NON-CLASSICAL DIFFUSION EQUATIONS
RELATED TO BIRTH-DEATH PROCESSES
WITH TWO BOUNDARIES

By

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Abstract. A pair of forward and backward diffusion equations is considered. In the forward equation, boundary values appear in the differential equation, and in the backward equation, boundary values are related to average values of the solution in the interior of the domain. The forward equation can be regarded as a diffusion approximation to a type of birth-death process with returns to the interior, or as a heat equation in one dimension where heat flowing out from the boundaries is returned to the interior. Existence and uniqueness theorems are proved, and some properties of the associated eigenvalues and eigenfunctions are deduced. An expression for the steady-state solution is obtained. Some information on the goodness of the diffusion approximation is also obtained.

1. Introduction. In [3], the second author studied a birth-death process with two boundaries. When the number of states becomes very large, the governing system of ordinary differential equations can be replaced by a single partial differential equation, the so-called diffusion approximation. For the equations in [3], the diffusion approximation takes the form

$$
\begin{aligned}
\frac{\partial \varphi}{\partial \tau}(x, \tau) = {} & \frac{\sigma^2}{2} \frac{\partial^2 \varphi}{\partial x^2}(x, \tau) - \beta \frac{\partial \varphi}{\partial x}(x, \tau) \\
& + \frac{\sigma^2}{2}\left[F_L'(x)\frac{\partial \varphi}{\partial x}(0, \tau) - F_R'(x)\frac{\partial \varphi}{\partial x}(1, \tau) \right], \quad 0 \le x \le 1, \ \tau \ge 0,
\end{aligned}
\tag{1.1}
$$

where F_L' and F_R' are derivatives of probability distribution functions and where the solution must satisfy the boundary conditions $\varphi(0, \tau) = \varphi(1, \tau) = 0$. If F_L or F_R has discontinuities, the corresponding derivative in (1.1) gives rise to a discontinuity condition on $\partial \varphi / \partial x$ which is described more precisely in Sec. 2. Equation (1.1) could also describe

a one-dimensional heat flow in which heat flowing out from the left and right boundaries is returned to the interior according to distribution functions F_L and F_R respectively.

The presence of the boundary values of $\partial\varphi/\partial x$ at $x = 0$ and $x = 1$ in (1.1) produces an equation of a type that seems not to have been considered in the literature. A special case of (1.1) was solved in [2]. The main object of this paper is to study the existence, uniqueness, and properties of solutions of (1.1) and its adjoint. Because both steady-state and time-dependent solutions are of interest in some applications, it is useful to examine the properties of the eigenvalues and eigenfunctions of the ordinary differential equation which is obtained from (1.1) by separating variables or by use of the Laplace transform. The convergence of the residue expansion that results from inversion of the Laplace transform has been established in [12].

We refer to (1.1) as the forward problem. The adjoint to (1.1), which we refer to as the backward problem, is associated with transition probabilities for the solution of a stochastic differential equation with jump returns from the boundary. Further details of this connection may be found in Mansourati's thesis [12]. Because of biorthogonality relations between eigenfunctions of the forward and backward problems, the eigenfunctions of the backward problem figure in the solution of the forward problem. Consequently we study the two problems together.

In the case of the backward problem, there have been investigations of similar problems. The earliest instance appears to be in a paper by Wilder [14]. Of the recent literature, the work of Cole [6], Day [7], and Friedman [10] is close. Friedman's work [10] might initially appear to include some of our results. However, in order to establish existence and uniqueness of a solution, Friedman had to make an assumption on F_L and F_R which would always be violated when F_L and F_R are probability distribution functions. We avoid Friedman's condition by using a different method to establish existence and uniqueness.

In a general way, our discussion can be regarded as a development of one of the threads in Feller's classic papers [8, 9]. A recent paper on another type of diffusion process with jump returns to the interior is that of Baccelli and Fayolle [1].

After explaining the connection with birth-death processes in Sec. 2, we establish the existence and uniqueness of solutions of the forward and backward problems in Sec. 3. In Sec. 4 we deduce some facts about the eigenvalues and eigenfunctions associated with the problem. In Sec. 5 we give two examples and in Sec. 6 we compare some results obtained from the diffusion approximation with exact results for a birth-death process with a finite number of states.

2. Diffusion approximation to a birth-death process.

In [3] Campbell treats a birth-death process which is governed by the system of ordinary differential equations

$$\frac{dp_k}{d\tau} = \mu L_k p_1 + \lambda p_{k-1} - (\lambda + \mu)p_k + \mu p_{k+1} + \lambda R_k p_m \quad (k = 1, \ldots, m), \qquad (2.1)$$

where we define $p_0(\tau) = p_{m+1}(\tau) = 0$. We interpret $p_k(\tau)$ as the probability that the system is in state k ($k = 1, 2, \ldots, m$) at time τ. Births occur at rate λ and deaths occur at rate μ, where λ and μ are positive constants. If the system is in state 1 and

a death occurs, it moves to state k with probability L_k, and if the system is in state m and a birth occurs, it moves to state k with probability R_k. Thus $L_k \geq 0, R_k \geq 0$, and $\sum L_k = \sum R_k = 1$. It is these boundary conditions at states 1 and m that distinguish this from a conventional birth-death process.

This process could describe an inventory in which the supplier holds a "fire sale" to dispose of stock if the warehouse is full or buys from another supplier when the warehouse is empty. It could describe a buffer at a node in a communications network, where some messages are removed and sent by an alternate route when the buffer fills, and where messages from other buffers are brought in for alternate routing whenever the buffer becomes empty. Other applications are possible.

When the number of states m is large, the system of ordinary differential equations can be approximated by a partial differential equation. Some idea of the goodness of the approximation involved can be obtained from [2], where a special case of (2.1) is solved and the corresponding partial differential equation is also solved. Additional results on this question appear in Sec. 6. For a more general discussion of diffusion approximations, see [5, 11, 13].

Let $(m+1)^{-1} = \Delta x$, $x_k = k\Delta x$, $p_k(\tau) = \varphi(x_k, \tau)\Delta x$, and define

$$F_L^m(x) = \sum_{k=1}^m L_k H(x - x_k), \qquad F_R^m(x) = \sum_{k=1}^m R_k H(x - x_k),$$

where H denotes the unit step function ($H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 0$) so that F_L^m and F_R^m are the probability distribution functions corresponding to $(L_1, \ldots, L_m)$ and $(R_1, \ldots, R_m)$. If we write

$$\Delta F_L^m(x_k) = F_L^m(x_k) - F_L^m(x_{k-1}) = L_k$$

and similarly for ΔF_R^m, and recall that $p_0(\tau) = p_{m+1}(\tau) = 0$, (2.1) becomes

$$
\begin{aligned}
\frac{d\varphi}{d\tau} = {}& \lambda(\Delta x)^2 \frac{[\varphi(x_{k+1}, \tau) - 2\varphi(x_k, \tau) + \varphi(x_{k-1}, \tau)]}{(\Delta x)^2} \\
& - (\lambda - \mu)\Delta x \frac{[\varphi(x_{k+1}, \tau) - \varphi(x_k, \tau)]}{\Delta x} + \mu(\Delta x)^2 \frac{\Delta F_L^m(x_k)}{\Delta x} \frac{[\varphi(x_1, \tau) - \varphi(0, \tau)]}{\Delta x} \\
& - \lambda(\Delta x)^2 \frac{\Delta F_R^m(x_k)}{\Delta x} \frac{[\varphi(1, \tau) - \varphi(x_m, \tau)]}{\Delta x}.
\end{aligned}
$$
(2.2)

