{x}<0, \dot{y}>0, \dot{z}<0$

63. $V16=\{(x,y,z): y<0, z>0, x-y>0, 1-xy>0\}: \dot{x}<0, \dot{y}>0, \dot{z}>0$

By the monotone regions from 1 to 63 we found, the paths of all the possible solutions are shown in Fig.5.1.1(b), detailed discussions are not given here. From Fig5.1.1(b), we can see that every of the 16 three dimensional monotone regions is in at least one loop and every loop is coupled with one or more other loops. A numerical solution is shown in Appendix C. From the numerical solution we can see that the solution switch from aroun one fixed point to the other. Both fixed points have the same eigenvalues: -1.3532, $0.1766+1.2028i$, $0.1766-1.2028i$. Fom Sprott B some special properties for a system with chaos can be concluded:

(1) There exist more than one loops with four three dimensional monotone regions which are coupled and there is always one component whose monotonicity does not change in each loop.

(2) Each loop with chaotic behavior has at least two 3-dimensional monotone regions that join other loops

(3) There must exist complex eigenvalues with nonzero imaginary part for the fixed point of its linearized system.

Sprott B is a simple 3-dimensional system with chaotic behavior. we believe that the chaotic behavior has close relation with the loops in a system.

Chapter 6.Conclusion and Future

This thesis proved that there is no chaos in any of the 3-D autonomous dissipative systems in S[4;1;0] except possibly systems (115) and (116). These two systems are both characterized by posessing coupled loops. There has not been a general way to determine the behaviour of coupled loops in this theory. The method to analyze the behaviour of higher dimensional autonomous dynamical systems introduced in this thesis is a beginning. Creating a monotone diagram is a lot of work, yet this procedure is algorithmic and so a computer program can be made to create the monotone diagram for each system. It is likely that chaos is due to certain configurations of loops. Subsequent work include further study of loops in a system and we are expecting an exciting result in this theory and we will continue working on the conjecture.

Appendix A. Table of the Sovability of the 138 Patterns

A.1 Classification of the Systems:

- I: Solvable by seperation of variables(28 patterns)

  The general form of the equations that can be solved by seperation of variables can be written as:

  $$f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$$

- II. nth order linear systems

  The general form of nth-order linear ordinery differetial equations can be written as:

  $$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = \varphi(x),$$

  where $a_i(x), i = 1,2,\ldots,n-1,$ are continuous functions.

  II.1. 1st order linear ODE(29 patterns)

  II.2. 2nd order linear ODE with constant coefficients(8 patterns)

  II.3. 2nd order linear ODE with variable coefficients(6 patterns)

- III. 1st order Riccati equation(6 patterns)

  The general first-order Riccati equation can be written as:

  $$\dot{x}(t) + p(t)x + q(t)x^2(t) = \varphi(t), \qquad '\cdot' = \frac{d}{dt}$$

  where $p(t)$ and $q(t)$ are arbitary functions.

- IV. Solution can be expressed by elliptic integrals(3 patterns)

  The first kind of elliptic integal can be written as:

  $$F(x,k) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad k^2 < 1$$

- V. Rayleigh Equations(3 patterns)

  The general form of the Rayleigh equations is

  $$y''(x) + f(y'(x)) + g(y(x)) = 0$$

- VI. 2nd order nonlinear autonomous ODE(13 patterns)

  Some of them can be transformed to the general form of Lienard equations:

  $$y''(x) + f(y(x))y'(x) + g(y(x)) = 0$$

  or the general form of Abel equations of the 2nd kind:

  $$y(x)y'(x) - y(x) = f(x)$$

- VII. 3rd order nonlinear autonomous ODE or 2nd order nonlinear nonautonomous ODE(24 patterns)
- S: The solution is listed in the table(18 patterns)

A.2 Table A: A List of the 138 Patterns

Table A List of the 138 patterns

No.	Systems	Solvable Form / Scalar Equation	Sol.?	Type
1.	$\dot{x} = ax^2 + bx,\ \dot{y} = cx,\ \dot{z} = rx$	$\dot{x} = ax^2 + bx$	y	I
2.	$\dot{x} = ax^2 + bx,\ \dot{y} = cy,\ \dot{z} = rx$	$\dot{x} = ax^2 + bx$	y	I
3.	$\dot{x} = ax^2 + bx,\ \dot{y} = cz,\ \dot{z} = rx$	$\dot{x} = ax^2 + bx$	y	I
4.	$\dot{x} = ax^2 + bx,\ \dot{y} = cy,\ \dot{z} = ry$	$\dot{x} = ax^2 + bx, y = Ae^{ct}$	y	I
5.	$\dot{x} = ax^2 + bx,\ \dot{y} = cy,\ \dot{z} = rz$	$\dot{x} = ax^2 + bx, y = Ae^{ct}, z = Be^{rt}$	y	I
6.	$\dot{x} = ax^2 + bx,\ \dot{y} = cz,\ \dot{z} = ry$	$\dot{x} = ax^2 + bx$	y	I
7.	$\dot{x} = ayz + bx,\ \dot{y} = cx,\ \dot{z} = rx$	$\dot{x} = bx + aAz + (ac / r)z^2,\ \dot{z} = rx$	n	VI
8.	$\dot{x} = ayz + bx,\ \dot{y} = cy,\ \dot{z} = rx$	$\ddot{z} - b\dot{z} - raAe^{ct}z = 0,\ y = Ae^{ct}$	y	II.3
9.	$\dot{x} = ayz + bx,\ \dot{y} = cz,\ \dot{z} = rx$	$\dot{y} = cz,\ \dot{z} = (ar / 2c)y^2 + bz + A$	n	VI
10.	$\dot{x} = ayz + bx,\ \dot{y} = cy,\ \dot{z} = ry$	$\dot{x} - bx = (ar / c)Be^{ct}(Be^{ct} - A)$	y	II.1
11.	$\dot{x} = ayz + bx,\ \dot{y} = cy,\ \dot{z} = rz$	$\dot{x} - bx = aABe^{(c+r)t},\ y = Ae^{ct}$	y	II.1
12.	$\dot{x} = ayz + bx,\ \dot{y} = cz,\ \dot{z} = ry$	$\dot{x} - bx = \lvert rc\rvert(a / 2c)(C_1e^{\sqrt{rc}t} - C_2e^{-\sqrt{rc}t})^2$	y	II.1
13.	$\dot{x} = ax^2 + by,\ \dot{y} = cx,\ \dot{z} = rx$	$\ddot{y} = (a / c)\dot{y}^2 + bcy$	y	V
14.	$\dot{x} = ay^2 + bx,\ \dot{y} = cx,\ \dot{z} = rx$	$\dot{x} = ay^2 + bx,\ \dot{y} = cx$	n	VI
15.	$\dot{x} = ay^2 + by,\ \dot{y} = cx,\ \dot{z} = rx$	$cx^2 = (2a / 3)y^3 + by^2 + A$	y	S
16.	$\dot{x} = axy + bx,\ \dot{y} = cx,\ \dot{z} = rx$	$cx = (a / 2)y^2 + by + A, y = (c / r)z + B$	y	S
17.	$\dot{x} = axy + by,\ \dot{y} = cx,\ \dot{z} = rx$	$2cx - 2cb[\ln(x + b / a)] = ay^2 + A$	y	S
18.	$\dot{x} = ay^2 + bz,\ \dot{y} = cx,\ \dot{z} = rx$	$3cx^2 = 2ay^3 + (3br / c)y^2 + 6bAy + 6A$	y	S
19.	$\dot{x} = axz + by,\ \dot{y} = cx,\ \dot{z} = rx$	$\dot{x} = (ad / c)xy + aAx + by,\ \dot{y} = cx$	n	VI
20.	$\dot{x} = ayz + bz,\ \dot{y} = cx,\ \dot{z} = rx$	$3c^2x^2 = 2ary^3 + 3(acA + br)y^2$ $+ 6cbAy + 6cB,\ z = (r / c)y + A$	y	S
21.	$\dot{x} = ax^2 + by,\ \dot{y} = cx,\ \dot{z} = ry$	$\ddot{y} = (a / c)\dot{y}^2 + bcy$	y	V
22.	$\dot{x} = ay^2 + bx,\ \dot{y} = cx,\ \dot{z} = ry$	$\dot{x} = ay^2 + bx,\ \dot{y} = cx$	n	VI
23.	$\dot{x} = ay^2 + by,\ \dot{y} = cx,\ \dot{z} = ry$	$3cx^2 = 2ay^3 + 3by^2 + A$	y	IV
24.	$\dot{x} = axy + bx,\ \dot{y} = cx,\ \dot{z} = ry$	$\dot{y} = (ca / 2)y^2 + cby + A$	y	I
25.	$\dot{x} = axy + by,\ \dot{y} = cx,\ \dot{z} = ry$	$2cx - 2cb[\ln(x + b / a)] = y^2 + A$	y	I*
26.	$\dot{x} = ax^2 + by,\ \dot{y} = cx,\ \dot{z} = rz$	$\ddot{y} = (a / c)\dot{y}^2 + bcy,\ z = Ae^{rt}$	y	V
27.	$\dot{x} = ay^2 + bx,\ \dot{y} = cx,\ \dot{z} = rz$	$\dot{x} = ay^2 + bx,\ \dot{y} = cx,\ z = Ae^{rt}$	n	VI
28.	$\dot{x} = ay^2 + by,\ \dot{y} = cx,\ \dot{z} = rz$	$3cx^2 = 2ay^3 + 3by^2 + A,\ z = Ae^{rt}$	y	S
29.	$\dot{x} = axy + bx,\ \dot{y} = cx,\ \dot{z} = rz$	$\dot{y} = (ca / 2)y^2 + cby + A$	y	I
30.	$\dot{x} = axy + by,\ \dot{y} = cx,\ \dot{z} = rz$	$2cx - 2cb[\ln(x + b / a)] = y^2 + A$	y	I

