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NON-CLASSICAL ORTHOGONAL POLYNOMIALS WITH EVEN WEIGHT FUNCTIONS ON SYMMETRIC INTERVALS

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NON-CLASSICAL ORTHOGONAL POLYNOMIALS WITH EVEN

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be a sequence of real polynomials orthogonal with respect to an even weight function $\rho(x)$ on a symmetric interval $[-a,a]$. If for each n (≥ 0) $\phi_n(x)$ is of degree exactly n and if the coefficient of x^n in $\phi_n(x)$ is chosen to be +1, then the polynomials of the sequence satisfy a three-term recurrence relation of the form (see Appendix A)

$$\left.\begin{aligned} \phi_0(x) &= 1, \\ \phi_1(x) &= x, \\ \phi_{n+1}(x) &= x\phi_n(x) - c_n\phi_{n-1}(x),\ n\geq 1. \end{aligned}\right\} \qquad (1)$$

In the pages which follow, two things are accomplished:

1. a formula is derived which expresses c_n in terms of the moments $\mu_n = \int_{-a}^{a} \rho(x)x^n dx$, $n\geq 0$, for any given even weight function $\rho(x)$ and any given positive a (including infinity if the moments remain finite);

2. for the weight function $\rho(x) = |x|^{\alpha}(1-x^2)^{\beta}$ $(\alpha>-1,\beta>-1)$ on the interval $[-1,1]$, it is shown (i) that

$$c_n = \frac{(n + \alpha\sin^2 \frac{n\pi}{2})(n + 2\beta + \alpha\sin^2 \frac{n\pi}{2})}{(2n - 1 + \alpha + 2\beta)(2n + 1 + \alpha + 2\beta)}, \quad n\geq 1,$$

and (ii) that if $\alpha\neq 0$ the orthogonal polynomials of the sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ are non-classical in the sense that they are not solutions of a second-order ordinary differential equation of the form

$$a_o(x)y'' + a_1(x)y' + \lambda_n y = 0, \quad \left(' \sim \frac{d}{dx}\right)$$

where a_o and a_1 are independent of n and λ_n is independent of x.

CHAPTER II

DERIVATION OF A FORMULA FOR c_n

The object of this chapter is to derive a formula which expresses c_n in terms of the moments

$$\mu_n = \int_{-a}^{a} \rho(x)x^n dx, \quad n\geq 0,$$

for any given even weight function $\rho(x)$ and any given positive a (including infinity if the moments remain finite). As a notational convenience, the symbol $\langle f,g\rangle$ is used to denote the inner product $\int_{-a}^{a} \rho(x)f(x)g(x)dx$.

As indicated in Equation (1), Chapter I, the orthogonal polynomials of the sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ corresponding to the weight function $\rho(x)$ satisfy the recurrence relation

$$\left.\begin{aligned} \phi_0(x) &= 1, \\ \phi_1(x) &= x, \\ \phi_{n+1}(x) &= x\phi_n(x) - c_n\phi_{n-1}(x), \quad n\geq 1. \end{aligned}\right\} \qquad (1)$$

Since for $n\geq 1$

$$0 = \langle\phi_{n+1},\phi_{n-1}\rangle = \langle x\phi_n - c_n\phi_{n-1},\phi_{n-1}\rangle$$

$$= \langle x\phi_n,\phi_{n-1}\rangle - c_n\langle\phi_{n-1},\phi_{n-1}\rangle ,$$

it follows that

$$c_n = \frac{\langle x\phi_n,\phi_{n-1}\rangle}{\langle\phi_{n-1},\phi_{n-1}\rangle} , \ n\geq 1, \tag{2}$$

a result which can be expressed more conveniently after a slight change. Replacing n by n - 1 in the third of Equations (1), multiplying the result by $\rho(x)\phi_n(x)$, and integrating from -a to a shows that

$$\langle\phi_n,\phi_n\rangle = \langle x\phi_{n-1},\phi_n\rangle - c_n\langle\phi_{n-2},\phi_n\rangle$$

$$= \langle x\phi_n,\phi_{n-1}\rangle, \quad n\geq 2;$$

and by direct verification for n = 1

$$\langle\phi_1,\phi_1\rangle = \langle x,x\rangle = \langle x^2,1\rangle = \langle x\phi_1,\phi_o\rangle.$$

Thus

$$\langle x\phi_n,\phi_{n-1}\rangle = \langle\phi_n,\phi_n\rangle, \quad n\geq 1. \tag{3}$$

Substituting (3) in (2) yields

$$c_n = \frac{\langle\phi_n,\phi_n\rangle}{\langle\phi_{n-1},\phi_{n-1}\rangle}, \quad n\geq 1. \tag{4}$$

The next few paragraphs explain how to compute $\langle\phi_n,\phi_n\rangle$.

In Appendix B it is shown that $\phi_n(x)$ can be represented in the following forms according as n is even or odd (vertical bars denote a determinant): if n is even and $n\geq 2$ (say, $n = 2k$, where $k\geq 1$),

$$\phi_{2k}(x) = K_{2k}\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\ 1 & x & \cdots & x^{2k} \end{vmatrix}, \tag{5}$$

where $K_{2k} =$

$$\frac{1}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}},$$

while if n is odd and $n\geq 3$ (say, $n = 2k + 1$, where $k\geq 1$),

$$\phi_{2k+1}(x) = K_{2k+1} \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \\ x & x^3 & \cdots & x^{2k+1} \end{vmatrix}, \tag{6}$$

where $K_{2k+1} = \dfrac{1}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}$.

The computation of $<\phi_n,\phi_n>$ when n is even is examined first. Thus the immediate problem is to evaluate

$$<\phi_{2k},\phi_{2k}> = \int_{-a}^{a} \phi_{2k}^2(x)\rho(x)dx, \quad k\geq 1.$$

In order to facilitate the evaluation of this integral, the integrand $\phi_{2k}^2(x)\rho(x)$ may be written as a determinant in which the variable x appears only in the last row. The steps in the transformation of $\phi_{2k}^2(x)\rho(x)$ into this form are now described.

Square $\phi_{2k}(x)$ by multiplying the determinant (5) by its transpose (here understood to mean the determinant obtained from (5) by interchanging rows and columns). In the resulting expression for $\phi_{2k}^2(x)$ the

elements involving x appear only in the last row and the last column.
Thus $\phi_{2k}^2(x) = K_{2k}^2 \times$

$$\begin{vmatrix} \sum_{j=0}^{k} \mu_{2j}^2 & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \cdots & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k} \mu_{2j}x^{2j} \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \sum_{j=0}^{k} \mu_{2j+2}^2 & \cdots & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \sum_{j=0}^{k} \mu_{2j+2}x^{2j} \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}^2 & \sum_{j=0}^{k} \mu_{2j+2k-2}x^{2j} \\ \sum_{j=0}^{k} \mu_{2j}x^{2j} & \sum_{j=0}^{k} \mu_{2j+2}x^{2j} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}x^{2j} & \sum_{j=0}^{k} x^{4j} \end{vmatrix} . \tag{7}$$

Now write (7) as a sum of k+1 determinants the pth of which ($1 \le p \le k + 1$) has the following properties: (i) its first k columns are identical with those of (7); (ii) its (k+1)st column consists of the pth addends in the (k+1)st column of (7). Then $\phi_{2k}^2(x) = K_{2k}^2 \sum_{m=0}^{k}$

$$\begin{vmatrix} \sum_{j=0}^{k} \mu_{2j}^2 & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \cdots & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \mu_{2m}x^{2m} \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \sum_{j=0}^{k} \mu_{2j+2}^2 & \cdots & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \mu_{2m+2}x^{2m} \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}^2 & \mu_{2m+2k-2}x^{2m} \\ \sum_{j=0}^{k} \mu_{2j}x^{2j} & \sum_{j=0}^{k} \mu_{2j+2}x^{2j} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}x^{2j} & x^{4m} \end{vmatrix} . \tag{8}$$

In each determinant which is an addend of the sum in (8), factor x^{2m} from the last column and multiply the last row by $\rho(x)x^{2m}$ to obtain

$$\rho(x)\phi_{2k}^2(x) = K_{2k}^2 \sum_{m=0}^{k} \tag{9}$$

$$\begin{vmatrix} \sum_{j=0}^{k} \mu_{2j}^2 & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \cdots & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \mu_{2m} \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \sum_{j=0}^{k} \mu_{2j+2}^2 & \cdots & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \mu_{2m+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}^2 & \mu_{2m+2k-2} \\ \sum_{j=0}^{k} \mu_{2j}x^{2j+2m}\rho(x) & \sum_{j=0}^{k} \mu_{2j+2}x^{2j+2m}\rho(x) & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}x^{2j+2m}\rho(x) & x^{4m}\rho(x) \end{vmatrix}.$$

Since all the entries that contain x now appear in the last row, it is possible to integrate $\rho(x)\phi_{2k}^2(x)$ from -a to a by performing the integration only on the last row. Thus (recall that $\int_{-a}^{a} \rho(x)x^n dx = \mu_n$), $\int_{-a}^{a}\rho(x)\phi_{2k}^2(x)dx = K_{2k}^2 \sum_{m=0}^{k}$

$$\begin{vmatrix} \sum_{j=0}^{k} \mu_{2j}^2 & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \cdots & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \mu_{2m} \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \sum_{j=0}^{k} \mu_{2j+2}^2 & \cdots & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \mu_{2m+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}^2 & \mu_{2m+2k-2} \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2m} & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2m} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}\mu_{2j+2m} & \mu_{4m} \end{vmatrix}, \quad k\geq 1.$$