In the diffusion approximation we replace the difference quotients by derivatives, $\lambda(\Delta x)^2$ and $\mu(\Delta x)^2$ by $\sigma^2/2$, and $(\lambda - \mu)\Delta x$ by β. We assume that, as $m \to \infty$, F_L^m and F_R^m approach limits F_L and F_R which are absolutely continuous with bounded derivatives f_L and f_R except at a finite number N (possibly zero) of interior points a_i ($i = 1, 2, \ldots, N$) where F_L jumps by an amount ρ_i and F_R jumps by an amount γ_i. The diffusion approximation then gives

$$
\begin{aligned}
\frac{\partial \varphi}{\partial \tau}(x, \tau) = {}& \frac{\sigma^2}{2} \frac{\partial^2 \varphi}{\partial x^2}(x, \tau) - \beta \frac{\partial \varphi}{\partial x}(x, \tau) \\
& + \frac{\sigma^2}{2} \left[f_L(x) \frac{\partial \varphi}{\partial x}(0, \tau) - f_R(x) \frac{\partial \varphi}{\partial x}(1, \tau) \right]
\end{aligned}
$$
(2.3)

for all $x \in (0,1)$ except $x = a_1, \ldots, a_N$. For notational convenience, we define $a_0 = 0$, $a_{N+1} = 1$, and assume $a_i < a_{i+1}$ for each i. Because p_0 and p_{m+1} vanish, we have boundary conditions

$$\varphi(0, \tau) = \varphi(1, \tau) = 0. \tag{2.4}$$

At points where F_L or F_R is discontinuous, replace $\varphi(x_{k+1}, \tau) - \varphi(x_k, \tau)$ by

$$\frac{\partial \varphi}{\partial x}(x_k, \tau)\Delta x,$$

recognize that $\partial \varphi / \partial x$ will be discontinuous there, and rewrite (2.2) as

$$\frac{\partial \varphi}{\partial \tau} = \lambda \Delta x \left[\frac{\partial \varphi}{\partial x}(x^+, \tau) - \frac{\partial \varphi}{\partial x}(x^-, \tau) \right] - (\lambda - \mu)\Delta x \frac{\partial \varphi}{\partial x}(x^+, \tau)$$
$$+ \mu \Delta x \Delta F_L'''(x) \frac{\partial \varphi}{\partial x}(0, \tau) - \lambda \Delta x \Delta F_R'''(x) \frac{\partial \varphi}{\partial x}(1, \tau).$$

Multiply by Δx, make the above replacements for $\lambda(\Delta x)^2$, etc., let $\Delta x \to 0$, and get at $x = a_i$,

$$\frac{\partial \varphi}{\partial x}(a_i^+, \tau) - \frac{\partial \varphi}{\partial x}(a_i^-, \tau) + \rho_i \frac{\partial \varphi}{\partial x}(0, \tau) - \gamma_i \frac{\partial \varphi}{\partial x}(1, \tau) = 0. \tag{2.5}$$

The *forward problem* is to find a function φ with domain $\{(x, \tau) : 0 \leq x \leq 1, \tau \geq 0\}$ which is such that

(i) $\varphi(x, \tau)$ is a continuous function of x for each τ and a continuous function of τ for each x,

(ii) on each strip $\{(x, \tau) : a_i < x < a_{i+1}, \tau \geq 0\}$, $i = 0, 1, \ldots, N$, $\varphi_x(x, \tau), \varphi_{xx}(x, \tau)$, and $\varphi_\tau(x, \tau)$ (where subscripts denote partial derivatives) are continuous functions of x for each τ and of τ for each x.

(iii) $\varphi_x(x, \tau)$ has right and left limits, $\varphi_x(a_i^+, \tau)$ and $\varphi_x(a_i^-, \tau)$ at a_i, for $i = 1, 2, \ldots, N$,

(iv) $\varphi_x(x, \tau)$ has a right limit $\varphi_x(0, \tau)$ at 0 and a left limit $\varphi_x(1, \tau)$ at 1.

(v) $\varphi(x, 0) = \varphi_0(x)$, where φ_0 is a given initial function.

(vi) $\varphi(x, \tau)$ satisfies (2.3)–(2.5).

In addition to the forward problem we consider the associated *backward problem*: to find a function Φ which, together with the derivatives Φ_x, Φ_{xx}, and Φ_τ, is continuous on the same domain $\{(x, \tau) : 0 \leq x \leq 1, \tau \geq 0\}$, and which satisfies

$$\frac{\partial \Phi}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 \Phi}{\partial x^2} + \beta \frac{\partial \Phi}{\partial x}, \tag{2.6}$$

$$\Phi(0, \tau) = \int_0^1 \Phi(\xi, \tau) \, dF_L(\xi). \tag{2.7}$$

$$\Phi(1, \tau) = \int_0^1 \Phi(\xi, \tau) \, dF_R(\xi). \tag{2.8}$$

As was mentioned in the Introduction, boundary conditions like (2.7) and (2.8) have been considered before [6, 7, 10, 14]. However, our existence and uniqueness results seem to be new. For a probabilistic interpretation of the backward problem, see Mansourati's thesis [12].

Equation (2.2) can be simplified by putting

$$t = \sigma^2 \tau / 2, \qquad \omega = \beta / \sigma^2,$$

and letting

$$V(x,t) = \exp[\omega^2 t - \omega x] \varphi\left(x, \frac{2t}{\sigma^2}\right), \tag{2.9a}$$

$$f_0(x) = e^{\omega x} f_L(x), \qquad f_1(x) = e^{\omega(1-x)} f_R(x), \tag{2.9b}$$

$$p_i = e^{-\omega a_i} \rho_i, \qquad q_i = e^{\omega(1-a_i)} \gamma_i, \tag{2.9c}$$

$$F_0(x) = \int_0^x e^{-\omega \xi} \, dF_L(\xi), \qquad F_1(x) = \int_0^x e^{\omega(1-\xi)} \, dF_R(\xi). \tag{2.10}$$

With this notation, the forward problem becomes

$$\frac{\partial V}{\partial \tau}(x,t) = \frac{\partial^2 V}{\partial x^2}(x,t) + f_0(x)\frac{\partial V}{\partial x}(0,t) - f_1(x)\frac{\partial V}{\partial x}(1,t), \tag{2.11}$$

$$V(0,t) = V(1,t) = 0, \tag{2.12}$$

$$V(x,0) = V_0(x), \tag{2.13}$$

and

$$V_x(a_i^+,t) - V_x(a_i^-,t) + p_i V_x(0,t) - q_i V_x(1,t) = 0, \tag{2.14}$$

for $i = 1, 2, \ldots, N$.

Similarly, we simplify (2.6) by setting

$$U(x,t) = \exp[\omega^2 t + \omega x]\Phi\left(x, \frac{2t}{\sigma^2}\right). \tag{2.15}$$

This gives the new form of the backward problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \tag{2.16}$$

$$U(0,t) = \int_0^1 U(x,t) \, dF_0(x), \tag{2.17}$$

$$U(1,t) = \int_0^1 U(x,t) \, dF_1(x), \tag{2.18}$$

and

$$U(x,0) = U_0(x). \tag{2.19}$$

Note that, in view of (2.10), when F_L and F_R are probability distribution functions it is not possible that both inequalities $F_0(1) < 1$ and $F_1(1) < 1$ hold, no matter what the sign of ω. This fact prevents us from applying Friedman's [10] existence theorem to our problem.