Table A Continued

No.	Systems	Solvable Form / Scalar Equation	Sol.?	Type
31.	$\dot{x} = ay^2 + bz, \quad \dot{y} = cx, \quad \dot{z} = rz$	$\ddot{y} = acy^2 + Abce^{rt}$	n	VII
32.	$\dot{x} = axz + by, \quad \dot{y} = cx, \quad \dot{z} = rz$	$\ddot{y} - aAe^{rt}\dot{y} - cby = 0, \ z = Ae^{rt}$	y	II.3
33.	$\dot{x} = ayz + bz, \quad \dot{y} = cx, \quad \dot{z} = rz$	$\ddot{y} - aAe^{rt}\dot{y} - cbAe^{rt} = 0, \ z = Ae^{rt}$	y	II.3
34.	$\dot{x} = ay^2 + bz, \quad \dot{y} = cx, \quad \dot{z} = ry$	$\dddot{z} - (ca / r)\dot{z}^2 - cbrz = 0$	n	VII
35.	$\dot{x} = axz + by, \quad \dot{y} = cx, \quad \dot{z} = ry$	$\dddot{z} - az\ddot{z} - cb\dot{z} = 0$	n	VII
36.	$\dot{x} = ayz + bz, \quad \dot{y} = cx, \quad \dot{z} = ry$	$\dddot{z} - acz\dot{z} - cbrz = 0$	n	VII
37.	$\dot{x} = ax^2 + by, \quad \dot{y} = cy, \quad \dot{z} = rx$	$\dot{x} = ax^2 + bAe^{ct}, \ y = Ae^{ct}$	y	III
38.	$\dot{x} = ay^2 + bx, \quad \dot{y} = cy, \quad \dot{z} = rx$	$\dot{x} - bx = aA^2e^{2ct}$	y	II.1
39.	$\dot{x} = ay^2 + by, \quad \dot{y} = cy, \quad \dot{z} = rx$	$x = (aA^2 / 2c)e^{2ct} + (bA / c)e^{ct} + B$	y	S
40.	$\dot{x} = axy + bx, \quad \dot{y} = cy, \quad \dot{z} = rx$	$\dot{x} = (aAe^{ct} + b)x, \ y = Ae^{ct}$	y	I
41.	$\dot{x} = axy + by, \quad \dot{y} = cy, \quad \dot{z} = rx$	$\dot{x} = aAe^{ct}x + bAe^{ct}, \ y = Ae^{ct}$	y	II.1
42.	$\dot{x} = ay^2 + bz, \quad \dot{y} = cy, \quad \dot{z} = rx$	$\ddot{z} - rbz = arA^2e^{2ct}$	y	II.2
43.	$\dot{x} = axz + by, \quad \dot{y} = cy, \quad \dot{z} = rx$	$\dot{z} - (a / 2)z^2 = rbAe^{ct} + B$	y	III
44.	$\dot{x} = ayz + bz, \quad \dot{y} = cy, \quad \dot{z} = rx$	$\ddot{z} - r(aAe^{ct} + b)z = 0, \ y = Ae^{ct}$	y	II.3
45.	$\dot{x} = ax^2 + by, \quad \dot{y} = cy, \quad \dot{z} = ry$	$\dot{x} = ax^2 + bAe^{ct}, \ z = (r / c) + B$	y	III
46.	$\dot{x} = ay^2 + bx, \quad \dot{y} = cy, \quad \dot{z} = ry$	$\dot{x} - bx = aA^2e^{2ct}, \ y = (c / r)z + B$	y	II.1
47.	$\dot{x} = ay^2 + by, \quad \dot{y} = cy, \quad \dot{z} = ry$	$x = (aA^2 / 2c)e^{2ct} + (bA / c)e^{ct} + B$	y	S
48.	$\dot{x} = axy + bx, \quad \dot{y} = cy, \quad \dot{z} = ry$	$\dot{x} = (aAe^{ct} + b)x, \ y = Ae^{ct}$	y	I
49.	$\dot{x} = axy + by, \quad \dot{y} = cy, \quad \dot{z} = ry$	$\dot{x} = aAe^{ct}x + bAe^{ct}, \ z = (r / c) + B$	y	II.1
50.	$\dot{x} = ay^2 + bz, \quad \dot{y} = cy, \quad \dot{z} = ry$	$x = (aA^2 / 2c)e^{2ct} + (brA / c^2)e^{ct}$ $+bBt + C, \ y = Ae^{ct}, z = (rA / c)e^{ct} + B$	y	S
51.	$\dot{x} = axz + by, \quad \dot{y} = cy, \quad \dot{z} = ry$	$\dot{x} - (a / c)(rAe^{ct} + cB)x = brAe^{ct}$	y	II.1
52.	$\dot{x} = ayz + bz, \quad \dot{y} = cy, \quad \dot{z} = ry$	$x = (arA^2 / 2c^2)e^{2ct}$ $+(aAB / c - bra / c^2)e^{ct} + bBt + C$	y	S
53.	$\dot{x} = ax^2 + by, \quad \dot{y} = cz, \quad \dot{z} = rx$	$\dddot{z} - (2a / r)\dot{z}\ddot{z} - rcbz = 0$	n	VII
54.	$\dot{x} = ay^2 + bx, \quad \dot{y} = cz, \quad \dot{z} = rx$	$\dddot{y} - b\ddot{y} - cray^2 = 0$	n	VII
55.	$\dot{x} = ay^2 + by, \quad \dot{y} = cz, \quad \dot{z} = rx$	$\dddot{y} - bcry - cray^2 = 0$	n	VII
56.	$\dot{x} = axy + bx, \quad \dot{y} = cz, \quad \dot{z} = rx$	$\dddot{y} - ay\ddot{y} - b\ddot{y} = 0$	n	VII
57.	$\dot{x} = axy + by, \quad \dot{y} = cz, \quad \dot{z} = rx$	$\dddot{y} - ay\ddot{y} - bcry = 0$	n	VII
58.	$\dot{x} = ay^2 + bz, \quad \dot{y} = cz, \quad \dot{z} = rx$	$\dddot{y} - acr\dot{y} - rby^2 = 0$	n	VII
59.	$\dot{x} = axz + by, \quad \dot{y} = cz, \quad \dot{z} = rx$	$\dddot{y} - (a / c)\dot{y}\ddot{y} - bcry = 0$	n	VII
60.	$\dot{x} = ayz + bz, \quad \dot{y} = cz, \quad \dot{z} = rx$	$\ddot{y} = (ar / 2)y^2 + bry + A$	y	IV