Observe that for m = 0,1,...,k-1, the determinant which is the mth addend in the sum is zero because the (m+1)st row is identical to the last row. So $\int_{-a}^{a} \rho(x)\phi_{2k}^2(x)dx = \langle\phi_{2k},\phi_{2k}\rangle =$

$$K_{2k}^2 \begin{vmatrix} \sum_{j=0}^{k} \mu_{2j}^2 & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \cdots & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \mu_{2k} \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \sum_{j=0}^{k} \mu_{2j+2}^2 & \cdots & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \mu_{2k+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}^2 & \mu_{4k-2} \\ \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k} & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}\mu_{2j+2k} & \mu_{4k} \end{vmatrix}$$

or, upon writing the determinant as a product of two determinants,

$$\langle\phi_{2k},\phi_{2k}\rangle = K_{2k}^2 \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} & 0 \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} & 0 \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} & 1 \end{vmatrix}.$$

Since the second determinant is simply $\frac{1}{K_{2k}}$,

$$\langle\phi_{2k},\phi_{2k}\rangle = K_{2k}\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}, \quad k\geq 1. \tag{10}$$

By a similar argument, when n is odd (say, n = 2k+1, where k≥1),

$$\langle\phi_{2k+1},\phi_{2k+1}\rangle = K_{2k+1}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}. \tag{11}$$

Now that $\langle\phi_{2k},\phi_{2k}\rangle$ and $\langle\phi_{2k+1},\phi_{2k+1}\rangle$ have been evaluated, it is possible to calculate c_n from (4). For n even (let n = 2k, where k≥2)

$$c_{2k} = \frac{\langle\phi_{2k},\phi_{2k}\rangle}{\langle\phi_{2k-1},\phi_{2k-1}\rangle} = \frac{K_{2k}\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}}{K_{2k-1}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}.$$

Thus,

$$c_{2k} = \frac{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k-2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6} \end{vmatrix} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix} \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}, \; k\geq 2. \qquad (12)$$

For n odd (let $n = 2k+1$, where $k\geq 1$)

$$c_{2k+1} = \frac{\langle\phi_{2k+1},\phi_{2k+1}\rangle}{\langle\phi_{2k},\phi_{2k}\rangle} = \frac{K_{2k+1}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}}{K_{2k}\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}} .$$

Thus,

$$c_{2k+1} = \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix} \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}}, \quad k\geq 1. \quad (13)$$

Since formulas (12) and (13) apply only for $k\geq 2$ and $k\geq 1$, respectively, c_1 and c_2 must be treated separately. By direct computation,

$$c_1 = \frac{\langle\phi_1,\phi_1\rangle}{\langle\phi_0,\phi_0\rangle} = \frac{\langle x,x\rangle}{\langle 1,1\rangle} = \frac{\mu_2}{\mu_0}$$

and

$$c_2 = \frac{\langle\phi_2,\phi_2\rangle}{\langle\phi_1,\phi_1\rangle} = \frac{\langle x\phi_1 - c_1\phi_0,\ x\phi_1 - c_1\phi_0\rangle}{\langle x,x\rangle}$$

$$= \frac{\langle x^2,x^2\rangle - 2c_1\langle x^2,1\rangle + c_1^2\langle 1,1\rangle}{\langle x,x\rangle}$$

$$= \frac{\mu_4 - 2\left(\frac{\mu_2}{\mu_0}\right)\mu_2 + \left(\frac{\mu_2}{\mu_0}\right)^2 \mu_0}{\mu_2} = \frac{\mu_4}{\mu_2} - \frac{\mu_2}{\mu_0} .$$

Thus c_n has been evaluated for all n ($n \geq 1$).

It is possible to give expressions for c_n which are different in form from (12) and (13). Since these alternative expressions are sometimes computationally more convenient, a derivation of them is now given.

From (4)

$$c_{n+1} = \frac{\langle \phi_{n+1}, \phi_{n+1} \rangle}{\langle \phi_n, \phi_n \rangle} .$$

By using the recurrence relation (1) and the identity (3) one finds that

$$c_{n+1} = \frac{\langle x\phi_n - c_n\phi_{n-1}, x\phi_n - c_n\phi_{n-1} \rangle}{\langle \phi_n, \phi_n \rangle}$$

$$= \frac{\langle x\phi_n, x\phi_n \rangle}{\langle \phi_n, \phi_n \rangle} - 2c_n + c_n^2 \cdot \frac{1}{c_n}$$

$$= \frac{\langle x\phi_n, x\phi_n \rangle}{\langle \phi_n, \phi_n \rangle} - c_n.$$

Now define $c_0 = 0$ so that

$$c_{n+1} + c_n = \frac{\langle x\phi_n, x\phi_n \rangle}{\langle \phi_n, \phi_n \rangle} , \quad n \geq 0.$$

Note that

$$c_n + c_{n-1} = \frac{\langle x\phi_{n-1}, x\phi_{n-1}\rangle}{\langle \phi_{n-1}, \phi_{n-1}\rangle},$$

$$c_{n-1} + c_{n-2} = \frac{\langle x\phi_{n-2}, x\phi_{n-2}\rangle}{\langle \phi_{n-2}, \phi_{n-2}\rangle},$$

$$\cdots \qquad \vdots \qquad \vdots$$

$$c_3 + c_2 = \frac{\langle x\phi_2, x\phi_2\rangle}{\langle \phi_2, \phi_2\rangle},$$

$$c_2 + c_1 = \frac{\langle x\phi_1, x\phi_1\rangle}{\langle \phi_1, \phi_1\rangle},$$

$$c_1 + c_0 = \frac{\langle x\phi_0, x\phi_0\rangle}{\langle \phi_0, \phi_0\rangle}.$$

By alternately adding and subtracting the above equations, $(c_n + c_{n-1}) - (c_{n-1} + c_{n-2}) + \cdots + (-1)^n(c_2 + c_1) + (-1)^{n+1}(c_1 + c_0) = \sum_{i=0}^{n-1}(-1)^{n+i+1} \frac{\langle x\phi_i, x\phi_i\rangle}{\langle \phi_i, \phi_i\rangle}$; or

$$c_n = \sum_{i=0}^{n-1} (-1)^{n+i+1} \frac{\langle x\phi_i, x\phi_i\rangle}{\langle \phi_i, \phi_i\rangle}, \quad n\geq 1, \tag{14}$$

since the left-hand side telescopes and $c_o = 0$. Since $\langle\phi_n,\phi_n\rangle$ has already been evaluated [see (10) and (11)], it is only necessary to compute $\langle x\phi_n,x\phi_n\rangle$ to be able to find c_n.

Consider first the case where n is even (let $n = 2k$, where $k \geq 1$). The procedure previously used to calculate $\langle\phi_{2k},\phi_{2k}\rangle$ can be used to calculate $\langle x\phi_{2k}, x\phi_{2k}\rangle$ if each entry in the last row in (9) is multiplied by x^2. Thus $\rho(x)x^2\phi_{2k}^2(x) = K_{2k}^2 \sum_{m=0}^{k}$

$$\begin{vmatrix}
\sum_{j=0}^{k}\mu_{2j}^2 & \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2} & \cdots & \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2k-2} & \mu_{2m} \\
\sum_{j=0}^{k}\mu_{2j}\mu_{2j+2} & \sum_{j=0}^{k}\mu_{2j+2}^2 & \cdots & \sum_{j=0}^{k}\mu_{2j+2}\mu_{2j+2k-2} & \mu_{2m+2} \\
\vdots & \vdots & & \vdots & \vdots \\
\sum_{j=0}^{k}\mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k}\mu_{2j+2}\mu_{2j+2k-2} & \cdots & \sum_{j=0}^{k}\mu_{2j+2k-2}^2 & \mu_{2m+2k-2} \\
\sum_{j=0}^{k}\mu_{2j}x^{2j+2m+2}\rho(x) & \sum_{j=0}^{k}\mu_{2j+2}x^{2j+2m+2}\rho(x) & \cdots & \sum_{j=0}^{k}\mu_{2j+2k-2}x^{2j+2m+2} & x^{4m+2}\rho(x)
\end{vmatrix}.$$

Again all the entries that contain x appear in the last row and $\rho(x)x^2\phi_{2k}^2(x)$ can be integrated from $-a$ to a by performing the integration on the last row only. Consequently, $\langle x\phi_{2k}, x\phi_{2k}\rangle = K_{2k}^2 \sum_{m=0}^{k}$

$$\begin{vmatrix} \sum_{j=0}^{k}\mu_{2j}^2 & \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2} & \cdots & \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2k-2} & \mu_{2m} \\ \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2} & \sum_{j=0}^{k}\mu_{2j+2}^2 & \cdots & \sum_{j=0}^{k}\mu_{2j+2}\mu_{2j+2k-2} & \mu_{2m+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k}\mu_{2j+2}\mu_{2j+2k-2} & \cdots & \sum_{j=0}^{k}\mu_{2j+2k-2}^2 & \mu_{2m+2k-2} \\ \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2m+2} & \sum_{j=0}^{k}\mu_{2j+2}\mu_{2j+2m+2} & \cdots & \sum_{j=0}^{k}\mu_{2j+2k-2}\mu_{2j+2m+2} & \mu_{4m+2} \end{vmatrix} .$$