We sometimes refer to (2.11)-(2.14) and (2.16)–(2.19) as the *scaled* forward and backward problems respectively.

3. Existence and uniqueness of solutions. The basic idea that we use to solve either the forward or backward equation is to find the solution first on the boundaries $x = 0$ and $x = 1$. Once the solution and its derivatives are known on the boundaries, the well-known theory of the one-dimensional heat equation provides the solution everywhere. We treat the scaled problems in this section, beginning with the slightly simpler backward problem (2.16)–(2.19).

If (2.17) and (2.18) are replaced by the conditions

$$U(0,t) = b_0(t), \qquad U(1,t) = b_1(t), \tag{3.1}$$

where b_0 and b_1 are continuous, then the unique bounded solution of the heat equation (2.16) satisfying (3.1) and (2.19) is [4]

$$
\begin{aligned}
U(x,t) = \int_0^1 G(x,\xi;t)U_0(\xi)\,d\xi - 2\int_0^t \frac{\partial\theta}{\partial x}(x, t-\tau)b_0(\tau)\,d\tau \\
+ 2\int_0^t \frac{\partial\theta}{\partial x}(x-1, t-\tau)b_1(\tau)\,d\tau,
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
G(x,\xi;t) &= 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\sin(n\pi x)\sin(n\pi\xi) \\
&= \theta(x-\xi, t) - \theta(x+\xi, t),
\end{aligned}
\tag{3.3}
$$

$$\theta(x,t) = \sum_{m=-\infty}^{+\infty} K(x+2m, t), \quad t > 0, \tag{3.4}$$

and

$$K(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{x^2}{4t}\right\}, \quad t > 0. \tag{3.5}$$

In order to obtain equations for b_0 and b_1, we integrate (3.2) with respect to dF_0 and dF_1 and use (2.17) and (2.18). If U, as given by (3.2), is a solution of (2.16)-(2.19), then b_0 and b_1 must satisfy the pair of integral equations

$$b_i(t) = \mathcal{G}_i(t) + \sum_{j=0}^{1}\int_0^t \mathcal{N}_{ij}(t-\tau)b_j(\tau)\,d\tau, \tag{3.6}$$

for $i = 0, 1$, where we set

$$\mathcal{G}_i(t) = \int_0^1 \left(\int_0^1 G(x,\xi;t)U_0(\xi)\,d\xi\right) dF_i(x),$$

$$\mathcal{N}_{i0}(t) = -2\int_0^1 \frac{\partial\theta}{\partial x}(x-t)\,dF_i(x),$$

$$\mathcal{N}_{i1}(t) = 2\int_0^1 \frac{\partial\theta}{\partial x}(x-1, t)\,dF_i(x).$$

We show first that the pair of Volterra integral equations (3.6) has a solution (b_0, b_1) and then verify that, for this b_0 and b_1, U in (3.2) is the desired solution. Some care is needed because of the singular behavior of K as $t \to 0$.

LEMMA 3.1. *If U_0 is bounded and measurable and if F_0 and F_1 are absolutely continuous with bounded derivatives f_0 and f_1 except at the interior points $a_1, a_2, \ldots, a_N$ where they have finite jumps, then $\mathcal{G}_i$ $(i = 0, 1)$ is continuous for $t > 0$ and*

$$\lim_{t \to 0^+} \mathcal{G}_i(t) = \int_0^1 U_0(x) \, dF_i(x). \tag{3.7}$$

Moreover, if U_0 is nonnegative, $\mathcal{G}_i$ is nonnegative.

Proof. If

$$h(x, t) = \int_0^1 G(x, \xi; t) U_0(\xi) \, d\xi$$

then it is well known [4] that h is continuous for $t > 0$ and that

$$\lim_{t \to 0^+} h(x, t) = U_0(x).$$

Thus (3.7) follows from the dominated convergence theorem. Nonnegativity follows from the fact that h, as a solution of the heat equation with nonnegative initial data, is known to be nonnegative. □

LEMMA 3.2. *If F_0 and F_1 are as described in Lemma 3.1, then $\mathcal{N}_{ij}$ is continuous for $t > 0$ and $t^{1/2}\mathcal{N}_{ij}(t)$ is bounded as $t \to 0^+$. Moreover, $\mathcal{N}_{ij}$ is nonnegative.*

Proof. Continuity for $t > 0$ follows from the uniform convergence of the series (3.4) and a corresponding one for the derivative. For $0 \leq x \leq 1$, it is clear that only one term in the series can be badly behaved as $t \to 0$. We have

$$-2\frac{\partial \theta}{\partial x}(x, t) = \frac{x}{2\sqrt{\pi}t^{3/2}} \exp[-x^2/(4t)] + R_0(x, t),$$

where R_0 is the sum of the terms for $m \neq 0$ in the derivative of (3.4). Integration with respect to $dF_i(x)$ on $(0, 1)$ and the change of variables $u = x/\sqrt{2t}$ yields

$$\mathcal{N}_{i0}(t) = \frac{1}{\sqrt{\pi t}} \int_0^{1/\sqrt{2t}} u \exp[-u^2/2] f_i(\sqrt{2t}u) \, du + o(t)$$

as $t \to 0$. Note that jumps in F_i occur at points a_i which are bounded away from zero. It follows that contributions from these jumps are $o(t)$ and only the contribution from f_i can affect the behavior near $t = 0$. Since f_i is, by hypothesis, bounded, it follows that $t^{1/2}\mathcal{N}_{i0}(t)$ is bounded as $t \to 0^+$. An exactly similar calculation shows that $t^{1/2}\mathcal{N}_{i1}(t)$ is bounded.

As noted earlier, $G(x, \xi; t) \geq 0$ for $0 \leq x, \xi \leq 1$. In view of (3.3), this implies that, for $x \in (0, 1)$ and $t > 0$,

$$\frac{\partial \theta}{\partial x}(x, t) = \lim_{\xi \to 0^+} \frac{\theta(x + \xi, t) - \theta(x - \xi, t)}{2\xi} \leq 0,$$

and from this $\mathcal{N}_{i0} \geq 0$. Also, by (3.4) and (3.5), θ is an even function of x, so that $\frac{\partial \theta}{\partial x}(x - 1, t) \geq 0$ for $x \in (0, 1)$. Consequently $\mathcal{N}_{i1} \geq 0$. □

LEMMA 3.3. Under the hypotheses of Lemma 3.1, there exists a unique continuous solution of the pair of integral equations (3.6) for $t > 0$. If U_0 is nonnegative, both b_0 and b_1 are nonnegative.