Table A Continued

No.	Systems	Solvable Form / Scalar Equation	Sol.?	Type
61.	$\dot{x} = ax^2 + by,\ \ \dot{y} = cz,\ \ \dot{z} = ry$	$\dot{x} - ax^2 = b(C_1e^{\sqrt{rct}} + C_2e^{-\sqrt{rct}})$	y	III
62.	$\dot{x} = ay^2 + bx,\ \ \dot{y} = cz,\ \ \dot{z} = ry$	$\dot{x} - bx = a(C_1e^{\sqrt{rct}} + C_2e^{-\sqrt{rct}})^2$	y	II.1
63.	$\dot{x} = ay^2 + by,\ \ \dot{y} = cz,\ \ \dot{z} = ry$	$x = (aC_1{}^2e^{2\sqrt{rct}} - aC_2{}^2e^{-2\sqrt{rct}} + 2bC_1e^{\sqrt{rct}}$ $- 2bC_2e^{-\sqrt{rct}}) / 2\sqrt{rc} + 2aCCt + B$	y	S
64.	$\dot{x} = axy + bx,\ \ \dot{y} = cz,\ \ \dot{z} = ry$	$\dot{x} = (aC_1e^{\sqrt{rct}} + aC_2e^{-\sqrt{rct}} + b)x$	y	I
65.	$\dot{x} = axy + by,\ \ \dot{y} = cz,\ \ \dot{z} = ry$	$\dot{x} = (C_1e^{\sqrt{rct}} + C_2e^{-\sqrt{rct}})(ax + b)$	y	I
66.	$\dot{x} = ay^2 + bz,\ \ \dot{y} = cz,\ \ \dot{z} = ry$	$\dot{x} = a(C_1e^{\sqrt{rct}} + C_2e^{-\sqrt{rct}})^2$ $+ (b / c)\sqrt{rc}(C_1e^{\sqrt{rct}} - C_2e^{-\sqrt{rct}})$	y	S
67.	$\dot{x} = axz + by,\ \ \dot{y} = cz,\ \ \dot{z} = ry$	$\dot{x} = (a / c)\sqrt{rc}(C_1e^{\sqrt{rct}} - C_2e^{-\sqrt{rct}})x$ $+ b(C_1e^{\sqrt{rct}} + C_2e^{-\sqrt{rct}})$	y	II.1
68.	$\dot{x} = ayz + bz,\ \ \dot{y} = cz,\ \ \dot{z} = ry$	$y = C_1e^{\sqrt{rct}} + C_2e^{-\sqrt{rct}}$, $z = (\sqrt{rc} / c)(C_1e^{\sqrt{rct}} - C_2e^{-\sqrt{rct}})$	y	I
69.	$\dot{x} = ax^2 + by,\ \ \dot{y} = cz,\ \ \dot{z} = rz$	$\dot{x} = ax^2 + (bcA / r)e^{rt} + bB$	y	III
70.	$\dot{x} = ay^2 + bx,\ \ \dot{y} = cz,\ \ \dot{z} = rz$	$\dot{x} - bx = a((cA / r)e^{rt} + B)^2$	y	II.1
71.	$\dot{x} = ay^2 + by,\ \ \dot{y} = cz,\ \ \dot{z} = rz$	$\dot{x} = a(cAe^{rt} / r + B)^2 + b(cAe^{rt} / r + B)$	y	I
72.	$\dot{x} = axy + bx,\ \ \dot{y} = cz,\ \ \dot{z} = rz$	$\dot{x} = (acAe^{rt} / r + aB + b)x$	y	I
73.	$\dot{x} = axy + by,\ \ \dot{y} = cz,\ \ \dot{z} = rz$	$\dot{x} = a(cAe^{rt} / r + B)x + b(cAe^{rt} / r + B)$	y	II.1
74.	$\dot{x} = ay^2 + bz,\ \ \dot{y} = cz,\ \ \dot{z} = rz$	$\dot{x} = a(cAe^{rt} / r + B)^2 + bAe^{rt}$	y	I
75.	$\dot{x} = axz + by,\ \ \dot{y} = cz,\ \ \dot{z} = rz$	$\dot{x} = aAe^{rt}x + b(cAe^{rt} / r + B)$	y	II.1
76.	$\dot{x} = ayz + bz,\ \ \dot{y} = cz,\ \ \dot{z} = rz$	$\dot{x} = Ae^{rt}(acAe^{rt} / r + aB + b)$	y	I
77.	$\dot{x} = ax^2 + by,\ \ \dot{y} = cy,\ \ \dot{z} = rz$	$\dot{x} = ax^2 + bAe^{ct}$, $y = Ae^{ct}$, $z = Be^{rt}$	y	III
78.	$\dot{x} = ay^2 + bx,\ \ \dot{y} = cy,\ \ \dot{z} = rz$	$\dot{x} - bx = aA^2e^{2ct}$	y	II.1
79.	$\dot{x} = ay^2 + by,\ \ \dot{y} = cy,\ \ \dot{z} = rz$	$x = (aA^2 / 2c)e^{2ct} + bAe^{ct} / c + B$	y	S
80.	$\dot{x} = axy + bx,\ \ \dot{y} = cy,\ \ \dot{z} = rz$	$\dot{x} = (aAe^{ct} + b)x$	y	I
81.	$\dot{x} = axy + by,\ \ \dot{y} = cy,\ \ \dot{z} = rz$	$\dot{x} = aAe^{ct}x + bAe^{ct}$	y	II.1
82.	$\dot{x} = ay^2 + bz,\ \ \dot{y} = cy,\ \ \dot{z} = rz$	$x = (aA^2 / 2c)e^{2ct} + bAe^{rt} / r + B$	y	S
83.	$\dot{x} = axz + by,\ \ \dot{y} = cy,\ \ \dot{z} = rz$	$\dot{x} = aBe^{rt}x + bAe^{ct}$	y	II.1
84.	$\dot{x} = ayz + bz,\ \ \dot{y} = cy,\ \ \dot{z} = rz$	$x = aABe^{(r+c)t} / (r + c) + bBe^{rt} / r + C$	y	S
85.	$\dot{x} = ax + by,\ \ \dot{y} = cx^2,\ \ \dot{z} = rx$	$\ddot{x} = a\dot{x} + bcx^2$	n	VI
86.	$\dot{x} = ax + bz,\ \ \dot{y} = cx^2,\ \ \dot{z} = rx$	$\ddot{x} - a\dot{x} - brx = 0$	y	II.2