Notice that for m = 0,1,2,...,k-2, the determinant which is the mth addend in the sum is zero, since the (m+2)nd row is identical to the last row. Hence $\langle x\phi_{2k}, x\phi_{2k}\rangle = K_{2k}^2 \times$

$$\begin{vmatrix} \sum_{j=0}^{k}\mu_{2j}^2 & \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2} & \cdots & \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2k-2} & \mu_{2k-2} \\ \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2} & \sum_{j=0}^{k}\mu_{2j+2}^2 & \cdots & \sum_{j=0}^{k}\mu_{2j+2}\mu_{2j+2k-2} & \mu_{2k} \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k}\mu_{2j+2}\mu_{2j+2k-2} & \cdots & \sum_{j=0}^{k}\mu_{2j+2k-2}^2 & \mu_{4k-4} \\ \sum_{j=0}^{k}\mu_{2j}\mu_{2j+2k} & \sum_{j=0}^{k}\mu_{2j+2}\mu_{2j+2k} & \cdots & \sum_{j=0}^{k}\mu_{2j+2k-2}\mu_{2j+2k} & \mu_{4k-2} \end{vmatrix}$$

$$
+ K_{2k}^2 \begin{vmatrix}
\sum_{j=0}^{k} \mu_{2j}^2 & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \cdots & \sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \mu_{2k} \\
\sum_{j=0}^{k} \mu_{2j}\mu_{2j+2} & \sum_{j=0}^{k} \mu_{2j+2}^2 & \cdots & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \mu_{2k+2} \\
\vdots & \vdots & & \vdots & \vdots \\
\sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k-2} & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k-2} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}^2 & \mu_{4k-2} \\
\sum_{j=0}^{k} \mu_{2j}\mu_{2j+2k+2} & \sum_{j=0}^{k} \mu_{2j+2}\mu_{2j+2k+2} & \cdots & \sum_{j=0}^{k} \mu_{2j+2k-2}\mu_{2j+2k+2} & \mu_{4k+2}
\end{vmatrix} .
$$

But each of the two determinants above is the product of two determinants as shown below. Therefore,

$$
\langle x\phi_{2k}, x\phi_{2k}\rangle = K_{2k}^2 \begin{vmatrix}
\mu_0 & \mu_2 & \cdots & \mu_{2k} \\
\mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\
\vdots & \vdots & & \vdots \\
\mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\
\mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k}
\end{vmatrix}
\begin{vmatrix}
\mu_0 & \mu_2 & \cdots & \mu_{2k-2} & 0 \\
\mu_2 & \mu_4 & \cdots & \mu_{2k} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
\mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-6} & 0 \\
\mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} & 1 \\
\mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} & 0
\end{vmatrix}
$$

$$
+ K_{2k}^2 \begin{vmatrix}
\mu_0 & \mu_2 & \cdots & \mu_{2k} \\
\mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\
\vdots & \vdots & & \vdots \\
\mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\
\mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2}
\end{vmatrix}
\begin{vmatrix}
\mu_0 & \mu_2 & \cdots & \mu_{2k-2} & 0 \\
\mu_2 & \mu_4 & \cdots & \mu_{2k} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
\mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} & 0 \\
\mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} & 1
\end{vmatrix}
$$

$$= K_{2k}^2 \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix} \qquad (15)$$

$$- K_{2k}^2 \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-6} \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}, \quad k\geq 2,$$

where the restriction $k\geq 2$ (instead of $k\geq 1$) has been imposed simply to avoid a possible misinterpretation of the nomenclature.

For the case where n is odd (let $n = 2k+1$, where $k\geq 1$), $\langle x\phi_{2k+1}, x\phi_{2k+1}\rangle$ is found by a similar technique to be

$\langle x\phi_{2k+1}, x\phi_{2k+1}\rangle =$

$$K_{2k+1}^2 \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \\ \mu_{2k+4} & \mu_{2k+6} & \cdots & \mu_{4k+4} \end{vmatrix} \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}$$

$$- K_{2k+1}^2 \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix} \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k} \end{vmatrix}, \quad k \geq 2, \quad (16)$$

where the restriction $k \geq 2$ has again been imposed for the same reason.

Now from (14)

$$c_n = \sum_{i=0}^{n-1} (-1)^{n+i+1} \frac{\langle x\phi_i,\ x\phi_i \rangle}{\langle \phi_i,\ \phi_i \rangle}, \quad n \geq 1.$$

By direct computation, since (15) and (16) are to be used only for $k \geq 2$,

$$\frac{\langle x\phi_0, x\phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{\mu_2}{\mu_0}, \quad (17)$$

$$\frac{\langle x\phi_1, x\phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} = \frac{\mu_4}{\mu_2}, \quad (18)$$

$$\frac{\langle x\phi_2, x\phi_2 \rangle}{\langle \phi_2, \phi_2 \rangle} = -\frac{\mu_2}{\mu_0} + \frac{\begin{vmatrix} \mu_0 & \mu_2 \\ \mu_4 & \mu_6 \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 \\ \mu_2 & \mu_4 \end{vmatrix}}, \quad (19)$$

and

$$\frac{\langle x\phi_3, x\phi_3\rangle}{\langle\phi_3,\phi_3\rangle} = -\frac{\mu_4}{\mu_2} + \frac{\begin{vmatrix}\mu_2 & \mu_4\\ \mu_6 & \mu_8\end{vmatrix}}{\begin{vmatrix}\mu_2 & \mu_4\\ \mu_4 & \mu_6\end{vmatrix}} . \tag{20}$$

From (10) and (15), for k≥2, $\frac{\langle x\phi_{2k}, x\phi_{2k}\rangle}{\langle\phi_{2k},\phi_{2k}\rangle} =$

$$\frac{K_{2k}^2\begin{vmatrix}\mu_0 & \mu_2 & \cdots & \mu_{2k}\\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2}\\ \vdots & \vdots & & \vdots\\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2}\\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2}\end{vmatrix}\begin{vmatrix}\mu_0 & \mu_2 & \cdots & \mu_{2k-2}\\ \mu_2 & \mu_4 & \cdots & \mu_{2k}\\ \vdots & \vdots & & \vdots\\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4}\end{vmatrix}}{K_{2k}^2\begin{vmatrix}\mu_0 & \mu_2 & \cdots & \mu_{2k}\\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2}\\ \vdots & \vdots & & \vdots\\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k}\end{vmatrix}\begin{vmatrix}\mu_0 & \mu_2 & \cdots & \mu_{2k-2}\\ \mu_2 & \mu_4 & \cdots & \mu_{2k}\\ \vdots & \vdots & & \vdots\\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4}\end{vmatrix}}$$

$$
- \frac{K_{2k}^{2}\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-6} \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}{K_{2k}^{2}\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}},
$$

or $\frac{\langle x\phi_{2k}, x\phi_{2k}\rangle}{\langle \phi_{2k}, \phi_{2k}\rangle} =$

$$
\frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}} - \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-6} \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}} . \qquad (21)
$$

From (11) and (16), for $k \geq 2$, $\dfrac{\langle x\phi_{2k+1}, x\phi_{2k+1}\rangle}{\langle \phi_{2k+1}, \phi_{2k+1}\rangle} =$

$$
\frac{K_{2k+1}^{2}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \\ \mu_{2k+4} & \mu_{2k+6} & \cdots & \mu_{4k+4} \end{vmatrix}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}{K_{2k+1}^{2}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}
$$

$$
- \frac{K_{2k+1}^{2}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k} \end{vmatrix}}{K_{2k+1}^{2}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}} .
$$

So $\dfrac{\langle x\phi_{2k+1}, x\phi_{2k+1}\rangle}{\langle\phi_{2k+1},\phi_{2k+1}\rangle} =$

$$\frac{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \\ \mu_{2k+4} & \mu_{2k+6} & \cdots & \mu_{4k+4} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}} - \frac{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}} . \qquad (22)$$

Thus, from (14),

$$c_{2k} = -\frac{\langle x\phi_o, x\phi_o\rangle}{\langle\phi_o,\phi_o\rangle} + \frac{\langle x\phi_1, x\phi_1\rangle}{\langle\phi_1,\phi_1\rangle} - \frac{\langle x\phi_2, x\phi_2\rangle}{\langle\phi_2,\phi_2\rangle}$$

$$+ \frac{\langle x\phi_3, x\phi_3\rangle}{\langle\phi_3,\phi_3\rangle} - \frac{\langle x\phi_4, x\phi_4\rangle}{\langle\phi_4,\phi_4\rangle} + \frac{\langle x\phi_5, x\phi_5\rangle}{\langle\phi_5,\phi_5\rangle}$$

$$- \cdots - \frac{\langle x\phi_{2k-4}, x\phi_{2k-4}\rangle}{\langle\phi_{2k-4},\phi_{2k-4}\rangle} + \frac{\langle x\phi_{2k-3}, x\phi_{2k-3}\rangle}{\langle\phi_{2k-3},\phi_{2k-3}\rangle}$$

$$- \frac{\langle x\phi_{2k-2}, x\phi_{2k-2}\rangle}{\langle\phi_{2k-2},\phi_{2k-2}\rangle} + \frac{\langle x\phi_{2k-1}, x\phi_{2k-1}\rangle}{\langle\phi_{2k-1},\phi_{2k-1}\rangle}$$