Proof. The proof is almost a standard successive approximation proof, slightly complicated by the mild singularity of the kernel functions. Let

$$b_i^{(n)}(t) = \mathcal{G}_i(t) + \sum_{j=0}^{1} \int_0^t \mathcal{N}_{ij}(t-\tau) b_j^{(n-1)}(\tau)\, d\tau. \tag{3.8a}$$

$$b_i^{(0)}(t) = \mathcal{G}_i(t), \tag{3.8b}$$

for $i = 0, 1$ and $n = 1, 2, \dots$. Then

$$b_i^{(n)}(t) = \mathcal{G}_i(t) + \sum_{m=1}^{n} \sum_{j=0}^{1} \int_0^t \mathcal{N}_{ij}^{(m)}(t-\tau) \mathcal{G}_j(\tau)\, d\tau, \tag{3.9}$$

where

$$\mathcal{N}_{ij}^{(n+1)}(t) = \sum_{k=0}^{1} \int_0^t \mathcal{N}_{ik}(t-\tau) \mathcal{N}_{kj}^{(n)}(\tau)\, d\tau, \quad n \geq 1,$$

$$\mathcal{N}_{ij}^{(1)}(t) = \mathcal{N}_{ij}(t).$$

Using Lemmas 3.1 and 3.2, we obtain an estimate

$$\left| \sum_{j=0}^{1} \int_0^t \mathcal{N}_{ij}^{(m)}(t-\tau) \mathcal{G}_j(\tau)\, d\tau \right| \leq c \left(\frac{\pi}{t} \right)^{1/2} \frac{(2k\sqrt{t\pi})^m}{\Gamma(\frac{m+1}{2})}, \tag{3.10}$$

for $0 < t < T$, where

$$k = \max_{i,j \in \{0,1\}} \sup_{0 < t < T} (t^{1/2} \mathcal{N}_{ij}(t)),$$

$$c = \max_{i \in \{0,1\}} \sup_{0 < t < T} \mathcal{G}_i(t),$$

and Γ denotes the gamma function. From (3.10) it follows that the sequence $b_i^{(n)}$ of (3.9) converges uniformly on any interval $[0, T]$. Thus there are limits $b_0(t)$ and $b_1(t)$ that satisfy (3.6). Uniqueness is easily demonstrated. By Lemma 3.2, $\mathcal{N}_{ij}^{(m)} \geq 0$ for each m, and consequently it is evident from (3.9) and Lemma 3.1 that b_0 and b_1 are nonnegative if U_0 is nonnegative. $\square$

THEOREM 3.4. If U_0 is bounded and measurable and if F_0 and F_1 are as described in Lemma 3.1, the scaled backward problem (2.16) (2.19) has a unique solution. If U_0 is nonnegative, the solution is nonnegative.

Proof. Let U be defined by (3.2) where (b_0, b_1) is the solution of (3.6) which exists by Lemma 3.3. By the standard theory of the heat equation, U satisfies (2.16) and (2.19). It is also known [4] that

$$\lim_{x \to 0^+} -2 \int_0^t \frac{\partial \theta}{\partial x}(x, t-\tau) b(\tau)\, d\tau = b(t), \quad \text{for } t > 0, \tag{3.11}$$

and

$$\lim_{x \to 1^-} -2 \int_0^t \frac{\partial \theta}{\partial x}(x, t - \tau)b(\tau)\,d\tau = 0, \quad \text{for } t > 0, \tag{3.12}$$

and that $G(0, \xi; t) = G(1, \xi; t) = 0$. From this we easily obtain

$$\lim_{x \to 0^+} U(x, t) = b_0(t), \qquad \lim_{x \to 1^-} U(x, t) = b_1(t),$$

so that (2.17) and (2.18) are also satisfied.

Uniqueness is shown by using Lemma 3.3 to show that $U(0, t)$ and $U(1, t)$ are uniquely defined and then by using the standard theory of the heat equation to show that U is defined elsewhere by (3.2). Nonnegativity when U_0 is nonnegative follows from the nonnegativity of b_0 and b_1 (Lemma 3.3), and from the nonnegativity of G, $-\theta_x(x, t)$, and $\theta_x(x - 1, t)$ (see the proof of Lemma 3.2). $\square$

The forward problem, (2.11)–(2.14), is treated in an analogous manner. We discuss those aspects that differ from the earlier treatment. Let

$$\begin{aligned}
V(x, t) &= \int_0^1 G(x, \xi; t)V_0(\xi)\,d\xi \\
&\quad + \int_0^t \psi_0(\tau)\left[\int_0^1 G(x, \xi; t - \tau)\,dF_0(\xi)\right]\,d\tau \\
&\quad + \int_0^t \psi_1(\tau)\left[\int_0^1 G(x, \xi; t - \tau)\,dF_1(\xi)\right]\,d\tau \\
&= \int_0^1 G(x, \xi; t)V_0(\xi)\,d\xi \\
&\quad + \int_0^t \psi_0(\tau)\left[\int_0^1 G(x, \xi; t - \tau)f_0(\xi)\,d\xi + \sum_{i=1}^N p_i G(x, a_i; t - \tau)\right]\,d\tau \\
&\quad + \int_0^t \psi_1(\tau)\left[\int_0^1 G(x, \xi; t - \tau)f_1(\xi)\,d\xi + \sum_{i=1}^N q_i G(x, a_i; t - \tau)\right]\,d\tau,
\end{aligned} \tag{3.13}$$

where G is defined by (3.3)–(3.5), V_0 is the initial function in (2.13), and ψ_0 and ψ_1 are to be determined so that (2.11)–(2.14) is satisfied.

For V as defined by (3.13), straightforward calculations using the properties of G show that, if $\psi_0(t)$ and $\psi_1(t)$ are continuous for $t > 0$ and are $O(t^{-1/2})$ as $t \to 0^+$, then

$$V(0, t) = V(1, t) = 0, \qquad V(x, 0^+) = V_0(x)$$

and, for $a_i < x < a_{i+1}$, $t > 0$,

$$V_t(x, t) = V_{xx}(x, t) + f_0(x)\psi_0(t) + f_1(x)\psi_1(t). \tag{3.14}$$

Furthermore, for reasons similar to those discussed in the proof of Lemma 3.2, the only terms of (3.13) that contribute to discontinuities of V_x at $x = a_j$ are those involving

$G(x, a_j; t - \tau)$. In fact,

$$V_x(a_j^+, t) - V_x(a_j^-, t)$$

$$= \int_0^t [p_j \psi_0(\tau) + q_j \psi_1(\tau)][\theta_x(0^+, t - \tau) - \theta_x(0^-, t - \tau)] \, d\tau \qquad (3.15)$$

$$= -[p_j \psi_0(t) + q_j \psi_1(t)],$$

where we have used (3.11) and the fact that $\theta(x, t)$ is even in x, so that $\theta_x(x, t)$ is odd in x. We see that (3.14) and (3.15) become (2.11) and (2.14) if $\psi_0(t) = V_x(0, t)$ and $\psi_1(t) = -V_x(1, t)$.

To find equations determining ψ_0 and ψ_1, differentiate (3.13) once with respect to x and set $x = 0$ and $x = 1$. Then ψ_0 and ψ_1 must satisfy

$$\psi_i(t) = \mathcal{H}_i(t) + \sum_{j=0}^{1} \int_0^t \mathcal{M}_{ij}(t - \tau)\psi_j(\tau) \, d\tau, \qquad (3.16)$$

where, for $i, j = 0, 1$,

$$\mathcal{H}_i(t) = (-1)^i \int_0^1 G_x(i, \xi; t) V_0(\xi) \, d\xi$$

and

$$\mathcal{M}_{ij}(t) = (-1)^i \int_0^1 G_x(i, \xi; t) \, dF_j(\xi).$$

Analogously to our earlier discussion, we can prove the following results.

LEMMA 3.5. If V_0 is bounded and measurable, then $\mathcal{H}_i$ is continuous for $t > 0$ and $t^{1/2}\mathcal{H}_i(t)$ is bounded as $t \to 0^+$. If V_0 is nonnegative, then $\mathcal{H}_i$ is nonnegative.