Table A Continued

No.	Systems	Solvable Form / Scalar Equation	Sol.?	Type
87.	$\dot{x} = ay + bz,\ \dot{y} = cx^2,\ \dot{z} = rx$	$\ddot{x} = brx + acx^2$	y	IV
88.	$\dot{x} = ax + by,\ \dot{y} = cy^2,\ \dot{z} = rx$	$\dot{x} = ax - b/(ct + A)$	y	II.1
89.	$\dot{x} = ax + bz,\ \dot{y} = cy^2,\ \dot{z} = rx$	$\ddot{x} - a\dot{x} - brx = 0$	y	II.2
90.	$\dot{x} = ay + bz,\ \dot{y} = cy^2,\ \dot{z} = rx$	$\ddot{x} - brx = ac/(ct + A)^2$	y	II.2
91.	$\dot{x} = ax + by,\ \dot{y} = cz^2,\ \dot{z} = rx$	$\dddot{z} - a\ddot{z} - cbrz^2 = 0$	n	VII
92.	$\dot{x} = ax + bz,\ \dot{y} = cz^2,\ \dot{z} = rx$	$\ddot{z} - a\dot{z} - brz = 0$	y	II.2
93.	$\dot{x} = ay + bz,\ \dot{y} = cz^2,\ \dot{z} = rx$	$\dddot{z} - a\dot{z} - carz^2 = 0$	n	VII
94.	$\dot{x} = ax + by,\ \dot{y} = cxy,\ \dot{z} = rx$	$\ddot{x} = a\dot{x} + cx\dot{x} - cax^2$	n	VI
95.	$\dot{x} = ax + bz,\ \dot{y} = cxy,\ \dot{z} = rx$	$\ddot{x} - a\dot{x} - brx = 0,\ y = Ae^{(c/r)z}$	y	II.2
96.	$\dot{x} = ay + bz,\ \dot{y} = cxy,\ \dot{z} = rx$	$x^2 = 2aAe^{(c/r)z}/c + bz^2/r + B$	y	S
97.	$\dot{x} = ax + by,\ \dot{y} = cxz,\ \dot{z} = rx$	$\dot{x} - ax = bcz^2/2r + bA,\ \dot{z} = rx$	n	VI
98.	$\dot{x} = ax + bz,\ \dot{y} = cxz,\ \dot{z} = rx$	$\ddot{z} - a\dot{z} - brz = 0$	y	II.2
99.	$\dot{x} = ay + bz,\ \dot{y} = cxz,\ \dot{z} = rx$	$rx^2 = acz^3/3r + bz^2 + 2Az + 2B,$ $y = cz^2/2r + A$	y	S
100.	$\dot{x} = ax + by,\ \dot{y} = cyz,\ \dot{z} = rx$	$\dddot{z} - a\ddot{z} - cz\ddot{z} + caz\dot{z} = 0$	n	VII
101.	$\dot{x} = ax + bz,\ \dot{y} = cyz,\ \dot{z} = rx$	$\ddot{z} - a\dot{z} - brz = 0$	y	II.2
102.	$\dot{x} = ay + bz,\ \dot{y} = cyz,\ \dot{z} = rx$	$\dddot{z} - cz\ddot{z} - rb\dot{z} + cbrz^2 = 0$	n	VII
103.	$\dot{x} = ax + by,\ \dot{y} = cx^2,\ \dot{z} = ry$	$\ddot{x} - a\dot{x} - cbx^2 = 0$	n	VI
104.	$\dot{x} = ax + bz,\ \dot{y} = cx^2,\ \dot{z} = ry$	$\dddot{x} - a\ddot{x} - crbx^2 = 0$	n	VII
105.	$\dot{x} = ay + bz,\ \dot{y} = cx^2,\ \dot{z} = ry$	$\dddot{x} - acx\dot{x} - crbx^2 = 0$	n	VII
106.	$\dot{x} = ax + by,\ \dot{y} = cy^2,\ \dot{z} = ry$	$\dot{x} - ax = -b/(ct + B)$	y	II.1
107.	$\dot{x} = ax + bz,\ \dot{y} = cy^2,\ \dot{z} = ry$	$\dot{x} - ax = (br/c)\ln(-1/A(ct + B))$	y	II.1
108.	$\dot{x} = ay + bz,\ \dot{y} = cy^2,\ \dot{z} = ry$	$\dot{x} = -a/(ct + B)$ $+ (br/c)\ln(-1/A(ct + B))$	y	I
109.	$\dot{x} = ax + by,\ \dot{y} = cz^2,\ \dot{z} = ry$	$\dot{y} = c((3r/2c)y^2 + 3A/c)^{2/3}$	y	I*
110.	$\dot{x} = ax + bz,\ \dot{y} = cz^2,\ \dot{z} = ry$	$\dot{x} - ax = b((3r/2c)y^2 + 3A/c)^{1/3}$ $\dot{y} = c((3r/2c)y^2 + 3A/c)^{2/3}$	y	I*
111.	$\dot{x} = ay + bz,\ \dot{y} = cz^2,\ \dot{z} = ry$	$\dot{x} = ay + b((3r/2c)y^2 + 3A/c)^{1/3}$ $\dot{y} = c((3r/2c)y^2 + 3A/c)^{2/3}$	y	I*
112.	$\dot{x} = ax + by,\ \dot{y} = cxy,\ \dot{z} = ry$	$\ddot{x} - (a + cx)\dot{x} + cax^2 = 0$	n	VI
113.	$\dot{x} = ax + bz,\ \dot{y} = cxy,\ \dot{z} = ry$	$\dddot{x} - (a + cx)\ddot{x} + cax\dot{x} = 0$	n	VII

Table A Continued

No.	Systems	Solvable Form / Scalar Equation	Sol.?	Type
114.	$\dot{x} = ay + bz,\ \ \dot{y} = cxy,\ \ \dot{z} = ry$	$(cax + br)\dddot{x} =$ $(ac^2x^2 + brcx + ca\dot{x})\ddot{x} = 0$	n	VII
115.	$\dot{x} = ax + by,\ \ \dot{y} = cxz,\ \ \dot{z} = ry$	$x\dddot{x} - (ax + 1)\ddot{x} + a\dot{x}^2 = crx^2\dot{x} + crax^3$	n	VII
116.	$\dot{x} = ax + bz,\ \ \dot{y} = cxz,\ \ \dot{z} = ry$	$\dddot{x} - a\ddot{x} = crx\dot{x} - crax^2$	n	VII
117.	$\dot{x} = ay + bz,\ \ \dot{y} = cxz,\ \ \dot{z} = ry$	$z\,\dddot{z} - \dot{z}\ddot{z} - acz^2\dot{z} - cbrz^3 = 0$	n	VII
118.	$\dot{x} = ax + by,\ \ \dot{y} = cyz,\ \ \dot{z} = ry$	$\dot{x} = ax + by,\ \ \dot{z} = cz^2 / 2 + rA$	y	II.1
119.	$\dot{x} = ax + bz,\ \ \dot{y} = cyz,\ \ \dot{z} = ry$	$\dot{x} = ax + bz,\ \ \dot{z} = cz^2 / 2 + rA$	y	II.1
120.	$\dot{x} = ay + bz,\ \ \dot{y} = cyz,\ \ \dot{z} = ry$	$\dot{x} = ay + bz,\ \ \dot{z} = cz^2 / 2 + rA$	y	I
121.	$\dot{x} = ax + by,\ \ \dot{y} = cx^2,\ \ \dot{z} = rz$	$\ddot{x} = a\dot{x} + bcx^2,\ \ z = Ae^{rt}$	n	VI
122.	$\dot{x} = ax + bz,\ \ \dot{y} = cx^2,\ \ \dot{z} = rz$	$\dot{x} - ax = bAe^{rt}$	y	II.1
123.	$\dot{x} = ay + bz,\ \ \dot{y} = cx^2,\ \ \dot{z} = rz$	$\ddot{x} - acx^2 = bAe^{rt}$	n	VII
124.	$\dot{x} = ax + by,\ \ \dot{y} = cy^2,\ \ \dot{z} = rz$	$\dot{x} = ax - b / (ct + A)$	y	II.1
125.	$\dot{x} = ax + bz,\ \ \dot{y} = cy^2,\ \ \dot{z} = rz$	$\dot{x} = ax + bAe^{rt}$	y	II.1
126.	$\dot{x} = ay + bz,\ \ \dot{y} = cy^2,\ \ \dot{z} = rz$	$x = -(a / c)\ln(t + B / c) + bAe^{rt} / r + C$	y	S
127.	$\dot{x} = ax + by,\ \ \dot{y} = cz^2,\ \ \dot{z} = rz$	$\dot{x} - ax = bcA^2e^{2rt} / 2r + bB$	y	II.1
128.	$\dot{x} = ax + bz,\ \ \dot{y} = cz^2,\ \ \dot{z} = rz$	$\dot{x} - ax = bAe^{rt}$	y	II.1
129.	$\dot{x} = ay + bz,\ \ \dot{y} = cz^2,\ \ \dot{z} = rz$	$x = acA^2e^{2rt} / 4r^2 + bAe^{rt} / r + aBt + C$	y	S
130.	$\dot{x} = ax + by,\ \ \dot{y} = cxy,\ \ \dot{z} = rz$	$\ddot{x} = a\dot{x} + cx\dot{x} - cax^2$	n	VI
131.	$\dot{x} = ax + bz,\ \ \dot{y} = cxy,\ \ \dot{z} = rz$	$\dot{x} - ax = bAe^{rt}$	y	II.1
132.	$\dot{x} = ay + bz,\ \ \dot{y} = cxy,\ \ \dot{z} = rz$	$\ddot{x} = cx\dot{x} - cbAe^{rt}x + brAe^{rt}$	n	VII
133.	$\dot{x} = ax + by,\ \ \dot{y} = cxz,\ \ \dot{z} = rz$	$\ddot{x} = a\dot{x} + bcAe^{rt}x$	y	II.3
134.	$\dot{x} = ax + bz,\ \ \dot{y} = cxz,\ \ \dot{z} = rz$	$\dot{x} = ax + bAe^{rt}$	y	II.1
135.	$\dot{x} = ay + bz,\ \ \dot{y} = cxz,\ \ \dot{z} = rz$	$\ddot{x} = acAe^{rt}x + rbAe^{rt}$	y	II.3
136.	$\dot{x} = ax + by,\ \ \dot{y} = cyz,\ \ \dot{z} = rz$	$\dot{x} = ax + bBe^{(cA/r)e^{rt}}$	y	I
137.	$\dot{x} = ax + bz,\ \ \dot{y} = cyz,\ \ \dot{z} = rz$	$\dot{x} = ax + bAe^{rt}$	y	II.1
138.	$\dot{x} = ay + bz,\ \ \dot{y} = cyz,\ \ \dot{z} = rz$	$\dot{x} = aBe^{(cA/r)e^{rt}} + bAe^{rt}$	y	I