$$= -\frac{\mu_2}{\mu_0} + \frac{\mu_4}{\mu_2} + \frac{\mu_2}{\mu_0} - \frac{\begin{vmatrix} \mu_0 & \mu_2 \\ \mu_4 & \mu_6 \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 \\ \mu_2 & \mu_4 \end{vmatrix}} - \frac{\mu_4}{\mu_2} + \frac{\begin{vmatrix} \mu_2 & \mu_4 \\ \mu_6 & \mu_8 \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{vmatrix}} - \frac{\begin{vmatrix} \mu_0 & \mu_2 & \mu_4 \\ \mu_2 & \mu_4 & \mu_6 \\ \mu_6 & \mu_8 & \mu_{10} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \mu_4 \\ \mu_2 & \mu_4 & \mu_6 \\ \mu_4 & \mu_6 & \mu_8 \end{vmatrix}}$$

$$+ \frac{\begin{vmatrix} \mu_0 & \mu_2 \\ \mu_4 & \mu_6 \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 \\ \mu_2 & \mu_4 \end{vmatrix}} + \frac{\begin{vmatrix} \mu_2 & \mu_4 & \mu_6 \\ \mu_4 & \mu_6 & \mu_8 \\ \mu_8 & \mu_{10} & \mu_{12} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 & \mu_6 \\ \mu_4 & \mu_6 & \mu_8 \\ \mu_6 & \mu_8 & \mu_{10} \end{vmatrix}} - \frac{\begin{vmatrix} \mu_2 & \mu_4 \\ \mu_6 & \mu_8 \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{vmatrix}} - \cdot \cdot \cdot$$

$$- \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-4} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-6} & \mu_{2k-4} & \cdots & \mu_{4k-10} \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-4} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-8} \end{vmatrix}} + \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-6} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-8} & \mu_{2k-6} & \cdots & \mu_{4k-14} \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-10} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-6} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-12} \end{vmatrix}}$$

$$+\frac{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k-2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-8} \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-4} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k-2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6} \end{vmatrix}} - \frac{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k-4} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k-2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-6} & \mu_{2k-4} & \cdots & \mu_{4k-12} \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-8} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k-4} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k-2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-10} \end{vmatrix}}$$

$$-\frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-6} \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}} + \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-4} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-6} & \mu_{2k-4} & \cdots & \mu_{4k-10} \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-4} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-8} \end{vmatrix}}$$

$$+\frac{\begin{vmatrix}\mu_2 & \mu_4 & \cdots & \mu_{2k}\\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2}\\ \vdots & \vdots & & \vdots\\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4}\\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k}\end{vmatrix}}{\begin{vmatrix}\mu_2 & \mu_4 & \cdots & \mu_{2k}\\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2}\\ \vdots & \vdots & & \vdots\\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2}\end{vmatrix}} - \frac{\begin{vmatrix}\mu_2 & \mu_4 & \cdots & \mu_{2k-2}\\ \mu_4 & \mu_6 & \cdots & \mu_{2k}\\ \vdots & \vdots & & \vdots\\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-8}\\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-4}\end{vmatrix}}{\begin{vmatrix}\mu_2 & \mu_4 & \cdots & \mu_{2k-2}\\ \mu_4 & \mu_6 & \cdots & \mu_{2k}\\ \vdots & \vdots & & \vdots\\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6}\end{vmatrix}}.$$

Finally, after a great deal of additive cancellation,

$$c_{2k} = \frac{\begin{vmatrix}\mu_2 & \mu_4 & \cdots & \mu_{2k}\\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2}\\ \vdots & \vdots & & \vdots\\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4}\\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k}\end{vmatrix}}{\begin{vmatrix}\mu_2 & \mu_4 & \cdots & \mu_{2k}\\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2}\\ \vdots & \vdots & & \vdots\\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2}\end{vmatrix}} - \frac{\begin{vmatrix}\mu_0 & \mu_2 & \cdots & \mu_{2k-2}\\ \mu_2 & \mu_4 & \cdots & \mu_{2k}\\ \vdots & \vdots & & \vdots\\ \mu_{2k-4} & \mu_{2k-2} & \cdots & \mu_{4k-6}\\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2}\end{vmatrix}}{\begin{vmatrix}\mu_0 & \mu_2 & \cdots & \mu_{2k-2}\\ \mu_2 & \mu_4 & \cdots & \mu_{2k}\\ \vdots & \vdots & & \vdots\\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4}\end{vmatrix}}, \quad k\geq 2. \qquad (23)$$

Similarly,

$$c_{2k+1} = \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}} - \frac{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}, \quad k \geq 2. \qquad (24)$$

Since (23) and (24) have been restricted to $k \geq 2$ for notational clarity, c_1, c_2, and c_3 must be computed separately [see (14) and (17) - (20)].

As mentioned previously, the expressions (23) and (24) for c_n, which are alternatives to (12) and (13), may have certain computational advantages--particularly if numerical approximations to the c_n's are wanted. Suppose that the determinants in the numerators in the four quotients that appear in (23) and (24) are replaced by their transposes. The three of these quotients which are composed of determinants of order k may then be interpreted as the expressions obtained by Cramer's Rule for the kth component of the solution to a system of k nonhomogeneous linear equations in k unknowns. A similar statement may be made about the quotient in (24) involving determinants of order k+1. Thus, since the evaluation of c_n from (23) or (24) reduces to operations which are equivalent to finding the kth or (k+1)st component of the solution of a system of k or k+1 linear equations, use of (23) and (24) makes it possible to take advantage of the variety of powerful computational techniques which have been developed for this purpose.

Perhaps as an afterthought, one may remark that the expressions for c_{2k} given by (12) and (23) are identical and that the expressions for c_{2k+1} given by (13) and (24) are identical. The equality of these expressions thus serves to verify and to generalize some extensional identities obtained by Aitken [1] by pivotal condensation.

CHAPTER III

AN APPLICATION OF THE FORMULA FOR c_n

The goal of this chapter is to show that for the two-parameter family of even weight functions

$$\rho(x) = |x|^{\alpha}(1-x^2)^{\beta}, \ \alpha > -1, \ \beta > -1, \tag{25}$$

on the interval [-1,1] the corresponding expression for c_n is

$$c_n = \frac{\left(n + \alpha\sin^2 \frac{n\pi}{2}\right)\left(n + 2\beta + \alpha\sin^2 \frac{n\pi}{2}\right)}{(2n + 1 + \alpha + 2\beta)(2n - 1 + \alpha + 2\beta)}, \quad n \geq 1.$$

It is also shown that if $\alpha \neq 0$ the polynomials of the sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ are non-classical.

The particular family of weight functions (25) was chosen for two reasons: (i) of many such families which were investigated, it is the only one which leads both to a simple expression for c_n and (for most choices of α and β) to non-classical orthogonal polynomials; (ii) for the special choice $\alpha=0$, it leads to the classical ultraspherical polynomials, and for the special choice $\beta=0$, it leads to non-classical polynomials previously studied by Law [4]--facts which serve as a partial check on the validity of the results.

For the weight function (25), the moments μ_n are

$$\mu_{2k+1} = 0, \qquad k\geq 0,$$

$$\mu_{2k} = \int_{-1}^{1} x^{2k}|x|^{\alpha}(1-x^2)^{\beta}dx$$

$$= 2\int_{0}^{1} x^{\alpha+2k}(1-x^2)^{\beta}dx, \qquad k\geq 0.$$

By using the substitution $x = \cos\theta$, μ_{2k} is found to be

$$\mu_{2k} = 2\int_{0}^{\frac{\pi}{2}} (\cos\theta)^{\alpha+2k}(\sin\theta)^{2\beta+1}d\theta$$

$$= \frac{\Gamma\left(\frac{\alpha+1}{2} + k\right)\Gamma(\beta+1)}{\Gamma\left(\frac{\alpha+1}{2} + k + \beta + 1\right)} = B\left(\frac{\alpha+1}{2} + k,\ \beta + 1\right), \qquad (26)$$

where Γ and B denote the gamma and beta functions. For notational convenience let

$$a = \frac{\alpha + 1}{2}, \qquad b = \frac{\alpha + 1}{2} + \beta. \qquad (27)$$

Then

$$\mu_{2k} = \frac{\Gamma(a + k)\Gamma(\beta + 1)}{\Gamma(b + k + 1)} = \left(\frac{a + k - 1}{b + k}\right) \frac{\Gamma(a + k - 1)\Gamma(\beta + 1)}{\Gamma(b + k)} \tag{28}$$

$$= \frac{a + k - 1}{b + k} \mu_{2k-2}.$$

From Equation (13)

$$c_{2k+1} = \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix} \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k+2} & \mu_{2k+4} & \cdots & \mu_{4k+2} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}}, \ k \geq 1.$$