LEMMA 3.6. If F_0 and F_1 are as described in Lemma 3.1, then $\mathcal{M}_{ij}$ is continuous for $t > 0$ and $t^{1/2}\mathcal{M}_{ij}(t)$ is bounded as $t \to 0$. Furthermore, $\mathcal{M}_{ij}$ is nonnegative.

LEMMA 3.7. Under the hypotheses of Lemmas 3.5 and 3.6, there exists a unique solution of the pair of integral equations (3.16) which is continuous for $t > 0$ and is $O(t^{-1/2})$ as $t \to 0$. If V_0 is nonnegative, both ψ_0 and ψ_1 are nonnegative.

The proofs are analogous to the proofs of Lemmas 3.1–3.3 and are omitted. Some further details are available in [12].

THEOREM 3.8. If V_0 is bounded and measurable and if F_0 and F_1 are as described in Lemma 3.1, the scaled forward problem (2.11)–(2.14) has a unique solution. If V_0 is nonnegative, the solution is nonnegative.

Proof. Let V be defined by (3.13) where (ψ_0, ψ_1) is the solution of (3.16). By the discussion leading up to (3.14) and (3.15), all that we need to establish is that $V_x(0, t) = \psi_0(t)$ and $V_x(1, t) = -\psi_1(t)$. But differentiation of (3.13), substitution of $x = 0$ and $x = 1$, and comparison with the right side of (3.16) shows that this is true. Uniqueness and nonnegativity follow in the same way as in Theorem 3.4. $\square$

Clearly, Theorems 3.4 and 3.8 establish existence and uniqueness of solutions φ and Φ of the original forward and backward problems (2.3)–(2.8) with their associated initial conditions.

4. Eigenvalues and eigenfunctions. Define differential operators M_α and L_α for each complex number α, by

$$M_\alpha[v](x) = v''(x) - \alpha^2 v(x) + f_0(x)v'(0) - f_1(x)v'(1)$$

and

$$L_\alpha[u](x) = u''(x) - \overline{\alpha}^2 u(x),$$

where the domain of M_α is the set

$$D = \{v : v \text{ continuous on } [0,1]; v'' \text{ continuous on each open interval } (a_i, a_{i+1})$$
$$\text{and } v' \text{ satisfying } v'(a_i^+) - v'(a_i^-) = q_i v'(1) - p_i v'(0) \text{ for } i = 1, \ldots, N;$$
$$v(0) = v(1) = 0\}$$

and the domain of L_α is the set $C^2[0,1]$ of functions with continuous second derivatives on $[0,1]$. Define also the functionals B_0 and B_1 by

$$B_k[u] = u(k) - \int_0^1 u(x)\,dF_k(x), \quad k = 0, 1.$$

If we attempt to solve the scaled forward problem (2.11)–(2.14) by the method of separating variables or by the Laplace transform method, we are led to the forward eigenvalue problem

$$M_\alpha[v] = 0, \quad v \in D.$$

Similarly, associated with the scaled backward problem (2.16)–(2.19) is the backward eigenvalue problem

$$L_\alpha[u] = 0, \qquad B_0[u] = B_1[u] = 0.$$

In this section we show that the eigenvalues are the same for the two problems, that eigenfunctions corresponding to distinct eigenvalues are biorthogonal, and that for any eigenvalue α^2, the eigenspaces for the two problems have the same dimension. For the original forward problem, (2.3)–(2.5), we obtain more information about the eigenvalues and the steady-state solution.

We begin with the version of Green's formula given by

LEMMA 4.1. If $v \in D$ and $u \in C^2[0,1]$, then

$$\int_0^1 \overline{u(x)} M_\alpha[v](x)\,dx - \int_0^1 v(x)\overline{L_\alpha[u]}(x)\,dx \tag{4.1}$$
$$= v'(1)\overline{B_1[u]} - v'(0)\overline{B_0[u]}.$$

The proof follows in the usual way from integration by parts on each interval (a_i, a_{i+1}).

To solve the backward eigenvalue problem, first replace α by $\overline{\alpha}$ and note that the general solution of $L_{\overline{\alpha}}[u] = 0$ is

$$u(x) = A \cosh \alpha x + \frac{B}{\alpha} \sinh \alpha x, \tag{4.2}$$

where for $\alpha = 0$ we use the limiting value as $\alpha \to 0$. Application of the conditions $B_k[u] = 0$ yields

$$AB_0[\cosh \alpha x] + BB_0[\alpha^{-1} \sinh \alpha x] = 0 \tag{4.3}$$

$$AB_1[\cosh \alpha x] + BB_1[\alpha^{-1} \sinh \alpha x] = 0. \tag{4.4}$$

The determinant of (4.3), (4.4) is

$$\Delta(\alpha) = B_0[\cosh \alpha x]B_1[\alpha^{-1} \sinh \alpha x] - B_0[\alpha^{-1} \sinh \alpha x]B_1[\cosh \alpha x]. \tag{4.5}$$

Thus if $\Delta(\alpha) = 0$, α^2 is an eigenvalue and (4.2)–(4.4) determine the eigenfunctions. Clearly $\Delta(\alpha)$ is an entire function of α; thus it has only a finite number of zeros inside any finite circle. Since $\Delta(-\alpha) = \Delta(\alpha)$ and $\Delta(\overline{\alpha}) = \overline{\Delta(\alpha)}$, α is a zero if and only if $-\alpha$ and $\overline{\alpha}$ are zeros. In particular, eigenvalues α^2 occur in conjugate pairs.

To solve the forward eigenvalue problem, we use (4.1) to determine the derivatives $v'(0)$ and $v'(1)$ of an eigenfunction v. Once these values are known, we can find v easily. That is, knowing $v(0) = 0$ and knowing $v'(0)$ and $v'(1)$ we can solve $M_\alpha[v] = 0$ on $(0, a_1)$. Then continuity gives $v(a_1^+) = v(a_1^-)$ and the jump condition gives $v'(a_1^+)$, once $v'(a_1^-)$ is known. With these values the differential equation is solved on (a_1, a_2), and so on. If $v'(0) = v'(1) = 0$, we obtain the trivial solution. To get the conditions on $v'(0)$ and $v'(1)$, let α^2 be an eigenvalue and let v be an eigenfunction. Apply (4.1) twice, once with $u(x) = \cosh \overline{\alpha} x$ and the second time with $u(x) = (\sinh \overline{\alpha} x)/\overline{\alpha}$ (so that in each case $L_\alpha[u] = 0$), to get

$$-v'(0)B_0[\cosh \alpha x] + v'(1)B_1[\cosh \alpha x] = 0 \tag{4.6}$$

and

$$-v'(0)B_0[\alpha^{-1} \sinh \alpha x] + v'(1)B_1[\alpha^{-1} \sinh \alpha x] = 0. \tag{4.7}$$

These equations, as equations for $(-v'(0), v'(1))$, are the transposes of (4.3), (4.4). If $\Delta(\alpha) \neq 0$, there is only the trivial solution; if $\Delta(\alpha) = 0$ and the rank of the matrix is one, each of (4.3), (4.4) and (4.6), (4.7) has one linearly independent solution, and if the rank is zero, each pair of equations has two linearly independent solutions. Thus we have proved

THEOREM 4.2. The forward and backward eigenvalue problems have the same eigenvalues. If α^2 is an eigenvalue, the number of linearly independent solutions of the forward problem equals the number of linearly independent solutions of the backward problem.