Note: * means that the integration is not easy.

Appendix B. Formulas

(1). $\int \frac{e^{ax}}{x}dx = \ln x + ax + \frac{(ax)^2}{2\times 2!} + \frac{(ax)^3}{3\times 3!} + \ldots$

(2). $\int \frac{e^{ax}}{x^n}dx = \frac{1}{n-1}\left(-\frac{e^{ax}}{x^{n-1}} + a\int \frac{e^{ax}}{x^{n-1}}dx\right),$ $\qquad n \geq 2$ integer.

(3). $\int \frac{1}{ax^2+bx+c}dx = \frac{1}{\sqrt{b^2-4ac}}\ln\left(\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right),$ $\qquad$ if $b^2-4ac>0$

$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{b^2-4ac}}\ln\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right),$ $\qquad$ if $b^2-4ac<0$

$\int \frac{1}{ax^2+bx+c}dx = -\frac{1}{2ax+b},$ $\qquad$ if $b^2-4ac=0$

(4). The cononical form of a special 2nd order o.d.e. is:

$$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + (\beta e^{\lambda x} + \delta)y = 0$$

The general solution is:

$$y = e^{-\frac{\alpha x}{2}}\left[C_1 J_\nu\left(\frac{2\sqrt{\beta}}{\lambda}e^{\frac{\lambda x}{2}}\right) + C_2 Y_\nu\left(\frac{2\sqrt{\beta}}{\lambda}e^{\frac{\lambda x}{2}}\right)\right], \qquad \nu = \frac{\sqrt{\alpha^2-4\delta}}{\lambda}$$

where J_ν *and* Y_ν are Bessel functions.

(5). A homogeneous linear equation of the 2nd order has the form:

$$f_2(x)y''(x) + f_1(x)y'(x) + f_0(x)y = 0 \qquad \text{(B.1)}$$

Assuming:

$$y(x) = u(x)e^{-\frac{1}{2}\int\frac{f_1(x)}{f_2(x)}dx}$$

there results from equation (B.1) the canonical (or normal) form

$$u''(x) + f(x)u(x) = 0 \qquad \text{(B.2)}$$

where $\quad f(x) = \frac{f_0(x)}{f_2(x)} - \frac{1}{4}(\frac{f_1(x)}{f_2(x)})^2 - \frac{1}{2}\frac{d}{dx}(\frac{f_1(x)}{f_2(x)})$

(6). The equation (B.2) is an another canonical form:

$$y''(x) - (\alpha e^{2\lambda x} + \beta e^{\lambda x} + \gamma)y(x) = 0 \qquad \text{(B.3)}$$

After the transformation: $z(x) = e^{\lambda x}$, $w(z) = z^{-\frac{\sqrt{\gamma}}{\lambda}y}$, where $k = \frac{\sqrt{\gamma}}{\lambda}$, eq. (B.3) becomes:

$$\lambda^2 zw''(z) + \lambda^2(2k+1)w'(z) - (\alpha z + \beta)w(z) = 0 \qquad \text{(B.4)}$$

(7).The general form:

$$(a_2x+b_2)y''(x) + (a_1x+b_1)y'(x) + (a_0x+b_0)y(x) = 0$$

where $a_i, b_i, i = 0,1,2$ are arbitrary constants. Let

$$D^2 = a_1^2 - 4a_0a_2,\ h = \frac{D - a_1}{2a_2},\ A(h) = 2a_2h + a_1,\ \sigma = -\frac{a_2}{A(h)}$$

$$\mu = -\frac{b_2}{a_2},\ B(h) = b_2h^2 + b_1h + b_0,\ \xi = \frac{x-\mu}{\sigma},\ a' = \frac{B(h)}{A(h)},\ b' = (a_2b_1 - a_1b_2)a_2^{-2}$$

Then the solution of the equation (B.4) can be written as:

$$y(x) = e^{hx}\Gamma(a, b; \xi)$$

here $\Gamma(a, b; \xi)$ be an arbitrary solution of the degenerate hypergeometric equation:

$$xy''(x) + (b - x)y' - ay = 0$$

(8). The canonical form:

$$y''(x) + (ae^x - b)y(x) = 0$$

the general solution of the equation is:

$$y(x) = C_1J_{2\sqrt{b}}(2\sqrt{a}e^{\frac{x}{2}}) + C_2Y_{2\sqrt{b}}(2\sqrt{a}e^{\frac{x}{2}})$$

where J_ν and Y_ν are Bessel function.

(9) The cononical form $y''(x) + \alpha e^{\lambda x}y(x) = 0$ has the general solution :

$$y(x) = C_1J_0(\frac{2\sqrt{\alpha}}{\lambda}e^{\frac{\lambda x}{2}}) + C_2Y_0(\frac{2\sqrt{\alpha}}{\lambda}e^{\frac{\lambda x}{2}})$$

where J_0 and Y_0 are Bessel functions

Appendix C. Numerical solutions of some chaotic systems

1. Sprott B

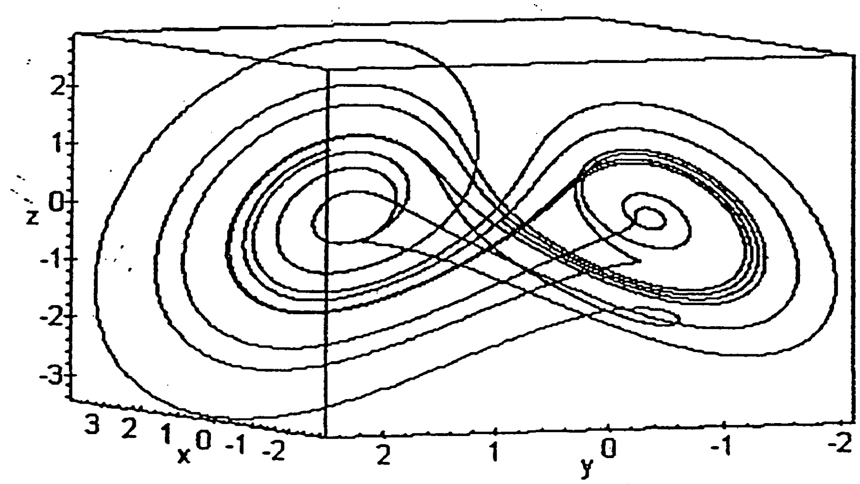
sprott-B.ms

```
> with(DEtools);
> fhn:=[diff(x(t),t)=y*z,
> diff(y(t),t)=x-y,
> diff(z(t),t)=1-x*y];
```