If (28) is used to express each moment in a given row in terms of the moment which is the first entry in that row, c_{2k+1} can be written as

$$c_{2k+1} = \frac{\left|\begin{array}{cccc|cccc}
\mu_0 & \dfrac{\prod_{i=0}^{0}(a+i)}{\prod_{j=1}^{1}(b+j)}\mu_0 & \cdots & \dfrac{\prod_{i=0}^{k-2}(a+i)}{\prod_{j=1}^{k-1}(b+j)}\mu_0 & \mu_2 & \dfrac{\prod_{i=1}^{1}(a+i)}{\prod_{j=2}^{2}(b+j)}\mu_2 & \cdots & \dfrac{\prod_{i=1}^{k}(a+i)}{\prod_{j=2}^{k+1}(b+j)}\mu_2 \\
\mu_2 & \dfrac{\prod_{i=1}^{1}(a+i)}{\prod_{j=2}^{2}(b+j)}\mu_2 & \cdots & \dfrac{\prod_{i=1}^{k-1}(a+i)}{\prod_{j=2}^{k}(b+j)}\mu_2 & \mu_4 & \dfrac{\prod_{i=2}^{2}(a+i)}{\prod_{j=3}^{3}(b+j)}\mu_4 & \cdots & \dfrac{\prod_{i=2}^{k+1}(a+i)}{\prod_{j=3}^{k+2}(b+j)}\mu_4 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\mu_{2k-2} & \dfrac{\prod_{i=k-1}^{k-1}(a+i)}{\prod_{j=k}^{k}(b+j)}\mu_{2k-2} & \cdots & \dfrac{\prod_{i=k-1}^{2k-3}(a+i)}{\prod_{j=k}^{2k-2}(b+j)}\mu_{2k-2} & \mu_{2k+2} & \dfrac{\prod_{i=k+1}^{k+1}(a+i)}{\prod_{j=k+2}^{k+2}(b+j)}\mu_{2k+2} & \cdots & \dfrac{\prod_{i=k+1}^{2k}(a+i)}{\prod_{j=k+2}^{2k+1}(b+j)}\mu_{2k+2}
\end{array}\right|}{\left|\begin{array}{cccc|cccc}
\mu_2 & \dfrac{\prod_{i=1}^{1}(a+i)}{\prod_{j=2}^{2}(b+j)}\mu_2 & \cdots & \dfrac{\prod_{i=1}^{k-1}(a+i)}{\prod_{j=2}^{k}(b+j)}\mu_2 & \mu_0 & \dfrac{\prod_{i=0}^{0}(a+i)}{\prod_{j=1}^{1}(b+j)}\mu_0 & \cdots & \dfrac{\prod_{i=0}^{k-1}(a+i)}{\prod_{j=1}^{k}(b+j)}\mu_0 \\
\mu_4 & \dfrac{\prod_{i=2}^{2}(a+i)}{\prod_{j=3}^{3}(b+j)}\mu_4 & \cdots & \dfrac{\prod_{i=2}^{k}(a+i)}{\prod_{j=3}^{k+1}(b+j)}\mu_4 & \mu_2 & \dfrac{\prod_{i=1}^{1}(a+i)}{\prod_{j=2}^{2}(b+j)}\mu_2 & \cdots & \dfrac{\prod_{i=1}^{k}(a+i)}{\prod_{j=2}^{k+1}(b+j)}\mu_2 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\mu_{2k} & \dfrac{\prod_{i=k}^{k}(a+i)}{\prod_{j=k+1}^{k+1}(b+j)}\mu_{2k} & \cdots & \dfrac{\prod_{i=k}^{2k-2}(a+i)}{\prod_{j=k+1}^{2k-1}(b+j)}\mu_{2k} & \mu_{2k} & \dfrac{\prod_{i=k}^{k}(a+i)}{\prod_{j=k+1}^{k+1}(b+j)}\mu_{2k} & \cdots & \dfrac{\prod_{i=k}^{2k-1}(a+i)}{\prod_{j=k+1}^{2k}(b+j)}\mu_{2k}
\end{array}\right|}.$$

For each row

(i) factor out the common moment;

(ii) multiply each row by the product necessary to eliminate all denominators in that row; and

(iii) use (28) to eliminate the moments that remain.

Then $c_{2k+1} = \frac{(a+k)(b+k)}{(b+2k)(b+2k+1)} \times$

$$\frac{\begin{vmatrix} \prod_{j=1}^{k-1}(b+j) & \prod_{i=0}^{0}(a+i)\prod_{j=2}^{k-1}(b+j) & \cdots & \prod_{i=0}^{k-2}(a+i) \\ \prod_{j=2}^{k}(b+j) & \prod_{i=1}^{1}(a+i)\prod_{j=3}^{k}(b+j) & \cdots & \prod_{i=1}^{k-1}(a+i) \\ \vdots & \vdots & & \vdots \\ \prod_{j=k}^{2k-2}(b+j) & \prod_{i=k-1}^{k-1}(a+i)\prod_{j=k+1}^{2k-2}(b+j) & \cdots & \prod_{i=k-1}^{2k-3}(a+i) \end{vmatrix} \begin{vmatrix} \prod_{j=2}^{k+1}(b+j) & \prod_{i=1}^{1}(a+i)\prod_{j=3}^{k+1}(b+j) & \cdots & \text{[illegible]} \\ \prod_{j=3}^{k+2}(b+j) & \prod_{i=2}^{2}(a+i)\prod_{j=4}^{k+2}(b+j) & \cdots & \text{[illegible]} \\ \vdots & \vdots & & \\ \prod_{j=k+2}^{2k+1}(b+j) & \prod_{i=k+1}^{k+1}(a+i)\prod_{j=k+3}^{2k+3}(b+j) & \cdots & \text{[illegible]} \end{vmatrix}}{\begin{vmatrix} \prod_{j=2}^{k}(b+j) & \prod_{i=1}^{1}(a+i)\prod_{j=3}^{k}(b+j) & \cdots & \prod_{i=1}^{k-1}(a+i) \\ \prod_{j=3}^{k+1}(b+j) & \prod_{i=2}^{2}(a+i)\prod_{j=4}^{k+1}(b+j) & \cdots & \prod_{i=2}^{k}(a+i) \\ \vdots & \vdots & & \vdots \\ \prod_{j=k+1}^{2k-1}(b+j) & \prod_{i=k}^{k}(a+i)\prod_{j=k+2}^{2k-1}(b+j) & \cdots & \prod_{i=k}^{2k-2}(a+i) \end{vmatrix} \begin{vmatrix} \prod_{j=1}^{k}(b+j) & \prod_{i=0}^{0}(a+i)\prod_{j=2}^{k}(b+j) & \cdots & \text{[illegible]} \\ \prod_{j=2}^{k+1}(b+j) & \prod_{i=1}^{1}(a+i)\prod_{j=3}^{k+1}(b+j) & \cdots & \text{[illegible]} \\ \vdots & \vdots & & \\ \prod_{j=k+1}^{2k}(b+j) & \prod_{i=k}^{k}(a+i)\prod_{j=k+2}^{2k}(b+j) & \cdots & \text{[illegible]} \end{vmatrix}}$$

In Appendix C it is shown that if

$$f_k(x,y) = \begin{vmatrix} \prod_{j=1}^{k-1}(y+j) & \prod_{i=0}^{0}(x+i)\prod_{j=2}^{k-1}(y+j) & \cdots & \prod_{i=0}^{k-2}(x+i) \\ \prod_{j=2}^{k}(y+j) & \prod_{i=1}^{1}(x+i)\prod_{j=3}^{k}(y+j) & \cdots & \prod_{i=1}^{k-1}(x+i) \\ \vdots & \vdots & & \vdots \\ \prod_{j=k}^{2k-2}(y+j) & \prod_{i=k-1}^{k-1}(x+i)\prod_{j=k+1}^{2k-2}(y+j) & \cdots & \prod_{i=k-1}^{2k-3}(x+i) \end{vmatrix}, \quad (29)$$

then $f_k(x,y) = \prod_{i=1}^{k-1} i!(y-x+i)^{k-i}$. Hence,

$$c_{2k+1} = \frac{(a+k)(b+k)}{(b+2k)(b+2k+1)} \cdot \frac{f_k(a,b) \cdot f_{k+1}(a+1,b+1)}{f_k(a+1,b+1)f_{k+1}(a,b)} . \quad (30)$$

But $f_k(x,y) = f_k(x+1,y+1)$ for all k, since $f_k(x,y)$ is a function of $y - x$. Therefore,

$$c_{2k+1} = \frac{(a+k)(b+k)}{(b+2k)(b+2k+1)}, \quad k\geq 1. \quad (31)$$

From (12)

$$c_{2k} = \frac{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k-2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6} \end{vmatrix} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix} \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}, \quad k \geq 2.$$