Biorthogonality of eigenfunctions also follows immediately from (4.1), as follows:

THEOREM 4.3. Let v be an eigenfunction of the forward problem with eigenvalue α^2 and let u be an eigenfunction of the backward problem with eigenvalue β^2. If $\alpha^2 \neq \beta^2$,

$$\int_0^1 \overline{u}(x)v(x)\,dx = 0.$$

Proof. Since

$$L_\alpha[u] = L_\beta[u] + (\overline{\beta}^2 - \overline{\alpha}^2)u = (\overline{\beta}^2 - \overline{\alpha}^2)u,$$

and $M_\alpha[v] = 0$, substitution in (4.1) gives

$$(\alpha^2 - \beta^2) \int_0^1 v(x)\overline{u}(x)\, dx = 0,$$

from which the result follows. $\square$

Biorthogonality of the eigenfunctions associated with the original forward and backward problems (2.3)–(2.8), also with weight function one, follows easily from (2.9) and (2.15), or directly from the differential equations (2.3) and (2.6).

For expansion theorems and stability theorems, we often need bounds on eigenvalues. A coarse bound is given by

THEOREM 4.4. *Under our general hypotheses about the functions F_0 and F_1, the zeros of $\Delta(\alpha)$ all lie in some strip $-a \le \mathrm{Re}(\alpha) \le a$ and the eigenvalues α^2 lie in a half-plane $\mathrm{Re}(\alpha^2) \le a^2$.*

Proof. A little algebra shows that

$$\alpha\Delta(\alpha) = \sinh\alpha - \int_0^1 \sinh\alpha(1-x)\, dF_0(x) - \int_0^1 \sinh\alpha x\, dF_1(x)$$
$$+ \int_0^1 \int_0^1 \sinh\alpha(y-x)\, dF_0(x)\, dF_1(y).$$

Recall that the jump-points, if any, of F_0 and F_1 are bounded away from 0 and 1, so that the measures $dF_0(x)$ and $dF_1(x)$ are of the form $f_0(x)\, dx$ and $f_1(x)\, dx$, with bounded f_0 and f_1 near the values $x = 0$ and $x = 1$. Thus, if we let $\alpha = \alpha_1 + i\alpha_2$,

$$\lim_{\alpha_1 \to \infty} [2e^{-\alpha}\alpha\Delta(\alpha)] = 1.$$

Consequently there is a positive number a such that $\Delta(\alpha) \ne 0$ for $\alpha_1 > a$. Since $\Delta(-\alpha) = \Delta(\alpha)$, it follows that $\Delta(\alpha) \ne 0$ for $\alpha_1 < -a$, and the zeros lie in the strip $-a \le \alpha_1 \le a$. $\square$

Theorem 4.4 holds whenever F_0 and F_1 have the properties listed in Lemma 3.1, i.e., they are absolutely continuous with bounded derivatives except at a finite number of points in the open interval $(0, 1)$, where they may have finite jumps. However, when they are also related to probability distribution functions by (2.10), more precise information is available. In this case it is more useful to deal with the unscaled equations.

THEOREM 4.5. *Consider the following eigenvalue problem associated with the backward problem (2.6)–(2.8) on $[0, 1]$ where F_L and F_R are probability distribution functions:*

$$\frac{\sigma^2}{2}\Phi''(x) + \beta\Phi'(x) = \alpha^2\Phi(x), \tag{4.8a}$$

$$\Phi(0) = \int_0^1 \Phi(x)\, dF_L(x), \tag{4.8b}$$

$$\Phi(1) = \int_0^1 \Phi(x)\, dF_R(x). \tag{4.8c}$$

Zero is an eigenvalue associated with the constant eigenfunction and, if α^2 is any eigen-value, $\mathrm{Re}(\alpha^2) \leq 0$.

Proof. For $\alpha^2 = 0$, direct calculations using the fact that

$$\int_0^1 dF_L(x) = \int_0^1 dF_R(x) = 1$$

show that the only solution of (4.8) is the constant function.

Next, given an arbitrary eigenfunction Φ, we show that $|\Phi|$ has a maximum at some interior point c in the interval $(0,1)$. Since $|\Phi|$ is continuous on $[0,1]$, it reaches its maximum on $[0,1]$. If the maximum is achieved only at 0 or 1, or both, then $|\Phi(x)| < |\Phi(0)|$ or $|\Phi(x)| < |\Phi(1)|$, or both, for all $x \in (0,1)$. Assuming that the first inequality holds, we have the following contradiction:

$$|\Phi(0)| = \left| \int_0^1 \Phi(\xi) \, dF_L(\xi) \right| \leq \int_{(0,1)} |\Phi(\xi)| \, dF_L(\xi)$$

$$< \int_{(0,1)} |\Phi(0)| \, dF_L(\xi) = |\Phi(0)|.$$

The strict inequality follows from the assumption that F_L has jump discontinuities at interior points only. A similar contradiction may be deduced from the inequality involving 1. Therefore, there exists an interior point c where $|\Phi|$ reaches its maximum.

Now, multiplying (4.8a) by the complex conjugate $\overline{\Phi}$ implies that

$$\frac{\sigma^2}{2}[\overline{\Phi}\Phi'' + |\Phi'|^2] - \frac{\sigma^2}{2}|\Phi'|^2 + \beta\overline{\Phi}\Phi' = \alpha^2|\Phi|^2. \tag{4.9}$$

If Φ satisfies (4.8a) and is constant, the first and second derivatives would be zero and, consequently, $\alpha^2\Phi(x) = 0$. If $\alpha \neq 0$, then $\Phi \equiv 0$ which, by definition, cannot be an eigenfunction. Thus, there are no constant eigenfunctions associated with nonzero eigenvalues. Now, at $x = c$, we have

$$|\Phi(c)|^2 > 0, \qquad \frac{d}{dx}|\Phi|^2 = 0, \qquad \text{and} \qquad \frac{d^2}{dx^2}|\Phi|^2 \leq 0.$$

Since

$$\frac{d}{dx}|\Phi(x)|^2 = 2[\mathrm{Re}(\overline{\Phi}\Phi')](x),$$

we have $[\mathrm{Re}(\overline{\Phi}\Phi')](c) = 0$. Moreover, since

$$\frac{d^2}{dx^2}|\Phi(x)|^2 = 2\mathrm{Re}[\overline{\Phi}\Phi'' + |\Phi'|^2](x),$$

we have $\mathrm{Re}[\overline{\Phi}\Phi'' + |\Phi'|^2](c) \leq 0$. Thus, taking the real part of (4.9) at $x = c$ implies that $\mathrm{Re}(\alpha^2) \leq 0$. $\quad\square$

Since zero is an eigenvalue for the backward problem, it is also an eigenvalue of the forward problem. The corresponding eigenfunction, called the steady-state solution, is

often important in applications. A steady-state solution is a continuous function g on the interval $[0, 1]$ that vanishes at the end-points and that satisfies

$$g''(x) - 2\omega g'(x) + [f_L(x)g'(0) - f_R(x)g'(1)] = 0 \tag{4.10a}$$

on each open interval (a_i, a_{i+1}) for $i = 0, 1, \ldots, N$, and that satisfies

$$g'(a_i^+) - g'(a_i^-) + \rho_i g'(0) - \gamma_i g'(1) = 0 \tag{4.10b}$$

for $i = 1, 2, \ldots, N$, where $\omega = \beta/\sigma^2$.