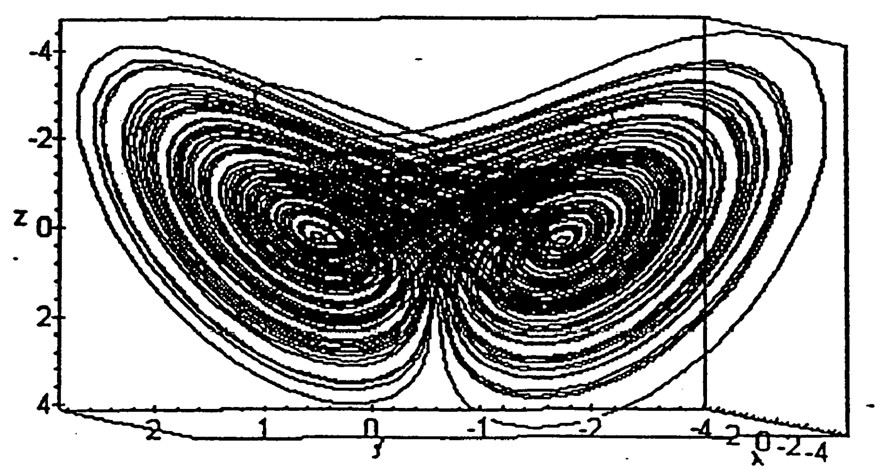
$$fhn := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = z\,y, \frac{\partial}{\partial t}\mathrm{y}(t) = x - y, \frac{\partial}{\partial t}\mathrm{z}(t) = 1 - x\,y \right]$$

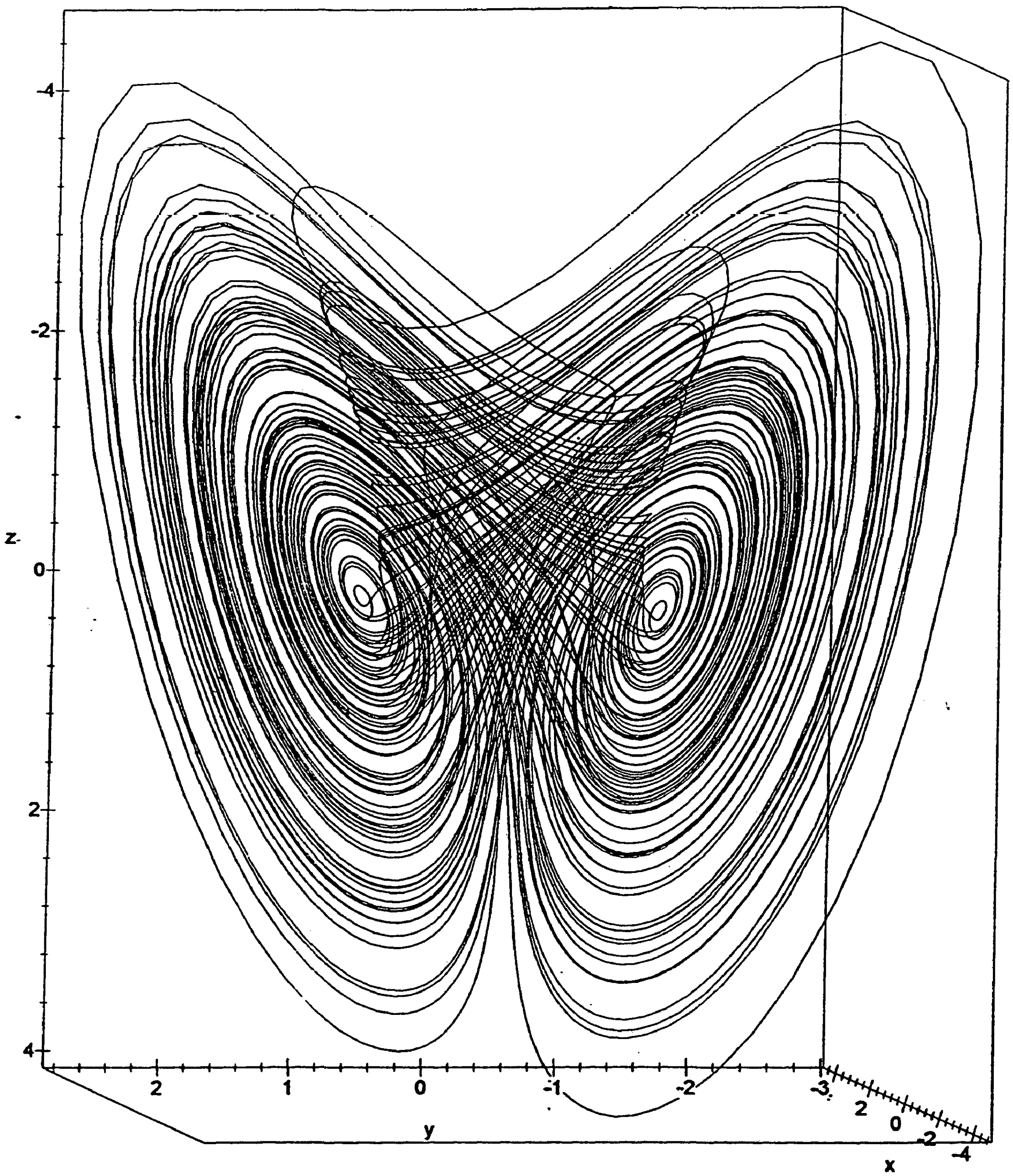
```
> DEplot(fhn,[x,y,z],990..1000,{[0,1.6,1,1]},stepsize=0.1);
```