By using the same technique which was used to compute c_{2k+1}, one finds that

$$c_{2k} = \frac{1}{(b+2k-1)(b+2k)} \times$$

$$\frac{\begin{vmatrix} \prod_{j=2}^{k-1}(b+j) & \prod_{i=1}^{1}(a+i)\prod_{j=3}^{k-1}(b+j) & \cdots & \prod_{i=1}^{k-2}(a+i) \\ \prod_{j=3}^{k}(b+j) & \prod_{i=2}^{2}(a+i)\prod_{j=4}^{k}(b+j) & \cdots & \prod_{i=2}^{k-1}(a+i) \\ \vdots & \vdots & & \vdots \\ \prod_{j=k}^{2k-3}(b+j) & \prod_{i=k-1}^{k-1}(a+i)\prod_{j=k+1}^{2k-3}(b+j) & \cdots & \prod_{i=k}^{2k-4}(a+i) \end{vmatrix} \begin{vmatrix} \prod_{j=1}^{k}(b+j) & \prod_{i=0}^{0}(a+i)\prod_{j=2}^{k}(b+j) & \cdots & \prod_{i=0}^{k-1}(a+i) \\ \prod_{j=2}^{k+1}(b+j) & \prod_{i=1}^{1}(a+i)\prod_{j=3}^{k+1}(b+j) & \cdots & \prod_{i=1}^{k}(a+i) \\ \vdots & \vdots & & \vdots \\ \prod_{j=k+1}^{2k}(b+j) & \prod_{i=k}^{k}(a+i)\prod_{j=k+2}^{2k}(b+j) & & \prod_{i=k}^{2k-1}(a+i) \end{vmatrix}}{\begin{vmatrix} \prod_{j=1}^{k-1}(b+j) & \prod_{i=0}^{0}(a+i)\prod_{j=2}^{k-1}(b+j) & \cdots & \prod_{i=0}^{k-2}(a+i) \\ \prod_{j=2}^{k}(b+j) & \prod_{i=1}^{1}(a+i)\prod_{j=3}^{k}(b+j) & \cdots & \prod_{i=1}^{k-1}(a+i) \\ \vdots & \vdots & & \vdots \\ \prod_{j=k}^{2k-2}(b+j) & \prod_{i=k-1}^{k-1}(a+i)\prod_{j=k+1}^{2k-2}(b+j) & \cdots & \prod_{i=k-1}^{2k-3}(a+i) \end{vmatrix} \begin{vmatrix} \prod_{j=2}^{k}(b+j) & \prod_{i=1}^{1}(a+i)\prod_{j=3}^{k}(b+j) & \cdots & \prod_{i=1}^{k-1}(a+i) \\ \prod_{j=3}^{k+1}(b+j) & \prod_{i=2}^{2}(a+i)\prod_{j=4}^{k+1}(b+j) & \cdots & \prod_{i=2}^{k}(a+i) \\ \vdots & \vdots & & \vdots \\ \prod_{j=k+1}^{2k-1}(b+j) & \prod_{i=k}^{k}(a+i)\prod_{j=k+2}^{2k-1}(b+j) & \cdots & \prod_{i=k}^{2k-2}(a+i) \end{vmatrix}}.$$

From (29)

$$c_{2k} = \frac{1}{(b+2k-1)(b+2k)} \cdot \frac{f_{k-1}(a+1,b+1)f_{k+1}(a,b)}{f_k(a,b)f_k(a+1,b+1)} .$$

By use of the identity (30)

$$c_{2k} = \frac{1}{(b+2k-1)(b+2k)} \frac{\prod_{i=1}^{k-2} i!(b-a+i)^{k-1-i} \prod_{i=1}^{k} i!(b-a+i)^{k+1-i}}{\prod_{i=1}^{k-1} i!(b-a+i)^{k-i} \prod_{i=1}^{k-1} i!(b-a+i)^{k-i}} .$$

After common factors in the several products are canceled and the fact that $b-a = \beta$ is used,

$$c_{2k} = \frac{k(k+\beta)}{(b+2k-1)(b+2k)} , \quad k\geq 2. \tag{32}$$

By direct computation $c_1 = \frac{a}{b + 1}$ and $c_2 = \frac{\beta + 1}{(b+1)(b+2)}$.
If a and b are replaced by their original values (see (27)) and if (31) and (32) are combined,

$$c_n = \frac{\left(n + \alpha\sin^2 \frac{n\pi}{2}\right)\left(n + 2\beta + \alpha\sin^2 \frac{n\pi}{2}\right)}{(\alpha + 2\beta + 2n - 1)(\alpha + 2\beta + 2n + 1)} , \quad n\geq 1.$$

Whether the polynomials of the sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ corresponding to the weight function

$$\rho(x) = |x|^{\alpha}(1-x^2)^{\beta}, \ \alpha > -1, \ \beta > -1, \ -1 \le x \le 1,$$

are non-classical or not remains to be discussed. If $\alpha = \beta = 0$, the $\phi_n(x)$ are simply the Legendre polynomials $P_n(x)$ (except for multiplicative factors which depend on n but not on x). If $\alpha = 0$ but $\beta \neq 0$, they are (again except for multiplicative factors) the ultraspherical polynomials $P_n^{(\beta,\beta)}(x)$. If $\beta = 0$ but $\alpha \neq 0$, they are the polynomials $S_n^{(\alpha)}(x)$ discussed by Law [4] and shown by him to be non-classical. If neither α nor β is zero, applying a test described by Jayne [3] shows that one of the necessary conditions for classicality is not satisfied (in Jayne's notation, $g_2(n) \neq 0$, $n \neq 1$). So, in summary, the polynomials $\phi_n(x)$ are non-classical if and only if $\alpha \neq 0$.

APPENDIX

APPENDIX A

SPECIALIZATION OF THE RECURRENCE RELATION TO EVEN WEIGHT FUNCTIONS AND SYMMETRIC INTERVALS

Any three consecutive members of a sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ of orthogonal polynomials in which $\phi_n(x)$ is of degree exactly n satisfy a three-term recurrence relation [2] of the form

$$\left.\begin{aligned} \phi_o(x) &= 1, \\ \phi_1(x) &= A_o x + B_o, \\ \phi_{n+1}(x) &= (A_n x + B_n)\phi_n(x) - C_n\phi_{n-1}(x), \quad n\geq 1. \end{aligned}\right\} \qquad (A.1)$$

If the coefficient of x^n in $\phi_n(x)$, $n\geq 0$, is required to be +1, it is clear from (A.1) that $A_n = 1$ $(n\geq 0)$; and (A.1) then takes the form

$$\left.\begin{aligned} \phi_o(x) &= 1, \\ \phi_1(x) &= x + B_o, \\ \phi_{n+1}(x) &= (x + B_n)\phi_n(x) - C_n\phi_{n-1}(x), \quad n\geq 1. \end{aligned}\right\} \qquad (A.2)$$

If, in addition, the weight function $\rho(x)$ is even and the interval of orthogonality is symmetric with respect to the origin, then $B_n = 0$ for all $n \ge 0$. To verify this fact, note that because of orthogonality

$$0 = \int_{-a}^{a} \rho(x)\phi_{n+1}(x)\phi_n(x)dx \tag{A.3}$$

for all $n \ge 0$. Replacing ϕ_{n+1} in (A.3) by its equivalent from (A.2) yields

$$0 = \int_{-a}^{a} \rho(x)\left[(x+B_n)\phi_n(x) - C_n\phi_{n-1}(x)\right]\phi_n(x)dx \tag{A.4}$$

$$= \int_{-a}^{a} x\rho(x)\phi_n^2(x)dx + B_n \int_{-a}^{a} \rho(x)\phi_n^2(x)dx, \quad n \ge 0.$$

If $n = 0$,

$$0 = \int_{-a}^{a} x\rho(x)\phi_o^2(x)dx + B_o \int_{-a}^{a} \rho(x)\phi_o^2(x)dx;$$

$$0 = \int_{-a}^{a} x\rho(x)dx + B_o \int_{-a}^{a} \rho(x)dx.$$

But $x\rho(x)$ is odd; so $\int_{-a}^{a} x\rho(x)dx = 0$. And, since $\int_{-a}^{a} \rho(x)dx \ne 0$, $B_o = 0$.

Suppose $B_n = 0$ for $n = 0,1,2,\ldots,k$. It then follows from (A.2) that $\phi_n(x)$ $(0 \le n \le k+1)$ is even or odd in x according as n is an even or odd integer. But now (A.4), with $n = k+1$, implies that $B_{k+1} = 0$, from which it follows by induction that $B_n = 0$ for every n.

Thus (A.2) may be written

$$\phi_o(x) = 1,$$

$$\phi_1(x) = x,$$

$$\phi_{n+1}(x) = x\phi_n(x) - C_n\phi_{n-1}(x), \quad n \geq 1,$$

which is the desired result if the change $C_n = c_n$ is made in order that the nomenclature used here will be consistent with that used in some of the references cited.

APPENDIX B

VERIFICATION THAT $\phi_n(x)$ CAN BE REPRESENTED AS A DETERMINANT

The purpose of this appendix is to show that if $\{\phi_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials as described in Chapter I, the $\phi_n(x)$ can be represented as follows (vertical bars denote a determinant):

if n is even and $n \geq 2$ (say $n = 2k$, where $k \geq 1$),

$$\phi_{2k}(x) = \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\ 1 & x^2 & \cdots & x^{2k} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}},$$

while if n is odd and $n \geq 3$ (say $n = 2k+1$, where $k \geq 1$),

$$\phi_{2k+1}(x) = \frac{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \\ x & x^3 & \cdots & x^{2k+1} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}},$$

where $\mu_n = \int_{-a}^{a} \rho(x)x^n dx$.

Since x^i can be written as a linear combination of ϕ_k $(k = 0,1,2,\ldots,i)$ and since ϕ_i is a linear combination of x^k $(k = 0,1,2,\ldots,i)$, the orthogonality condition

$$\int_{-a}^{a} \rho(x)\phi_i(x)\phi_m(x)dx = 0, \quad 0 \le i \le m-1 \text{ (m fixed, } m \ge 1) \tag{B.1}$$

is equivalent to the condition

$$\int_{-a}^{a} \rho(x)x^i\phi_m(x)dx = 0, \quad 0 \le i \le m-1 \text{ (m fixed, } m \ge 1). \tag{B.2}$$

So if ϕ_m satisfies (B.2), it has the desired orthogonality property (B.1).