THEOREM 4.6. Define functions L and R by

$$L(x) = \int_0^x e^{2\omega(x-\xi)}[1 - F_L(\xi)] \, d\xi \tag{4.11}$$

and

$$R(x) = \int_0^x e^{2\omega(x-\xi)} F_R(\xi) \, d\xi. \tag{4.12}$$

Then g_0, defined by

$$g_0(x) = L(x)R(1) - L(1)R(x), \tag{4.13}$$

is a steady-state solution. Moreover, $g_0(x) \geq 0$ for each x in $[0, 1]$.

Proof. It is straightforward to verify that g_0 satisfies (4.10). Also,

$$L(x)R(y) - L(y)R(x) = \int_0^y d\eta \int_0^x d\xi \, e^{2\omega(x+y-\xi-\eta)}[Q(\xi)F_R(\eta) - Q(\eta)F_R(\xi)],$$

where

$$Q(x) = 1 - F_L(x).$$

Using the antisymmetry of the integrand about the line $\eta = \xi$, we have, for $y > x$,

$$L(x)R(y) - L(y)R(x) = \int_x^y d\eta \int_0^x d\xi \, e^{2\omega(x+y-\xi-\eta)}[Q(\xi)F_R(\eta) - Q(\eta)F_R(\xi)].$$

Since, for $\eta > \xi, Q(\xi) > Q(\eta)$ and $F_R(\eta) > F_R(\xi)$, the integrand is positive. Hence the integral is positive for $y > x$. The particular case $y = 1$ yields the result $g_0(x) \geq 0$. $\square$

Positivity of $g_0(x)$ means that g_0 can be normalized and converted into a probability density function.

5. Examples. To illustrate the theory, we discuss two examples briefly. The first example corresponds to the diffusion problem where F_L and F_R each concentrate their masses at $x = 1/2$. In this example the birth rate is not necessarily equal to the death rate; so we do not assume that $\omega = \beta/\sigma^2 = 0$. When $\omega = 0$ we have the case treated in [2]. The second example corresponds to uniform distributions on $[0, 1]$ with no jumps and with $\omega = 0$.

EXAMPLE 1. Let $F_L(x) = H(x - \frac{1}{2})$ and $F_R(x) = H(x - \frac{1}{2})$, where H denotes the unit step function as before. Then the eigenfunction problem associated with the original forward problem (2.3)–(2.5) is

$$\frac{\sigma^2}{2}[\varphi''(x) - 2\omega\varphi'(x)] = \alpha^2\varphi(x),$$

$$\varphi'\left(\frac{1}{2}+\right) - \varphi'\left(\frac{1}{2}-\right) = \varphi'(1) - \varphi'(0),$$

$$\varphi(0) = \varphi(1) = 0,$$

while the eigenfunction problem for the backward problem (2.6)–(2.8) is

$$\frac{\sigma^2}{2}[\Phi''(x) + 2\omega\Phi'(x)] = \alpha^2\Phi(x),$$

$$\Phi(0) = \Phi\left(\frac{1}{2}\right) = \Phi(1).$$

If $\omega \neq 0$, the eigenvalues are simple. It can be verified that the eigenvalues and eigenfunctions separate naturally into two sets, which we label with superscripts (1) and (2), as follows:

$$(\alpha_n^{(1)})^2 = \frac{\sigma^2}{2}(-4\pi^2 n^2 - \omega^2) \quad (n = 1, 2, \dots),$$

$$\varphi_n^{(1)}(x) = \begin{cases} e^{\omega x}[1 - (-1)^n e^{-\omega/2}] \sin 2\pi nx, & 0 \leq x \leq \frac{1}{2}, \\ e^{\omega(x-1)}[1 - (-1)^n e^{\omega/2}] \sin 2\pi nx, & \frac{1}{2} \leq x < 1, \end{cases}$$

$$\Phi_n^{(1)}(x) = e^{-\omega x} \sin 2\pi nx,$$

and

$$(\alpha_n^{(2)})^2 = (-8\pi^2 n^2 - 4\pi n\omega i)\sigma^2 \quad (n = 0, \pm 1, \pm 2, \dots),$$

$$\varphi_n^{(2)}(x) = \begin{cases} e^{\omega x} \sinh(\omega - 4\pi ni)x, & 0 \leq x \leq \frac{1}{2}, \\ e^{\omega x} \sinh(\omega - 4\pi ni)(1 - x), & \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$\Phi_n^{(2)}(x) = e^{-i4n\pi x}.$$

As $\omega \to 0$, the eigenvalues $(\alpha_n^{(2)})^2$ and $(\alpha_{-n}^{(2)})^2$ merge with $(\alpha_{2n}^{(1)})^2$. The end result for $\omega = 0$ is the two sets of eigenvalues and eigenfunctions:

$$(\alpha_n^{(1)})^2 = -2\pi^2(2n + 1)^2\sigma^2 \quad (n = 1, 2, \dots),$$

$$\varphi_n^{(1)}(x) = \sin 2(2n + 1)\pi x,$$

$$\Phi_n^{(1)}(x) = \sin 2(2n + 1)\pi x,$$

and

$$(\alpha_0^{(2)})^2 = 0,$$

$$\varphi_0^{(2)}(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1 - x, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$\Phi_0^{(2)}(x) = 1,$$

$$(\alpha_n^{(2)})^2 = -8n^2\pi^2\sigma^2 \quad (n = 1, 2, \dots),$$

$$\varphi_n^{(2,1)}(x) = \begin{cases} \sin 4n\pi x, & 0 \le x \le \frac{1}{2}, \\ 0, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$\varphi_n^{(2,2)}(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ \sin 4n\pi x, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$\Phi_n^{(2,1)}(x) = \sin 4n\pi x,$$

$$\Phi_n^{(2,2)}(x) = \cos 4n\pi x.$$

Thus the eigenspaces corresponding to eigenvalues $(\alpha_n^{(2)})^2$, $n = 1, 2, \dots$, have dimension 2.

EXAMPLE 2. Let $F_L(x) = F_R(x) = x$ and let $\omega = 0$. Then $F_0 = F_L$, $F_1 = F_R$, and we can use (4.2)–(4.7). From (4.5) the eigenvalues are zeros of

$$\Delta(\alpha) = \frac{2}{\alpha} \sinh \frac{\alpha}{2} \left(\cosh \frac{\alpha}{2} - \frac{2}{\alpha} \sinh \frac{\alpha}{2} \right),$$

multiplied by $\sigma^2/2$. The eigenvalues and eigenfunctions again separate into two sets:

$$(\alpha_n^{(1)})^2 = -2n^2\pi^2\sigma^2 \quad (n = 1, 2, \dots),$$

$$\varphi_n^{(1)}(x) = \sin 2n\pi x,$$

$$\Phi_n^{(1)}(x) = \sin 2n\pi x,$$

and

$$(\alpha_n^{(2)})^2 = -(\beta_n)^2\sigma^2/2 \quad (n = 0, 1, 2, \dots),$$

where $\beta_0, \beta_1, \beta_2, \dots$ are the nonnegative solutions of

$$\tan \frac{\beta}{2} = \frac{\beta}{2}$$

arranged in increasing order with $\beta_0 = 0$,

$$\varphi_0^{(2)}(x) = x(1 - x),$$

$$\Phi_0^{(2)}(x) = 1,$$

$$\varphi_n^{(2)}(x) = \sin(\beta_n x/2) \sin(\beta_n[1 - x]/2) \quad (n = 1, 2, \dots),$$

$$\Phi_n^{(2)}(x) = \cos \beta_n \left(x - \frac{1}{2} \right) \quad (n = 1, 2, \dots).$$

6. Accuracy of diffusion approximation. One motivation for studying these equations was that they represent, in some sense, an approximation to (2.1) when the number of states is large. In this section we compare the steady-state solution $g_0(x)$ given by (4.13) with the steady-state solution of (2.1). We also compare the eigenvalues and eigenfunctions of Example 1, Sec. 5, with the eigenvalues and eigenvectors of the appropriate special case of (2.1). The comparisons provide some evidence that, in cases where (2.1) cannot be solved explicitly, solutions of (2.3)–(2.5) can be used to give good approximations.