case1: t=0..100,step=0.01

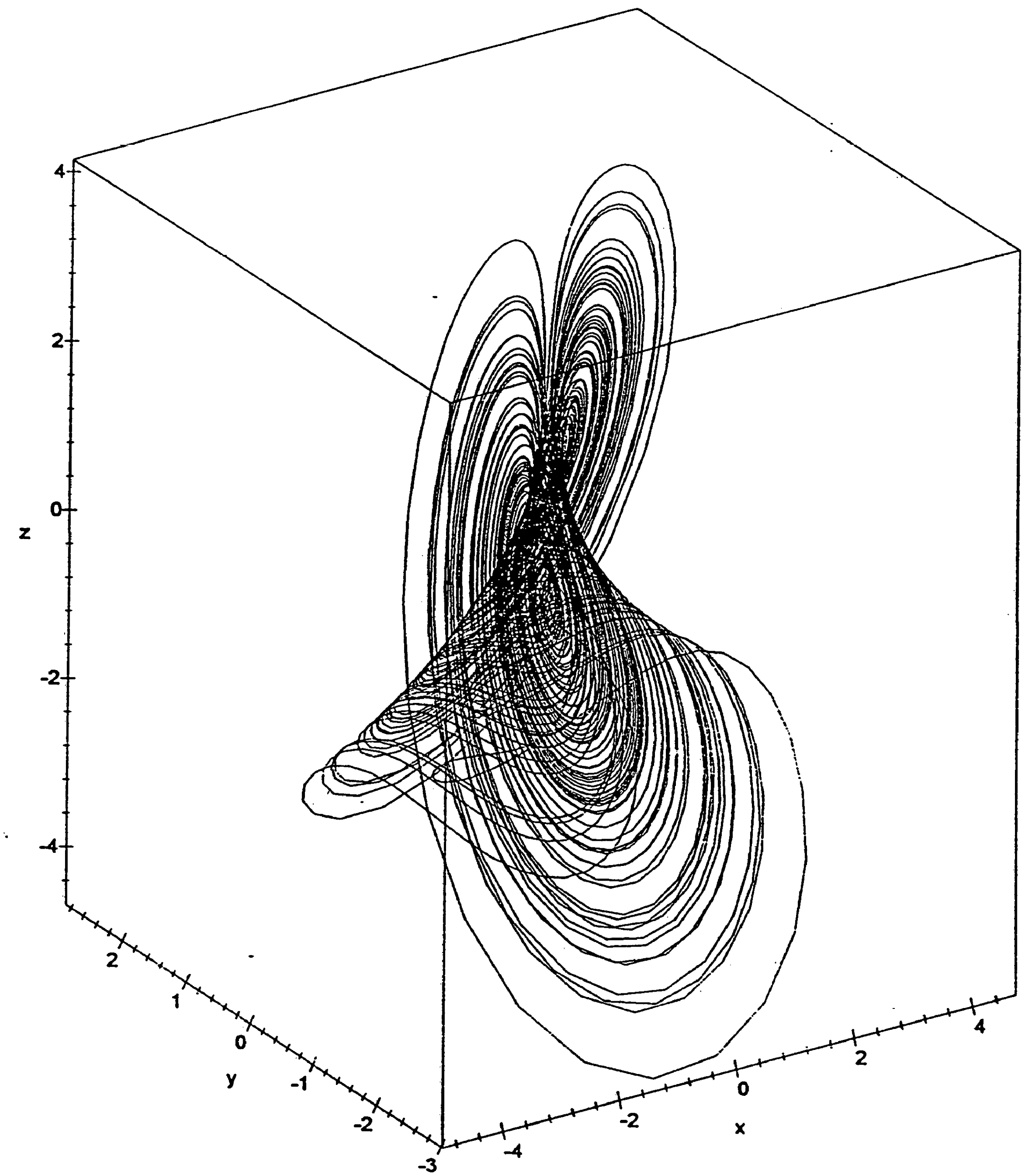


case2: t=0..1000,step=0.1





-4
-2
z
0
2
4
2
1
0
-1
-2
-3
y
2
0
-2
-4
x


4
2
0
z
-2
-4
2
1
0
y
-1
-2
-3
-4
-2
0
x
2
4

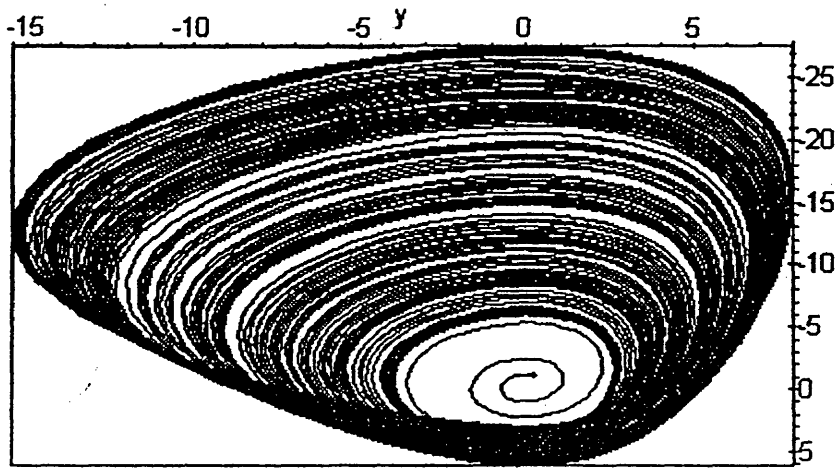
2. Sprott N

```
> with(DEtools);
> fhn:=[diff(x(t),t)=-2*y,
> diff(y(t),t)=x+z*z,
> diff(z(t),t)=1+y-2*z];
```

$$fhn := \left[\frac{\partial}{\partial t} x(t) = -2\,y, \frac{\partial}{\partial t} y(t) = x + z^2, \frac{\partial}{\partial t} z(t) = 1 + y - 2\,z \right]$$

```
> DEplot(fhn,[x,y,z],0..1000,{[0,-1.1,0.2,0.1]},stepsize=0.1);
```

case1: t=0..1000,step=0.1 [0,-1.1,0.2,0.1]