Now let N be a fixed but arbitrary positive integer and suppose that ϕ_N is written in the form

$$\phi_N(x) = x^N + \gamma_{N-1}x^{N-1} + \ldots + \gamma_1 x + \gamma_o,$$

where $\gamma_{N-1},\gamma_{N-2},\ldots,\gamma_1,\gamma_o$ are functions of N but not of x. Since ϕ_N has the property (B.2),

$$\int_{-a}^{a} \rho(x)x^{N+i}dx + \gamma_{N-1} \int_{-a}^{a} \rho(x)x^{N+i-1}dx + \ldots \tag{B.3}$$

$$+ \gamma_1 \int_{-a}^{a} \rho(x)x^{i+1}dx + \gamma_o \int_{-a}^{a} \rho(x)x^{i}dx = 0, \quad 0\le i\le N-1.$$

But $\int_{-a}^{a} \rho(x)x^n dx = \mu_n$; so (B.3) becomes

$$\mu_{N+i} + \mu_{N+i-1}\gamma_{N-1} + \mu_{N+i-2}\gamma_{N-2} + \cdots + \mu_{i+1}\gamma_1 + \mu_i\gamma_o = 0, \quad 0\le i\le N-1. \tag{B.4}$$

Since $\rho(x)$ is even and the interval of orthogonality is symmetric about $x = 0$, the moments with odd subscripts are zero. Also since the subscripts on the moments in (B.4) depend on two indices (N and i), it is desirable to consider two cases according as N is even or odd.

Consider first the case where N is even (say, $N = 2k$, where $k\ge 1$). With $N = 2k$ (B.4) can be written as

$$\mu_{2k+i} + \mu_{2k+i-1}\gamma_{2k-1} + \mu_{2k+i-2}\gamma_{2k-2} + \cdots + \mu_{i+1}\gamma_1 + \mu_i\gamma_0 = 0, \qquad 0 \le i \le 2k-1. \tag{B.5}$$

Those equations in (B.5) for which $i = 1,3,\ldots,2k-1$—namely

$$\mu_{2k+i-1}\gamma_{2k-1} + \mu_{2k+i-3}\gamma_{2k-3} + \cdots + \mu_{i+3}\gamma_3 + \mu_{i+1}\gamma_1 = 0, \qquad i=1,3,\ldots,2k-1\text{—}$$

represent a system of k homogeneous equations in the k unknowns γ_i, $i = 1,3,\ldots,2k-1$. If there exists a nontrivial solution to the system, then the value of the determinant of the coefficients must be zero. But by an argument given in [2, page 19], the value of the determinant of coefficients is nonzero. Hence $\gamma_i = 0$ for $i = 1,3,\ldots,2k-1$. Now the remaining equations in (B.5) have the form

$$\mu_{2k+i} + \mu_{2k+i-2}\gamma_{2k-2} + \cdots + \mu_{i+2}\gamma_2 + \mu_i\gamma_0 = 0, \qquad i = 0,2,4,\ldots,2k-2. \tag{B.6}$$

Transposing the leading term on the left-hand side in (B.6) to the right-hand side yields

$$\mu_{2k+i-2}\gamma_{2k-2} + \mu_{2k+i-4}\gamma_{2k-4} + \cdots + \mu_{i+2}\gamma_2 + \mu_i\gamma_0 = -\mu_{2k+i}, \qquad i = 0,2,\ldots,2k-2,$$

which is a system of k nonhomogeneous equations in the k unknowns γ_i, $i = 0,2,\ldots,2k-2$. Since the value of the determinant of coefficients is nonzero [2], the system has a unique solution. By Cramer's Rule the solution is

$$\gamma_o = \frac{\begin{vmatrix} -\mu_{2k} & \mu_2 & \mu_4 & \cdots & \mu_{2k-2} \\ -\mu_{2k+2} & \mu_4 & \mu_6 & \cdots & \mu_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ -\mu_{4k-2} & \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-4} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}} = (-1)^k \frac{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}},$$

$$\gamma_j = \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{j-2} & -\mu_{2k} & \mu_{j+2} & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_j & -\mu_{2k+2} & \mu_{j+4} & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{j+2k-4} & -\mu_{4k-2} & \mu_{j+2k} & \cdots & \mu_{4k-4} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}}$$

$$= (-1)^{k-\frac{j}{2}} \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{j-2} & \mu_{j+2} & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_j & \mu_{j+4} & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{j+2k-4} & \mu_{j+2k} & \cdots & \mu_{4k-4} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}}, \quad j=2,4,\ldots,2k-4,$$

$\gamma_{2k-2} =$

$$\frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-4} & -\mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-2} & -\mu_{2k+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6} & -\mu_{4k-2} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}} = - \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-4} & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-2} & \mu_{2k+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6} & \mu_{4k-2} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}} .$$

Let

$$K_{2k} = \frac{1}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}} . \tag{B.7}$$

When the previous results are substituted into

$$\phi_{2k}(x) = x^{2k} + \gamma_{2k-1}x^{2k-1} + \gamma_{2k-2}x^{2k-2} + \ldots + \gamma_1 x + \gamma_o ,$$

$\phi_{2k}(x) =$

$$x^{2k} - x^{2k-2}K_{2k} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-4} & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-2} & \mu_{2k+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6} & \mu_{4k-2} \end{vmatrix}$$

$$+ \sum_{j=1}^{k-2} (-1)^{k-j} K_{2k} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2j-2} & \mu_{2j+2} & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2j} & \mu_{2j+4} & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{2j+2k-4} & \mu_{2j+2k} & \cdots & \mu_{4k-2} \end{vmatrix} x^{2j}$$

$$+ K_{2k}(-1)^k \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}, \quad k \geq 1.$$

Since

$$K_{2k} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix} = 1,$$

$$\phi_{2k}(x) = K_{2k} x^{2k} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}$$

$$- K_{2k} x^{2k-2} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-4} & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k-2} & \mu_{2k+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-6} & \mu_{4k-2} \end{vmatrix}$$

$$+ \sum_{j=1}^{k-2} (-1)^{k-j} K_{2k} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2j-2} & \mu_{2j+2} & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2j} & \mu_{2j+4} & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{2j+2k-4} & \mu_{2j+2k} & \cdots & \mu_{4k-2} \end{vmatrix} x^{2j}$$

$$+ K_{2k}(-1)^k \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}, \quad k\geq 1. \qquad (B.8)$$

The right-hand side of (B.8) is the expansion of the determinant

$$K_{2k} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\ 1 & x^2 & \cdots & x^{2k} \end{vmatrix}$$

by cofactors of the last row. Thus

$$\phi_{2k}(x) = K_{2k} \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\ 1 & x^2 & \cdots & x^{2k} \end{vmatrix}, \quad k\geq 1,$$

or

$$\phi_{2k}(x) = \frac{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k} \\ \mu_2 & \mu_4 & \cdots & \mu_{4k-2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-2} \\ 1 & x^2 & \cdots & x^{2k} \end{vmatrix}}{\begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2k-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \vdots & \vdots & & \vdots \\ \mu_{2k-2} & \mu_{2k} & \cdots & \mu_{4k-4} \end{vmatrix}}, \quad k \geq 1.$$

By an analogous procedure for N odd and N≥3 (say, N=2k+1, where k≥1),

$$\phi_{2k+1}(x) = K_{2k+1} \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k+2} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+4} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k} \\ x & x^3 & \cdots & x^{2k+1} \end{vmatrix},$$

where

$$K_{2k+1} = \frac{1}{\begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2k} \\ \mu_4 & \mu_6 & \cdots & \mu_{2k+2} \\ \vdots & \vdots & & \vdots \\ \mu_{2k} & \mu_{2k+2} & \cdots & \mu_{4k-2} \end{vmatrix}}.$$

APPENDIX C

VERIFICATION THAT $f_k(x,y)$ IS A FUNCTION OF y-x ONLY

The goal of this appendix is to show that if $f_k(x,y)$ =

$$\begin{vmatrix}
\prod_{j=1}^{k-1}(y+j) & \prod_{i=0}^{0}(x+i)\prod_{j=2}^{k-1}(y+j) & \prod_{i=0}^{1}(x+i)\prod_{j=3}^{k-1}(y+j) & \cdots & \prod_{i=0}^{k-3}(x+i)\prod_{j=k-1}^{k-1}(y+j) & \prod_{i=0}^{k-2}(x+i) \\
\prod_{j=2}^{k}(y+j) & \prod_{i=1}^{k}(x+i)\prod_{j=3}^{k}(y+j) & \prod_{i=1}^{2}(x+i)\prod_{j=4}^{k}(y+j) & \cdots & \prod_{i=1}^{k-2}(x+i)\prod_{j=k}^{k}(y+j) & \prod_{i=1}^{k-1}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\prod_{j=m}^{k+m-2}(y+j) & \prod_{i=m-1}^{m-1}(x+i)\prod_{j=m+1}^{k+m-2}(y+j) & \prod_{i=m-1}^{m}(x+i)\prod_{j=m+2}^{k+m-2}(y+j) & \cdots & \prod_{i=m-1}^{m+k-4}(x+i)\prod_{j=m+k-2}^{m+k-2}(y+j) & \prod_{i=m-1}^{m+k-3}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\prod_{j=k}^{2k-2}(y+j) & \prod_{i=k-1}^{k-1}(x+i)\prod_{j=k+1}^{2k-2}(y+j) & \prod_{i=k-1}^{k}(x+i)\prod_{j=k+2}^{2k-2}(y+j) & \cdots & \prod_{i=k-1}^{2k-4}(x+i)\prod_{j=2k-2}^{2k-2}(y+j) & \prod_{i=k-1}^{2k-3}(x+i)
\end{vmatrix}, \quad (C.1)$$

then $f_k(x,y) = \prod_{i=1}^{k-1} i!(y-x+i)^{k-i}$; hence $f_k(x,y)$ is a function of y-x.