(a) *Steady-state solutions.* To begin, note that integration by parts in (4.11) and (4.12) yields the expressions

$$2\omega L(x) = e^{2\omega x} - 1 - \int_0^x [e^{2\omega(x-\xi)} - 1]\, dF_L(\xi) \tag{6.1}$$

and

$$2\omega R(x) = \int_0^x [e^{2\omega(x-\xi)} - 1]\, dF_R(\xi), \tag{6.2}$$

where we assume $\omega \neq 0$. Now replace F_L and F_R in (6.1) and (6.2) by F_L^m and F_R^m, the discrete distribution functions which are used in (2.2), and denote the resulting functions by L^m and R^m, so that

$$2\omega L^m(x) = e^{2\omega x} - 1 - \sum_{\{\xi_j \leq x\}} L_j [e^{2\omega(x-\xi_j)} - 1] \tag{6.3}$$

and

$$2\omega R^m(x) = \sum_{\{\xi_j \leq x\}} R_j [e^{2\omega(x-\xi_j)} - 1]. \tag{6.4}$$

Now write $\xi_j = j/(m+1)$, let $x \to x_k = k/(m+1)$ from the left, and define

$$w = \frac{\omega}{m+1}. \tag{6.5}$$

Then

$$2\omega L^m(x_k^-) = e^{2kw} - 1 - \sum_{j=1}^{k-1} L_j [e^{2(k-j)w} - 1] \tag{6.6}$$

and

$$2\omega R^m(x_k^-) = \sum_{j=1}^{k-1} R_j [e^{2(k-j)w} - 1], \tag{6.7}$$

where $k \in \{0, 1, \ldots, m+1\}$ and empty sums are defined to be zero. Corresponding to (4.13), we put

$$g_0^m(x_k^-) = L^m(x_k^-)R^m(1) - L^m(1)R^m(x_k^-). \tag{6.8}$$

To complete the correspondence between steady-state solutions of (2.1) and (2.3)–(2.5) we recall that our assumptions about λ and μ could be written in the form

$$\lim_{m\to\infty} \frac{\lambda(m)}{(m+1)^2} = \frac{\sigma^2}{2}$$

and

$$\lim_{m \to \infty} \frac{\lambda(m) - \mu(m)}{m + 1} = \beta.$$

From this, a standard calculation shows that

$$\lim_{m \to \infty} \left[\frac{\lambda(m)}{\mu(m)}\right]^{m+1} = e^{2\omega},$$

so that, for large m,

$$\frac{\lambda}{\mu} \approx e^{2\omega/(m+1)} = e^{2w},$$

where w is defined by (6.5).

If we replace e^{2w} by (λ/μ) in (6.6)–(6.8) and do the necessary algebraic simplifications, we find that the resulting expression for $g_0^m(x_k^-)$ is just the same (apart from a normalizing constant and necessary adjustments to notation) as the component π_k of the steady-state solution of (2.1) which is to be found in [3, Eq. (13)]. Thus, replacement of F_L and F_R by the corresponding discrete distribution functions, replacement of $e^{2\omega/(m+1)}$ by λ/μ, and evaluation of $g_0(x)$ at points $k/(m+1)$, $k = 0, 1, \ldots, m+1$, give the corresponding steady-state solution in the discrete case.

(b) *Example 1 and the corresponding discrete case.* We consider the special case of (2.1) where the number of states m is replaced by $2N + 1$, where N is an even number, and where $L_N = R_N = 1$, with the other L_k and R_k equal to zero. This means that returns are always to the midpoint, just as in the diffusion of Example 1. The solution of this problem is given in [2]. As in Example 1 the eigenvalues and eigenvectors divide naturally into two sets. There are N eigenvalues

$$r_n^{(1)} = -\lambda - \mu + 2\sqrt{\lambda\mu} \cos \frac{\pi n}{N + 1} \quad (n = 1, 2, \ldots, N),$$

and $N + 1$ eigenvalues

$$r_n^{(2)} = -2(\lambda + \mu) \sin^2 \frac{\pi n}{N + 1} - i(\lambda - \mu) \sin \frac{2\pi n}{N + 1} \quad \left(n = 0, \pm 1, \ldots, \pm \frac{N}{2}\right).$$

For $n \ll N$ we approximate the sine and cosine by Taylor series to order $(N + 1)^{-2}$, getting

$$r_n^{(1)} \approx -(\sqrt{\lambda} - \sqrt{\mu})^2 - \sqrt{\lambda\mu} \frac{\pi^2 n^2}{(N + 1)^2}$$

and

$$r_n^{(2)} \approx -2(\lambda + \mu) \frac{\pi^2 n^2}{(N + 1)^2} - i(\lambda - \mu) \frac{2\pi n}{N + 1}.$$

For large N, the approximations for λ and μ imply that

$$\lambda \approx \mu \approx \sqrt{\lambda\mu} \approx 2\sigma^2 (N + 1)^2, \qquad \lambda - \mu \approx 2\beta(N + 1),$$

and, after a little algebra, that

$$(\sqrt{\lambda} - \sqrt{\mu})^2 \approx \frac{\sigma^2}{2} \omega^2.$$

With these replacements, $r_n^{(1)}$ and $r_n^{(2)}$ become the same as the eigenvalues $(\alpha_n^{(1)})^2$ and $(\alpha_n^{(2)})^2$ of Example 1. Thus, the diffusion approximation gives good estimates of the lower (i.e., $n \ll N$) eigenvalues.

The comparison of eigenvectors resembles the comparison that was done for the steady-state solutions. We provide some detail for the first N components of the eigenvectors corresponding to $r_n^{(1)}$. The situation for the other $N + 1$ components and for the eigenvectors corresponding to $r_n^{(2)}$ is essentially the same. The first N components of the right eigenvector are [2, Eq. (11)]

$$\varphi_n(k) = e^{-(N+1-k)w}[(-1)^n - e^{(N+1)w}]\sin\frac{k\pi n}{N+1}, \quad k = 1, \dots, N,$$

while the $2N + 1$ components of the left eigenvector are [2, Eq. (12)]

$$\Phi_n(k) = (-1)^n e^{(N+1-k)w}\sin\frac{k\pi n}{N+1},$$

where $w = \frac{1}{2}\ln(\lambda/\mu)$. If we now write x_k for $k/(2(N+1))$ and, for the reasons given earlier in this section, replace w by $\omega/(2(N+1))$, we get

$$\varphi_n(k) = -\varphi_n^{(1)}(x_k), \quad 0 < x_k < \frac{1}{2},$$

and

$$\Phi_n(k) = (-1)^n e^{\omega/2}\Phi_n^{(1)}(x_k), \quad 0 < x_k < 1.$$

Thus, evaluation of the eigenfunctions φ_n and Φ_n at the points $x_k = k/(2(N+1))$ yields, up to multiplying constants, the components of the right and left eigenvectors respectively in [2]. Unlike the situation with eigenvalues, this result does not depend on any assumption that $n \ll N$.

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