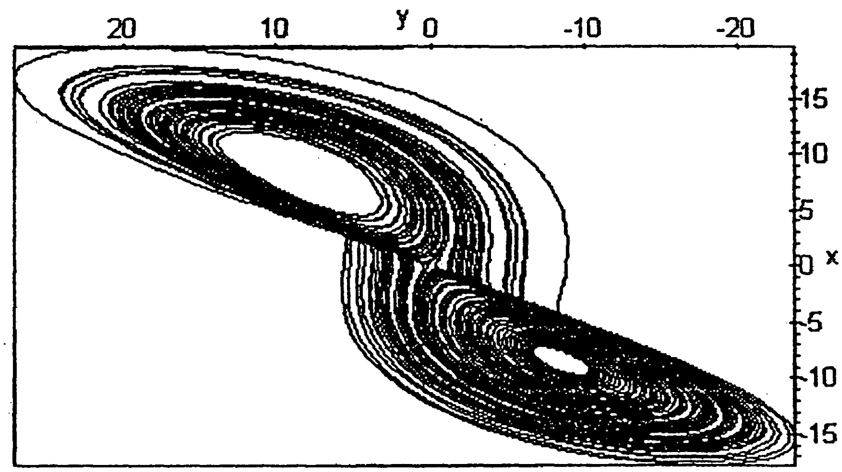


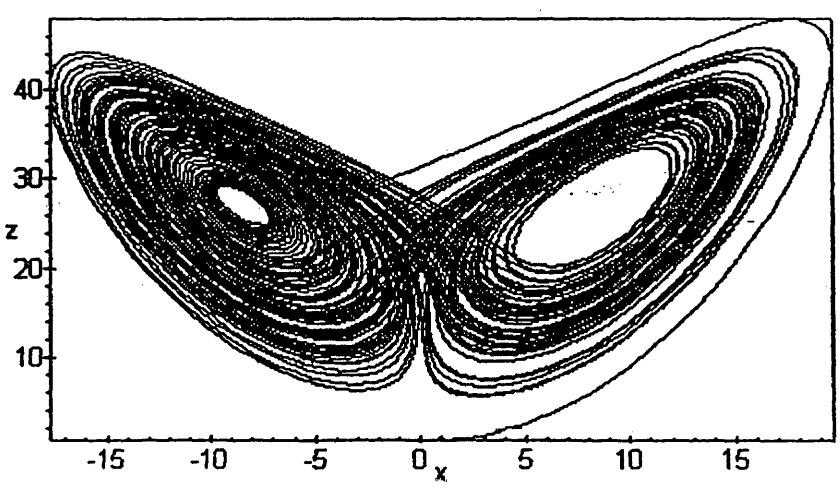
3. Lorenz system

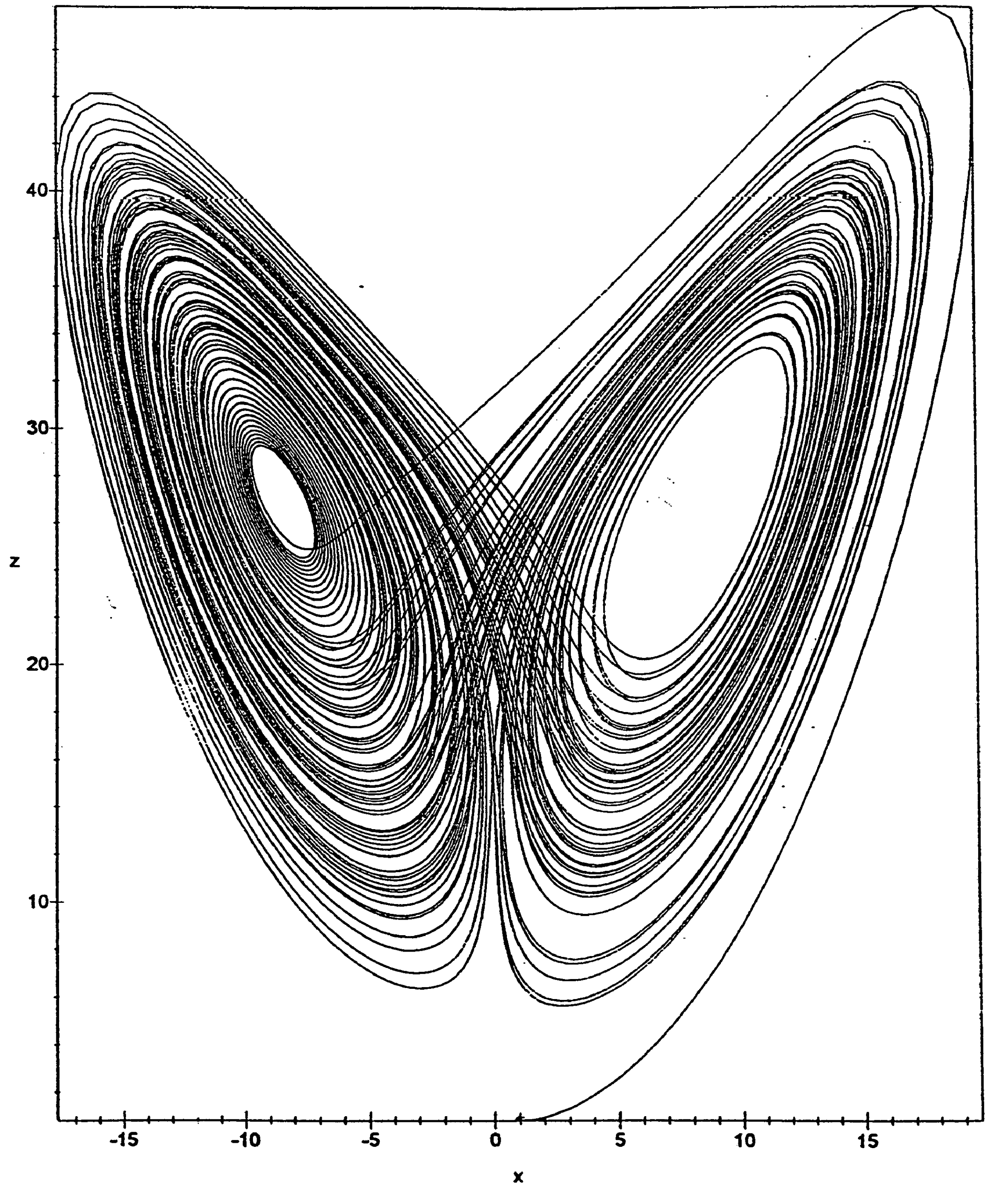
```
> with(DEtools);
> fhn:=[diff(x(xi),xi)=-sigma*x+sigma*y,
> diff(y(xi),xi)=-x*z+r*x-y,
> diff(z(xi),xi)=x*y-b*z];
```

$$fhn := \left[\frac{\partial}{\partial\xi}\mathrm{x}(\xi) = -10\,x + 10\,y, \frac{\partial}{\partial\xi}\mathrm{y}(\xi) = -x\,z + 28\,x - y, \frac{\partial}{\partial\xi}\mathrm{z}(\xi) = x\,y - \frac{8}{3}\,z\right]$$

```
> sigma:=10:
> b:=8/3:
> r:=28:
> DEplot(fhn,[x,y,z],0..100,{[0,1,0,1]},stepsize=0.01);
```






40
30
z
20
10
-15
-10
-5
0
5
10
15
x

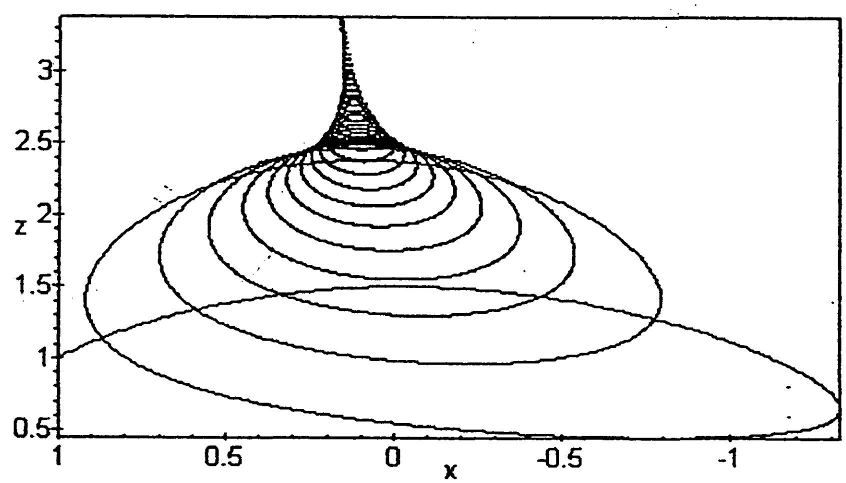
4. Rössler system

```
> with(DEtools);
> a:=0.1:
> b:=1.5:
> c:=0.3:

> fhn:=[diff(x(t),t)=-y-z,

> diff(y(t),t)=x+a*y,

> diff(z(t),t)=b+x*z-c*z];
```

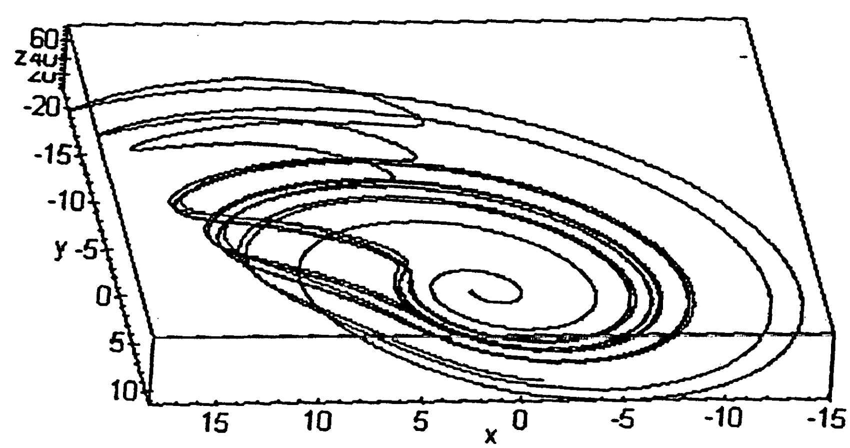
$$fhn := \left[\frac{\partial}{\partial t} \mathrm{x}(t) = -y - z, \frac{\partial}{\partial t} \mathrm{y}(t) = x + .1\, y, \frac{\partial}{\partial t} \mathrm{z}(t) = 1.5 + x\, z - .3\, z \right]$$

```
> DEplot(fhn,[x,y,z],0..100,{[0,1,0,1]},stepsize=0.01);
```

case1: a = 0.05, b = 0.5, c = 0.3



case2: a = 0.4, b = 2.5 ,c = 10



References:

[1] E. N. Lorenz, Deterministic Nonperiodic Flow, J. Atmos. Sci.. 20, 130(1963).

[2] O.E.Rössler, An Equation for Continuous Chaos, Physics. Letters. 57A, 397(1976)

[3] J. C. Sprott, Some simple chaotic flows, Physical Review E, V50(2), R647(1994).

[4] J. Heidel, F. Zhang, Chaos in three Dimensional Quadratic Systems, 24th Middle American Mathematics Conference Lincohn NE, Oct, 1995.

[5] J.H.Hubbard, B.H.West, Differential Equations: A Dynamical System Approach-High Dimensional Systems. Springer-Verlag, 1995

[6] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Spinger-Verlag, 1990.

[7] Martin Braun, Differential Equations and Their Applications, Fourth Edition, Springer- Verlag, 1993

[8] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, EMA, Vol.54, Cambridge, 1995.

[9] J.M. Nese,Quantifying Local Predictability in Phase Space, Physica D 35 North Holland, Amsterdam (1989) P237-250

[10] Polianin, A.D.(Andrei Dmitrievich), Handbook of exact solutions for ordinary differential equations /Andrei D. Polyanin, Valentin F. Zaitsev, Boca Raton:CRC Press[1995]