The property of $f_k(x,y)$ that will be used in this verification is the following: in any two adjacent columns the corresponding row entries have (k-2) factors in common and one unlike factor in each. The proof that $f_k(x,y)$ is a function of y-x and that $f_k(x,y) = \prod_{i=1}^{k-1} i!(y-x+i)^{k-i}$ will consist of an explanation of the evaluation of the defining determinant of $f_k(x,y)$.

In the first column of (C.1) rewrite the entries as follows:

$$\prod_{j=m}^{k+m-2}(y+j) = (y-x+1)\prod_{j=m+1}^{k+m-2}(y+j) + \prod_{i=m-1}^{m-1}(x+i)\prod_{j=m-1}^{k+m-2}(y+j), \quad m = 1,2,\ldots,k. \qquad (C.2)$$

Substitute (C.2) into $f_k(x,y)$ and subtract the second column from the first so that $f_k(x,y) =$

$$\left| \begin{array}{ll} (y-x+1) \prod_{j=2}^{k-1} (y+j) & \\ (y-x+1) \prod_{j=3}^{k} (y+j) & \text{(The remaining k-1 columns} \\ \vdots & \text{are the same as in (C.1).)} \\ (y-x+1) \prod_{j=m+1}^{k+m-2} (y+j) & \\ \vdots & \\ (y-x+1) \prod_{j=k+1}^{2k-2} (y+j) & \end{array} \right| .$$

Now rewrite the entries in the second column as

$$\prod_{i=m-1}^{m-1} (x+i) \prod_{j=m+1}^{k+m+2} (y+j) = (y-x+1) \prod_{i=m-1}^{m-1} (x+i) \prod_{j=m+2}^{k+m+2} (y+j) + \prod_{i=m-1}^{m} (x+i) \prod_{j=m+2}^{k+m+2} (y+j)$$

and subtract the third column from the second column, continuing this process for the third through the (k-1)st columns to get (after factoring (y-x+1) from the first (k-1) columns)

$$f_k(x,y) =$$

$$(y-x+1)^{k-1}\begin{vmatrix}
\prod_{j=2}^{k-1}(y+j) & \prod_{i=0}^{0}(x+i)\prod_{j=3}^{k-1}(y+j) & \prod_{i=0}^{1}(x+i)\prod_{j=4}^{k-1}(y+j) & \cdots & \prod_{i=0}^{k-3}(x+i) & \prod_{i=0}^{k-2}(x+i) \\
\prod_{j=3}^{k}(y+j) & \prod_{i=1}^{1}(x+i)\prod_{j=4}^{k}(y+j) & \prod_{i=1}^{2}(x+i)\prod_{j=5}^{k}(y+j) & \cdots & \prod_{i=1}^{k-2}(x+i) & \prod_{i=1}^{k-1}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\prod_{j=m-1}^{k+m-2}(y+j) & \prod_{i=m-1}^{m-1}(x+i)\prod_{j=m+2}^{k+m-2}(y+j) & \prod_{i=m-1}^{m}(x+i)\prod_{j=m+3}^{k+m-2}(y+j) & \cdots & \prod_{i=m-1}^{m+k-4}(x+i) & \prod_{i=m-1}^{m+k-3}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\prod_{j=k+1}^{2k-2}(y+j) & \prod_{i=k-1}^{k-1}(x+i)\prod_{j=k+2}^{2k-2}(y+j) & \prod_{i=k-1}^{k}(x+i)\prod_{j=k+3}^{2k-2}(y+j) & \cdots & \prod_{i=k-1}^{2k-4}(x+i) & \prod_{i=k-1}^{2k-3}(x+i)
\end{vmatrix}.$$

Note that the effect of these operations on (C.1) is (i) to raise the lower index by one in all the products that contain $(y+j)$ factors and (ii) to factor $(y-x+1)^{k-1}$ from the determinant.

This procedure is repeated $(k-1)$ times, the first time as shown on the first $(k-1)$ columns (the common factor being $(y-x+1)$), the second time on the first $(k-2)$ columns (the common factor being $(y-x+2)$), and so on, until at the $(k-1)$th time, the operations are performed on the first column only $\{$the common factor being $(y-x+(k-1))\}$. Thus the final result of these operations is

$$f_k(x,y) = (y-x+1)^{k-1}(y-x+2)^{k-2}(y-x+3)^{k-3}\ldots(y-x+(k-2))^2(y-x+(k-1)) \times$$

$$\begin{vmatrix}
1 & \prod_{i=0}^{0}(x+i) & \prod_{i=0}^{1}(x+i) & \cdots & \prod_{i=0}^{k-3}(x+i) & \prod_{i=0}^{k-2}(x+i) \\
1 & \prod_{i=1}^{1}(x+i) & \prod_{i=1}^{2}(x+i) & \cdots & \prod_{i=1}^{k-2}(x+i) & \prod_{i=1}^{k-1}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & \prod_{i=m-1}^{m-1}(x+i) & \prod_{i=m-1}^{m}(x+i) & \cdots & \prod_{i=m-1}^{m+k-4}(x+i) & \prod_{i=m-1}^{m+k-3}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & \prod_{i=k-1}^{k-1}(x+i) & \prod_{i=k-1}^{k}(x+i) & & \prod_{i=k-1}^{2k-4}(x+i) & \prod_{i=k-1}^{2k-3}(x+i)
\end{vmatrix}. \quad (C.3)$$

It will now be shown that the determinant in (C.3) is independent of x. In this determinant, subtract the mth row from the (m+1)th row in the order m = k-1,k-2,...,1. Thus $f_k(x,y) =$

$$\prod_{i=1}^{k-1} (y-x+i)^{k-i} \times$$

$$\begin{vmatrix}
1 & \prod_{i=0}^{0}(x+i) & \prod_{i=0}^{1}(x+i) & \cdots & \prod_{i=0}^{k-3}(x+i) & \prod_{i=0}^{k-2}(x+i) \\
0 & 1 & 2\prod_{i=1}^{1}(x+i) & \cdots & (k-2)\prod_{i=1}^{k-3}(x+i) & (k-1)\prod_{i=1}^{k-2}(x+i) \\
0 & 1 & 2\prod_{i=2}^{2}(x+i) & \cdots & (k-2)\prod_{i=2}^{k-2}(x+i) & (k-1)\prod_{i=2}^{k-1}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 1 & 2\prod_{i=m-1}^{m-1}(x+i) & \cdots & (k-2)\prod_{i=m-1}^{m+k-5}(x+i) & (k-1)\prod_{i=m-1}^{m+k-4}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 1 & 2\prod_{i=k-1}^{k-1}(x+i) & & (k-2)\prod_{i=k-1}^{2k-5}(x+i) & (k-1)\prod_{i=k-1}^{2k-4}(x+i)
\end{vmatrix}.$$

Again subtract the mth row from the (m+1)th row for m = k-1,k-2,...,2 so that

$$f_k(x,y) = \prod_{i=1}^{k-1} (y-x+i)^{k-i} \times$$

$$\begin{vmatrix}
1 & \prod_{i=0}^{0}(x+i) & \prod_{i=0}^{1}(x+i) & \cdots & \prod_{i=0}^{k-3}(x+i) & \prod_{i=0}^{k-2}(x+i) \\
0 & 1 & 2\prod_{i=1}^{1}(x+i) & \cdots & (k-2)\prod_{i=1}^{k-3}(x+i) & (k-1)\prod_{i=1}^{k-2}(x+i) \\
0 & 0 & 2 & \cdots & (k-2)(k-3)\prod_{i=2}^{k-3}(x+i) & (k-1)(k-2)\prod_{i=2}^{k-2}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 2 & \cdots & (k-2)(k-3)\prod_{i=m-1}^{m+k-6}(x+i) & (k-1)(k-2)\prod_{i=m-1}^{m+k-5}(x+i) \\
0 & 0 & 2 & \cdots & (k-2)(k-3)\prod_{i=k-1}^{2k-6}(x+i) & (k-1)(k-2)\prod_{i=k-1}^{2k-5}(x+i)
\end{vmatrix}$$

Proceed in this manner (k-1) times, where at the ith step the mth row is subtracted from the (m+1)th row for m = k-1,k-2,...,i.

Thus $f_k(x,y) = \prod_{i=1}^{k-1} (y-x+i)^{k-i} \times$

$$\begin{vmatrix}
1 & \prod_{i=0}^{0}(x+i) & \prod_{i=0}^{1}(x+i) & \cdots & \prod_{i=0}^{k-3}(x+i) & \prod_{i=0}^{k-2}(x+i) \\
0 & 1 & 2\prod_{i=1}^{1}(x+i) & \cdots & (k-2)\prod_{i=0}^{k-3}(x+i) & (k-1)\prod_{i=1}^{k-2}(x+i) \\
0 & 0 & 2 & \cdots & (k-2)(k-3)\prod_{i=2}^{k-3}(x+i) & (k-1)(k-2)\prod_{i=2}^{k-2}(x+i) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (k-2)! & (k-1)(k-2)\ldots 2\prod_{i=k-2}^{k-2}(x+i) \\
0 & 0 & 0 & \cdots & 0 & (k-1)!
\end{vmatrix}$$

Hence the value of the determinant is the product $\prod_{i=0}^{k-1} i!$ of the diagonal elements, and $f_k(x,y) = \prod_{i=1}^{k-1} i!\ (y-x+i)^{k-i}$.

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